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Adaptive Quantum Measurements in Future Gravitational-wave Detectors

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1 Introduction and motivation

Contemporary so-called second-generation gravitational-wave detectors, such as Advanced LIGO [1, 2], Advanced VIRGO [3], and LCGT [4], which are under construction now, will be quantum noise limited. Two quantum noises will limit the sensitivity: the phase fluctuations of the light inside the interferometer (shot noise) at high gravitational wave frequencies and the random force created by the amplitude fluctuations of the light (radiation-pressure noise) at low frequencies. For balanced detector the best sensitivity point - where these two noise sources become equal - is known as the Standard Quantum Limit (SQL) [5]. In the linear position meter (the gravitational-wave interferometer is special case of it) the shot noise corresponds to the measurement noise and radiation pressure noise - to the back-action noise. Spectral densities of these noises obey the Heisenberg uncertainty relation [5]. The SQL is not an absolute limit for measurement precision: there are few methods of overcoming it in gravitational-wave detectors. The most well-known examples are Quantum Non-Demolition (QND) measurements and Back-Action Evading (BAE) measurements (see, *e.g.*, [6, 7, 8, 9]). The first method supposes using the Hamiltonian of interaction the test body and measurement device, which commutes with operator of measured quantity [10, 6, 11, 5]. The second method uses the correlation between the measurement noise and the back-action noise.[12, 13, 14, 15, 16, 9]

In planned third generation detectors,like Einstein Telescope gravitational-wave detector [17] using these methods is supposed, that should provide sensitivity better than the SQL. However, both of them require sufficient modification of existing experimental schemes and have some technical difficulties (for example, variational-readout scheme is sensitive to optical losses).

In our work we want to investigate the other approach - adaptive linear measurements. The idea is to change the parameters of the scheme depending on the results of previous measurements. Our aim was to create the effective algorithm for quantum adaptive measurements, check whether we can overcome SQL and compare it to the existing methods.

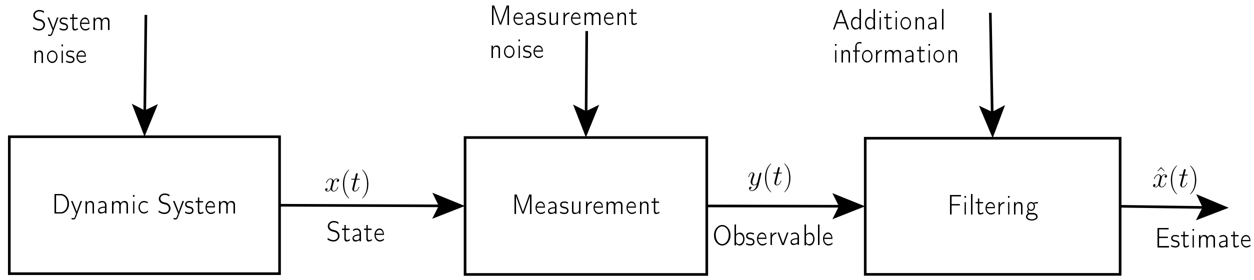


Figure 1: Classical filtering problem: $x(t)$ is values we want to measure, $y(t)$ is measurement result, $\hat{x}(t)$ is estimaiton

2 Adaptive measurement theory

2.1 Conception

The majority of experiments obey the following principal scheme: the "black box" (our system) produces some output and we try to find parameters of this system using just this signal (see Fig.1). There are two contributions to it: dynamic processes in our system and noises acting on it. All the information about values we want to know contains in the signal part of the output and noises only disturb us - obviously, we wish to get rid of the them. This task can be solved by means of filtering theory: we estimate desired parameters from noisy signal, applying so-called filtering operator to the output. The choice of this estimation procedure depends on the particular problem, but the conception stays mostly the same - we choose it to satisfy some criteria, such as minimal mean square error.

Adaptive measurements in general and adaptive filtering in particular involve using this filtering approach[18, 19]. The simple adaptive procedure is: measurement - estimation the parameters - adjust measurement device - new measurement. Many researches apply this idea in classical experiments, but is quantum physics there are only few research groups working on this task[20, 21]. Wiseman and colleagues in made significant contribution to the adaptive filtering and control theory [22, 23, 24, 25]. By the moment there are a few works on adaptive research in GW detectors [26, 27].

In this research we want to apply the same approach to the optomechanical system.

