Matrix theory - Assignment 7

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***Abstract*—This document illustrates on finding foot of perpendicular from plane using SVD**

Download all python codes from

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https://github.com/shreeprasadbhat/matrix-theory/
    blob/master/assignment7/
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and latex-tikz from

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https://github.com/shreeprasadbhat/matrix-theory/
    blob/master/assignment7/
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1 Problem

Determine the distance from the Y-axis to the plane $5x - 2z - 3 = 0$

2 Solution

Equation of plane can be expressed as

$$\mathbf{n}^T\mathbf{x} = c \tag{2.0.1}$$

Rewriting given equation of plane in (2.0.1) form

$$\begin{pmatrix} 5 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \tag{2.0.2}$$

where : $\mathbf{n} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $c = 3$

We need to represent equation of plane in parametric form,

$$\mathbf{x} = \mathbf{p} + \lambda_1\mathbf{q} + \lambda_2\mathbf{r} \tag{2.0.3}$$

Here $\mathbf{p}$ is any point on plane and $\mathbf{q}, \mathbf{r}$ are two vectors parallel to plane and hence $\perp$ to $\mathbf{n}$. Find two vectors that are $\perp$ to $\mathbf{n}$

$$\implies \begin{pmatrix} 5 & 0 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{2.0.4}$$

Put $a = 0$ and $b = 1$ in (2.0.4), $\implies c = 0$
Put $a = 1$ and $b = 0$ in (2.0.4), $\implies c = \frac{5}{2}$

Hence $\mathbf{q} = \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix}, \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Let us find point $\mathbf{p}$ on the plane. Put $x = 1, y = 0$ in (2.0.2), we get $\mathbf{p} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Since given plane is parallel to Y-axis, we can use any point $\mathbf{P}$ on Y-axis to compute shortest distance.

$$\text{Let, } \mathbf{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.5}$$

Let $\mathbf{Q}$ be the point on plane with shortest distance to $\mathbf{P}$. $\mathbf{Q}$ can be expressed in (2.0.3) form as

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_1\begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{2.0.6}$$

Equate $\mathbf{P}$ and $\mathbf{Q}$, and compute λ_1, λ_2 using *pseudoinverse* from *SVD* (since the given plane and Y-axis never intersect, pseudoinverse should give the points which are closest)

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_1\begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.7}$$

$$\lambda_1\begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \tag{2.0.8}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \tag{2.0.9}$$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{2.0.10}$$

$$\implies \mathbf{x} = \mathbf{M}^{+}\mathbf{b} \tag{2.0.11}$$

where $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$

Diagonalize $\mathbf{M}\mathbf{M}^T$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix}1 & 0\\ 0 & 1\\ \frac{5}{2} & 0\end{pmatrix}\begin{pmatrix}1 & 0 & \frac{5}{2}\\ 0 & 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & \frac{5}{2}\\ 0 & 1 & 0\\ \frac{5}{2} & 0 & \frac{25}{4}\end{pmatrix} \tag{2.0.12}$$

$$= \begin{pmatrix}0 & \frac{2}{5} & -\frac{5}{2}\\ 1 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}\frac{29}{4} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ \frac{2}{5} & 0 & 1\\ -\frac{5}{2} & 0 & 1\end{pmatrix} \tag{2.0.13}$$

$$= \mathbf{U}\Sigma^T\Sigma\mathbf{U}^T \tag{2.0.14}$$

Verify (2.0.13) from,

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codes/diagonalize1.py
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Diagonalize $\mathbf{M}^T\mathbf{M}$

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}1 & 0 & \frac{5}{2}\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\\ \frac{5}{2} & 0\end{pmatrix} = \begin{pmatrix}\frac{29}{4} & 0\\ 0 & 1\end{pmatrix} \tag{2.0.15}$$

$$= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{29}{4} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{2.0.16}$$

$$= \mathbf{V}\Sigma^T\Sigma\mathbf{V}^T \tag{2.0.17}$$

Verify (2.0.16) from,

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codes/diagonalize2.py
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Compute *SVD* of $\mathbf{M}$ from (2.0.13) and (2.0.18),

$$\mathbf{M} = \mathbf{U}\Sigma\mathbf{V}^T \tag{2.0.18}$$

$$\begin{pmatrix}1 & 0\\ 0 & 1\\ \frac{5}{2} & 0\end{pmatrix} = \begin{pmatrix}0 & \frac{2}{5} & -\frac{5}{2}\\ 1 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}\frac{\sqrt{29}}{2} & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{2.0.19}$$

$$\mathbf{M}^+ = \mathbf{V}\Sigma^T\mathbf{U}^T \tag{2.0.20}$$

$$= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{\sqrt{29}}{2} & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ \frac{2}{5} & 0 & 1\\ -\frac{5}{2} & 0 & 1\end{pmatrix} \tag{2.0.21}$$

$$= \begin{pmatrix}\frac{4}{29} & 0 & \frac{10}{29}\\ 0 & 1 & 0\end{pmatrix} \tag{2.0.22}$$

Verify (2.0.22) from,

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codes/pseudo_inverse.py
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Substitute (2.0.22) in (2.0.11),

$$\mathbf{x} = \begin{pmatrix}\frac{4}{29} & 0 & \frac{10}{29}\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}-1\\ 0\\ -1\end{pmatrix} = \begin{pmatrix}-\frac{14}{29}\\ 0\end{pmatrix} = \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \tag{2.0.23}$$

Substituting λ_1, λ_2 in (2.0.6)

$$\mathbf{Q} = \begin{pmatrix}\frac{15}{29}\\ 0\\ -\frac{6}{29}\end{pmatrix} \tag{2.0.24}$$

Distance between point $\mathbf{P}$ and $\mathbf{Q}$ is

$$\|\mathbf{P} - \mathbf{Q}\| = \sqrt{\left(\frac{15}{29}\right)^2 + 0 + \left(-\frac{6}{29}\right)^2} = \frac{3}{\sqrt{29}} \tag{2.0.25}$$

Hence, distance from Y-axis to $5x-2z-3 = 0$ is $\frac{3}{\sqrt{29}}$.

Verifying solution to (2.0.10) with *least squares* method

$$\mathbf{M}^T(\mathbf{b} - \mathbf{M}\mathbf{x}) = 0 \tag{2.0.26}$$

$$\implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.27}$$

Substituting $\mathbf{M}, \mathbf{b}$ from (2.0.9) in (2.0.27)

$$\begin{pmatrix}1 & 0 & \frac{5}{2}\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\\ \frac{5}{2} & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 0 & \frac{5}{2}\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}-1\\ 0\\ -1\end{pmatrix} \tag{2.0.28}$$

$$\begin{pmatrix}\frac{29}{4} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \begin{pmatrix}-\frac{7}{2}\\ 0\end{pmatrix} \tag{2.0.29}$$

$$\implies \frac{29}{4}\lambda_1 = -\frac{7}{2} \tag{2.0.30}$$

$$\lambda_1 = -\frac{7}{2} \times \frac{4}{29} = -\frac{14}{29} \tag{2.0.31}$$

$$\text{and } \lambda_2 = 0 \tag{2.0.32}$$

$$\implies \mathbf{x} = \begin{pmatrix}-\frac{14}{29}\\ 0\end{pmatrix} \tag{2.0.33}$$

Comparing (2.0.23) and (2.0.33) solution is verified.