Multi-Armed Bandits

Efficient Algorithms for Learning with Semi-Bandit Feedback

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Overview

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Motivation and Introduction

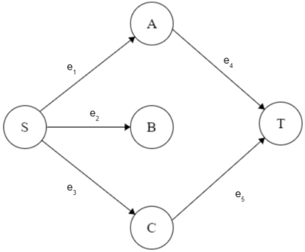
Goals of this section

- Motivate using Online Least Cost Path problem.
- Understanding the Semi-Bandits setting.
- Understanding the Combinatorial problem.
- Understanding Linear Generalization.
- Formally introducing the problem.

Motivation
The Online Least Cost Path problem

- We have a (directed or undirected) graph G with
  - $L =$ number of edge,
  - $w_i =$ cost associate with the edge i.
  - $S =$ start node and,
  - $T =$ target node.
- The goal: Find the cheapest path from S to T w.r.to the cost associated with the path.

Motivation
The Online Least Cost Path problem - An example



- In the above graph $L = 5$.
- A path is denoted as a $5 - D$ vector.
  - Path SAT, $A_1 = \boxed{1 \mid 0 \mid 0 \mid 1 \mid 0}$
  - Path SCT, $A_2 = \boxed{0 \mid 0 \mid 1 \mid 0 \mid 1}$
- Set of valid paths, $\mathcal{A} = \{A_1, A_2\} \subseteq \{0, 1\}^L$

The Online Least Cost Path problem - Cost computation

- Let us say we have decided to go with the path SAT. $\boxed{1}\boxed{0}\boxed{0}\boxed{1}\boxed{0}$
- To compute the cost, we may get one of the following:
  1. Full information setting:
  $$w = \boxed{0.3}\boxed{0.2}\boxed{0.7}\boxed{0.5}\boxed{0.4}$$
  2. Semi-Bandits setting:
  $$w = \boxed{0.3}\boxed{\text{x}}\boxed{\text{x}}\boxed{0.5}\boxed{\text{x}}$$
  3. Full-Bandits setting:
  $$l = \boxed{0.8}$$
- For the Full information and Semi-Bandits setting, the loss is $l = A \cdot w$

Motivation
How is this different from the familiar MAB setting?

- In the MAB setting, we choose $i \epsilon [L]$.
  - i can take one of L values.
- However, here we choose $A \epsilon \mathcal{A} \subseteq \{0, 1\}^L$.
  - A can take one of potentially 2^L values.

- That is, we want to choose one or more arms in each round.
- In the Online Least Cost path problem,
  Edge $\equiv$ Arm
  Path $\equiv$ multiple Arms.
- This is an example of Combinatorial Optimization problem.

Motivation

- An immediate issue with this setting is that applying algorithms such as $EXP - 3$ may not have a good regret bound, since both L and $|\mathcal{A}|$ can be very large in real life settings.
- Can we make it independent of L?
    - Here's a sneak peak...

- In this setting (and many other real life settings), we have access to features of edges, such as,
  1. Length,
  2. Traffic, and
  3. Road Quality.

	Length	Traffic	Road Quality
e_1	10	3	7
e_2	4	6	3
e_3	8	1	4
e_4	6	2	5
e_5	3	7	2

- Hence, we know a (possibly imperfect) generalization matrix Φ.
  In this case,

$$\Phi = \begin{bmatrix} 10 & 3 & 7 \\ 4 & 6 & 3 \\ 8 & 1 & 4 \\ 6 & 2 & 5 \\ 3 & 7 & 2 \end{bmatrix}$$

- We assume that the cost of an edge is a linear combination of its feature values.

$$cost(e) = \theta_1 \; length(e) + \theta_2 \; traffic(e) + \theta_3 \; quality(e) = \vec{\theta} \cdot \phi_e$$

- Now, the task is to estimate the entries in $\vec{\theta}$.
- Notice the advantage:
  The values to be estimated equal the number of features, and not L.

