

Matrix theory - Challenge

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Abstract—This document illustrates proving properties of traingle using linear algebra

Download all latex-tikz from

https://github.com/shreeprasadbhat/matrix-theory/blob/master//challenging_problems/challenge_03_10_2020/

1 PROBLEM

In conic sections you have seen that

$V = PDP^T$, with $P^T P = I$. So P is an orthogonal matrix. For what matrices V do you get this kind of decomposition where P is an orthogonal ? Can you prove this result?

2 ANSWER

We get this kind of result for symmetric matrix. Any symmetric matrix can always be decomposed as $V = PDP^T$, with $P^T P = I$.

3 PROOF

Let V be an 2×2 symmetric matrix. Its characteristic equation is

$$\lambda^2 - (a + b)\lambda + (ab - h^2) = 0 \quad (3.0.1)$$

So the eigenvalues are given by

$$\frac{(a + b) \pm \sqrt{(a - b)^2 + 4h^2}}{2} \quad (3.0.2)$$

Since discriminant $(a - b)^2 + 4h^2$ is always positive, **eigenvalues of symmetric matrices will always be real.**

Also, these eigenvalues are equal only if

$$(a - b)^2 + 4h^2 = 0 \implies a = b, h = 0 \quad (3.0.3)$$

Let λ_1 and λ_2 be any two eigenvalue of V. Let \mathbf{u} and \mathbf{v} be two corresponding eigenvectors. We have

$$\lambda_1(\mathbf{u}^T \mathbf{v}) = (\lambda_1 \mathbf{u})^T \mathbf{v} = (V\mathbf{u})^T \mathbf{v} \quad (3.0.4)$$

$$= \mathbf{u}^T (V\mathbf{v}) = \mathbf{u}^T (\lambda_2 \mathbf{v}) \quad (3.0.5)$$

$$= \lambda_2(\mathbf{u}^T \mathbf{v}) \quad (3.0.6)$$

If $\lambda_1 \neq \lambda_2$, then it follows that

$$\mathbf{u}^T \mathbf{v} = 0 \quad (3.0.7)$$

If $\lambda_1 = \lambda_2$, then from (3.0.3),

$$V = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \implies V = aI \quad (3.0.8)$$

It follows that **any $n \times n$ symmetric matrix must possess n mutually orthogonal eigenvectors.**

Let P be eigenvector matrix, in which very two eigenvectors are orthogonal,

$$P = \begin{pmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{pmatrix} \quad (3.0.9)$$

By definition of orthogonal matrix, we have,

$$PP^T = P^T P = I \quad (3.0.10)$$

Consider,

$$VP = \begin{pmatrix} | & & | \\ V\mathbf{v}_1 & \cdots & V\mathbf{v}_n \\ | & & | \end{pmatrix} \quad (3.0.11)$$

$$= \begin{pmatrix} | & & | \\ \lambda_1 \mathbf{v}_1 & \cdots & \lambda_n \mathbf{v}_n \\ | & & | \end{pmatrix} \quad (3.0.12)$$

$$= \begin{pmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \quad (3.0.13)$$

$$VP = PD \quad (3.0.14)$$

$$VPP^T = PDP^T \quad (3.0.15)$$

$$V = PDP^T \quad (3.0.16)$$

Hence proved the result.