Matrix theory - Challenge

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Abstract—This document illustrates proving properties of traingle using linear algebra

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1 PROBLEM

In conic sections you have seen that
$V = PDP^T$, with $P^T P = I$. So P is an orthogonal matrix. For what matrices V do you get this kind of decomposition where P is an orthogonal ? Can you prove this result?

2 ANSWER

We get this kind of result for real symmetric matrix. Any real symmetric matrix can always be decomposed as $V = PDP^T$, with $P^T P = I$.

3 PROOF

Let **V** be an 2×2 real symmetric matrix. Its characteristic equation is

$$\lambda^2 - (a + b)\lambda + (ab - h^2) = 0 \qquad (3.0.1)$$

So the eigenvalues are given by

$$\frac{(a + b) \pm \sqrt{(a - b)^2 + 4h^2}}{2} \qquad (3.0.2)$$

These eigenvalues are equal only if

$$(a - b)^2 + 4h^2 = 0 \implies a = b, h = 0 \qquad (3.0.3)$$

Let λ_1 and λ_2 be any two eigenvalue of V. Let **u** and **v** be two corresponding eigenvectors. We have

$$\lambda_1(\mathbf{u}^T\mathbf{v}) = (\lambda_1\mathbf{u})^T\mathbf{v} \qquad (3.0.4)$$
$$= (V\mathbf{u})^T\mathbf{v} \qquad (3.0.5)$$
$$= \mathbf{u}^T(V\mathbf{v}) \qquad (3.0.6)$$
$$= \mathbf{u}^T(\lambda_2\mathbf{v}) \qquad (3.0.7)$$
$$= \lambda_2(\mathbf{u}^T\mathbf{v}) \qquad (3.0.8)$$

If $\lambda_1 \neq \lambda_2$, then it follows that

$$\mathbf{u}^T\mathbf{v} = 0 \qquad (3.0.9)$$

If $\lambda_1 = \lambda_2$, then from (**??**),

$$\mathbf{V} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \implies \mathbf{V} = aI \qquad (3.0.10)$$

Thus, for real symmetric matrix every pair of eigenvectors will be orthogonal to each other.
Let **P** be eigenvector matrix, in which very two eigenvectors are orthogonal,

$$\mathbf{P} = \begin{pmatrix} | & & | \\ \mathbf{v_1} & \cdots & \mathbf{v_n} \\ | & & | \end{pmatrix} \qquad (3.0.11)$$

By definition of orthogonal matrix, we have,

$$\mathbf{PP^T} = \mathbf{P^T P} = I \qquad (3.0.12)$$

Consider,

$$\mathbf{VP} = \begin{pmatrix} | & & | \\ \mathbf{Vv_1} & \cdots & \mathbf{Vv_n} \\ | & & | \end{pmatrix} \qquad (3.0.13)$$

$$= \begin{pmatrix} | & & | \\ \lambda_1\mathbf{v_1} & \cdots & \lambda_n\mathbf{v_n} \\ | & & | \end{pmatrix} \qquad (3.0.14)$$

$$= \begin{pmatrix} | & & | \\ \mathbf{v_1} & \cdots & \mathbf{v_n} \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \qquad (3.0.15)$$

$$\mathbf{VP} = \mathbf{PD} \qquad (3.0.16)$$
$$\mathbf{VPP^T} = \mathbf{PDP^T} \qquad (3.0.17)$$
$$\mathbf{V} = \mathbf{PDP^T} \qquad (3.0.18)$$

Hence proved the result.