Matrix theory - Assignment 10

Shreeprasad Bhat
AI20MTECH14011

***Abstract*—This document illustrates applications of Rank-Nullity Theorem**

Download latex-tikz from

https://github.com/shreeprasadbhat/matrix−theory/blob/master/assignment10/

1 Problem

Let $\mathbf{A}$ be an $m \times n$ matrix with entries in F and let T be the linear transformation from $F^{n\times 1}$ into $F^{m\times l}$ defined by $T(\mathbf{X}) = \mathbf{AX}$. Show that

1) if $m < n$ it may happen that T is onto without being non-singular
2) if $m > n$ we may have T non-singular but not onto.

2 Definitions

A linear transformation $T(\mathbf{X}) : \mathbb{V} \to \mathbb{V}$ is said to be *singular* if $\exists$ some $\mathbf{X} \in \mathbb{V}$ s.t

$$T(\mathbf{X}) = \mathbf{0} \text{ and } \mathbf{X} \neq \mathbf{0} \tag{2.0.1}$$

$$\text{i.e } Nullity(T) \neq 0 \tag{2.0.2}$$

A linear transformation $T(\mathbf{X}) : \mathbb{V} \to \mathbb{V}$ is said to be *non-singular* if $\exists$ some $\mathbf{X} \in \mathbb{V}$ s.t

$$T(\mathbf{X}) = \mathbf{0} \implies \mathbf{X} = \mathbf{0} \tag{2.0.3}$$

$$\text{i.e } Nullity(T) = 0 \tag{2.0.4}$$

A linear transformation $T(\mathbf{X}) : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *onto* if for every $\mathbf{b} \in \mathbb{R}^m$, $T(\mathbf{X}) = \mathbf{b}$ has atleast one solution $\mathbf{X} \in \mathbb{R}^n$

$$\text{i.e } Rank(T) = m \ \& \tag{2.0.5}$$

$$Rank(T) \leq n \tag{2.0.6}$$

3 Solution

Let $\mathbf{A}$ be an $m \times n$ matrix with entries in F and let T be the linear transformation from $F^{n\times 1}$ into $F^{m\times l}$ defined by $T(\mathbf{X}) = \mathbf{AX}$	
$m < n$	$m > n$
$\implies rank(\mathbf{A}) < n$	$\implies rank(\mathbf{A}) > n$
Hence $T(\mathbf{X})$ can be onto.	If $T(\mathbf{X}) = \mathbf{0}, \mathbf{AX} = \mathbf{0} \implies \mathbf{X} = \mathbf{0}$ Hence T(X) may be a non-singular.
	$\implies Nullity(\mathbf{A}) = 0$
From Rank-Nullity Theorem,	But Rank-Nullity Theorem is not satisfied,
$Rank(\mathbf{A}) + Nullity(\mathbf{A}) = n$	$\because Rank(\mathbf{A}) + Nullity(\mathbf{A}) > n.$
$\implies Nullity(\mathbf{A}) \neq 0$	$\implies T(\mathbf{X})$ do not span $F^{m\times 1}$.
$\therefore T(\mathbf{X})$ is a singular.	$\therefore T$ is not onto.