Let's formalize what all we have seen till now.

A Combinatorial Optimization Problem

Can be represented as a triple $(E, \mathcal{A}, w)$ where,

- E = Set of L arms.
- $\mathcal{A}$ = Set of (some or all) subsets A of E with $|A| \leq K$.
- $w : E \rightarrow \mathbb{R}$ is the weight function.

- The total weight of $A \in \mathcal{A}$ is $\sum_{e \in A} w_e$.

Semi-Bandit Problem

The losses associated to only the chosen arms are seen.

Linear Generalization

For $e \epsilon [L], w(e) = \Phi_e \theta$,
where Φ_e is the feature vector of the arm e, and θ indicates the weights of the features.

The Combinatorial semi-bandits

It is an online learning problem where at each step the learning agent chooses a subset of arms A subject to combinatorial constraints, and then observes weights of the selected arms, $\{\mathrm{w}(e) : e \in A\}$, and gets their sum $f(A, \mathrm{w})$ as a payoff.

The Goal

- The learner chooses a subset of arms A^t at round t. The loss suffered at this round $R_t = f(A^*, \mathrm{w_t}) - f(A^t, w)$ (In the Reward setup).
- We want the cumulative Regret to be *good*,
  And the algorithm to be efficient when applied in Large scale settings.

Efficient Algorithm for Learning with Semi-Bandit Feedback (Paper 1)

Learning with Semi-Bandit Feedback

- The paper considers the problem of online Combinatorial Optimization under semi-bandit feedback. (No linear Generalization)
- The paper assumes:
  - Finite Decision Set (potentially very large)
  - Efficient offline combinatorial optimization is possible
  - Elements of the decision set can be described with d-dimensional arrays with at most m non-zero entries.
  - For example a decision vector in which 3 arms out of possible 7 arms are played will look like $\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

General Protocol for online Combinatorial Optimization

- **Parameters:** set of decision vectors $\mathcal{S} = \{v(1), v(2), \ldots, v(N)\} \subseteq \{0,1\}^d$, number of rounds T;
  For all $t = 1, 2, \ldots, T$, **repeat**
  1. The learner chooses a probability distribution p_t over $\{1, 2, \ldots, N\}$.
  2. The learner draws an action I_t randomly according to p_t. Consequently, the learner plays decision vector $V_t = v(I_t)$.
  3. The environment chooses loss vector ℓ_t.
  4. The learner suffers loss $V_t^\top \ell_t$.
  5. The learner observes some feedback based on ℓ_t and V_t.
- Generic framework to accommodate a number of interesting problem instances such as path planning, ranking and matching problems , finding minimum weight spanning trees and cut sets etc.

Normal Bandit Setting

- Total N arms and we chose only one arm.
- A distribution $\mathsf{p_t}$ over the arms where $p_{t,i} = \mathcal{P}[\mathsf{I_t} = i | \mathcal{F}_{t-1}]$ where $\mathcal{F}_{t-1}$ is the history of the learners observation and choice made upto step $t-1$.
- Most bandit algorithms rely on feeding some loss estimates to black box algorithm like Hedge etc. A common loss estimate used is
  $\hat{\ell}_{t,i} = \frac{\ell_{t,i}}{p_{t,i}} \mathbb{I}[\mathsf{I_t} = \mathsf{i}]$ (An unbiased estimate)
- Almost all existing algorithms use some form of the above loss estimate. But $\mathsf{p_t}$ is not readily available for all algorithms like FPL.
- Following loss estimate is proposed

Geometric Resampling

- In round t the learner draws $\mathrm{I_t} \sim \mathrm{p_t}$
- For n = 1, 2 , ...
  - Let $n \leftarrow n + 1$
  - Draw $I'(n) \sim \mathrm{p_t}$
  - If $I'(n) = I_t$, break
- Let $K_t = n$

- $\hat{\ell_{t,i}} = \ell_{t,i}\mathbb{I}[\mathrm{I}_t = i]K_t$ (Also unbiased)
- The number samples can be unbounded, so we cap the number of samples by M and use $\hat{K}_t = min(K_t, M)$
- Introduces some bias, but if M is chosen appropriately the performance does not hurt much.