2.2 System

We consider simple optomechanical system of oscillator and homodyne detector (see Fig.2) The oscillator can be described with following equation of motion:

$$\begin{cases} \dot{\hat{x}}(t) &= \frac{\hat{p}(t)}{m}, \\ \dot{\hat{p}}(t) &= -2\gamma\hat{p}(t) - m\omega_m^2\hat{x}(t) + \alpha\hat{a}_1(t), \end{cases} \tag{1}$$

where $\alpha\hat{a}_1(t)$ corresponds to back-action noise, α is coupling constant between the oscillator and optical field. So, for the ingoing optical fields $\hat{a}_{1,2}$ and outgoing $\hat{b}_{1,2}$ (see Fig. 2) we get following input-output relations:

$$\begin{cases} \hat{b}_1(t) &= \hat{a}_1(t), \\ \hat{b}_2(t) &= \hat{a}_2(t) + \frac{\alpha}{\hbar}\hat{x}(t). \end{cases} \tag{2}$$

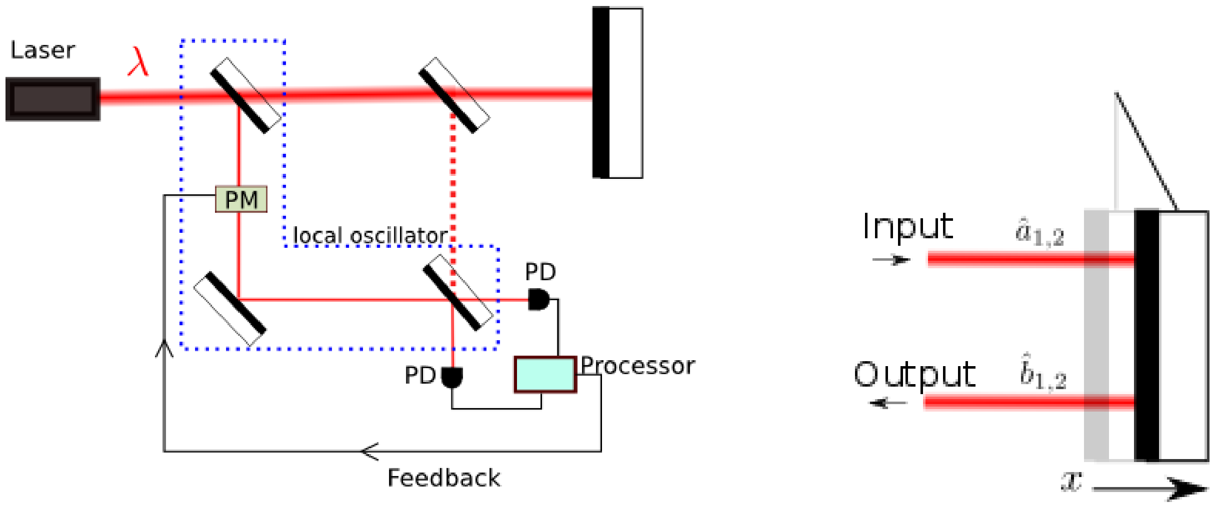


Figure 2: Optomechanical system with laser beam, falling on the oscillator (*right*) and homodyne detector, measuring position of this oscillator(*left*) with feedback

Here we introduced ingoing and outgoing optical fields. In the paraxial and narrowband approximation they are related with strength of electrical field as:

$$\hat{E}(t) = \sqrt{\frac{4\pi\hbar\omega_0}{\mathcal{S}c}}\left[\left(\bar{a} + \hat{a}_1(t)\right)\cos\omega_0 t + \hat{a}_2(t)\sin\omega_0 t\right], \tag{3}$$

where $\bar{a}$ is classical amplitude and $\mathcal{S}$ is effective cross-section area of the laser beam. A similar equation holds for the outgoing fields $\hat{b}_{1,2}$. Other important fact is commutators of the field operators:

$$\left[\hat{a}_1(t), \hat{a}_2(t')\right] = \left[\hat{b}_1(t), \hat{b}_2(t')\right] = i\delta(t - t'). \tag{4}$$

Then we can write the correlation functions:

$$\langle \hat{a}_i(t)\hat{a}_j(t')\rangle_{sym} = \frac{1}{2}\delta_{ij}\delta(t - t'), \quad i, j = 1, 2. \tag{5}$$

In Eq.2 we inroduced $\hat{x}(t)$ as general solution for harmonic oscillator:

$$\hat{x} = \hat{x}(0)\cos\omega_m t + \frac{\hat{p}(0)}{m\omega_m}\sin\omega_m t + \int\limits_0^\infty dt'\, G(t - t')\Theta(t - t')(F(t) + \alpha\hat{a}_1(t')), \tag{6}$$

where $F(t)$ is external force acting on the oscillator and we define Green's function of the oscillator as $G(t) = \frac{\sin\omega_m t}{m\omega_m}$ and $\Theta(t)$ is Heaviside function. In general case the shape of the external force is unknown.

For detection a signal we use homodyne detector with local oscillator's phase changing in time. The output optical field

$$\hat{B}_{out}(t) = \hat{b}_1(t)\cos\omega_0 t + \hat{b}_2(t)\sin\omega_0 t \tag{7}$$

mixes with local oscillator light

$$L(t) = L_c(t)\cos\omega_0 t + L_s(t)\sin\omega_0 t = L_0\cos(\omega_0 t + \zeta(t)). \tag{8}$$

The resulting photocurrent is

$$i(t) = i_1(t) - i_2(t) = \frac{1}{2}\Big(\overline{(L(t)+\hat{B}_{out}(t))^2} - \overline{(L(t)-\hat{B}_{out}(t))^2}\Big) = 2\overline{L(t)\hat{B}_{out}(t)} = \\ = L_0\hat{b}_1(t)\cos\zeta(t) + L_0\hat{b}_2(t)\sin\zeta(t). \tag{9}$$

So, we can rewrite the output as:

$$\hat{y}(t) = \hat{b}_1(t)\cos\zeta(t) + \hat{b}_2(t)\sin\zeta(t), \tag{10}$$

and our main task is search of the optimal homodyne angle $\zeta(t)$ for any moment of time. We can do this using adaptive measurements.

