Matrix theory - Assignment 10

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***Abstract*—This document illustrates applications of Rank-Nullity Theorem**

Download latex-tikz from

https://github.com/shreeprasadbhat/matrix−theory/blob/master/assignment10/

1 Problem

Let $\mathbf{A}$ be an $m \times n$ matrix with entries in F and let T be the linear transformation from $F^{n\times 1}$ into $F^{m\times l}$ defined by $T(\mathbf{X}) = \mathbf{AX}$. Show that if $m < n$ it may happen that T is onto without being non-singular. Similarly, show that if $m > n$ we may have T non-singular but not onto.

2 Definition

A linear transformation $T(\mathbf{X}) : V \to V$ is said to be *singular* if $\exists$ some $\mathbf{X} \in V$ s.t

$$T(\mathbf{X}) = \mathbf{0} \text{ and } \mathbf{X} \neq \mathbf{0} \tag{2.0.1}$$
$$\text{i.e } Null(T) \neq \phi \tag{2.0.2}$$

A linear transformation $T(\mathbf{X}) : V \to V$ is said to be *non-singular* if $\exists$ some $\mathbf{X} \in V$ s.t

$$T(\mathbf{X}) = \mathbf{0} \implies \mathbf{X} = \mathbf{0} \tag{2.0.3}$$
$$\text{i.e } Null(T) = \phi \tag{2.0.4}$$

A linear transformation $T(\mathbf{X}) : F^{n\times 1} \to F^{m\times 1}$ is said to be *onto* if for every $\mathbf{b} \in F^{m\times 1}$, $T(\mathbf{X})$ has atleast one solution $\mathbf{X} \in F^{n\times 1}$

$$\text{i.e } dim\ Col(T) = m\ \& \tag{2.0.5}$$
$$dim\ Col(T) \leq n \tag{2.0.6}$$

3 Solution

1) if $m < n$

$$\implies dim\ Col(\mathbf{A}) < n \tag{3.0.1}$$

Hence $T(\mathbf{X})$ can be onto.

From Rank-Nullity Theorem,

$$dim\ Col(\mathbf{A}) + dim\ Null(\mathbf{A}) = n \tag{3.0.2}$$

From (3.0.1) and (3.0.2)

$$dim\ Null(\mathbf{A}) \neq 0 \tag{3.0.3}$$

$\therefore$ By definition $T(\mathbf{X})$ is a singular transformation.

Hence if $m < n$ it may happen that T is onto without being non-singular.

2) if $m > n$

$$\implies dim\ col(\mathbf{A}) > n \tag{3.0.4}$$

Assume $\mathbf{A}$ is not singular.

$$\text{If } T(\mathbf{X}) = \mathbf{0} \tag{3.0.5}$$
$$\mathbf{AX} = \mathbf{0} \tag{3.0.6}$$
$$\implies \mathbf{X} = \mathbf{0} \tag{3.0.7}$$

Hence T(X) may be a non-singular.

$$\implies dim\ Null(\mathbf{A}) = 0 \tag{3.0.8}$$

But Rank-Nullity Theorem is not satisfied,

$$\because dim\ Col(\mathbf{A}) + dim\ Null(\mathbf{A}) \neq n \tag{3.0.9}$$

From (3.0.4) and (3.0.9), we can say $T(\mathbf{X})$ do not span $F^{m\times 1}$ and $T(\mathbf{X})$ is not onto.

Hence if $m > n$ it may have that T is non-singular but not onto.