Generalizing Geometric Sampling to Semi-Bandit Feedback

- In round t again the learner draws $\mathsf{I_t} \sim \mathsf{p_t}$
- For each arm draw M samples (because anyway we are capping by M). We can effectively draw only M samples and use them for all the arms being played. So, draw M additional indices $I'_t(1), I'_t(2) \dots I'_t(M) \sim \mathsf{p_t}$
- Define $K_{t,j} = min\{1 \leq s \leq M : v_j(I'_t(s)) = v_j(I_t)\}$ for all arms.
- Use Loss estimate for each arm as $\hat{\ell}_{t,j} = K_{t,j} V_{t,j} \ell_{t,j}$

- Since $V_{t,j}$ is non-zero only for the arms being played the estimates are well defined.

FPL with Geometric Sampling

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```
Input: S = {v(1), v(2), ..., v(N)} ⊆ {0,1}^d, η ∈ R+, M ∈ Z+;
Initialization: L̂(1) = ··· = L̂(d) = 0;
for t=1,...,T do
    Draw Z(1), ..., Z(d) independently from distribution Exp(η);
    Choose action I = argmin_{i∈{1,2,...,N}} { v(i)^T (L̂ − Z) };
    K(1) = ··· = K(d) = M;
    k = 0;                                   /* Counter for reoccurred indices */
    for n=1,...,M-1 do                                 /* Geometric Resamplig */
        Draw Z'(1), ..., Z'(d) independently from distribution Exp(η);
        I(n) = argmin_{i∈{1,2,...,N}} { v(i)^T (L̂ − Z') };
        for j=1,...,d do
            if v(I(n))(j) = 1 & K(j) = M then
                K(j) = n;
                k = k + 1;
                if k = ||v(I)||_1 then break;    /* All indices reoccurred */
            end
        end
    end
    for j=1,...,d do  L̂(j) = L̂(j) + K(j)v(I)(j)ℓ(j) ;              /* Update */
end
```

Computational Efficiency under Semi Bandit Feedback

- The expected number of times the algorithm draws an action up to time step T can be upper bounded by dT. This can be shown as

- For each co-ordinate that the original arm had 1, we take sampling until we get 1 in the same co-ordinate again. (Here we do not assume cutoff)
- Let $q_{t,k} = \mathbb{E}[V_{t,k} | \mathcal{F}_{t-1}]$
- At time step t for a given co-ordinate k with 1, the expected number of samples is $\frac{1}{q_{t,k}}$ while the probability of co-ordinate k being 1 is $q_{t,k}$
- So, expected number of samples is $\sum_{k=1}^{d} q_{t,k} \frac{1}{q_{t,k}} = d$. (For one round)
- For T rounds it will be dT.
- The expected running time is comforting.

Regret Bound

- The total expected regret of FPL with Geometric Resampling satisfies $R_n \leq \frac{m(\log d+1)}{\eta} + \eta m d T + \frac{dT}{eM}$ under semi-bandit information.
- For the full information setting if $C_T = \sum_{t=1}^{T} \mathbb{E}[\mathrm{V}_\mathrm{t}^\mathrm{T} \ell_\mathrm{t}]$ then the expected regret of FPL satisfies $R_n \leq \frac{m(\log d+1)}{\eta} + \eta m C_t$ under full information.

- For $\eta = \sqrt{\frac{\log d+1}{dT}}$ and $M \geq \frac{\sqrt{dT}}{eM(\sqrt{\log d+1})}$, $R_n = \mathcal{O}(m\sqrt{dT \log d})$
- For $\eta = \sqrt{\frac{(\log d+1)}{mT}}$ $R_n = \mathcal{O}(m^{\frac{3}{2}}\sqrt{T(\log d + 1)})$.

