Matrix theory - Assignment 8

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***Abstract*—This document illustrates finding subspaces of a vector spaces**

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https://github.com/shreeprasadbhat/matrix−theory/blob/master/assignment8/

1 Problem

a. Prove that only subspace of $\mathbb{R}^1$ are $\mathbb{R}^1$ and the zero subspace
b. Prove that a subspace of $\mathbb{R}^2$ is $\mathbb{R}^2$, or the zero subspace, or consists of all scalar multiples of some fixed vector in $\mathbb{R}^2$. (The last type of subspace is, intuitively, a straight line through the origin.)
c. Can you describe the subspaces of $\mathbb{R}^3$?

2 Solution

a. Let $W \neq 0$ be subspace of $\mathbb{R}^1$. Then W is a nonempty subset of $\mathbb{R}^1$ and there exist $w \in W$ such that $w \neq 0$ which gives us that there exist w^{-1}.

   Let $x \in \mathbb{R}^1$. Since W is in $\mathbb{R}^1$ we have that it is closed under scalar multiplication which gives us that $(xw^{-1})w = x(w^{-1}w) = x.1 = x \in W$

   Hence $\mathbb{R}^1 \subset W$ and therefore $W = \mathbb{R}^1$

   Thus the only subspace of $\mathbb{R}^1$ distinct of 0 is $\mathbb{R}^1$ and therefore only subspaces of $\mathbb{R}^1$ are 0 and $\mathbb{R}^1$.

b. Clearly, 0 and $\mathbb{R}^2$ itself are subspaces of $\mathbb{R}^2$. If $u \neq 0$ and $u \in \mathbb{R}^2$ then span$\{\mathbf{u}\} = c\mathbf{u} : c \in \mathbb{R}$ = set of all scalar multiples of $\mathbf{u}$ is a subspace of $\mathbb{R}^2$.

   To show that these are the only subspaces of $\mathbb{R}^2$, assume that $W \subset \mathbb{R}^2$ is any subspace of $\mathbb{R}^2$. Since $W \subset \mathbb{R}^2$ is a subspace of $\mathbb{R}^2$, we have that $\mathbf{0} \in W$. If $W \neq \mathbf{0}$ then there is a vector $\mathbf{u} \neq 0$ and $\mathbf{u} \in W$, and hence W contains $c\mathbf{u}$ for every $c \in \mathbb{R}$. If $W \neq span\{\mathbf{u}\}$, then there is a vector $v \in W$ so that $\mathbf{v} \neq k\mathbf{u}$ for any $k \in \mathbb{R}$.

Then $\mathbf{z} = c\mathbf{u} + d\mathbf{v} \in span\{\mathbf{u},\mathbf{v}\}$ for any $c, d \in \mathbb{R}$. Since W is a subspace $c\mathbf{u}$ and $d\mathbf{v} \in W$ for any $c, d \in \mathbb{R}$, and hence so does $\mathbf{z} = c\mathbf{u} + d\mathbf{v}$. Thus $\mathbf{z} \in span\{\mathbf{u},\mathbf{v}\} \implies z \in W$, and so $span\{\mathbf{u},\mathbf{v}\} \subset W \subset \mathbb{R}^2$.

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ be any vector in $\mathbb{R}^2$, and let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and let $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. We show that there are real numbers c and d so that $c\mathbf{u} + d\mathbf{v} = \mathbf{x}$

$$\begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix} + \begin{pmatrix} dv_1 \\ dv_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.0.1}$$

$$\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.0.2}$$

Since $\mathbf{v} \neq k\mathbf{u}$ for any $k \in \mathbb{R}$ and since $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ assume that $u_1 \neq 0$, and since $k\mathbf{u} \neq \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ assume that $v_2 \neq 0$. Then

$$A = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.3}$$

Hence A is row equivalent to I_2 and so A is invertible and so (2.0.2) has unique solution for c and d. Thus for any $\mathbf{x} \in \mathbb{R}^2$ we can find real numbers c and d such that $\mathbf{x} = c\mathbf{u} + d\mathbf{v}$. Hence $\mathbf{x} \in \mathbb{R}^2 \implies x \in span\{\mathbf{u},\mathbf{v}\}$. Thus $\mathbb{R}^2 \subset span\{\mathbf{u},\mathbf{v}\} \subset W \subset \mathbb{R}^2$.

Hence $span\{\mathbf{u},\mathbf{v}\}$ = W = $\mathbb{R}^2$, and so the only subspace of $\mathbb{R}^2$ are $\mathbf{0}$, $\mathbb{R}^2$, and $L = c\mathbf{u} : \mathbf{u} \neq 0, c \in \mathbb{R}$.

c. The following are the subspaces of $\mathbb{R}^3$:

1. Origin is a trivial subspace of $\mathbb{R}^3$.

2. $\mathbb{R}^3$ itself is a trivial subspace of $\mathbb{R}^3$.

3. Every line through origin is subspace of $\mathbb{R}^3$.

4. Every plane in $\mathbb{R}^3$ passing through origin is a subspace $\mathbb{R}^3$.
   Proof : Let W be a plane passing through origin. We need $\mathbf{0} \in W$, but we have that since we're only considering planes that contain origin. Next, we need W is closed under vector addition. If $\mathbf{w_1}$ and $\mathbf{w_2}$ both belong to W, then so does $\mathbf{w_1} + \mathbf{w_2}$ because it's found by constructing a parallelogram, and the whole parallelogram lies in the plane W. Finally, we need W is closed under scalar products, but it is since scalar multiples lie in a straight line through the origin, and that line lies in W. Thus, each plane W passing through the origin is a subspace of $\mathbb{R}^3$.

5. The intersection of any of the above subspaces will also be a subspace of $\mathbb{R}^3$. Because intersection of subspaces of a vector space is also a subspace of vector space.
   Proof : Let W be a collection of subspaces of V, and let $W = \cap W_i$ be their intersection. Since each W_i is a subspace, each of it contains the zero vector. Thus the zero vector is in the intersection W, and W is non-empty. Let α and β be vectors in W and let c be a scalar. By definition of W, both α and β belong to each W_i, and because each W_i is a subspace, the vector $(c\alpha + \beta)$ is again in W. Hence by definition of subspace, W is a subspace of V.

These 5 are only subspaces of $\mathbb{R}^3$ possible.