2.3 Adaptive measurements: general approach

In our work we consider simple case of impulse force with unknown amplitude $\bar{F}$, acting on the system at unknown time τ:

$$F(t) = \bar{F}\delta(t-\tau). \tag{11}$$

Then for detected signal we get simple relation (see Eq.10):

$$y(t) = \hat{a}_1(t)\cos\zeta(t) + \Big[\hat{a}_2(t) + \frac{\alpha}{\hbar}\Big(\hat{x}(0)\cos\omega_m t + \frac{\hat{p}(0)}{m\omega_m}\sin\omega_m t + \alpha\int\limits_0^\infty dt'\,G(t-t')\Theta(t-t')\,\hat{a}(t') + \\ + \frac{\bar{F}}{m\omega_m}\sin\omega_m(t-\tau)\Theta(t-\tau)\Big)\Big]\sin\zeta(t), \tag{12}$$

or, in other notations:

$$y(t) = n_1(t)\cos\zeta(t) + n_2(t)\sin\zeta(t) + I(t)\sin\zeta(t) + \frac{\alpha}{\hbar}(A_c\sin\omega_m t - A_s\cos\omega_m t)\sin\zeta(t), \tag{13}$$

where:

$$n_1(t) = \hat{a}_1(t), \tag{14}$$

$$n_2(t) = \hat{a}_2(t) + \frac{\alpha^2}{\hbar}\int\limits_0^\infty dt'\,G(t-t')\,\hat{a}(t')\Theta(t-t'), \tag{15}$$

$$A_c = \frac{\bar{F}}{m\omega_m}\cos\omega_m\tau\,\Theta(t-t'), \tag{16}$$

$$A_s = \frac{\bar{F}}{m\omega_m}\sin\omega_m\tau\,\Theta(t-t'), \tag{17}$$

$$I(t) = \frac{\alpha}{\hbar}(\hat{x}(0)\cos\omega_m t + \frac{\hat{p}(0)}{m\omega_m}\sin\omega_m t). \tag{18}$$

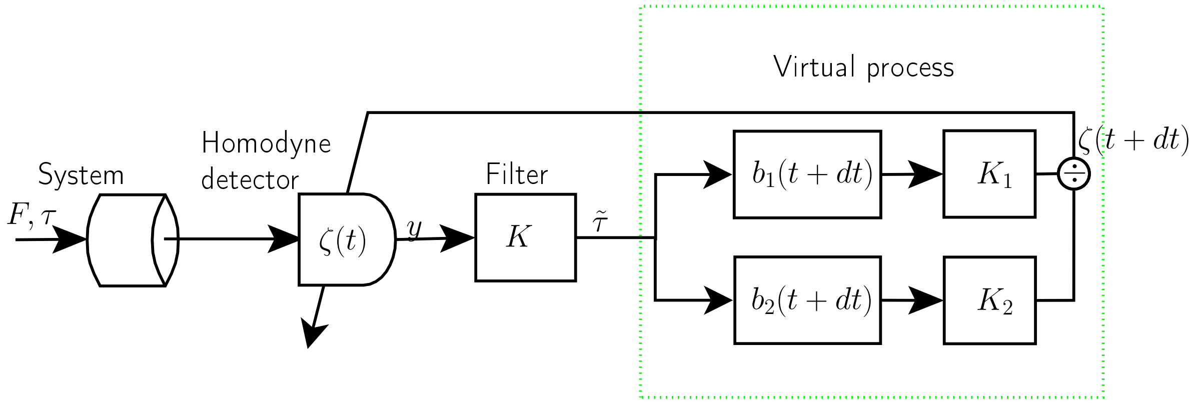


Figure 3: Adaptive measurement scheme: F, τ are parameters we want to estimate; $\zeta(t)$ is adaptively changing homodyne angle; $y(t)$ is result of the measurement; $\tilde{\tau}$ is estimation for arrival time; $b_{1,2}$ are output optical quadratures; $K, K_{1,2}$ are filters

In furter calculations we do not take into account initial state $\hat{x}(0), \hat{p}(0)$ for simplicity.

The important thing is, that, how it is shown in Appendix A, if estimation for two quadratures $A_{c,s}$ is optimal then we can do any mapping to get desired value of force and arrival time and this optimality conserves. That is why we are solving not non-linear task, described by Eq.12, but linear one, described by Eq.13.

