Matrix theory - Assignment 15

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***Abstract*—This document proves properties on transpose of linear transformation.**

Download latex-tikz from

https://github.com/shreeprasadbhat/matrix−theory/blob/master/assignment15/

1 Problem

Let V be a finite-dimensional vector space over the field F. Show that $T \to T^t$ is an isomorphism of $L(V,V)$ onto $L(V^*,V^*)$.

2 Defintions

isomorphism	Any $one-one$ linear trans- -formation $T : V \to W$ is *isormorphism of V onto W*
one-one	$T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *one-one* if for every $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{AX} = \mathbf{b}$ has at most one solution in $\mathbb{R}^n$. By definition, all *invertible* transformations are *one-one*
invertible	$T : V \to W$ is *invertible* if there exists another linear transformation $U : W \to V$ such that UT is the *identity* transformation on V and TU is the identity transformation on W.

3 Solution

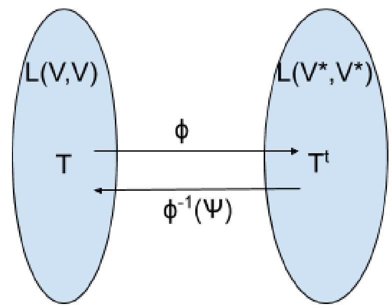

Fig. 0: $\phi : L(V,V) \to L(V^*,V^*)$

Given	$\phi(T) = T'$ where $\phi : L(V,V) \to L(V^*,V^*)$ $T \in L(V,V)$ and $T' \in L(V^*,V^*)$
To prove	ϕ is an *isomorphism* of $L(V,V)$ *onto* $L(V^*,V^*)$ *i.e* ϕ is $one-one \implies \phi$ is *invertible*
Proof	Consider a linear transformation $\psi : L(V^*,V^*) \to L((V^*)^*,(V^*)^*)$ We know, $L((V^*)^*,(V^*)^*) = L(V,V)$ Hence $\psi : L(V^*,V^*) \to L(V,V)$ We have $\phi \circ \psi = I$ and $\psi \circ \phi = I$ $\therefore \phi$ is invertible with ψ its inverse. Hence ϕ is *isomorphism*

4 Example

Consider, $T \in L(\mathbb{R}^3, \mathbb{R}^3)$

$$T = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \tag{4.0.1}$$

$$T' = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{4.0.2}$$

If $T \in L(\mathbb{R}^3, \mathbb{R}^3)$ we can show that $T' \in L(\mathbb{R}^{3^*}, \mathbb{R}^{3^*})$ as follows.

Consider linear functional $f : \mathbb{R}^{3^*} \to F$

$$f(x, y, z) = 2x + 3y + 4z \tag{4.0.3}$$

Let $g : \mathbb{R}^3 \to F$. By definition of transpose,

$$g = T' f \tag{4.0.4}$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 6 \end{pmatrix} \tag{4.0.5}$$

$$g(x, y, z) = 7x + 5y + 6z \tag{4.0.6}$$

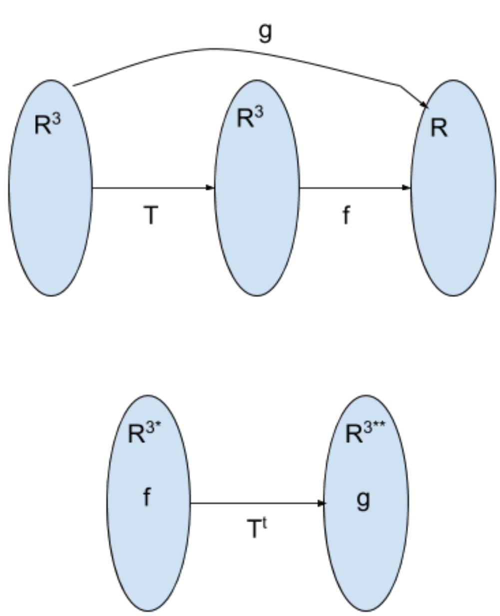

Fig. 0: $T : \mathbb{R}^3 \to \mathbb{R}^3$ and $T' : \mathbb{R}^{3^*} \to \mathbb{R}^{3^{**}}$

Consider vector in $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^3$,

$$Tv = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 3 \end{pmatrix} \tag{4.0.7}$$

$$f(Tv) = f(7, 3, 3) \tag{4.0.8}$$

$$= 2 * 7 + 3 * 3 + 4 * 3 = 35 \tag{4.0.9}$$

$$g(1, 2, 3) = 7 * 1 + 5 * 2 + 6 * 3 = 35 \tag{4.0.10}$$

Hence verified $g = T' f$, and T' is transpose of T. Given, $\phi T = T'$

$$\phi = T' T^{-1} \tag{4.0.11}$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \tag{4.0.12}$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.0.13}$$

$$\phi = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.0.14}$$

$$\phi^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.0.15}$$

Since ϕ^{-1} exists ϕ is isomorphism of $L(V, V)$ onto $L(V^*, V^*)$