Matrix theory - Assignment 12

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***Abstract*—This document solves problem on representations of linear transformations**

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https://github.com/shreeprasadbhat/matrix−theory/blob/master/assignment12/

1 Problem

Let V and W be finite-dimensional vector spaces over the field F and let T be a linear transformation from V into W. If

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\} \text{ and } \mathcal{B}' = \{\beta_1, \ldots, \beta_m\} \tag{1.0.1}$$

are ordered bases for V and W, respectively, define the linear transformation $E^{p,q}$ as in the proof of Theorem 5: $E^{p,q}(\alpha_i) = \delta_{iq}\beta_p$. Then the $E^{p,q}$, $1 \le p \le m$, $1 \le q \le n$, form a basis for $L(V, W)$ and so

$$T = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q} \tag{1.0.2}$$

for certain scalars A_{pq} (the coordinates of T in this basis for $L(V, W)$). Show that the matrix A with entries $A(p, q) = A_{pq}$ is precisely the matrix of T relative to the pair $\mathcal{B}, \mathcal{B}'$.

2 Solution

Given,

$$T = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q} \tag{2.0.1}$$

where

$$E^{p,q}(\alpha_i) = \begin{cases} \beta_p & p = i \\ 0 & \text{otherwise} \end{cases} \tag{2.0.2}$$

$$= \delta_{iq}\beta_p \tag{2.0.3}$$

where $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ is basis of V and $\mathcal{B}' = \{\beta_1, \ldots, \beta_n\}$ is basis of W.

Consider a vector $\mathbf{x} = \{x_1, x_2, \ldots, x_n\} \in V$,

$$\mathbf{x} = \sum_{q=1}^{n} x_q \alpha_q \tag{2.0.4}$$

$$\therefore E^{p,q}(\mathbf{x}) = \sum_{q=1}^{n} x_q E^{p,q}(\alpha_q) \tag{2.0.5}$$

$$= x_q \delta_{iq} \beta_p \tag{2.0.6}$$

Consider $T(\mathbf{x})$, from (2.0.1)

$$T(\mathbf{x}) = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q}(\mathbf{x}) \tag{2.0.7}$$

Substitute (2.0.6) in (2.0.7)

$$T(\mathbf{x}) = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} x_p \delta_{iq} \beta_q \tag{2.0.8}$$

From (2.0.3), $\delta_{iq}\beta_q$ is the transformation of basis of V to W. Hence $T : V \to W$ is

$$T = \begin{pmatrix} \sum_{p=1}^{n} A_{p1} x_p \\ \sum_{p=1}^{n} A_{p2} x_p \\ \vdots \\ \sum_{p=1}^{n} A_{pm} x_p \end{pmatrix} \tag{2.0.9}$$

$$T = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ A_{m1} & A_{m2} & \ldots & A_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{2.0.10}$$

$\therefore$ We showed that the matrix A with entries $A(p, q) = A_{pq}$ is precisely the matrix of T relative to the pair $\mathcal{B}, \mathcal{B}'$.