The procedure of estimation is similar to described in previous section, but there are some significant differences. Each time step we make two stage procedure (see Fig.3):

1. Estimation of arrival time.

   On this stage we use all previous collected data for estimation the arrival time. At time t we get some output from homodyne detector $y(t)$. Then we can use filter to get some estimation of arrival time $\tilde{\tau}$. This procedure is in some sense classical, because we use here only data that we have already collected for estimation the arrival time (the strict mathematical description we will provide below)

2. Choice of optimal homodyne angle

   In order to understand this stage we should turn to so-called variational measurements [28]. It uses correlation between measurement noise and back-action noise to evade the back-action. It can be shown mathematically (see Appendix C) that in this case there's filtering function that lets us avoid the back-action. The only restriction of this procedure is that we have to know arrival time. The same idea lyes in our procedure.

   We use estimated arrival time to proceed back-action evading measurements. First we predict two quadratures $\hat{b}_{1,2}$ for next two time steps. We have to use two-steps because back-action affects on the system with time delay, so we can choose the optimal homodyne angle $\zeta(t+dt)$ for evading the back-action at time $t+2dt$. This procedure: prediction of quadratures, filtering and calculating the homodyne angle for the next step we do "in our minds" and we assume that it takes no time to do this, so it's like some virtual process (see Fig. 4)

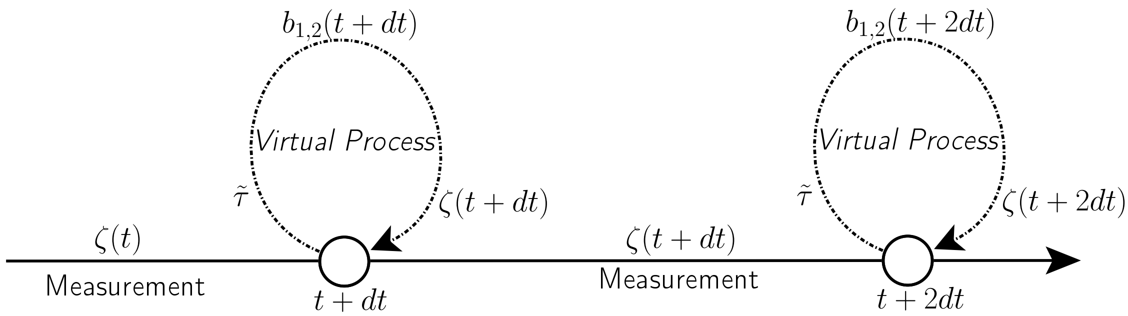


Figure 4: Scheme for understanding the second stage.$\zeta(t)$ is adaptively changing homodyne angle; $\tilde{\tau}$ is estimation for arrival time; $b_{1,2}$ are output optical quadratures

This procedure can be described with a strict mathematics.

2.3.1 First stage

In our case we can write output as

$$y(t) = \mathbb{C}(t)x + n(t) \tag{19}$$

Here:

$$\mathbb{C}(t) = \sin\zeta(t)\begin{bmatrix}\sin\omega_m t, & -\cos\omega_m t\end{bmatrix} \tag{20}$$

$$x = \begin{bmatrix}A_c\\A_s\end{bmatrix} \tag{21}$$

$$n(t) = \begin{bmatrix}\cos\zeta(t), & \sin\zeta(t)\end{bmatrix}\begin{bmatrix}n_1(t)\\n_2(t)\end{bmatrix} \tag{22}$$

We want to estimate two parameters $A_{c,s}$, using the filtering approach, as we mentioned before. In Appendix B we show, how to get filtering function and derive estimation (see Eq.55):

$$\tilde{x} = \begin{bmatrix}\tilde{A}_c\\\tilde{A}_s\end{bmatrix} = (\mathbb{D}^T\mathbb{N}^{-1}\mathbb{D})^{-1}\mathbb{D}^T\mathbb{N}^{-1}\mathbf{y} \tag{23}$$

Here measurement matrix is

$$\mathbb{D} = \begin{bmatrix}\mathbb{C}(t_0)\\\mathbb{C}(t_1)\\\vdots\\\mathbb{C}(t)\end{bmatrix}, \tag{24}$$

signal vector is

$$\mathbf{y} = \begin{bmatrix}y(t_0)\\y(t_1)\\\vdots\\y(t)\end{bmatrix} \tag{25}$$

and noise matrix is

$$\mathbb{N} = \begin{bmatrix} \langle n(t_0), n(t_0)\rangle & \langle n(t_0), n(t_1)\rangle & \dots & \langle n(t_0), n(t)\rangle \\ \langle n(t_1), n(t_0)\rangle & \langle n(t_1), n(t_1)\rangle & \dots & \langle n(t_1), n(t)\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle n(t), n(t)\rangle & \langle n(t_1), n(t)\rangle & \dots & \langle n(t), n(t)\rangle \end{bmatrix} \tag{26}$$

where

$$\langle n(t_i)n(t_j)\rangle = \begin{bmatrix}\cos\zeta(t), & \sin\zeta(t)\end{bmatrix} \begin{bmatrix} \langle n_1(t_i), n_1(t_j)\rangle & \langle n_1(t_i), n_2(t_j)\rangle \\ \langle n_2(t_i), n_1(t_j)\rangle & \langle n_2(t_i), n_2(t_j)\rangle \end{bmatrix} \begin{bmatrix}\cos\zeta(t_j) \\ \sin\zeta(t_j)\end{bmatrix} \tag{27}$$

