

# Matrix theory - Assignment 10

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**Abstract—**This document illustrates applications of Rank-Nullity Theorem

Download latex-tikz from

<https://github.com/shreeprasadbhat/matrix-theory/blob/master/assignment10/>

## 1 PROBLEM

Let  $\mathbf{A}$  be an  $m \times n$  matrix with entries in  $F$  and let  $T$  be the linear transformation from  $F^{n \times 1}$  into  $F^{m \times 1}$  defined by  $T(\mathbf{X}) = \mathbf{AX}$ . Show that

- 1) if  $m < n$  it may happen that  $T$  is onto without being non-singular
- 2) if  $m > n$  we may have  $T$  non-singular but not onto.

## 2 DEFINITIONS

A linear transformation $T(\mathbf{X}) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be	
singular	<p>if <math>\exists</math> some <math>\mathbf{X} \in \mathbb{R}^m</math> s.t</p> $T(\mathbf{X}) = \mathbf{0} \text{ and } \mathbf{X} \neq \mathbf{0}$ <p>i.e <math>\text{Nullity}(T) \neq 0</math></p>
non-singular	<p>if <math>\exists</math> some <math>\mathbf{X} \in \mathbb{R}^m</math> s.t</p> $T(\mathbf{X}) = \mathbf{0} \implies \mathbf{X} = \mathbf{0}$ <p>i.e <math>\text{Nullity}(T) = 0</math></p>
onto	<p>if for every <math>\mathbf{b} \in \mathbb{R}^m</math>, <math>T(\mathbf{X}) = \mathbf{b}</math> has atleast one solution <math>\mathbf{X} \in \mathbb{R}^n</math> and <math>m \leq n</math></p> <p>i.e <math>\dim(\text{Col}(T)) = m \implies \text{Rank}(T) = m</math></p> <p>If <math>m &gt; n</math>, then <math>T(\mathbf{X}) = \mathbf{b}</math> will not have solution because Rank-Nullity theorem is not satisfied.</p>

## 3 SOLUTION

Let $\mathbf{A}$ be an $m \times n$ matrix with entries in $F$ and let $T$ be the linear transformation from $F^{n \times 1}$ into $F^{m \times 1}$ defined by $T(\mathbf{X}) = \mathbf{AX}$ . If,		
	$m < n$	$m > n$
singular	$\implies \text{rank}(\mathbf{A}) < n$ From Rank-Nullity Theorem, $\text{Rank}(\mathbf{A}) + \text{Nullity}(\mathbf{A}) = n$ $\implies \text{Nullity}(\mathbf{A}) \neq 0$ $\therefore T(\mathbf{X})$ is a singular.	$\implies \text{rank}(\mathbf{A}) > n$ If $T(\mathbf{X}) = \mathbf{0}, \mathbf{AX} = \mathbf{0}$ $\implies \mathbf{X} = \mathbf{0}$ Hence $T(\mathbf{X})$ is non-singular. $\implies \text{Nullity}(\mathbf{A}) = 0$
onto	$\implies \text{rank}(\mathbf{A}) < n$ Hence $T(\mathbf{X})$ can be onto.	$\text{Rank}(\mathbf{A}) + \text{Nullity}(\mathbf{A}) > n$ . Rank-Nullity Theorem is, $T(\mathbf{X})$ do not span $F^{m \times 1}$ . $\therefore T$ is not onto.

## 4 EXAMPLE

Below is an example in which linear transformation is both onto and singular.

$$\text{Let, } T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \quad (4.0.1)$$

$$T(\mathbf{X}) = \mathbf{AX} = \mathbf{b} \quad (4.0.2)$$

$$\text{Let, } \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad (4.0.3)$$

$$\text{Consider, } \mathbf{X} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \quad (4.0.4)$$

$$\implies \mathbf{AX} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \quad (4.0.5)$$

$$= \begin{pmatrix} 6 \\ 5 \end{pmatrix} \quad (4.0.6)$$

Since  $\text{Rank}(\mathbf{A}) = 2$  and  $\text{Rank}(\mathbf{A}) < 3$ ,  $T$  is onto.

$$\text{Consider, } \mathbf{X} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (4.0.7)$$

$$\implies \mathbf{AX} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (4.0.8)$$

$$= \mathbf{0} \quad (4.0.9)$$

Since  $\exists \mathbf{X} \neq \mathbf{0}$  such that  $\mathbf{AX} = \mathbf{0}$ ,  $T$  is singular.

$\therefore T$  is both onto and singular.