

Matrix theory - Assignment 10

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Abstract—This document illustrates applications of Rank-Nullity Theorem

Download latex-tikz from

<https://github.com/shreeprasadbhat/matrix-theory/blob/master/assignment10/>

1 PROBLEM

Let \mathbf{A} be an $m \times n$ matrix with entries in F and let T be the linear transformation from $F^{n \times 1}$ into $F^{m \times 1}$ defined by $T(\mathbf{X}) = \mathbf{A}\mathbf{X}$. Show that if $m < n$ it may happen that T is onto without being non-singular. Similarly, show that if $m > n$ we may have T non-singular but not onto.

2 DEFINITION

A linear transformation $T(\mathbf{X}) : V \rightarrow V$ is said to be *singular* if \exists some $\mathbf{X} \in V$ s.t

$$T(\mathbf{X}) = \mathbf{0} \text{ and } \mathbf{X} \neq \mathbf{0} \quad (2.0.1)$$

$$\text{i.e } \text{Null}(T) \neq \phi \quad (2.0.2)$$

A linear transformation $T(\mathbf{X}) : V \rightarrow V$ is said to be *non-singular* if \exists some $\mathbf{X} \in V$ s.t

$$T(\mathbf{X}) = \mathbf{0} \implies \mathbf{X} = \mathbf{0} \quad (2.0.3)$$

$$\text{i.e } \text{Null}(T) = \phi \quad (2.0.4)$$

A linear transformation $T(\mathbf{X}) : F^{n \times 1} \rightarrow F^{m \times 1}$ is said to be *onto* if $\forall \mathbf{b} \in F^{m \times 1}$, $T(\mathbf{X}) = \mathbf{b}$ has at least one solution in \mathbf{X} in $F^{n \times 1}$

$$\text{i.e } \dim \text{Col}(T) = m \text{ \& } \quad (2.0.5)$$

$$\dim \text{Col}(T) \leq n \quad (2.0.6)$$

3 SOLUTION

1) if $m < n$

$$\implies \dim \text{Col}(\mathbf{A}) < n \quad (3.0.1)$$

Hence $T(\mathbf{X})$ can be onto.

From Rank-Nullity Theorem,

$$\dim \text{Col}(\mathbf{A}) + \dim \text{Null}(\mathbf{A}) = n \quad (3.0.2)$$

From (3.0.1) and (3.0.2)

$$\dim \text{Null}(\mathbf{A}) \neq 0 \quad (3.0.3)$$

\therefore By definition $T(\mathbf{X})$ is a singular transformation.

Hence if $m < n$ it may happen that T is onto without being non-singular.

2) if $m > n$

$$\implies \dim \text{col}(\mathbf{A}) > n \quad (3.0.4)$$

Assume \mathbf{A} is not singular.

$$\text{If } T(\mathbf{X}) = \mathbf{0} \quad (3.0.5)$$

$$\mathbf{A}\mathbf{X} = \mathbf{0} \quad (3.0.6)$$

$$\implies \mathbf{X} = \mathbf{0} \quad (3.0.7)$$

Hence $T(\mathbf{X})$ may be a non-singular.

$$\implies \dim \text{Null}(\mathbf{A}) = 0 \quad (3.0.8)$$

But Rank-Nullity Theorem is not satisfied,

$$\therefore \dim \text{Col}(\mathbf{A}) + \dim \text{Null}(\mathbf{A}) \neq n \quad (3.0.9)$$

From (3.0.4) and (3.0.9), we can say $T(\mathbf{X})$ do not span $F^{m \times 1}$ and $T(\mathbf{X})$ is not onto.

Hence if $m > n$ it may have that T is non-singular but not onto.