Matrix theory - Challenge

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***Abstract*—This document solves problem of quadric forms using matrix method.**

Download latex-tikz from

https://github.com/shreeprasadbhat/matrix−theory/blob/master/challengeing_problems/challenge_23_11_20/

1 Problem

Find pair of planes represented by equation

$$9x^2 - 4y^2 + z^2 - 6xz - 4y - 1 = 0$$

2 Solution

General second degree equation in 3D is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + f = 0 \tag{2.0.1}$$

Can be written in matrix form as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \mathbf{V} = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}, \mathbf{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{2.0.3}$$

Given equation can be written in (2.0.2) form as

$$\mathbf{x}^T \begin{pmatrix} 9 & 0 & -3 \\ 0 & -4 & 0 \\ -3 & 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}^T \mathbf{x} - 1 = 0 \tag{2.0.4}$$

Find determinant of V,

$$\left|\mathbf{V}\right| = \begin{vmatrix} 9 & 0 & -3 \\ 0 & -4 & 0 \\ -3 & 0 & 1 \end{vmatrix} \tag{2.0.5}$$

$$= 9(-4) - 3(-12) \tag{2.0.6}$$

$$= 0 \tag{2.0.7}$$

Hence the given equation represents pair of planes.

Do $affine\ transformation$ of the given object.

Find a point on the line of intersection of planes, we have

$$\begin{pmatrix} 9 & 0 & -3 \\ 0 & -4 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \tag{2.0.8}$$

find (α, β, γ) using row reduction,

$$\begin{pmatrix} 9 & 0 & -3 & 0 \\ 0 & -4 & 0 & 2 \\ -3 & 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_3 + \frac{R_1}{3}} \begin{pmatrix} 9 & 0 & -3 & 0 \\ 0 & -4 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.0.9}$$

$$\begin{pmatrix} 9 & 0 & -3 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \tag{2.0.10}$$

$$\implies \beta = \frac{-1}{2} \tag{2.0.11}$$

$$\text{choose } \gamma = 0, \implies \alpha = 0 \tag{2.0.12}$$

$$\therefore (\alpha, \beta, \gamma) = (0, -\frac{1}{2}, 0) \tag{2.0.13}$$

To move the given object to origin, set

$$(x, y, z) = (x, y, z) - (\alpha, \beta, \gamma) \tag{2.0.14}$$

$$= (x, \quad y - \frac{1}{2}, \quad z) \tag{2.0.15}$$

Substitute (2.0.15) in given equation

$$9x^2 - 4\left(y - \frac{1}{2}\right)^2 + z^2 - 6xz - 4\left(y - \frac{1}{2}\right) - 1 \quad = 0 \tag{2.0.16}$$

$${}^2 - 4y^2 + z^2 - 6xz = 0 \tag{2.0.17}$$

(2.0.17) can be written in matrix form as

$$\mathbf{x}^T \begin{pmatrix} 9 & 0 & -3 \\ 0 & -4 & 0 \\ -3 & 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.0.18}$$

To make the coordinate axis parallel to object axis, perform *spectral decomposition* on (2.0.18), find eigenvectors,

$$\begin{pmatrix} 9-\lambda & 0 & -3 \\ 0 & -4-\lambda & 0 \\ -3 & 0 & 1-\lambda \end{pmatrix} = 0 \quad (2.0.19)$$

$$(9-\lambda)(-4-\lambda)(1-\lambda) - 3(-4-\lambda)(3) = 0 \quad (2.0.20)$$

$$(-4-\lambda)(9 - 9\lambda - \lambda + \lambda^2 - 9) = 0 \quad (2.0.21)$$

$$(-4-\lambda)\lambda(\lambda - 10) = 0 \quad (2.0.22)$$

$$\implies \text{Eigen values are } \lambda = 0, -4, 10 \quad (2.0.23)$$

Find eigen vectors,
Put $\lambda = 0$

$$\begin{pmatrix} 9 & 0 & -3 \\ 0 & -4 & 0 \\ -3 & 0 & 1 \end{pmatrix} \mathbf{e_1} = 0 \implies \mathbf{e_1} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \quad (2.0.24)$$

Put $\lambda = -4$

$$\begin{pmatrix} 13 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 5 \end{pmatrix} \mathbf{e_2} = 0 \implies \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (2.0.25)$$

Put $\lambda = 10$

$$\begin{pmatrix} -1 & 0 & -3 \\ 0 & -14 & 0 \\ -3 & 0 & -9 \end{pmatrix} \mathbf{e_3} = 0 \implies \mathbf{e_3} = \begin{pmatrix} \frac{1}{3} \\ 0 \\ 1 \end{pmatrix} \quad (2.0.26)$$

Normalized eigen vectors are

$$\mathbf{e_1} = \begin{pmatrix} \frac{-3}{\sqrt{10}} \\ 0 \\ \frac{1}{\sqrt{10}} \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e_3} = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ 0 \\ \frac{3}{\sqrt{10}} \end{pmatrix} \quad (2.0.27)$$

Hence eigen decomposition of matrix is

$$\begin{pmatrix} 9 & 0 & -3 \\ 0 & -4 & 0 \\ -3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{-3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 10 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \end{pmatrix} \quad (2.0.28)$$

Substitute (2.0.28) in (2.0.18)

$$\mathbf{x}^T \begin{pmatrix} \frac{-3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 10 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \end{pmatrix} \mathbf{x} = 0 \quad (2.0.29)$$

Put $x' = \frac{-3x+z}{\sqrt{10}}$, $y' = y$, $z' = \frac{x+3z}{\sqrt{10}}$ in (2.0.29), we get

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}^T \begin{pmatrix} 10 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = 0 \quad (2.0.30)$$

$$\implies 10x'^2 - 4y'^2 = 0 \quad (2.0.31)$$

$$y' = \pm\sqrt{\frac{5}{2}}x' \quad (2.0.32)$$

Substitute back $x' = \frac{-3x+z}{\sqrt{10}}$, $y' = y - \frac{1}{2}$ in (2.0.32)

$$\left(y - \frac{1}{2}\right) = \pm\sqrt{\frac{5}{2}}\left(\frac{-3x+z}{\sqrt{10}}\right) \quad (2.0.33)$$

$$\frac{2y-1}{2} = \pm\sqrt{\frac{5}{2\times 10}}(-3x+z) \quad (2.0.34)$$

$$\frac{2y-1}{2} = \pm\frac{(-3x+z)}{2} \quad (2.0.35)$$

$$2y - 1 = \pm(-3x+z) \quad (2.0.36)$$

$$(2.0.37)$$

Hence the planes are

$$\implies 3x - 2y - z - 1 = 0, \quad 3x + 2y - z + 1 = 0 \quad (2.0.38)$$