4.3.1

$T(n) = T(n-1) + n \quad \Rightarrow O(n^2)$

$\because T(n-1) = T(n-1-1) + n-1$

$= T(n-2) + n-1$

$T(n) = T(n-2) + n-1 + n$

$= T(n-2) + 2n-1$

$= T(n-2) + 2n - \frac{2}{2}(2-1)$

$= T(n-k) + kn - \frac{k(k-1)}{k}$

$k = n-1$

$= T(n-(n-1)) + (n-1)n - \frac{(n-1)((n-1)-1)}{(n-1)}$

$= T(n-n+1) + (n-1)n - ((n-1)-1)$

$= T(1) + n^2 - n - (n-2)$

Assume $T(1) = 1$

$= 1 + n^2 - n - n + 2$

$T(n) = n^2 - 2n + 2 + 1$

$\therefore O(n^2)$

4.3.7

$T(n) = 4T(n/3) + n$

$T(n) = \Theta(n^{\log_3 4})$ — using master's thm case 1

$n^{\log_3 4}$ vs n

$n^{\log_3 4-1} \Rightarrow n^{\log_3 3}$, $\epsilon = 1$

$\Rightarrow n^1 \Rightarrow n$

So case 1

$\therefore T(n) = \Theta(n^{\log_3 4})$

Using substitution

assumption $T(n) \leq cn^{\log_3 4}$

$T(n) = 4T(n/3) + dn$

$= 4c\left(\frac{n}{3}\right)^{\log_3 4} + dn$

d is constant d ~~>0~~ is positive

$= 4c\left(\frac{n^{\log_3 4}}{3^{\log_3 4}}\right) + dn$

$= \cancel{4}c\left(\frac{n^{\log_3 4}}{\cancel{4}}\right) + dn$

$= c(n^{\log_3 4}) + dn$

we need to show

$c(n^{\log_3 4}) + dn \leq cn^{\log_3 4}$

which is ~~not pos~~ fail

because then $d \leq 0$

but d has to be positive

So it fails

subtract off lower order term to make it work

assume $T(n) \leq c\,n^{\log_3 4} - dn$

$T(n) = 4T(n/3) + dn$ (d is positive constant)

$= 4\left(c\left(\frac{n}{3}\right)^{\log_3 4} - d\left(\frac{n}{3}\right)\right) + dn$

$= 4c\left(\frac{n^{\log_3 4}}{3^{\log_3 4}}\right) - 4d\left(\frac{n}{3}\right) + dn$

$= \cancel{4}c\left(\frac{n^{\log_3 4}}{\cancel{4}}\right) - \frac{4dn}{3} + dn$

$= c\left(n^{\log_3 4}\right) - \left(\frac{4dn + 3dn}{3}\right)$

$= c\left(n^{\log_3 4}\right) - \left(\frac{7dn}{3}\right)$

we have to show

$c\left(n^{\log_3 4}\right) - \frac{7dn}{3} \leq c\,n^{\log_3 4} - dn$

which is true.

4·4-9

$T(n) = T(\alpha n) + T((1-\alpha)n) + cn$

(if α is constant, $0 < \alpha < 1$, $c > 0$

cn

$c(\alpha n)$ $\quad$ $c(1-\alpha)n$

$T(\alpha(\alpha n))$ $\quad$ $c(\alpha(1-\alpha))n$ $\quad$ $c(1-\alpha)(\alpha n)$ $\quad$ $c(1-\alpha)((1-\alpha)n)$

$\Downarrow$ $\quad$ $\Downarrow$ $\quad$ $\Downarrow$ $\quad$ $\Downarrow$

$c(\alpha^2 n)$ $\quad$ $c(\alpha(1-\alpha))n$ $\quad$ $c(1-\alpha)(\alpha n)$ $\quad$ $c(1-\alpha)^2 n$

So we will eventually hit the base case

- left most branch will become base case after $\log_{(1/1-\alpha)} n$
- Right most branch will become base case after $\log_{(1/\alpha)} n$

$\log_{(1/\alpha)} n > \log_{(1/1-\alpha)} n$

$T(n) \geq cn \log_{(1/1-\alpha)} n$

$= \Omega(n \log n)$

This is lower bound.

