Matrix theory - Assignment 10

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***Abstract*—This document illustrates applications of Rank-Nullity Theorem**

Download latex-tikz from

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https://github.com/shreeprasadbhat/matrix−theory/
    blob/master/assignment10/
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1 Problem

Let $\mathbf{A}$ be an $m\times n$ matrix with entries in F and let T be the linear transformation from $F^{n\times 1}$ into $F^{m\times l}$ defined by $T(\mathbf{X}) = \mathbf{AX}$. Show that

1) if $m < n$ it may happen that T is onto without being non-singular
2) if $m > n$ we may have T non-singular but not onto.

2 Definitions

A linear transformation $T(\mathbf{X}) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be	
singular	if $\exists$ some $\mathbf{X} \in \mathbb{R}^m$ s.t $T(\mathbf{X}) = \mathbf{0}$ and $\mathbf{X} \neq \mathbf{0}$ i.e $Nullity(T) \neq 0$
non-singular	if $\exists$ some $\mathbf{X} \in \mathbb{R}^m$ s.t $T(\mathbf{X}) = \mathbf{0} \implies \mathbf{X} = \mathbf{0}$ i.e $Nullity(T) = 0$
onto	if for every $\mathbf{b} \in \mathbb{R}^m$, $T(\mathbf{X}) = \mathbf{b}$ has atleast one solution $\mathbf{X} \in \mathbb{R}^n$ i.e $dim(Col(T)) = m \implies Rank(T) = m$ and $m \leq n$ If $m > n$, then $T(\mathbf{X}) = \mathbf{b}$ will not have solution becuase Rank-Nullity theorem is not satisfied.

3 Solution

Let $\mathbf{A}$ be an $m \times n$ matrix with entries in F and let T be the linear transformation from $F^{n\times 1}$ into $F^{m\times l}$ defined by $T(\mathbf{X}) = \mathbf{AX}$. If,		
	$m < n$	$m > n$
singular	$\implies rank(\mathbf{A}) < n$ From Rank-Nullity Theorem, $Rank(\mathbf{A}) + Nullity(\mathbf{A}) = n$ $\implies Nullity(\mathbf{A}) \neq 0$ $\therefore T(\mathbf{X})$ is a singular.	$\implies rank(\mathbf{A}) > n$ If $T(\mathbf{X}) = \mathbf{0}, \mathbf{AX} = \mathbf{0}$ $\implies \mathbf{X} = \mathbf{0}$ Hence T(X) is non-singular. $\implies Nullity(\mathbf{A}) = 0$
onto	$\implies rank(\mathbf{A}) < n$ Hence $T(\mathbf{X})$ can be onto.	$Rank(\mathbf{A}) + Nullity(\mathbf{A}) > n.$ Rank-Nullity Theorem is, $T(\mathbf{X})$ do not span $F^{m\times 1}$. $\therefore T$ is not onto.

4 Example

Below is an example in which linear transformation is both onto and singular.

$$\text{Let, } T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \tag{4.0.1}$$

$$T(\mathbf{X}) = \mathbf{AX} = \mathbf{b} \tag{4.0.2}$$

$$\text{Let, } \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{4.0.3}$$

$$\text{Consider, } \mathbf{X} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \tag{4.0.4}$$

$$\implies \mathbf{AX} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \tag{4.0.5}$$

$$= \begin{pmatrix} 6 \\ 5 \end{pmatrix} \tag{4.0.6}$$

Since $Rank(\mathbf{A}) = 2$ and $Rank(\mathbf{A}) < 3$, T is onto.

$$\text{Consider, } \mathbf{X} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{4.0.7}$$

$$\implies \mathbf{AX} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{4.0.8}$$

$$= \mathbf{0} \tag{4.0.9}$$

Since $\exists \quad \mathbf{X} \neq \mathbf{0}$ such that $\mathbf{AX} = \mathbf{0}$, T is singular.

$\therefore$ T is both onto and singular.