Matrix theory - Assignment 11

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***Abstract*—This document illustrates isomorphism properties of linear transformations**

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https://github.com/shreeprasadbhat/matrix−theory/blob/master/assignment11/

1 Problem

Let V and W be finite-dimensional vector spaces over the field F and let U be an isomorphism of V and W. Prove that $T \to UTU^{-1}$ is an isomorphism of $L(V, V)$ onto $L(W, W)$.

2 Definitions

linear transformation	Let V and W be vector spaces over field F. A **linear transformation** V into W is a function T from V into W such that $T(c\alpha + \beta) = c(T\alpha) + T\beta$ for all α and β in V and all scalars in c in F.
isomorphism	If V and W are vector spaces over the field F, any $one - one$ linear transformation $T : V \to W$ is called **isormorphism of** V **onto** W
one-one	A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-one** if for every $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{AX} = \mathbf{b}$ has atmost one solution in $\mathbb{R}^n$. Equivalently, if $T(\mathbf{u}) = T(\mathbf{v})$, then $u = v$. By definition, all *invertible* transformations are **one-one**
invertible	A linear transformation $T : V \to W$ is **invertible** if there exists another linear transformation $U : W \to V$ such that UT is the *identity* transformation on V and TU is the identity transformation on W. T is **invertible** if and only if T is $one - one$ and *onto*

3 Proof

Given,

$$\mathcal{T}(T) : T \to UTU^{-1} \tag{3.0.1}$$

where U is isomorphism of V onto W and,

$$\mathcal{T} : L(V,V) \to L(W,W) \tag{3.0.2}$$

We have to prove $\mathcal{T}$ is isomorphism of $L(V,V)$ onto $L(W,W)$. i.e it is same as proving $\mathcal{T}$ is invertible, because $isomorphim \implies one - one \implies invertible$ by definition.

Consider inverse transformation of (3.0.1)

$$\mathcal{S} : L(W,W) \to L(V,V) \tag{3.0.3}$$

$$\mathcal{S} : S \to U^{-1}SU \tag{3.0.4}$$

where $U^{-1}SU$ is a composition of 3 linear transformations

$$V \xrightarrow{U} W \xrightarrow{S} W \xrightarrow{U^{-1}} V \tag{3.0.5}$$

Now consider $\mathcal{S}(UTU^{-1})$,

$$\mathcal{S}(UTU^{-1}) = U^{-1}(UTU^{-1})U = T \tag{3.0.6}$$

Similarly consider $\mathcal{T}(U^{-1}SU)$,

$$\mathcal{T}(U^{-1}SU) = U(U^{-1}SU)U^{-1} = S \tag{3.0.7}$$

From (3.0.6) and (3.0.7)

$$TS = I \text{ and } ST = I \tag{3.0.8}$$

From (3.0.8) we can say $\mathcal{T}$ is invertible since we have found an inverse $\mathcal{S}$.

Hence $\mathcal{T}$ is one-one implies $\mathcal{T}$ isomorphism of V onto W.

4 Example

Let

$$U = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \tag{4.0.1}$$

here U is an isomorphism from $\mathbb{R}^{2*2}$ to $\mathbb{R}^{2*2}$ since inverse of U exists and

$$U^{-1} = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{1}{2} \end{pmatrix} \tag{4.0.2}$$

Consider

$$T = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix} \in \mathbb{R}^{2\times2} \tag{4.0.3}$$

Now

$$UTU^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix} \tag{4.0.4}$$

$$= \begin{pmatrix} -16 & -7 \\ -33 & -14 \end{pmatrix} \in \mathbb{R}^{2\times2} \tag{4.0.5}$$

Also inverse exists for T

$$S = T^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} \end{pmatrix} \tag{4.0.6}$$

Since T inverse exists $\mathcal{T}(T) = UTU^{-1}$ is an isomorphism from $\mathbb{R}^{2\times2}$ onto $\mathbb{R}^{2\times2}$.