Matrix theory - Assignment 13

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***Abstract*—This document proves properties on linear functionals**

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https://github.com/shreeprasadbhat/matrix−theory/
    blob/master/assignment13/
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1 Problem

Show that the trace functional on $n \times n$ matrices is unique in the following sense. If W is the space of $n \times n$ matrices over the field F and if f is a linear functional on W such that $f(AB) = f(BA)$ for each A and B in W, then f is a scalar multiple of the trace function. If, in addition, $f(I) = n$, then f is the trace function.

2 Solution

Given, W is space of $n \times n$ matrices over field F and f is a linear functional on W such that

$$f(AB) = f(BA) \tag{2.0.1}$$

for each $A, B \in W$.

Let $\{E^{pq}, 1 \leq p, q \leq n\}$ be the basis of W, where E^{pq} is a $n \times n$ matrix such that

$$E^{pq} = \begin{cases} 1 & pq^{th}\text{-position} \\ 0 & \text{everywhere else} \end{cases} \tag{2.0.2}$$

Let $A \in W$, then

$$A = \sum_{p,q=1}^{n} c^{pq} E^{pq}, \quad \text{for some scalars } c^{pq} \tag{2.0.3}$$

Since f is linear,

$$f(A) = \sum_{p,q=1}^{n} c^{pq} f(E^{pq}) \tag{2.0.4}$$

Also,

$$\text{trace}(A) = \sum_{p,q=1}^{n} c^{pq} \text{trace}(E^{pq}) \tag{2.0.5}$$

Now let's find $f(E^{pq})$.
If $p = q$, we know

$$I = \sum_{p,q=1}^{n} E^{pp} \tag{2.0.6}$$

$$f(I) = \sum_{p,q=1}^{n} f(E^{pp}) \tag{2.0.7}$$

$$= n f(E^{pp}) \qquad \text{(by (2.0.2))} \tag{2.0.8}$$

$$\implies f(E^{pp}) = \frac{f(I)}{n} \tag{2.0.9}$$

If $p \neq q$, then

$$f(E^{pq}) = f(E^{p1} E^{1q}) \tag{2.0.10}$$

$$= f(E^{1q} E^{p1}) \qquad \text{(by (2.0.1))} \tag{2.0.11}$$

$$= \mathbf{0} \qquad \text{(by (2.0.2))} \tag{2.0.12}$$

Consider trace functional on $n \times n$ matrices, then

$$\text{trace}(E^{pq}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{2.0.13}$$

Hence from (2.0.9),(2.0.12) and (2.0.14) we can write $f(E^{pq})$ as

$$f(E^{pq}) = \frac{f(I)}{n} \text{trace}(E^{pq}) \tag{2.0.14}$$

Substitute (2.0.14) in (2.0.4)

$$f(A) = \sum_{p,q=1}^{n} c^{pq} \left(\frac{f(I)}{n} \text{trace}(E^{pq}) \right) \tag{2.0.15}$$

$$= \frac{f(I)}{n} \sum_{p,q=1}^{n} c^{pq} \text{trace}(E^{pq}) \tag{2.0.16}$$

$$f(A) = \frac{f(I)}{n} \text{trace}(A) \tag{2.0.17}$$

Hence f is a scalar multiple of trace functional.

If $f(I) = n$, then f is the trace function

$$f(A) = \text{trace}(A) \tag{2.0.18}$$