Matrix theory - Challenge

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***Abstract*—This document solves problem of quadric forms using matrix method.**

Download latex-tikz from

https://github.com/shreeprasadbhat/matrix−theory/blob/master/challengeing_problems/challenge_23_11_20/

1 Problem

Find pair of planes represented by equation

$$9x^2 - 4y^2 + z^2 - 6xz - 4y - 1 = 0$$

2 Solution

General second degree equation in 3D is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + f = 0 \tag{2.0.1}$$

Can be written in matrix form as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \mathbf{V} = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}, \mathbf{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{2.0.3}$$

Given equation can be written in (2.0.2) form as

$$\mathbf{x}^T\begin{pmatrix} 9 & 0 & -3 \\ 0 & -4 & 0 \\ -3 & 0 & 1 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}^T \mathbf{x} - 1 = 0 \tag{2.0.4}$$

Find determinant of V,

$$\left|\mathbf{V}\right| = \begin{vmatrix} 9 & 0 & -3 \\ 0 & -4 & 0 \\ -3 & 0 & 1 \end{vmatrix} \tag{2.0.5}$$

$$= 9(-4) - 3(-12) \tag{2.0.6}$$

$$= 0 \tag{2.0.7}$$

Hence the given equation represents pair of planes.

To move the figure to origin, put $y = y - \frac{1}{2}$, (2.0.4) becomes

$$\mathbf{x}^T\begin{pmatrix} 9 & 0 & -3 \\ 0 & -4 & 0 \\ -3 & 0 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{2.0.8}$$

To make coordinate axis parallel to figure axis, perform *spectral decomposition* on (2.0.8),

$$\mathbf{x}^T\begin{pmatrix} \frac{-3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} 10 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{-3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \end{pmatrix}\mathbf{x} = 0 \tag{2.0.9}$$

Put $x' = \frac{-3x+z}{\sqrt{10}}$, $y' = y$, $z' = \frac{x+3z}{\sqrt{10}}$ in (2.0.9), we get

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}^T \begin{pmatrix} 10 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = 0 \tag{2.0.10}$$

$$\implies 10x'^2 - 4y'^2 = 0 \tag{2.0.11}$$

$$y' = \pm\sqrt{\frac{5}{2}}x' \tag{2.0.12}$$

Substitute back $x' = \frac{-3x+z}{\sqrt{10}}$, $y' = y - \frac{1}{2}$ in (2.0.12)

$$\left(y - \frac{1}{2}\right) = \pm\sqrt{\frac{5}{2}}\left(\frac{-3x+z}{\sqrt{10}}\right) \tag{2.0.13}$$

$$\frac{2y-1}{2} = \pm\sqrt{\frac{5}{2\times 10}}(-3x+z) \tag{2.0.14}$$

$$\frac{2y-1}{2} = \pm\frac{(-3x+z)}{2} \tag{2.0.15}$$

$$2y - 1 = \pm(-3x+z) \tag{2.0.16}$$

$$\tag{2.0.17}$$

Hence the planes are

$$\implies 3x - 2y - z - 1 = 0, \;\; 3x + 2y - z + 1 = 0 \tag{2.0.18}$$