Comparison with State of Art

- Full information setting $\rightarrow$ Optimal Regret $\rightarrow \mathcal{O}(m\sqrt{T\log d})$
  FPL-with-GR achives $\mathcal{O}(m^{\frac{3}{2}}\sqrt{T(\log d + 1)})$ off by a factor of $\sqrt{m}$
- Semi-bandit setting $\rightarrow$ Optimal Regret $\mathcal{O}(\sqrt{mdT})$.
  FPL-with-GR achives $\mathcal{O}(m\sqrt{dT\log d})$ off by a factor of $\sqrt{m\log d}$

Conclusion and Future work

- The paper has introduced the first general efficient algorithm for online combinatorial optimization under semi-bandit feedback.
- It remains as an open problem whether the gaps we have shown can be closed for FPL-style algorithms.
- The most important open problem is the development efficient online linear optimization with full bandit feedback.

Efficient Learning in Large Scale Combinatorial Semi-Bandits (Paper 2)

Large Scale Combinatorial Semi-Bandits

- This paper looks at efficient algorithms for stochastic combinatorial semi-bandits with **linear generalization**
- We look at algorithms that have regret bound **INDEPENDENT OF L** (hence large scale)

Setup Used for Combinatorial optimization

Combinatorial setup

$$(E, \mathcal{A}, P)$$

- $E = \{1, \ldots, L\} \equiv$ arms (the ground set)
- $\mathcal{A} \subseteq \{A \subseteq E : |A| \leq K\} \equiv$ allowed combinations of arms
- $P \equiv$ probability distribution over weights $\mathrm{w} \in \mathbb{R}^L$ on E. ($\bar{\mathrm{w}} = \mathbb{E}[\mathrm{w}]$)
- After the arms A are pulled, we observe the individual return of each arm $\{\mathrm{w}(e) : e \in A\}$
- $f(A, \mathrm{w}) \equiv$ loss on pulling $A \in \mathcal{A}$

We assume linear generalization(possibly imperfect) and that the agent knows the generalization matrix $\Phi \in \mathbb{R}^{L \times d}$.If $\bar{\mathrm{w}} \in span[\Phi]$ we call it as coherent learning case otherwise agnostic. WLOG rank$[\Phi] = d$.

Performance Metrics

- $R_t = f(A^*, \mathrm{w_t}) - f(A^t, w)$ where $A^* = ORACLE(E, \mathcal{A}, \bar{w})$
- For fixed $\bar{\mathrm{w}}$,
  $R(T) = \sum_{t=1}^{T} \mathbb{E}[R_t | \bar{\mathrm{w}}] \equiv$ **EXPECTED CUMULATIVE REGRET** where expectation is over random weights and possible randomization in the algorithm.
- for randomly generated $\bar{\mathrm{w}}$,
  $R_{bayes}(T) = \sum_{t=1}^{T} \mathbb{E}[R_t] \equiv$ **BAYES CUMULATIVE REGRET** where expectation is over random weights, possible randomization in the algorithm and also over $\bar{\mathrm{w}}$.
- $\theta^* = argmin_{\theta} || \bar{\mathrm{w}} - \Phi\theta ||$. Since $rank[\Phi] = d$, θ^* is uniquely defined. Also in coherent learning case $\bar{\mathrm{w}} = \Phi\theta^*$.

ALGORITHMS

KALMAN FILTERING for updating parameters

Algorithm 1

Input: $\bar{\theta}_t$, Σ_t, σ, and feature-observation pairs $\{(\phi_e, \mathrm{w}_t(e)) : e \in A^t\}$
Initialize $\bar{\theta}_{t+1} \leftarrow \bar{\theta}_t$ and $\Sigma_{t+1} \leftarrow \Sigma_t$
