

# Matrix theory - Assignment 10

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**Abstract—This document illustrates applications of Rank-Nullity Theorem**

Download latex-tikz from

<https://github.com/shreeprasadbhat/matrix-theory/blob/master/assignment10/>

## 1 PROBLEM

Let  $\mathbf{A}$  be an  $m \times n$  matrix with entries in  $F$  and let  $T$  be the linear transformation from  $F^{n \times 1}$  into  $F^{m \times 1}$  defined by  $T(\mathbf{X}) = \mathbf{A}\mathbf{X}$ . Show that if  $m < n$  it may happen that  $T$  is onto without being non-singular. Similarly, show that if  $m > n$  we may have  $T$  non-singular but not onto.

## 2 DEFINITION

A linear transformation  $T(\mathbf{X}) : V \rightarrow V$  is said to be *singular* if  $\exists$  some  $\mathbf{X} \in V$  s.t

$$T(\mathbf{X}) = \mathbf{0} \text{ and } \mathbf{X} \neq \mathbf{0} \quad (2.0.1)$$

$$\text{i.e } \text{Null}(T) \neq \phi \quad (2.0.2)$$

A linear transformation  $T(\mathbf{X}) : V \rightarrow V$  is said to be *non-singular* if  $\exists$  some  $\mathbf{X} \in V$  s.t

$$T(\mathbf{X}) = \mathbf{0} \implies \mathbf{X} = \mathbf{0} \quad (2.0.3)$$

$$\text{i.e } \text{Null}(T) = \phi \quad (2.0.4)$$

A linear transformation  $T(\mathbf{X}) : F^{n \times 1} \rightarrow F^{m \times 1}$  is said to be *onto* if

$$\dim \text{Col}(T) \leq n \quad (2.0.5)$$

## 3 SOLUTION

1) if  $m < n$

$$\implies \dim \text{Col}(\mathbf{A}) < n \quad (3.0.1)$$

Hence  $T(\mathbf{X})$  can be onto.

From Rank-Nullity Theorem,

$$\dim \text{Col}(\mathbf{A}) + \dim \text{Null}(\mathbf{A}) = n \quad (3.0.2)$$

From (3.0.1) and (3.0.2)

$$\dim \text{Null}(\mathbf{A}) \neq 0 \quad (3.0.3)$$

$\therefore$  By definition  $T(\mathbf{X})$  is a singular transformation.

Hence if  $m < n$  it may happen that  $T$  is onto without being non-singular.

2) if  $m > n$

$$\implies \dim \text{col}(\mathbf{A}) > n \quad (3.0.4)$$

Assume  $\mathbf{A}$  is not singular.

$$\text{If } T(\mathbf{X}) = \mathbf{0} \quad (3.0.5)$$

$$\mathbf{A}\mathbf{X} = \mathbf{0} \quad (3.0.6)$$

$$\implies \mathbf{X} = \mathbf{0} \quad (3.0.7)$$

Hence  $T(\mathbf{X})$  may be a non-singular.

$$\implies \dim \text{Null}(\mathbf{A}) = 0 \quad (3.0.8)$$

But Rank-Nullity Theorem is not satisfied,

$$\therefore \dim \text{Col}(\mathbf{A}) + \dim \text{Null}(\mathbf{A}) \neq n \quad (3.0.9)$$

From (3.0.4) and (3.0.9), we can say  $T(\mathbf{X})$  do not span  $F^{m \times 1}$  and  $T(\mathbf{X})$  is not onto.

Hence if  $m > n$  it may have that  $T$  is non-singular but not onto.