and

$$\begin{cases} \langle \hat{n}_1(t)\hat{n}_1(t')\rangle_{sym} = & \frac{1}{2}\delta(t-t'), \\ \langle \hat{n}_1(t)\hat{n}_2(t')\rangle_{sym} = & \frac{\alpha^2}{2\hbar}G(t'-t)\Theta(t'-t), \\ \langle \hat{n}_2(t)\hat{n}_1(t')\rangle_{sym} = & \frac{\alpha^2}{2\hbar}G(t-t')\Theta(t-t'), \\ \langle \hat{n}_2(t)\hat{n}_2(t')\rangle_{sym} = & \frac{1}{2}\delta(t-t') + \frac{\alpha^4}{4\hbar^2 m^2\omega_m^2}\left(t'\cos\omega_m(t-t') - \frac{1}{\omega_m}\cos\omega_m t\sin\omega_m t'\right)\Theta(t-t') + \\ & + \frac{\alpha^4}{4\hbar^2 m^2\omega_m^2}\left(t\cos\omega_m(t-t') - \frac{1}{\omega_m}\cos\omega_m t'\sin\omega_m t\right)\Theta(t'-t) + \\ & + \frac{\alpha^2}{2mw\hbar}\cos\omega_m(t-t'). \end{cases} \tag{28}$$

Using this estimation,we get the estimated arrival time:

$$\tilde{\tau} = \frac{1}{\omega_m}\arctan\frac{\tilde{A}_s}{\tilde{A}_c} \tag{29}$$

This equation describes the estimation for arrival time that we get on each time step.

2.3.2 Second stage

In previous step we used the knowledge about measurement results in the past. Now we want to make prediction about next measurement in order to find optimal homodyne angle. Suppose, that we can make prediction about both quadratures $b_{1,2}$. Then:

$$\begin{cases} \hat{b}_1(t) = \hat{a}_1(t) \\ \hat{b}_2(t) = \hat{a}_2(t) + \frac{\alpha^2}{\hbar}\int\limits_0^\infty dt'\, G(t-t')\,\Theta(t-t')\hat{a}(t') + \frac{\alpha}{\hbar}x_0\sin\omega_m(t-\tilde{\tau}) \end{cases} \tag{30}$$

If we take into account this prediction, we can assume, that in the moments after time t we can measure both quadratures at the same time:

$$\mathbf{y}\begin{bmatrix} y(t_1) \\ \vdots \\ y(t) \\ \hat{b}_1(t+dt) \\ \hat{b}_2(t+dt) \\ \hat{b}_1(t+2dt) \\ \hat{b}_2(t+2dt) \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{\hbar}\sin\omega_m(t_1) \\ \vdots \\ \frac{\alpha}{\hbar}\sin\omega_m(t) \\ 0 \\ \frac{\alpha}{\hbar}\sin\omega_m(t+dt) \\ 0 \\ \frac{\alpha}{\hbar}\sin\omega_m(t+2dt) \end{bmatrix} x_0 + \begin{bmatrix} \hat{n}_1(t_1)\cos(\omega_m t_1) + \hat{n}_2(t_1)\sin\omega_m t_1 \\ \vdots \\ \hat{n}_1(t)\cos(\omega_m t) + \hat{n}_2(t_1)\sin\omega_m t \\ \hat{n}_1(t+dt) \\ \hat{n}_2(t+dt) \\ \hat{n}_1(t+2dt) \\ \hat{n}_2(t+2dt) \end{bmatrix} = \mathbb{D}x_0 + \mathbf{n} \tag{31}$$

Then we construct the estimator in the same way, as before:

$$\begin{aligned} \tilde{x}_0 &= (\mathbb{D}^T\mathbb{N}^{-1}\mathbb{D})^{-1}\mathbb{D}^T\mathbb{N}^{-1}\mathbf{y} = \\ &= \begin{bmatrix} L_1, & L_2, & \dots & L_N, & L_{N+1}, & L'_{N+1}, & L_{N+2}, & L'_{N+2} \end{bmatrix} \begin{bmatrix} y(t_1) \\ \vdots \\ y(t) \\ \hat{b}_1(t+dt) \\ \hat{b}_2(t+dt) \\ \hat{b}_1(t+2dt) \\ \hat{b}_2(t+2dt) \end{bmatrix} = \\ &= Y_N + L_{N+1}\hat{b}_1(t+dt) + L'_{N+1}\hat{b}_2(t+dt) + L_{N+1}\hat{b}_1(t+2dt) + L'_{N+1}\hat{b}_2(t+2dt) \end{aligned} \tag{32}$$

From the other hand, if we apply some filtering function K to real data $\mathbf{y}_r$ at time step $t+2t$,we get

$$\begin{aligned} \tilde{x}_0 = \mathbb{K}\mathbf{y}_r = \mathbb{K}_N\mathbf{y}_N + K_{N+1}\hat{b}_1(t+dt)\cos\zeta(t+dt) + K_{N+1}\hat{b}_2(t+dt)\sin\zeta(t+dt) + \\ + K_{N+2}\hat{b}_1(t+2dt)\cos\zeta(t+dt) + K_{N+2}\hat{b}_2(t+dt)\sin\zeta(t+2dt) \end{aligned} \tag{33}$$