for $k = 1, \ldots, |A^t|$ **do**
Update $\bar{\theta}_{t+1}$ and Σ_{t+1} as follows, where a_k^t is the kth element in A^t

$$\bar{\theta}_{t+1} \leftarrow \left[I - \frac{\Sigma_{t+1}\phi_{a_k^t}\phi_{a_k^t}^T}{\phi_{a_k^t}^T \Sigma_{t+1}\phi_{a_k^t} + \sigma^2} \right] \bar{\theta}_{t+1} + \left[\frac{\Sigma_{t+1}\phi_{a_k^t}}{\phi_{a_k^t}^T \Sigma_{t+1}\phi_{a_k^t} + \sigma^2} \right] \mathrm{w}_t\left(a_k^t\right) \quad \text{and} \quad \Sigma_{t+1} \leftarrow \Sigma_{t+1} - \frac{\Sigma_{t+1}\phi_{a_k^t}\phi_{a_k^t}^T \Sigma_{t+1}}{\phi_{a_k^t}^T \Sigma_{t+1}\phi_{a_k^t} + \sigma^2},$$

end for
Output: $\bar{\theta}_{t+1}$ and Σ_{t+1}

Combinatorial Linear Thompson Sampling

CombLinTS

Input: Combinatorial structure $(E, \mathcal{A})$, generalization matrix $\Phi \in \mathbb{R}^{L \times d}$, algorithm parameters $\lambda, \sigma > 0$, oracle `ORACLE`

Initialize $\Sigma_1 \leftarrow \lambda^2 I \in \mathbb{R}^{d \times d}$ and $\bar{\theta}_1 = 0 \in \mathbb{R}^d$

for all $t = 1, 2, \ldots, n$ **do**

  Sample $\theta_t \sim N\left(\bar{\theta}_t, \Sigma_t\right)$

  Compute $A^t \leftarrow \texttt{ORACLE}(E, \mathcal{A}, \Phi\theta_t)$

  Choose set A^t, and observe $\mathrm{w}_t(e), \forall e \in A^t$

  Compute $\bar{\theta}_{t+1}$ and Σ_{t+1} based on Algorithm 1

end for

CombLinTS: Key Points

- λ $\rightarrow$inverse regularization parameter.
- smaller λ makes the covariance matrix $\sum_t$ closer to 0.
- smaller $\lambda \Rightarrow$ narrower prior$\Rightarrow$ insufficient exploration $\Rightarrow$ degraded performance of CombLinTS.
- σ controls the decrease rate of covariance matrix Σ_t.
- Large σ will lead to slow learning and a smaller σ will make the algorithm quickly converge to some sub-optimal coefficient vector.

CombLinTS: Regret bound

Regret Bound

- If $\bar{w} = \Phi\theta^*$, the prior on θ^* is $\mathcal{N}(0, \lambda^2 I)$, the noises $(w_t(e) - \bar{w}(e))$ are i.i.d sampled from $\mathcal{N}(0, \sigma^2)$ then CombLinTS guarantees: $R_{Bayes}(T) = \tilde{O}(K\lambda\sqrt{dTmin\{\ln(L), d\}})$

- The conditions ensure it is a coherent gaussian case.
- The $\tilde{O}$ notation hides the logarithmic factors.
- The regret bound is a minimum of two bounds. The first bound is L-dependent as $O(\sqrt{\ln(L)})$. The second bound is L-independent but is $\tilde{O}(d)$ instead of $\tilde{O}(\sqrt{d})$.

Combinatorial Linear UCB

CombLinUCB

Input: Combinatorial structure $(E, \mathcal{A})$, generalization matrix $\Phi \in \mathbb{R}^{L \times d}$, algorithm parameters $\lambda, \sigma, c > 0$, oracle `ORACLE`

Initialize $\Sigma_1 \leftarrow \lambda^2 I \in \mathbb{R}^{d \times d}$ and $\bar{\theta}_1 = 0 \in \mathbb{R}^d$

for all $t = 1, 2, \ldots, n$ **do**

Define the UCB weight vector $\hat{\mathrm{w}}_t$ as

$$\hat{\mathrm{w}}_t(e) = \langle \phi_e, \bar{\theta}_t \rangle + c\sqrt{\phi_e^T \Sigma_t \phi_e} \quad \forall e \in E$$

