Matrix theory - Assignment 5

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***Abstract*—This document illustrates solving pair of straight lines using linear algebra**

Download all python codes from

```
https://github.com/shreeprasadbhat/matrix−theory/
    tree/master/assignment5/codes
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and latex codes from

```
https://github.com/shreeprasadbhat/matrix−theory/
    blob/master/assignment5/
```

1 Problem

Find the value of h so that the equation

$$6x^2 + 2hxy + 12y^2 + 22x + 31y + 20 = 0$$

may represent two straight lines.

2 Construction

The general equation second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

(2.0.1) represents pair of straight lines if

$$\begin{vmatrix} a & h & d \\ h & c & e \\ d & e & f \end{vmatrix} = 0 \tag{2.0.2}$$

3 Solution

From (2.0.2), given equation represents pair of straight lines if

$$\begin{vmatrix} 6 & h & 11 \\ h & 12 & \frac{31}{2} \\ 11 & \frac{31}{2} & 20 \end{vmatrix} = 0 \tag{3.0.1}$$

$$\implies h = \frac{17}{2} \text{ or } h = \frac{171}{20} \tag{3.0.2}$$

Verify (3.0.2) using python code from

```
https://github.com/shreeprasadbhat/matrix−theory/
    tree/master/assignment5/codes/
    solve_determinant.py
```

4 Find equation of pair of straight lines

4.1 Construction

The general equation second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{4.1.1}$$

Let (α, β) be their point of intersection, then

$$\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -d \\ -e \end{pmatrix} \tag{4.1.2}$$

Under *Affine transformation*,

$$\mathbf{x} = \mathbf{M}\mathbf{y} + c \tag{4.1.3}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{4.1.4}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X + \alpha \\ Y + \beta \end{pmatrix} \tag{4.1.5}$$

(4.1.1) under transformation (4.1.5) will become,

$$aX^2 + 2bXY + cY^2 = 0 \tag{4.1.6}$$

$$\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{4.1.7}$$

$$\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{4.1.8}$$

$$\begin{pmatrix} X' & Y' \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} X' \\ Y' \end{pmatrix} = 0 \tag{4.1.9}$$

$$\text{where } X' = Xu_1 + Yu_2 \text{ and } Y' = Xv_1 + Yv_2 \tag{4.1.10}$$

$$\implies \lambda_1(X')^2 + \lambda_2(Y')^2 = 0 \tag{4.1.11}$$

(4.1.11) is called *Spectral decomposition* of matrix

$$\implies X' = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}Y' \tag{4.1.12}$$

$$u_1X + u_2Y = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1X + v_2Y) \tag{4.1.13}$$

$$u_1(x-\alpha) + u_2(y-\beta) = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1(x-\alpha) + v_2(y-\beta)) \tag{4.1.14}$$

4.2 Solution

Given equation is

$$6x^2 + 17xy + 12y^2 + 22x + 31y + 20 = 0 \quad (4.2.1)$$

Substituting in (4.1.2)

$$\begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -11 \\ -\frac{31}{2} \end{pmatrix} \quad (4.2.2)$$

$$\implies \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (4.2.3)$$

Verify (4.2.3) using python code from

```
https://github.com/shreeprasadbhat/matrix−theory/
    tree/master/assignment5/codes/
    find_intersection.py
```

4.2.1 Taking $h = \frac{17}{2}$:

$$\text{From } \textit{Spectral theorem}, \ \mathbf{V} = \mathbf{PDP}^T \quad (4.2.4)$$

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \quad (4.2.5)$$

$$\mathbf{P} = \begin{pmatrix} \frac{-5\sqrt{13}-6}{17} & \frac{-6+5\sqrt{13}}{17} \\ 1 & 1 \end{pmatrix} \quad (4.2.6)$$

$$\mathbf{D} = \begin{pmatrix} 9 - \frac{5\sqrt{13}}{2} & 0 \\ 0 & 9 + \frac{5\sqrt{13}}{2} \end{pmatrix} \quad (4.2.7)$$

Verify (4.2.6) and (4.2.7) using python code from