$T(n) \leq cn \log_{(1/\alpha)} n$

$T(n) \leq n\log n$

$= O(n\log n)$

so,

we can write,

$T(n) = \Theta(n\log n)$

4.5.2 – $T(n) = aT\left(\frac{n}{4}\right) + \Theta(n^2)$

a is no. of subproblems

a=?

algorithm has to be faster than strassen's algorithm,

for strassen's algorithm, $\Theta(n^{\log 7})$

we can get the Θ of professor caesar's algorithm using master's method.

subproblem determines assymptotic running time of problem.

$n^{\log_4 a}$ vs n^2

case 1 applies, ie. in worst case,

so in worst case,

$T(n) = \Theta(n^{\log_4 a}) = \Theta(n^{\log_2 \sqrt{a}})$

so $\Theta(n^{\log_2 \sqrt{a}}) < \Theta(n^{\log_2 7})$

$n^{\log_2 \sqrt{a}} < n^{\log_2 7}$

$\log_2 \sqrt{a} < \log_2 7$

$\sqrt{a} < 7$

so $a < 49$

so largest integer value of $a = 48$

4.1 a) $T(n) = 2T(n/2) + n^4$

$n^{\log_2 2}$ vs n^4

$n^{\log_2 2+14}$ $(\epsilon = 14)$ $= n^{\log_2 16} = n^{\log_2 2^4}$

$= n^4$

So case 3 applies

if $2(n/2)^4 \leq c(n)^4$

$\cancel{2}\left(\frac{n^4}{\cancel{16}\,8}\right) \leq c n^4$

$\frac{1}{8} n^4 \leq c n^4$

for $c = \frac{1}{8} < 1$

So $T(n) = \Theta(n^4)$

b) $T(n) = T(7n/10) + n$

$n^{\log_{\frac{10}{7}} 1}$ vs n

$n^{\log_{\frac{10}{7}} \cancel{0+}1}$ vs n^1

So by case 2 $T(n) = \Theta(n \log n)$

~~if $c(n/(10/7)) \leq c(n)$~~

~~$\frac{7n}{10} < cn$~~

~~for $c = \frac{7}{10} < 1$~~

~~So case 3 applies~~

~~$T(n) = \Theta(n)$~~

by Akra bazzi,

$\left(\frac{10}{7}\right)^p = 1$

$\therefore p = 0$

$\Theta\left(n^0 + 1 + \int_1^n \frac{u}{u^{0+1}} du\right)$

$\Theta\left(\int_1^n u^{-p} du\right)$

$\Theta\left(\frac{u^{1-p}}{1-p}\right)_{u=1}^{u=n}$

$= \Theta\left(\frac{n^{1-p} - 1}{1-p}\right)$

$= \Theta(n^{1-p}) = \Theta(n^{1-0})$

$= \Theta(n^1)$

$= \Theta(n)$

c) $T(n) = 16T(n/4) + n^2$

$n^{\log_4 16}$ vs n^2

$n^{\log_4 4^2}$ vs n^2

So case 2 applies

$T(n) = \Theta(n^{\log_4 16} \log n) = \Theta(n^2 \log n)$

d) $T(n) = 7T(n/3) + n^2$

$n^{\log_3 7}$ ~~for~~ vs n^2

$n^{\log_3 7+2} = n^{\log_3 9}$ $(\epsilon = 2)$ $= n^{\log_3 3^2} = n^2$

So case 3 applies

$7(n/3)^2 \le cn^2$

$\frac{7}{9} n^2 \le cn^2$

for $c = \frac{7}{9} < 1$

So $T(n) = \Theta(n^2)$

e) $T(n) = 7T(n/2) + n^2$

$n^{\log_2 7}$ vs n^2

$n^{\log_2 \cancel{7-3}} = n^{\log_2 4} = n^{\log_2 2^2} = n^2$

$(\epsilon = 3)$