Compute $A^t \leftarrow \texttt{ORACLE}(E, \mathcal{A}, \hat{\mathrm{w}}_t)$

Choose set A^t, and observe $\mathrm{w}_t(e), \forall e \in A^t$

Compute $\bar{\theta}_{t+1}$ and Σ_{t+1} based on Algorithm 1

end for

CombLinUCB: Key Points

- $\lambda \rightarrow$inverse regularization parameter.
- σ controls the decrease rate of covariance matrix Σ_t..
- The constant c controls the degree of optimism.
- Small $c \Rightarrow$ algorithm might converge to some sub-optimal coefficient vector due to insufficient exploration.
- Large $c \Rightarrow$ excessive exploration and slow learning.

CombLinUCB: Regret Bound

Assuming P is a subset of $[0,1]^L$, the stochastic item weights are statistically independent under P. Then for the coherent learning case i.e. $\bar{\mathrm{w}} = \Phi\theta$ we have: For any λ , σ and $\delta \in (0,1)$ and any c satisfying $c \geq \frac{1}{\sigma}\sqrt{d\ln(1+\frac{Tk\lambda^2}{d\sigma^2}) + 2\ln(\frac{1}{\delta})} + \frac{||\theta^*||_2}{\lambda}$ we have:

$$R(T) \leq 2cK\lambda\sqrt{\frac{dT\ln(1+\frac{Tk\lambda^2}{d\sigma^2})}{\ln(1+\frac{\lambda^2}{\sigma^2})}} + TK\delta$$

Specifically if we chose $\lambda = \sigma = 1$, $\delta = \frac{1}{nK}$ and c is the lower bound on its above condition then:

Regret Bound

$$R(T) = \tilde{O}(Kd\sqrt{T})$$

Comparison with state of art

Standard Result

Standard results have been set for the no generalisation case.

No generalization $\Rightarrow$ lower bound of $\Omega(\sqrt{LKT})$

$\Rightarrow \Phi = \mathrm{I} \Rightarrow L = d \Rightarrow$ No generalization lower bound = $\Omega(\sqrt{KdT})$

CombLinTS

$\tilde{O}(\sqrt{Kmin\{\ln(L), d\}})$ larger than the $\Omega(\sqrt{KdT})$

CombLinUCB

$\tilde{O}(\sqrt{Kd})$ larger than the $\Omega(\sqrt{KdT})$

Note

The $\Omega(\sqrt{d})$ and $\Omega(\sqrt{K})$ factors are due to linear generalization. Full tightness analysis has been left to future work.

Conclusion and Future Work

- This paper has introduced two learning algorithms CombLinTS and CombLinUCB for stochastic combinatorial semi bandits with linear genrelariation.
- The main contribution has been that the paper has successfully introduced L independent bounds which is highly useful for real life problems e.g Online advertisements(millions of users and products).
- This paper has left it open to derive bounds for the agnostic learning case.
- Another open problem is how to extend the results to combinatorial semi bandits with non-linear generalization.

TAKEAWAYS

What have we seen so far

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- **Geometric Resampling:**
  - Don't need p_t explicitly;
  - Computationally efficient;
  - Fits into any semi bandit algorithm

What have we seen so far

- **Geometric Resampling:**
  - Don't need p_t explicitly;
  - Computationally efficient;
  - Fits into any semi bandit algorithm
- **Linear Generalization:**
  - Makes regret bound L independent

THANK YOU