How to get analytical formula for filter?

Let's consider simple case, without any particular measuring device (see Fig.1). All we know about it is than it measures position of the oscillator and it has spectral densities of noises: S_x, S_F, S_{xF} then we can calculate the optimal unbiased estimate minimizing the variation.

Our system is described by the following equation:

$$\mathbf{D}\hat{x}(t) = m\ddot{\hat{x}}(t) + m\omega_m^2\hat{x}(t) = F_s(t) + \hat{F}_{fl}(t), \tag{1}$$

where $F_s(t) = F_0\delta(t-\tau)$ is our signal force and we want to estimate it's amplitude F_0

We can write down the general solution for this equation:

$$\hat{x}(t) = \hat{x}(0)\cos\omega_m t + \frac{\hat{p}(0)}{m\omega_m}\sin\omega_m t + \mathbf{D}^{-1}(F_s(t) + \hat{F}_{fl}(t)) \tag{2}$$

Here we notice, that $\hat{x}(t)$ is coordinate of the oscillator, and after measuring device it is $\tilde{x}(t) = \hat{x}(t) + \hat{x}_{fl}(t)$ Then we have to get rid of initial conditions. Normally it can be done by applying operator $\mathbf{D}$ to the output signal $\tilde{x}(t)$. But in this case we get

$$\tilde{F}(t) = \mathbf{D}\tilde{x}(t) = F_s(t) + \hat{F}_{fl}(t) + \mathbf{D}\hat{x}_{fl}(t) \tag{3}$$

and we get the delta function with unknown parameter, that can't be linearized. More specifically, even if we apply the spectral approach here, we get:

$$F_s(\omega) = \int_{-\infty}^{\infty} dt\, F_0\delta(t-\tau)e^{-\imath\omega t} = F_0 e^{-\imath\omega\tau} \tag{4}$$

and we can't linearize it.

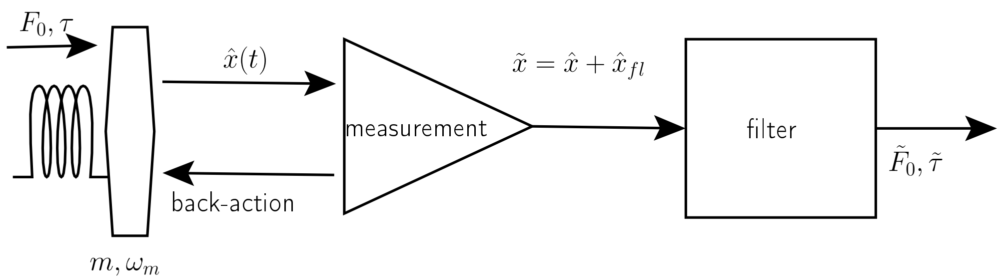

FIG. 1. General system

That's why we now can say just "let's consider in insufficient" and throw these terms:

$$\tilde{x}(t) = \hat{x}_{fl}(t) + \frac{1}{m\omega_m}\int\limits_0^t dt' \,(F_s(t') + \hat{F}_{fl}(t'))\sin\omega_m(t-t') =$$

$$= \hat{x}_{fl}(t) + \frac{1}{m\omega_m}(F_0\cos\omega_m\tau\sin\omega_m t - F_0\sin\omega_m\tau\cos\omega_m t) + \frac{1}{m\omega_m}\int\limits_0^t dt'\,\hat{F}_{fl}(t')\sin\omega_m(t-t') \quad (5)$$

or we can change $A_c = \frac{1}{m\omega_m}F_0\cos\omega_m\tau$ and $A_s = \frac{1}{m\omega_m}F_0\sin\omega_m\tau$ and get linear equation:

$$\tilde{x}(t) = \hat{x}_{fl}(t) + A_c\sin\omega_m t - A_s\cos\omega_m t + \frac{1}{m\omega_m}\int\limits_0^t dt'\,\hat{F}_{fl}(t')\sin\omega_m(t-t') \quad (6)$$

then we proceed filtering:

$$\begin{cases} \tilde{A}_c = \int\limits_{-\infty}^{\infty} g_1(t')\tilde{x}(t')\,dt' \\ \tilde{A}_s = \int\limits_{-\infty}^{\infty} g_2(t')\tilde{x}(t')\,dt' \end{cases} \quad (7)$$

in order to get unbiased estimation, which means that $\tilde{A}_{c,s} = \bar{A}_{c,s} = A_{c,s}$, we should have constraints for the filtering functions:

$$\int\limits_{-\infty}^{\infty} g_1(t')\sin\omega_m t'\,dt' = 1 \quad (8)$$

$$\int\limits_{-\infty}^{\infty} g_2(t')\cos\omega_m t'\,dt' = 1 \quad (9)$$

Then for variations (we omit the formulas for A_s,because they are similar to that with A_c):

$$\langle(A_c-\tilde{A}_c)^2\rangle = \int\limits_{-\infty}^{\infty} dt\, g_1(t)\,\Big(\langle\hat{x}^2_{fl}\rangle + \frac{1}{m^2\omega_m^2}\iint\limits_0^t dt'\,dt''\,\langle F_{fl}(t'), F_{fl}(t'')\rangle\sin\omega_m(t-t')\sin\omega_m(t-t'') +$$

$$+ \frac{2}{m\omega_m}\int\limits_0^t dt'\,\langle F_{fl}(t'), x_{fl}(t)\rangle\sin\omega_m(t-t')\Big) \quad (10)$$

We assume, that knowing spectral densities we can calculate correlation functions and all the big term inside parentheses in previous equation (let's call it $K(t)$):

$$\langle(A_c-\tilde{A}_c)^2\rangle = \int\limits_{-\infty}^{\infty} dt\, g_1(t)\,K(t) \quad (11)$$

then we construct the Lagrange function:

$$L[g_1(t)] = \int\limits_{-\infty}^{\infty} dt\, g_1(t)\,K(t) + \lambda(\int\limits_{-\infty}^{\infty} g_1(t')\sin\omega_m t'\,dt' - 1) \quad (12)$$

and variate it over g_1:

$$\delta L[g_1(t)] = \int\limits_{-\infty}^{\infty} dt(2g_1(t)\,K(t) + \lambda \sin\omega_m t)\delta g_1(t) = 0 \tag{13}$$

Then, using the main lemma of calculus of variations, we get:

$$\begin{cases} 2g_1(t)K(t) + \lambda \sin\omega_m t = 0 \\ \int\limits_{-\infty}^{\infty} g_1(t')\sin\omega_m t'\,dt' = 1 \end{cases} \tag{14}$$

if we know the the $K(t)$ this system can be simply solved and we get the filtering function.

What is good about this approach: we have analytical formula for continuous case and we can calculate all variations and other stuff we want to. What is bad: really huge formula, trowing the initial conditions.