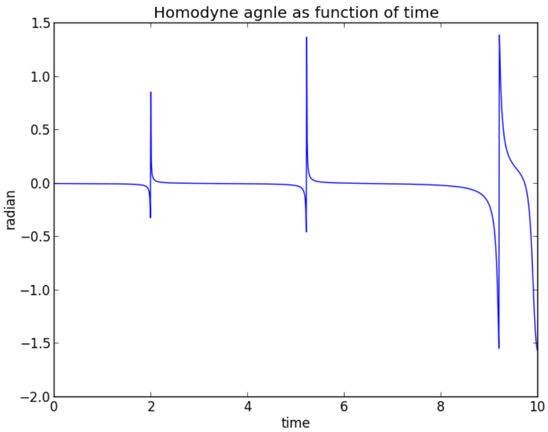
We are interested in only the part of this equation with time $t+dt$. Obviously, from last two equation we get:

$$\mathrm{tg}\,\zeta(t+dt) = \frac{L_{N+1}}{L'_{N+1}} \tag{34}$$

And we simply get the homodyne angle at step $t+2dt$ that minimizes the back-action on the step $t+2dt$

2.3.3 Alternative second stage

Alternatively to this method we can just use the formula for homodyne angle, that we got in Appendix C (see Eq.69,68,70)



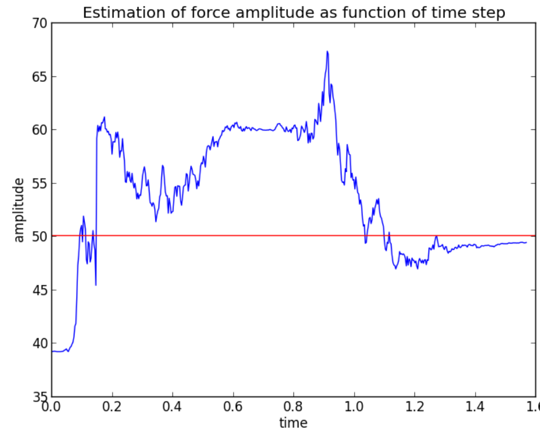


Figure 5: Homodyne angle $\zeta(t)$ dependence of time *(left)* and estimation of delay time depending on time *(right)* (one period).

	$\bar{F}_{est}$	ΔF	$\bar{\tau}_{est}$	$\Delta\tau$
Variational readout	–	0.0015	–	–
Adaptive measurement	51.122	0.022	0.9	0.0004
Fixed homodyne angle $\zeta = \pi/2$	51.035	0.064	0.9023	0.0023

Table 1: Comparison of precision of estimations: force and arrival time for the following parameters: arrival time $\tau = 0.9$,amplitude of the force $F = 51.1$,measurement time $T = 6.6$

3 Results and discussion

By the moment this project is still running, so there are some results, that we can present,but there are a lot of issues to be studied. Main results we achieved:

- created adaptive approach for quantum measurements,
- applied this approach to model task and got estimation of two parameters of the force (see Fig.),
- compared the precision of the adaptive measurements with variation readout and measurements with fixed homodyne angle (see Tab. 1).

Summarizing these results we can say, that precision of adaptive measurements is better, than for fixed homodyne angle, but still worse than for variational measurements (with known arrival time). So, we can conclude, that adaptive measurements can be better than conventional one and can provide more flexibility than variational (because they provide an opportunity to estimate both arrival time and amplitude of the force). But these conclusions are just speculation about the result. To get precise answer on the questions we arose in the introduction section, we should continue our work.

Of course, these preliminary results can only inspire our future investigation in this field. Our next step in this work is understanding of all mathematical properties of the algorithm and making more statistics on it. This will help us to realize possible improvements and problems in the approach. In prospect we want to generalize the algorithm and apply it to some more realistic case.

A Conservation of optimal estimation while mapping

If the estimation is optimal it means that dispersion is minimal and then we can say that von Neumann entropy is maximal:

$$S = -\mathbf{Tr}\{\rho(x)\ln\rho(x)\}, \tag{35}$$
$$\delta S = 0, \tag{36}$$
$$\delta^2 S < 0, \tag{37}$$

where $\rho(x)$ is density matrix that depends on parameter vector $x = \{x_1, x_2, ..., x_n\}$.

