Matrix theory - Challenge

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***Abstract*—This document illustrates proving properties of traingle using linear algebra**

Download all latex-tikz from

https://github.com/shreeprasadbhat/matrix−theory/blob/master//challenging_problems/challenge_03_10_2020/

1 Problem

In conic sections you have seen that
$V = PDP^T$, with $P^T P = I$. So P is an orthogonal matrix. For what matrices V do you get this kind of decomposition where P is an orthogonal ? Can you prove this result?

2 ANSWER

We get this kind of result for symmetric matrix. Any symmetric matrix can always be decomposed as $V = PDP^T$, with $P^T P = I$.

3 PROOF

Let $\mathbf{V}$ be an 2×2 symmetric matrix. Its characteristic equation is

$$\lambda^2 - (a+b)\lambda + (ab - h^2) = 0 \tag{3.0.1}$$

So the eigenvalues are given by

$$\frac{(a+b) \pm \sqrt{(a-b)^2 + 4h^2}}{2} \tag{3.0.2}$$

Since discriminant $(a-b)^2 + 4h^2$ is always positive, **eigenvalues of symmetric matrices will always be real**.
Also, these eigenvalues are equal only if

$$(a-b)^2 + 4h^2 = 0 \implies a = b, h = 0 \tag{3.0.3}$$

Let λ_1 and λ_2 be any two eigenvalue of V. Let $\mathbf{u}$ and $\mathbf{v}$ be two corresponding eigenvectors. We have

$$\lambda_1(\mathbf{u}^T\mathbf{v}) = (\lambda_1\mathbf{u})^T\mathbf{v} = (V\mathbf{u})^T\mathbf{v} \tag{3.0.4}$$
$$= \mathbf{u}^T(\mathbf{V}\mathbf{v}) = \mathbf{u}^T(\lambda_2\mathbf{v}) \tag{3.0.5}$$
$$= \lambda_2(\mathbf{u}^T\mathbf{v}) \tag{3.0.6}$$

If $\lambda_1 \neq \lambda_2$, then it follows that

$$\mathbf{u}^T\mathbf{v} = 0 \tag{3.0.7}$$

If $\lambda_1 = \lambda_2$, then from (3.0.3),

$$\mathbf{V} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \implies \mathbf{V} = aI \tag{3.0.8}$$

It follows that **any** $n \times n$ **symmetric matrix must possess** n **mutually orthogonal eigenvectors**.

Let $\mathbf{P}$ be eigenvector matrix, in which very two eigenvectors are orthogonal,

$$\mathbf{P} = \begin{pmatrix} | & & | \\ \mathbf{v_1} & \cdots & \mathbf{v_n} \\ | & & | \end{pmatrix} \tag{3.0.9}$$

By definition of orthogonal matrix, we have,

$$\mathbf{P}\mathbf{P^T} = \mathbf{P}^T\mathbf{P} = I \tag{3.0.10}$$

Consider,

$$\mathbf{VP} = \begin{pmatrix} | & & | \\ \mathbf{V}\mathbf{v_1} & \cdots & \mathbf{V}\mathbf{v_n} \\ | & & | \end{pmatrix} \tag{3.0.11}$$
$$= \begin{pmatrix} | & & | \\ \lambda_1\mathbf{v_1} & \cdots & \lambda_n\mathbf{v_n} \\ | & & | \end{pmatrix} \tag{3.0.12}$$
$$= \begin{pmatrix} | & & | \\ \mathbf{v_1} & \cdots & \mathbf{v_n} \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \tag{3.0.13}$$
$$\mathbf{VP} = \mathbf{PD} \tag{3.0.14}$$
$$\mathbf{VP}\mathbf{P}^T = \mathbf{PD}\mathbf{P}^T \tag{3.0.15}$$
$$\mathbf{V} = \mathbf{PD}\mathbf{P}^T \tag{3.0.16}$$

Hence **for every symmetric matrix V there exists a diagonal matrix D and orthogonal matrix P such that V = PDP*T*. The diagonal entries of D are the eigenvalues of V and the columns of P are the corresponding eigenvectors**.