so this is case 1

$T(n) = \Theta(n^{\log_2 7})$

f) $T(n) = 2T(n/4) + \sqrt{n}$

$n^{\log_4 2}$ vs $\sqrt{n} = n^{1/2}$

$n^{\log_2 \sqrt{2}}$ vs $n^{1/2}$

$n^{\log_2 2^{1/2}}$ vs $n^{1/2}$

$n^{1/2}$ vs $n^{1/2}$

so this is case 2

$T(n) = \Theta(n^{\log_4 2} \log n) = \Theta(\sqrt{n} \log n)$

9) $T(n) = T(n-2) + n^2$

Master's theorem doesn't apply

So we can solve it by substitution method

To guess -

$$n \left\{ \begin{array}{ccc} n & - & n \\ \vdots & & \\ n-1 & - & n-1 \\ \vdots & & \\ n-2 & - & n-2 \\ \vdots & & \\ 2 & - & 2 \\ \vdots & & \\ 1 & - & 1 \end{array} \right.$$

So assume $T(n) \leq cn^2$ for upper bound

$T(n) = T(n-1) + cn$

$= c(n-1)^2 + dn$ d is positive

$= c(n^2 - 2n + 1) + cn \leq cn^2$

$= cn^2 - 2cn + c + cn \leq cn^2$

$= -2cn + c + cn \leq 0$

$= n(-2c + d) + c \leq 0$

$= n(d - 2c) + c \leq 0$

for $c = 1$ & $n \geq 1$

So $O(n^2)$

for lower bound Assume $T \geq cn^2$

$T(n) = T(n-1) + dn$ d is positive

$= c(n-1)^2 + dn$

$= c(n^2 - 2n + 1) + dn \geq cn^2$

$= cn^2 - 2cn + c + dn \geq cn^2$

$= -2cn + c + dn \geq 0$

$= n(-2c + d) + c \geq 0$

$= n(d - 2c) + c \geq 0$

for $n \geq 0$ & $0 < c < 0.5$

$\therefore O(n^2)$

so $\Theta(n^2)$

$T(n) = 1$ if $x <= 5$

$T(n) = T(n/5) + T(2n/5) + T(3n/5) + cn$ otherwise

$$\left(\frac{1}{5}\right)^p + \left(\frac{2}{5}\right)^p + \left(\frac{3}{5}\right)^p = 1$$

$\therefore p = 1.224463$

$$T(n) = \Theta\left(n^p\left(1 + \int_1^n \frac{u}{u^{p+1}} du\right)\right)$$

$$= \Theta\left(n^p\left(1 + \int_1^n u^1 \cdot u^{-p} \cdot u^{-1} du\right)\right)$$

$$= \Theta\left(n^p\left(1 + \int_1^n u^{-p} du\right)\right)$$

$$= \Theta\left(n^p\left(1 + \left(\frac{u^{1-p}}{1-p}\right)_{u=1}^{u=n}\right)\right)$$

$$= \Theta\left(n^p\left(1+\left(\frac{n^{1-p}-1}{1-p}\right)\right)\right)$$

$$= \Theta\left(n^p\left(1+\Theta\left(n^{1-p}\right)\right)\right)$$

$$= \Theta\left(n^{1.224463}\left(1+\Theta\left(n^{1-1.224463}\right)\right)\right)$$

$$= \Theta\left(n^{1.224463}\left(1+\Theta\left(n^{-0.224463}\right)\right)\right)$$

$$= \Theta\left(n^{1.224463} + n^{1.224463-0.224463}\right)$$

$$= \Theta\left(n^{1.224463} + n^{1}\right)$$

$$= \Theta\left(n^{1.224463+1}\right)$$

$$= \Theta\left(n^{2.224463}\right)$$

$$\simeq \Theta\left(n^{2}\right)$$