Matrix theory - Assignment 9

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***Abstract*—This document proves result on linear transformations**

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https://github.com/shreeprasadbhat/matrix−theory/blob/master/assignment9/

1 Problem

Let $\mathbf{V}$ be the space of $n \times 1$ matrices over F and let $\mathbf{W}$ be the space of $m \times 1$ matrices over F. Let $\mathbf{A}$ be a fixed $m \times n$ matrix over F and let T be the linear transformation from $\mathbf{V}$ into $\mathbf{W}$ defined by $T(\mathbf{X}) = \mathbf{AX}$. Prove that T is the zero transformation if and only if $\mathbf{A}$ is the zero matrix.

2 Proof

If $\mathbf{A}_{m\times n}$ is a zero transformation and $\mathbf{X}_{n\times 1}$ is a vector, then

$$\mathbf{AX} = \mathbf{0}_{m\times 1} \tag{2.0.1}$$

Let,

$$\mathbf{A} = \begin{pmatrix} \mathbf{A_1} & \dots & \mathbf{A_j} & \dots & \mathbf{A_n} \end{pmatrix}_{1\times n} \text{ and} \tag{2.0.2}$$

$$\mathbf{X_j} = \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix}, \text{ where } x_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{2.0.3}$$

If $\mathbf{A}_{m\times n}$ is zero transformation, then for any vector $\mathbf{X}_{n\times 1}$, $\mathbf{AX} = \mathbf{0}$.
Consider ,

$$\mathbf{AX_j} = \mathbf{0}_{m\times 1} \tag{2.0.4}$$

$$\begin{pmatrix} \mathbf{A_1} \dots \mathbf{A_j} \dots \mathbf{A_n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} = \mathbf{0}_{m\times 1} \tag{2.0.5}$$

From (2.0.3) and (2.0.5)

$$\mathbf{A_j} = \mathbf{0}_{m\times 1} \text{ for } j = 1, 2, ...n \tag{2.0.6}$$

Substitute (2.0.6) in (2.0.2)

$$\mathbf{A} = \begin{pmatrix} \mathbf{0}_{m\times 1} & \mathbf{0}_{m\times 1} & \dots & \mathbf{0}_{m\times 1} \end{pmatrix}_{1\times n} \tag{2.0.7}$$

$$\therefore \mathbf{A} = \mathbf{0}_{m\times n} \tag{2.0.8}$$

Hence $\mathbf{A}$ is zero matrix.

Let us assume $\mathbf{A}_{m\times n}$ is a zero matrix

$$\mathbf{A} = \mathbf{0}_{m\times n} \tag{2.0.9}$$

Then,

$$T(\mathbf{X}) = \mathbf{AX} \tag{2.0.10}$$

$$= \mathbf{0}.\mathbf{X} \tag{2.0.11}$$

$$= \mathbf{0}_{m\times 1} \ , \forall \mathbf{X} \in F \tag{2.0.12}$$

Hence $T(\mathbf{X}) = \mathbf{AX}$ is the zero transformation.

From (2.0.8) and (2.0.12) it is proved that T is the zero transformation if and only if $\mathbf{A}$ is the zero matrix.