

Matrix theory - Challenge

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Abstract—This document illustrates proving properties of traingle using linear algebra

Download all latex-tikz from

https://github.com/shreeprasadbhat/matrix-theory/blob/master//challenging_problems/challenge_03_10_2020/

1 PROBLEM

In conic sections you have seen that

$V = PDP^T$, with $P^T P = I$. So P is an orthogonal matrix. For what matrices V do you get this kind of decomposition where P is an orthogonal ? Can you prove this result?

2 ANSWER

We get this kind of result for real symmetric matrix. Any real symmetric matrix can always be decomposed as $V = PDP^T$, with $P^T P = I$.

3 PROOF

Consider an 2×2 real symmetric matrix

$$\mathbf{V} = \begin{pmatrix} a & h \\ h & b \end{pmatrix} \quad (3.0.1)$$

Its characteristic equation is

$$\lambda^2 - (a + b)\lambda + (ab - h^2) = 0 \quad (3.0.2)$$

So the eigenvalues are given by

$$\frac{(a + b) \pm \sqrt{(a - b)^2 + 4h^2}}{2} \quad (3.0.3)$$

These eigenvalues are equal only if

$$(a - b)^2 + 4h^2 = 0 \implies a = b, h = 0 \quad (3.0.4)$$

Let λ_1 and λ_2 be two eigenvalue of V. Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be two corresponding eigenvectors. We have

$$\lambda_1(\mathbf{u}^T \mathbf{v}) = (\lambda_1 \mathbf{u})^T \mathbf{v} \quad (3.0.5)$$

$$= (\mathbf{V} \mathbf{u})^T \mathbf{v} \quad (3.0.6)$$

$$= au_1 v_1 + h(u_2 v_1 + u_1 v_2) + bu_2 v_2 \quad (3.0.7)$$

$$= \mathbf{u}^T (\mathbf{V} \mathbf{v}) \quad (3.0.8)$$

$$= \mathbf{u}^T (\lambda_2 \mathbf{v}) \quad (3.0.9)$$

$$= \lambda_2 (\mathbf{u}^T \mathbf{v}) \quad (3.0.10)$$

If $\lambda_1 \neq \lambda_2$, then it follows that

$$\mathbf{u}^T \mathbf{v} = 0 \quad (3.0.11)$$

If $\lambda_1 = \lambda_2$, then from (3.0.4),

$$\mathbf{V} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \implies \mathbf{V} = aI \quad (3.0.12)$$

Thus, for real symmetric matrix eigenvectors will be orthogonal.

If $\mathbf{P} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$. By definition of orthogonal matrix, we have,

$$\mathbf{P} \mathbf{P}^T = \mathbf{P}^T \mathbf{P} = I \quad (3.0.13)$$

Consider,

$$\mathbf{V} \mathbf{P} = \begin{pmatrix} \mathbf{V} u_1 & \mathbf{V} v_1 \\ \mathbf{V} u_2 & \mathbf{V} v_2 \end{pmatrix} \quad (3.0.14)$$

$$= \begin{pmatrix} \lambda_1 u_1 & \lambda_2 v_1 \\ \lambda_1 u_2 & \lambda_2 v_2 \end{pmatrix} \quad (3.0.15)$$

$$= \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (3.0.16)$$

$$= \mathbf{P} \mathbf{D} \quad (3.0.17)$$

$$\mathbf{V} \mathbf{P} \mathbf{P}^T = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad (3.0.18)$$

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad (3.0.19)$$

Hence proved the result.