Then we can write for other coordinates $y = y(x)$

$$\delta S = dS = \sum_{i=1}^{n}\frac{\partial S}{\partial\rho(x_i)}d\rho(x_i) = \sum_{i=1}^{n}\frac{\partial S}{\partial\rho(x_i)}\frac{d\rho(x_i)}{dx_i}dx_i = \sum_{i=1}^{n}\frac{\partial S}{\partial\rho(y_i)}\frac{d\rho(y_i)}{dy_i}\frac{dy_i}{dx_i}dx_i = 0 \tag{38}$$

Then

$$\frac{\partial S}{\partial\rho(x_i)} = \frac{\partial S}{\partial\rho(y_i)} = 0 \tag{39}$$

if $\frac{\rho(y_i)}{\rho(x_i)} \neq 0$. So, if $dS(x) = 0$ then $dS(y) = 0$ And, for the second derivative:

$$\delta^2 S = d^2 S(x) = \sum_{i,j=1}^{n}\frac{\partial^2 S}{\partial\rho(x_i)\partial\rho(x_j)}d\rho(x_i)d\rho(x_j) \tag{40}$$

This equation can be simplified using Eq.39: all the mixed therms would be equal to zero. Then we get from Eq.40:

$$d^2 S(x) = \sum_{i=1}^{n}\frac{\partial^2 S}{\partial\rho(x_i)^2}(d\rho(x_i))^2 < 0 \tag{41}$$

Because of the positive definition of $(d\rho(x_i))^2$ we can write, that $\frac{\partial^2 S}{\partial\rho(x_i)^2} < 0$ and for other coordinates:

$$d^2 S(y) = \sum_{i=1}^{n}\frac{\partial^2 S}{\partial\rho(y_i)^2}(d\rho(y_i))^2 = \sum_{i=1}^{n}\frac{\partial^2 S}{\partial\rho(x_i)^2}(d\rho(x_i))^2 < 0 \tag{42}$$

We proved that for any change of coordinates the maximum of entropy conserves and then conserves the optimum of the estimation.

B Linear estimation[29]

Consider output signal to be

$$\mathbf{y} = \mathbb{D}x + \mathbf{n}, \tag{43}$$

where $\mathbf{y}$ is a $N \times 1$ vector, $\mathbb{D}$ is a known $N \times p$ matrix, x is a $p \times 1$ vector of parameters, $\mathbf{n}$ is a $N \times 1$ noise vector with zero mean and covariance $\mathbb{N}$.

We want to get unbiased estimation of x, assuming no prior knowledge about this signal. This estimation would be:

$$x = \mathbb{L}\mathbf{y}, \tag{44}$$

where $\mathbb{L}$ is filtering function.

The condition of unbiasedness leads to the following:

$$E[\hat{x}] = \mathbb{L}E[\mathbf{y}] = x \tag{45}$$

and noticing the zero mean of the noise:

$$E[\mathbf{y}] = \mathbb{D}x, \tag{46}$$

then we get the constraint for filtering function:

$$\mathbb{L}\mathbb{D} = \mathbb{I}. \tag{47}$$

Our aim is to find optimal in sense of minimal of variation filtering function. Variation of the estimation is:

$$var(\hat{x}) = E[(x - \hat{x})^2] = E[(\mathbb{L}(E[\mathbf{y}] - \mathbf{y}))^2] = E[\mathbb{L}(\mathbf{y} - E[\mathbf{y}])(\mathbf{y} - E[\mathbf{y}])^T\mathbb{L}^T] = \mathbb{L}\mathbb{N}\mathbb{L}^T \tag{48}$$

We should take into account the constraint and construct Lagrange function:

$$J[\mathbb{L}] = \mathbb{L}\mathbb{N}\mathbb{L}^T + \lambda(\mathbb{L}\mathbb{D} - \mathbb{I}) \tag{49}$$

Then we calculate functional derivative over filtering function:

$$\delta J[\mathbb{L}] = \delta\mathbb{L}\big(2\mathbb{N}\mathbb{L}^T + \mathbb{D}\lambda\big) = 0 \tag{50}$$

Then

$$\mathbb{L}^T = -\frac{1}{2}\mathbb{N}^{-1}\mathbb{D}\lambda \tag{51}$$

and

$$\mathbb{D}^T\mathbb{L}^T = -\frac{1}{2}\mathbb{D}^T\mathbb{N}^{-1}\mathbb{D}\lambda = \mathbb{I}, \tag{52}$$

So we get an equation for λ:

$$-\frac{1}{2}\lambda = (\mathbb{D}^T\mathbb{N}^{-1}\mathbb{D})^{-1}, \tag{53}$$

$$\mathbb{L}^T = \mathbb{N}^{-1}\mathbb{D}(\mathbb{D}^T\mathbb{N}^{-1}\mathbb{D})^{-1} \tag{54}$$

and finally we get an equation for filtering function:

$$\mathbb{L} = (\mathbb{D}^T\mathbb{N}^{-1}\mathbb{D})^{-1}\mathbb{D}^T\mathbb{N}^{-1} \tag{55}$$

We can also simply calculate the variation for this estimation (see Eq.48).

$$\tilde{x} = \mathbb{L}\mathbb{N}\mathbb{L}^T = (\mathbb{D}^T\mathbb{N}^{-1}\mathbb{D})^{-1}\mathbb{D}^T\mathbb{N}^{-1}\mathbb{N}\mathbb{N}^{-1}\mathbb{D}(\mathbb{D}^T\mathbb{N}^{-1}\mathbb{D})^{-1} = (\mathbb{D}^T\mathbb{N}^{-1}\mathbb{D})^{-1} \tag{56}$$

C Variational measurements

Let's derive how homodyne angle changes in time.