```
https://github.com/shreeprasadbhat/matrix−theory/
    tree/master/assignment5/codes/diagonalize1.py
```

Substituting (4.2.3), (4.2.6) and (4.2.7) in (4.1.15),

$$\frac{-5\sqrt{13}-6}{17}(x+1) + (y-2) = \pm\sqrt{-\frac{9+\frac{5\sqrt{13}}{2}}{9-\frac{5\sqrt{13}}{2}}}\left(\frac{-6+5\sqrt{13}}{17}(x+1) + (y+2)\right) \quad (4.2.8)$$

Simplifying (4.2.8),

$$2x + 3y + 4 = 0 \text{ and } 3x + 4y + 5 = 0 \quad (4.2.9)$$

$$\implies (2x + 3y + 4)(3x + 4y + 5) = 0 \quad (4.2.10)$$

Verify (4.2.9) using python code from

```
https://github.com/shreeprasadbhat/matrix−theory/
    tree/master/assignment5/codes/calculate1.py
```

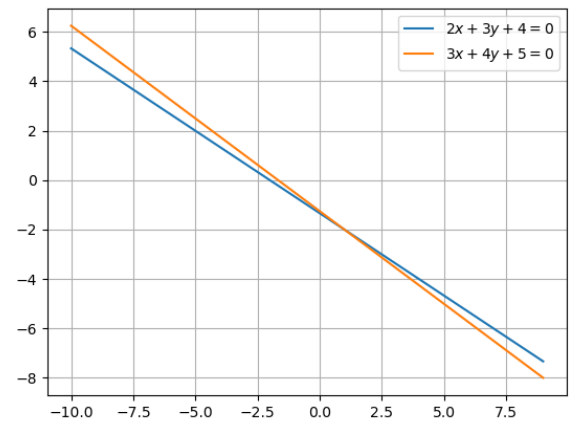

Fig. 1: Pair of straight lines $3x + 4y + 5 = 0$ and $2x + 3y + 4 = 0$

4.2.2 Taking $h = \frac{171}{20}$:

$$\text{From } \textit{Spectral theorem}, \ \mathbf{V} = \mathbf{PDP}^T \quad (4.2.11)$$

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{171}{2} \\ \frac{171}{2} & 12 \end{pmatrix} \quad (4.2.12)$$

$$\mathbf{P} = \begin{pmatrix} \frac{-\sqrt{3649}-20}{57} & \frac{-20+\sqrt{3649}}{57} \end{pmatrix} \quad (4.2.13)$$

$$\mathbf{D} = \begin{pmatrix} 9 - \frac{3\sqrt{3649}}{20} & 0 \\ 0 & 9 + \frac{3\sqrt{3649}}{20} \end{pmatrix} \quad (4.2.14)$$

Verify (4.2.13) and (4.2.14) using python code from

```
https://github.com/shreeprasadbhat/matrix−theory/
    tree/master/assignment5/codes/diagonalize2.py
```

Substituting (4.2.3),(4.2.13) and (4.2.14) in (4.1.15),

$$\frac{-\sqrt{3649}-20}{57}(x+1) + (y-2) = \pm\sqrt{-\frac{9+\frac{3\sqrt{3649}}{20}}{9-\frac{3\sqrt{3649}}{20}}}\left(\frac{-20+\sqrt{3649}}{57}(x+1) + (y+2)\right) \quad (4.2.15)$$

Simplifying (4.2.14),

$$2x + 3y + 4 = 0 \text{ and } 3x + 4y + 5 = 0 \quad (4.2.16)$$

$$\implies (2x + 3y + 4)(3x + 4y + 5) = 0 \quad (4.2.17)$$

Verify (4.2.15) using python code from

