

Matrix theory - Assignment 15

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Abstract—This document proves properties on transpose of linear transformation.

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<https://github.com/shreeprasadbhat/matrix-theory/blob/master/assignment15/>

1 PROBLEM

Let V be a finite-dimensional vector space over the field F . Show that $T \rightarrow T'$ is an isomorphism of $L(V, V)$ onto $L(V^*, V^*)$.

2 DEFINITIONS

<i>isomorphism</i>	A linear transformation $T : V \rightarrow W$ is <i>isomorphism of V onto W</i> if T is <i>one - one</i>
<i>one-one</i>	A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is <i>one-one</i> if $\forall \mathbf{b} \in \mathbb{R}^m, \mathbf{A}\mathbf{X} = \mathbf{b}$ has at most one solution in \mathbb{R}^n By definition, all <i>invertible</i> linear transformations are <i>one-one</i>
<i>invertible</i>	A linear transformation $T : V \rightarrow W$ is <i>invertible</i> if there exists another linear transformation $U : W \rightarrow V$ such that UT is identity transformation on V and TU is identity transformation on W

3 SOLUTION

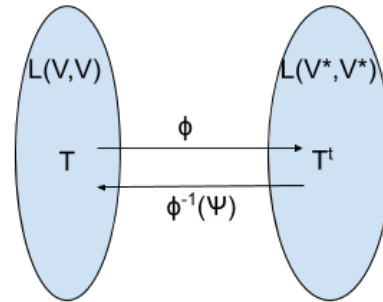


Fig. 0: $\phi : L(V, V) \rightarrow L(V^*, V^*)$

Given	$\phi(T) = T'$ where $\phi : L(V, V) \rightarrow L(V^*, V^*)$ $T \in L(V, V)$ and $T' \in L(V^*, V^*)$
To prove	ϕ is an <i>isomorphism of $L(V, V)$ onto $L(V^*, V^*)$</i> <i>i.e ϕ is one - one $\implies \phi$ is invertible</i>
Proof	Consider a linear transformation $\psi : L(V^*, V^*) \rightarrow L((V^*)^*, (V^*)^*)$ We know, $L((V^*)^*, (V^*)^*) = L(V, V)$ Hence $\psi : L(V^*, V^*) \rightarrow L(V, V)$ We have $\phi \circ \psi = I$ and $\psi \circ \phi = I$ $\therefore \phi$ is invertible with ψ its inverse. Hence ϕ is <i>isomorphic</i> .

4 EXAMPLE

Consider, $T \in L(\mathbb{R}^3, \mathbb{R}^3)$

$$T = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and} \quad (4.0.1)$$

$$T' = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad (4.0.2)$$

If $T \in L(\mathbb{R}^3, \mathbb{R}^3)$ we can show that $T' \in L(\mathbb{R}^{3*}, \mathbb{R}^{3*})$ as follows.

Consider linear functional $f : \mathbb{R}^{3*} \rightarrow F$

$$f(x, y, z) = 2x + 3y + 4z \quad (4.0.3)$$

Let $g : \mathbb{R}^3 \rightarrow F$. By definition of transpose,

$$g = T'f \quad (4.0.4)$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 6 \end{pmatrix} \quad (4.0.5)$$

$$g(x, y, z) = 7x + 5y + 6z \quad (4.0.6)$$

Consider vector in $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^3$,

$$Tv = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 3 \end{pmatrix} \quad (4.0.7)$$

$$f(Tv) = f(7, 3, 3) \quad (4.0.8)$$

$$= 2 * 7 + 3 * 3 + 4 * 3 = 35 \quad (4.0.9)$$

$$g(1, 2, 3) = 7 * 1 + 5 * 2 + 6 * 3 = 35 \quad (4.0.10)$$

Hence verified $g = T'f$, and T' is transpose of T .
Given, $\phi T = T'$

$$\phi = T'T^{-1} \quad (4.0.11)$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \quad (4.0.12)$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.0.13)$$

$$\phi = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.0.14)$$

$$\phi^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.0.15)$$

Since ϕ^{-1} exists ϕ is isomorphism of $L(V, V)$ onto $L(V^*, V^*)$

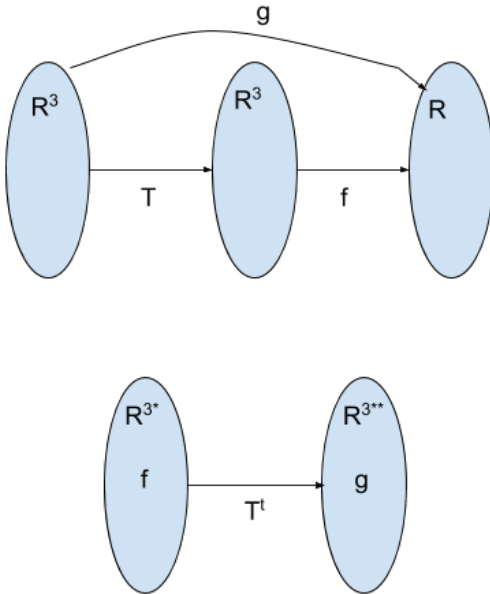


Fig. 0: $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $T' : \mathbb{R}^{3*} \rightarrow \mathbb{R}^{3**}$