

Matrix theory - Assignment 7

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Abstract—This document illustrates on finding foot of perpendicular from plane using SVD

Download all python codes from

<https://github.com/shreeprasadbhat/matrix-theory/blob/master/assignment7/>

and latex-tikz from

<https://github.com/shreeprasadbhat/matrix-theory/blob/master/assignment7/>

1 PROBLEM

Determine the distance from the Y-axis to the plane $5x - 2z - 3 = 0$

2 SOLUTION

Equation of plane can be expressed as

$$\mathbf{n}^T \mathbf{x} = c \quad (2.0.1)$$

Rewriting given equation of plane in (2.0.1) form

$$\begin{pmatrix} 5 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \quad (2.0.2)$$

where : $\mathbf{n} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $c = 3$

We need to represent equation of plane in parametric form,

$$\mathbf{x} = \mathbf{p} + \lambda_1 \mathbf{q} + \lambda_2 \mathbf{r} \quad (2.0.3)$$

Here p is any point on plane and \mathbf{q}, \mathbf{r} are two vectors parallel to plane and hence \perp to \mathbf{n} . Find two vectors that are \perp to \mathbf{n}

$$\begin{pmatrix} 5 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad (2.0.4)$$

Put $a = 0$ and $b = 1$ in (2.0.3), $\implies c = 0$

Put $a = 1$ and $b = 0$ in (2.0.3), $\implies c = \frac{5}{2}$

$$\text{Hence } \mathbf{q} = \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix}, \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Let us find point \mathbf{p} on the plane. Put $x = 1, y = 0$ in

$$(2.0.2), \text{ we get } \mathbf{p} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Since given plane is parallel to y-axis, we can use any point P on y-axis to compute shortest distance.

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.5)$$

Let \mathbf{Q} be the point on plane with shortest distance to \mathbf{P} . \mathbf{Q} can be expressed in (2.0.4) form as

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (2.0.6)$$

Equation \mathbf{P} and \mathbf{Q} , and computing pseudo inverse using SVD should give the value of λ_1 and λ_2 (since plane and y-axis never intersect pseudo inverse should give the points which are closest)

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.7)$$

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \quad (2.0.8)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \quad (2.0.9)$$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (2.0.10)$$

$$\mathbf{x} = \mathbf{M}^+ \mathbf{b} \quad (2.0.11)$$

$$\text{where } \mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

Diagonalize $\mathbf{M}\mathbf{M}^T$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & 0 \\ \frac{5}{2} & 0 & \frac{25}{4} \end{pmatrix} \quad (2.0.12)$$

$$= \begin{pmatrix} 0 & \frac{2}{5} & -\frac{5}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{29}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{5} & 0 & 1 \\ -\frac{5}{2} & 0 & 1 \end{pmatrix} \quad (2.0.13)$$

$$= \mathbf{U}\mathbf{\Sigma}^T\mathbf{\Sigma}\mathbf{U}^T \quad (2.0.14)$$

Verify (2.0.13) from,

codes/diagonalize1.py

Diagonalize $\mathbf{M}^T\mathbf{M}$

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{29}{4} & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.15)$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{29}{4} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.0.16)$$

$$= \mathbf{V}\mathbf{\Sigma}^T\mathbf{\Sigma}\mathbf{V}^T \quad (2.0.17)$$

Verify (2.0.16) from,

codes/diagonalize2.py

Compute SVD of \mathbf{M} from (2.0.13) and (2.0.18),

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \quad (2.0.18)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{5} & -\frac{5}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{29}}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.19)$$

$$\mathbf{M}^+ = \mathbf{V}\mathbf{\Sigma}^T\mathbf{U}^T \quad (2.0.20)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{29}}{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{5} & 0 & 1 \\ -\frac{5}{2} & 0 & 1 \end{pmatrix} \quad (2.0.21)$$

$$= \begin{pmatrix} \frac{4}{29} & 0 & \frac{10}{29} \\ 0 & 1 & 0 \end{pmatrix} \quad (2.0.22)$$

Verify (2.0.22) from,

codes/pseudo_inverse.py

Substitute (2.0.22) in (2.0.11),

$$\mathbf{x} = \begin{pmatrix} \frac{4}{29} & 0 & \frac{10}{29} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{14}{29} \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \quad (2.0.23)$$

Substituting λ_1, λ_2 in (2.0.6)

$$\mathbf{Q} = \begin{pmatrix} \frac{15}{29} \\ 0 \\ -\frac{6}{29} \end{pmatrix} \quad (2.0.24)$$

Distance between point \mathbf{P} and \mathbf{Q} is

$$\|\mathbf{P} - \mathbf{Q}\| = \sqrt{\left(\frac{15}{29}\right)^2 + 0 + \left(-\frac{6}{29}\right)^2} = \frac{3}{\sqrt{29}} \quad (2.0.25)$$

Hence the distance from y-axis to $5x - 2z - 3 = 0$ is $\frac{3}{\sqrt{29}}$