Matrix theory - Assignment 11

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***Abstract*—This document illustrates isomorphism properties of linear transformations**

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https://github.com/shreeprasadbhat/matrix−theory/
    blob/master/assignment11/
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1 Problem

Let V and W be finite-dimensional vector spaces over the field F and let U be an isomorphism of V and W. Prove that $T \to UTU^{-1}$ is an isomorphism of $L(V,V)$ onto $L(W,W)$.

2 Definitions

linear transformation	Let V and W be vector spaces over field F. A **linear transformation** V into W is a function T from V into W such that $T(c\alpha + \beta) = c(T\alpha) + T\beta$ for all α and β in V and all scalars in c in F.
isomorphism	If V and W are vector spaces over the field F, any $one-one$ linear transformation $T : V \to W$ is called **isormorphism of** V **onto** W
one-one	A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-one** if for every $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{AX} = \mathbf{b}$ has atmost one solution in $\mathbb{R}^n$. Equivalently, if $T(\mathbf{u}) = T(\mathbf{v})$, then $u = v$. By definition, all *invertible* transformations are **one-one**
invertible	A linear transformation $T : V \to W$ is **invertible** if there exists another linear transformation $U : W \to V$ such that UT is the *identity* transformation on V and TU is the identity transformation on W. T is **invertible** if and only if T is $one-one$ and *onto*

3 Proof

Given	$\mathcal{T}(T) : T \to UTU^{-1}$ U is isomorphism of V onto W that means U is $one - one$ $\mathcal{T} : L(V, V) \to L(W, W)$
To prove	$\mathcal{T}$ is isomorphism of $L(V, V)$ onto $L(W, W)$ It is same as proving $\mathcal{T}$ is invertible, because $isomorphim \implies one - one \implies invertible$ by definition
Proof	Consider inverse transformation $\mathcal{S} : L(W, W) \to L(V, V)$ $\mathcal{S} : S \to U^{-1}SU$ where $U^{-1}SU$ is a composition of 3 linear transformations $V \xrightarrow{U} W \xrightarrow{S} W \xrightarrow{U^{-1}} V$ Now consider $\mathcal{S}(UTU^{-1})$, $\mathcal{S}(UTU^{-1}) = U^{-1}(UTU^{-1})U = T$ Similarly consider $\mathcal{T}(U^{-1}SU)$, $\mathcal{T}(U^{-1}SU) = U(U^{-1}SU)U^{-1} = S$ $\implies TS = I$ and $ST = I$ we can say $\mathcal{T}$ is invertible since we have found an inverse $\mathcal{S}$ Hence $\mathcal{T}$ is one-one implies $\mathcal{T}$ isomorphism of V onto W

4 Example

Let

$$U = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \tag{4.0.1}$$

here U is an isomorphism from $\mathbb{R}^{2*2}$ to $\mathbb{R}^{2*2}$ since inverse of U exists and

$$U^{-1} = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{1}{2} \end{pmatrix} \tag{4.0.2}$$

Consider

$$T = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix} \in \mathbb{R}^{2\times2} \tag{4.0.3}$$

Now

$$UTU^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix} \tag{4.0.4}$$

$$= \begin{pmatrix} -16 & -7 \\ -33 & -14 \end{pmatrix} \in \mathbb{R}^{2\times2} \tag{4.0.5}$$

Also inverse exists for T

$$S = T^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} \end{pmatrix} \tag{4.0.6}$$

Since T inverse exists $\mathcal{T}(T) = UTU^{-1}$ is an isomorphism from $\mathbb{R}^{2\times2}$ onto $\mathbb{R}^{2\times2}$.