

# Matrix theory - Assignment 8

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**Abstract—This document illustrates finding subspaces of a vector spaces**

Download latex-tikz from

<https://github.com/shreeprasadbhat/matrix-theory/blob/master/assignment8/>

## 1 PROBLEM

- Prove that only subspace of  $\mathbb{R}^1$  are  $\mathbb{R}^1$  and the zero subspace
- Prove that a subspace of  $\mathbb{R}^2$  is  $\mathbb{R}^2$ , or the zero subspace, or consists of all scalar multiples of some fixed vector in  $\mathbb{R}^2$ . (The last type of subspace is, intuitively, a straight line through the origin.)
- Can you describe the subspaces of  $\mathbb{R}^3$  ?

## 2 SOLUTION

- Let  $W \neq 0$  be subspace of  $\mathbb{R}^1$ . Then  $W$  is a nonempty subset of  $\mathbb{R}^1$  and there exist  $w \in W$  such that  $w \neq 0$  which gives us that there exist  $w^{-1}$ .

Let  $x \in \mathbb{R}^1$ . Since  $W$  is in  $\mathbb{R}^1$  we have that it is closed under scalar multiplication which gives us that  $(xw^{-1})w = x(w^{-1}w) = x \cdot 1 = x \in W$

Hence  $\mathbb{R}^1 \subset W$  and therefore  $W = \mathbb{R}^1$

Thus the only subspace of  $\mathbb{R}^1$  distinct of 0 is  $\mathbb{R}^1$  and therefore only subspaces of  $\mathbb{R}^1$  are 0 and  $\mathbb{R}^1$ .

- Clearly, 0 and  $\mathbb{R}^2$  itself are subspaces of  $\mathbb{R}^2$ . If  $u \neq 0$  and  $u \in \mathbb{R}^2$  then  $\text{span}\{\mathbf{u}\} = \{c\mathbf{u} : c \in \mathbb{R}\} = \text{set of all scalar multiples of } \mathbf{u}$  is a subspace of  $\mathbb{R}^2$ .

To show that these are the only subspaces of  $\mathbb{R}^2$ , assume that  $W \subset \mathbb{R}^2$  is any subspace of  $\mathbb{R}^2$ . Since  $W \subset \mathbb{R}^2$  is a subspace of  $\mathbb{R}^2$ , we have

that  $\mathbf{0} \in W$ . If  $W \neq \mathbf{0}$  then there is a vector  $\mathbf{u} \neq \mathbf{0}$  and  $\mathbf{u} \in W$ , and hence  $W$  contains  $c\mathbf{u}$  for every  $c \in \mathbb{R}$ . If  $W \neq \text{span}\{\mathbf{u}\}$ , then there is a vector  $\mathbf{v} \in W$  so that  $\mathbf{v} \neq k\mathbf{u}$  for any  $k \in \mathbb{R}$ .

Then  $\mathbf{z} = c\mathbf{u} + d\mathbf{v} \in \text{span}\{\mathbf{u}, \mathbf{v}\}$  for any  $c, d \in \mathbb{R}$ . Since  $W$  is a subspace  $c\mathbf{u}$  and  $d\mathbf{v} \in W$  for any  $c, d \in \mathbb{R}$ , and hence so does  $\mathbf{z} = c\mathbf{u} + d\mathbf{v}$ . Thus  $\mathbf{z} \in \text{span}\{\mathbf{u}, \mathbf{v}\} \implies \mathbf{z} \in W$ , and so  $\text{span}\{\mathbf{u}, \mathbf{v}\} \subset W \subset \mathbb{R}^2$ .

Let  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$  be any vector in  $\mathbb{R}^2$ , and let  $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  and let  $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ . We show that there are real numbers  $c$  and  $d$  so that  $c\mathbf{u} + d\mathbf{v} = \mathbf{x}$

$$\begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix} + \begin{pmatrix} dv_1 \\ dv_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (2.0.1)$$

$$\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (2.0.2)$$

Since  $\mathbf{v} \neq k\mathbf{u}$  for any  $k \in \mathbb{R}$  and since  $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  assume that  $u_1 \neq 0$ , and since  $k\mathbf{u} \neq \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  assume that  $v_2 \neq 0$ . Then

$$A = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.3)$$

Hence  $A$  is row equivalent to  $I_2$  and so  $A$  is invertible and so (2.0.2) has unique solution for  $c$  and  $d$ . Thus for any  $\mathbf{x} \in \mathbb{R}^2$  we can find real numbers  $c$  and  $d$  such that  $\mathbf{x} = c\mathbf{u} + d\mathbf{v}$ . Hence  $\mathbf{x} \in \mathbb{R}^2 \implies \mathbf{x} \in \text{span}\{\mathbf{u}, \mathbf{v}\}$ . Thus  $\mathbb{R}^2 \subset \text{span}\{\mathbf{u}, \mathbf{v}\} \subset W \subset \mathbb{R}^2$ .

Hence  $\text{span}\{\mathbf{u}, \mathbf{v}\} = W = \mathbb{R}^2$ , and so the only subspace of  $\mathbb{R}^2$  are  $\mathbf{0}$ ,  $\mathbb{R}^2$ , and  $L = \{c\mathbf{u} : \mathbf{u} \neq \mathbf{0}, c \in \mathbb{R}\}$ .

c. The following are the subspaces of  $\mathbb{R}^3$  :

1. Origin is a trivial subspace of  $\mathbb{R}^3$ .
2.  $\mathbb{R}^3$  itself is a trivial subspace of  $\mathbb{R}^3$ .
3. Every line through origin is subspace of  $\mathbb{R}^3$ .
4. Every plane in  $\mathbb{R}^3$  passing through origin is a subspace  $\mathbb{R}^3$ .

*Proof :* Let  $W$  be a plane passing through origin. We need  $\mathbf{0} \in W$ , but we have that since we're only considering planes that contain origin. Next, we need  $W$  is closed under vector addition. If  $\mathbf{w}_1$  and  $\mathbf{w}_2$  both belong to  $W$ , then so does  $\mathbf{w}_1 + \mathbf{w}_2$  because it's found by constructing a parallelogram, and the whole parallelogram lies in the plane  $W$ . Finally, we need  $W$  is closed under scalar products, but it is since scalar multiples lie in a straight line through the origin, and that line lies in  $W$ . Thus, each plane  $W$  passing through the origin is a subspace of  $\mathbb{R}^3$ .

5. The intersection of any of the above subspaces will also be a subspace of  $\mathbb{R}^3$

*Proof :* Let  $W$  be a collection of subspaces of  $V$ , and let  $W = \cap W_i$  be their intersection. Since each  $W_i$  is a subspace, each contains the zero vector. Thus the zero vector is in the intersection  $W$ , and  $W$  is non-empty. Let  $\alpha$  and  $\beta$  be vectors in  $W$  and let  $c$  be a scalar. By definition of  $W$ , both  $\alpha$  and  $\beta$  belong to each  $W_i$ , and because each  $W_i$  is a subspace, the vector  $(c\alpha + \beta)$  is again in  $W$ . Hence by definition of subspace,  $W$  is a subspace of  $V$ .

These 5 are only subspaces of  $\mathbb{R}^3$  possible.