

Matrix theory - Assignment 8

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Abstract—This document illustrates finding subspaces of a vector spaces

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<https://github.com/shreeprasadbhat/matrix-theory/blob/master/assignment8/>

1 PROBLEM

- Prove that only subspace of \mathbb{R}^1 are \mathbb{R}^1 and the zero subspace
- Prove that a subspace of \mathbb{R}^2 is \mathbb{R}^2 , or the zero subspace, or consists of all scalar multiples of some fixed vector in \mathbb{R}^2 . (The last type of subspace is, intuitively, a straight line through the origin.)
- Can you describe the subspaces of \mathbb{R}^3 ?

2 SOLUTION

- Let $W \neq 0$ be subspace of \mathbb{R}^1 . Then W is a nonempty subset of \mathbb{R}^1 and there exist $w \in W$ such that $w \neq 0$ which gives us that there exist w^{-1} .

Let $x \in \mathbb{R}^1$. Since W is in \mathbb{R}^1 we have that it is closed under scalar multiplication which gives us that $(xw^{-1})w = x(w^{-1}w) = x \cdot 1 = x \in W$

Hence $\mathbb{R}^1 \subset W$ and therefore $W = \mathbb{R}^1$

Thus the only subspace of \mathbb{R}^1 distinct of 0 is \mathbb{R}^1 and therefore only subspaces of \mathbb{R}^1 are 0 and \mathbb{R}^1 .

- Clearly, 0 and \mathbb{R}^2 itself are subspaces of \mathbb{R}^2 . If $u \neq 0$ and $u \in \mathbb{R}^2$ then $\text{span}\{\mathbf{u}\} = \{c\mathbf{u} : c \in \mathbb{R}\} = \text{set of all scalar multiples of } \mathbf{u}$ is a subspace of \mathbb{R}^2 .

To show that these are the only subspaces of \mathbb{R}^2 , assume that $W \subset \mathbb{R}^2$ is any subspace of \mathbb{R}^2 . Since $W \subset \mathbb{R}^2$ is a subspace of \mathbb{R}^2 , we have

that $\mathbf{0} \in W$. If $W \neq \mathbf{0}$ then there is a vector $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{u} \in W$, and hence W contains $c\mathbf{u}$ for every $c \in \mathbb{R}$. If $W \neq \text{span}\{\mathbf{u}\}$, then there is a vector $\mathbf{v} \in W$ so that $\mathbf{v} \neq k\mathbf{u}$ for any $k \in \mathbb{R}$.

Then $\mathbf{z} = c\mathbf{u} + d\mathbf{v} \in \text{span}\{\mathbf{u}, \mathbf{v}\}$ for any $c, d \in \mathbb{R}$. Since W is a subspace $c\mathbf{u}$ and $d\mathbf{v} \in W$ for any $c, d \in \mathbb{R}$, and hence so does $\mathbf{z} = c\mathbf{u} + d\mathbf{v}$. Thus $\mathbf{z} \in \text{span}\{\mathbf{u}, \mathbf{v}\} \implies \mathbf{z} \in W$, and so $\text{span}\{\mathbf{u}, \mathbf{v}\} \subset W \subset \mathbb{R}^2$.

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ be any vector in \mathbb{R}^2 , and let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and let $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. We show that there are real numbers c and d so that $c\mathbf{u} + d\mathbf{v} = \mathbf{x}$

$$\begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix} + \begin{pmatrix} dv_1 \\ dv_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (2.0.1)$$

$$\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (2.0.2)$$

Since $\mathbf{v} \neq k\mathbf{u}$ for any $k \in \mathbb{R}$ and since $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ assume that $u_1 \neq 0$, and since $k\mathbf{u} \neq \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ assume that $v_2 \neq 0$. Then

$$A = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.3)$$

Hence A is row equivalent to I_2 and so A is invertible and so (2.0.2) has unique solution for c and d . Thus for any $\mathbf{x} \in \mathbb{R}^2$ we can find real numbers c and d such that $\mathbf{x} = c\mathbf{u} + d\mathbf{v}$. Hence $\mathbf{x} \in \mathbb{R}^2 \implies \mathbf{x} \in \text{span}\{\mathbf{u}, \mathbf{v}\}$. Thus $\mathbb{R}^2 \subset \text{span}\{\mathbf{u}, \mathbf{v}\} \subset W \subset \mathbb{R}^2$.

Hence $\text{span}\{\mathbf{u}, \mathbf{v}\} = W = \mathbb{R}^2$, and so the only subspace of \mathbb{R}^2 are $\mathbf{0}$, \mathbb{R}^2 , and $L = \{c\mathbf{u} : \mathbf{u} \neq \mathbf{0}, c \in \mathbb{R}\}$.

c. The following are the subspaces of \mathbb{R}^3 :

1. Origin is a trivial subspace of \mathbb{R}^3 .
2. \mathbb{R}^3 itself is a trivial subspace of \mathbb{R}^3 .
3. Every line through origin is subspace of \mathbb{R}^3 .
4. Every plane in \mathbb{R}^3 passing through origin is a subspace \mathbb{R}^3 .

Proof : Let W be a plane passing through origin. We need $\mathbf{0} \in W$, but we have that since we're only considering planes that contain origin. Next, we need W is closed under vector addition. If \mathbf{w}_1 and \mathbf{w}_2 both belong to W , then so does $\mathbf{w}_1 + \mathbf{w}_2$ because it's found by constructing a parallelogram, and the whole parallelogram lies in the plane W . Finally, we need W is closed under scalar products, but it is since scalar multiples lie in a straight line through the origin, and that line lies in W . Thus, each plane W passing through the origin is a subspace of \mathbb{R}^3 .

5. The intersection of any of the above subspaces will also be a subspace of \mathbb{R}^3

These 5 are only subspaces of \mathbb{R}^3 possible.