The output signal would be:

$$y(t) = \hat{b}_1(t)\cos\zeta(t) + \hat{b}_2(t)\sin\zeta(t) =$$
$$= \hat{a}_1(t)\cos\zeta(t) + \Big[\hat{a}_2(t) + \frac{\alpha}{\hbar}\big(\alpha\int_0^T dt'\, G(t-t')\,\hat{a}(t') + \frac{\bar{F}}{m\omega_m}\sin\omega_m(t-\tau)\big)\Big]\sin\zeta(t) \quad (57)$$

And after filtering with filtering function $g(t)$:

$$Y = \int_0^T dt'\,(g(t')\hat{b}_1(t')\cos\zeta(t') + g(t')\hat{b}_2(t')\sin\zeta(t')) = \int_0^T dt'(g_1(t')\hat{b}_1(t') + g_2(t')\hat{b}_2), \quad (58)$$

where $g_1(t) = g(t)\cos\zeta(t), g_2(t) = g(t)\sin\zeta(t)$. Or, we can write:

$$Y = \int_0^T dt'\big(g_1(t')a_1(t') + g_2(t')\frac{\alpha^2}{\hbar}\int_0^\infty dt''G(t'-t'')\Theta(t'-t'')a_1(t'') + g_2(t')a_2(t')\big) + \frac{\alpha}{\hbar}\int_0^T dt'\, g_2(t')x_0\sin\omega_m(t'-\tau) \quad (59)$$

Then, after changing the oder of interals, we can write down the equation for back-action evaison:

$$g_1(t) + \frac{\alpha^2}{\hbar}\int_t^T dt'\, g_2(t')G(t'-t) = 0 \quad (60)$$

and then for Y we get:

$$Y_{BAE} = \int_0^T dt'\big(g_2(t')a_2(t') + \frac{\alpha}{\hbar}g_2(t')x_0\sin\omega_m(t'-\tau)\big) \quad (61)$$

We can assume for maximizing the signal,that we have constraint:

$$\frac{\alpha}{\hbar}\int_0^T dt'\, g_2(t')\sin\omega_m(t'-\tau) = 1 \quad (62)$$

Then we construct Lagrangian function:

$$\mathcal{L}[g_2(t)] = \langle Y^2\rangle + \mu(\frac{\alpha}{\hbar}\int_0^T dt'\, g_2(t')\sin\omega_m(t'-\tau) - 1) \quad (63)$$

Here

$$\langle Y^2\rangle = \frac{1}{2}\int_0^T dt' g_2^2(t') + x_0^2 \quad (64)$$

because $\langle a_2(t)a_2(t')\rangle = \frac{1}{2}\delta(t-t')$. Then from Eq.63 we can calculate the variation of Lagrangian:

$$\delta\mathcal{L}[g_2(t)] = \frac{1}{2}\delta\int\limits_0^T dt' g_2^2(t') + \mu\delta(\frac{\alpha}{\hbar}\int\limits_0^T dt'\, g_2(t')\sin\omega_m(t'-\tau) - 1) \tag{65}$$

After calculation the variation:

$$\delta\mathcal{L}[g_2(t)] = \int\limits_0^T dt'\delta g_2(t')(g_2(t') + \mu\frac{\alpha}{\hbar}\sin\omega_m(t'-\tau)) = 0 \tag{66}$$

or, as to the fact that $\delta g_2(t') \neq 0$

$$\begin{cases} g_2(t) + \mu\frac{\alpha}{\hbar}\sin\omega_m(t-\tau) = 0 \\ \frac{\alpha}{\hbar}\int\limits_0^T dt'\, g_2(t')\sin\omega_m(t'-\tau) = 1 \end{cases} \tag{67}$$

Solving this system of equations, we get:

$$g_2(t) = \frac{4\omega_m\hbar}{\alpha}\frac{\sin\omega_m(t-\tau)}{2\omega_m T - \sin 2\omega_m(T-\tau) - \sin 2\omega_m\tau} \tag{68}$$

Then we can substitute Eq.68 to Eq.60 and get equation for $g_1(t)$

$$g_1(t) = \frac{\alpha}{m\omega_m}\frac{-2\omega_m(t-T)\cos\omega_m(t-\tau) + \sin\omega_m(t-\tau) + \sin\omega_m(t-2T+\tau)}{-2T\omega_m + \sin 2\omega_m(T-\tau) + \sin 2\omega_m\tau} \tag{69}$$

So,we can get analytical equation for homodyne angle from Eq.69,68:

$$\zeta(t) = \arctan\frac{g_2(t)}{g_1(t)} \tag{70}$$

Finally an error in estimation the force would be (see Eq.64 and 68)

$$\Delta F = \frac{2\omega_m\hbar^2}{\alpha^2}\frac{1}{2\omega_m T - \sin 2\omega_m(T-\tau) - \sin 2\omega_m\tau} \tag{71}$$

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