```
https://github.com/shreeprasadbhat/matrix−theory/
    tree/master/assignment5/codes/calculate2.py
```

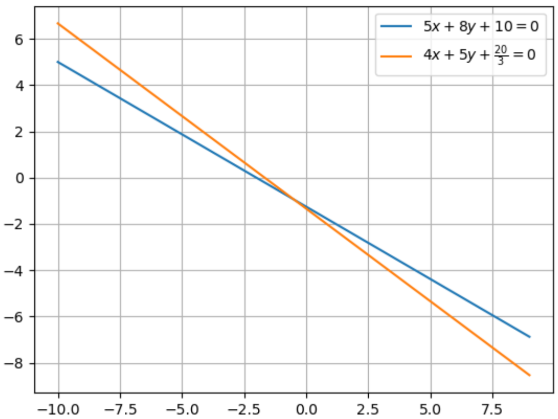

Fig. 2: Pair of straight lines $4x + 5y + \frac{20}{3} = 0$ and $5x + 8y + 10 = 0$

5 QR decomposition

5.1 Construction

$$\text{Let } \mathbf{V} = \begin{pmatrix} \mathbf{c_1} & \mathbf{c_2} \end{pmatrix} = \begin{pmatrix} a & h \\ h & b \end{pmatrix} \tag{5.1.1}$$

Apply *QR decomposition* on **V**,

$$\mathbf{V} = \mathbf{QR} \tag{5.1.2}$$

where :

$$\mathbf{Q} = \begin{pmatrix} \mathbf{q_1} & \mathbf{q_2} \end{pmatrix}, \ \mathbf{R} = \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} \tag{5.1.3}$$

$$\mathbf{c_1} = \mathbf{q_1} r_{11} \tag{5.1.4}$$

$$r_{11} = \|\mathbf{c_1}\| = \sqrt{a^2 + h^2} \tag{5.1.5}$$

$$\mathbf{q_1} = \frac{\mathbf{c_1}}{r_{11}} \tag{5.1.6}$$

$$\mathbf{c_2} = \mathbf{q_1} r_{12} + \mathbf{q_2} r_{22} \tag{5.1.7}$$

$$r_{21} = \mathbf{q_1}^T c_2 \tag{5.1.8}$$

$$r_{22} = \|\mathbf{c_2} - \mathbf{q_1} r_{12}\| \tag{5.1.9}$$

$$\mathbf{q_2} = \frac{\mathbf{c_2} - \mathbf{q_1} r_{12}}{r_2 2} \tag{5.1.10}$$

5.2 Solution

Given,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \tag{5.2.1}$$

From (5.1.5),

$$r_{11} = \sqrt{6^2 + \left(\frac{17}{2}\right)^2} = \frac{\sqrt{433}}{2} \tag{5.2.2}$$

Substitute (5.2.2) in (5.1.6)

$$\mathbf{q_1} = \begin{pmatrix} \frac{12}{\sqrt{433}} \\ \frac{17}{\sqrt{433}} \end{pmatrix} \tag{5.2.3}$$

Substitute (5.2.3) in (5.1.8)

$$r_{21} = \frac{306}{\sqrt{433}} \tag{5.2.4}$$

Substitute (5.2.4) in (5.1.9)

$$r_{22} = \frac{\sqrt{433}}{866} \tag{5.2.5}$$

Substitute (5.2.4) in (5.1.10)

$$\mathbf{q_2} = \begin{pmatrix} \frac{17}{\sqrt{433}} \\ -\frac{12}{\sqrt{433}} \end{pmatrix} \tag{5.2.6}$$

Hence QR decomposition of V is,

$$\mathbf{V} = \begin{pmatrix} \frac{12}{\sqrt{433}} & \frac{17}{\sqrt{433}} \\ \frac{17}{\sqrt{433}} & -\frac{12}{\sqrt{433}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{433}}{2} & \frac{306}{\sqrt{433}} \\ 0 & \frac{\sqrt{433}}{866} \end{pmatrix} \tag{5.2.7}$$

Verify (5.2.7) using python code from

```
https://github.com/shreeprasadbhat/matrix−theory/
    tree/master/assignment5/codes/
    find_QR_decomposition.py
```