Matrix theory - Challenge

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***Abstract*—This document illustrates proving properties of traingle using linear algebra**

Download all latex-tikz from

https://github.com/shreeprasadbhat/matrix−theory/blob/master//challenging_problems/challenge_03_10_2020/

1 Problem

In conic sections you have seen that

$V = PDP^T$, with $P^TP = I$. So P is an orthogonal matrix. For what matrices V do you get this kind of decomposition where P is an orthogonal ? Can you prove this result?

2 ANSWER

We get this kind of result for real symmetric matrix. Any real symmetric matrix can always be decomposed as $V = PDP^T$, with $P^TP = I$.

3 PROOF

Consider an 2×2 real symmetric matrix

$$\mathbf{V} = \begin{pmatrix} a & h \\ h & b \end{pmatrix} \tag{3.0.1}$$

Its characteristic equation is

$$\lambda^2 - (a+b)\lambda + (ab - h^2) = 0 \tag{3.0.2}$$

So the eigenvalues are given by

$$\frac{(a+b) \pm \sqrt{(a-b)^2 + 4h^2}}{2} \tag{3.0.3}$$

These eigenvalues are equal only if

$$(a-b)^2 + 4h^2 = 0 \implies a = b, h = 0 \tag{3.0.4}$$

Let λ_1 and λ_2 be two eigenvalue of V. Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be two corresponding eigenvectors. We have

$$\lambda_1(\mathbf{u}^T\mathbf{v}) = (\lambda_1\mathbf{u})^T\mathbf{v} \tag{3.0.5}$$
$$= (V\mathbf{u})^T\mathbf{v} \tag{3.0.6}$$
$$= au_1v_1 + h(u_2v_1 + u_1v_2) + bu_2v_2 \tag{3.0.7}$$
$$= \mathbf{u}^T(\mathbf{V}\mathbf{v}) \tag{3.0.8}$$
$$= \mathbf{u}^T(\lambda_2\mathbf{v}) \tag{3.0.9}$$
$$= \lambda_2(\mathbf{u}^T\mathbf{v}) \tag{3.0.10}$$

If $\lambda_1 \neq \lambda_2$, then it follows that

$$\mathbf{u}^T\mathbf{v} = 0 \tag{3.0.11}$$

If $\lambda_1 = \lambda_2$, then from (3.0.4),

$$\mathbf{V} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \implies \mathbf{V} = aI \tag{3.0.12}$$

Thus, for real symmetric matrix eigenvectors will be orthogonal. By definition of orthogonal matrix, we have,

$$\mathbf{PP^T} = \mathbf{P}^T\mathbf{P} = I \tag{3.0.13}$$

Consider,

$$\mathbf{VP} = \begin{pmatrix} \mathbf{V}u_1 & \mathbf{V}v_1 \\ \mathbf{V}u_2 & \mathbf{V}v_2 \end{pmatrix} \tag{3.0.14}$$
$$= \begin{pmatrix} \boldsymbol{\lambda_1}u_1 & \boldsymbol{\lambda_2}v_1 \\ \boldsymbol{\lambda_1}u_2 & \boldsymbol{\lambda_2}v_2 \end{pmatrix} \tag{3.0.15}$$
$$= \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{3.0.16}$$
$$= \mathbf{PD} \tag{3.0.17}$$
$$\mathbf{VPP}^T = \mathbf{PDP}^T \tag{3.0.18}$$
$$\mathbf{V} = \mathbf{PDP}^T \tag{3.0.19}$$

Hence proved the result.