Matrix theory - Assignment 13

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***Abstract*—This document proves properties on linear functionals**

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https://github.com/shreeprasadbhat/matrix−theory/blob/master/assignment13/

1 Problem

Show that the trace functional on $n \times n$ matrices is unique in the following sense. If W is the space of $n \times n$ matrices over the field F and if f is a linear functional on W such that $f(AB) = f(BA)$ for each A and B in W, then f is a scalar multiple of the trace function. If, in addition, $f(I) = n$, then f is the trace function.

2 Solution

Given	W is the space of $n \times n$ matrices over the field F f is a linear functional on W, that $f(AB) = f(BA)$
To prove	f is scalar multiple of trace $f(A) = c \times \text{trace}(A)$ and If $f(I) = n$, then f is the trace $f(A) = \text{trace}(A)$
Basis of W	Let $\{E^{pq}, 1 \le p, q \le n\}$ be the basis of W where E^{pq} is a $n \times n$ matrix such that, $E^{pq} = \begin{cases} 1 & pq^{th}\text{-position} \\ 0 & \text{everywhere else} \end{cases}$

Find $f(E^{pq})$	If $p = q$, we know $I = \sum_{p,q=1}^{n} E^{pp}$ then
	$f(I) = \sum_{p,q=1}^{n} f(E^{pp}) \implies f(I) = nf(E^{pp})$
	$\implies f(E^{pp}) = \frac{f(I)}{n}$
	If $p \neq q$, then
	$f(E^{pq}) = f(E^{p1}E^{1q}) = f(E^{1q}E^{p1}) = \mathbf{0}$
	$\text{trace}(E^{pq}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$
	$\therefore f(E^{pq}) = \frac{f(I)}{n}\text{trace}(E^{pq})$

Proof	Let $A \in W$, then $A = \sum_{p,q=1}^{n} c^{pq}E^{pq}$ Also, $\text{trace}(A) = \sum_{p,q=1}^{n} c^{pq}\text{trace}(E^{pq})$ Since f is linear, $f(A) = \sum_{p,q=1}^{n} c^{pq}f(E^{pq})$ $f(A) = \sum_{p,q=1}^{n} c^{pq}\left(\frac{f(I)}{n}\text{trace}(E^{pq})\right)$ $f(A) = \frac{f(I)}{n}\text{trace}(A)$ Hence f is a scalar multiple of trace. If $f(I) = n$, $f(A) = \text{trace}(A)$ Hence f is the trace function