

Matrix theory - Challenge

Shreeprasad Bhat
AI20MTECH14011

Abstract—This document illustrates proving properties of traingle using linear algebra

Download all latex-tikz from

https://github.com/shreeprasadbhat/matrix-theory/blob/master//challenging_problems/challenge_03_10_2020/

1 PROBLEM

In conic sections you have seen that $V = PDP^T$, with $P^T P = I$. So P is an orthogonal matrix. For what matrices V do you get this kind of decomposition where P is an orthogonal ? Can you prove this result?

2 ANSWER

We get this kind of result for real symmetric matrix. Any real symmetric matrix can always be decomposed as $V = PDP^T$, with $P^T P = I$.

3 PROOF

Let V be an $n \times n$ real symmetric matrix. Its characteristic equation is

$$\lambda^2 - (a+b)\lambda + (ab - h^2) = 0 \quad (3.0.1)$$

So the eigenvalues are given by

$$\frac{(a+b) \pm \sqrt{(a-b)^2 + 4h^2}}{2} \quad (3.0.2)$$

These eigenvalues are equal only if

$$(a-b)^2 + 4h^2 = 0 \implies a = b, h = 0 \quad (3.0.3)$$

Let λ_1 and λ_2 be any two eigenvalue of V . Let \mathbf{u} and \mathbf{v} be two corresponding eigenvectors. We have

$$\lambda_1(\mathbf{u}^T \mathbf{v}) = (\lambda_1 \mathbf{u})^T \mathbf{v} \quad (3.0.4)$$

$$= (\mathbf{V} \mathbf{u})^T \mathbf{v} \quad (3.0.5)$$

$$= \mathbf{u}^T (\mathbf{V} \mathbf{v}) \quad (3.0.6)$$

$$= \mathbf{u}^T (\lambda_2 \mathbf{v}) \quad (3.0.7)$$

$$= \lambda_2 (\mathbf{u}^T \mathbf{v}) \quad (3.0.8)$$

If $\lambda_1 \neq \lambda_2$, then it follows that

$$\mathbf{u}^T \mathbf{v} = 0 \quad (3.0.9)$$

If $\lambda_1 = \lambda_2$, then from (3.0.3),

$$\mathbf{V} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \implies \mathbf{V} = aI \quad (3.0.10)$$

Thus, for real symmetric matrix every pair of eigenvectors will be orthogonal to each other.

Let \mathbf{P} be eigenvector matrix, in which very two eigenvectors are orthogonal,

$$\mathbf{P} = \begin{pmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{pmatrix} \quad (3.0.11)$$

By definition of orthogonal matrix, we have,

$$\mathbf{P} \mathbf{P}^T = \mathbf{P}^T \mathbf{P} = I \quad (3.0.12)$$

Consider,

$$\mathbf{V} \mathbf{P} = \begin{pmatrix} | & & | \\ \mathbf{V} \mathbf{v}_1 & \cdots & \mathbf{V} \mathbf{v}_n \\ | & & | \end{pmatrix} \quad (3.0.13)$$

$$= \begin{pmatrix} | & & | \\ \lambda_1 \mathbf{v}_1 & \cdots & \lambda_n \mathbf{v}_n \\ | & & | \end{pmatrix} \quad (3.0.14)$$

$$= \begin{pmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \quad (3.0.15)$$

$$\mathbf{V} \mathbf{P} = \mathbf{P} \mathbf{D} \quad (3.0.16)$$

$$\mathbf{V} \mathbf{P} \mathbf{P}^T = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad (3.0.17)$$

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad (3.0.18)$$

Hence proved the result.