Matrix theory - Assignment 10

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***Abstract*—This document illustrates applications of Rank-Nullity Theorem**

Download latex-tikz from

https://github.com/shreeprasadbhat/matrix−theory/blob/master/assignment10/

1 Problem

Let $\mathbf{A}$ be an $m\times n$ matrix with entries in F and let T be the linear transformation from $F^{n\times 1}$ into $F^{m\times l}$ defined by $T(\mathbf{X}) = \mathbf{AX}$. Show that

1) if $m < n$ it may happen that T is onto without being non-singular
2) if $m > n$ we may have T non-singular but not onto.

2 Definitions

singular	A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be singular if $\exists$ some non-zero $\mathbf{X} \in \mathbb{R}^n$ s.t $\mathbf{AX} = \mathbf{0}$ i.e $Nullity(A) \neq 0$. From rank-nullity theorem we can say $rank(A) < n$
non-singular	A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be non-singular if $\mathbf{AX} = \mathbf{0}$ implies $\mathbf{X} = \mathbf{0}$ i.e $Nullity(A) = 0$
onto	A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, $m \leq n$ is said to be onto if for every $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{AX} = \mathbf{b}$ has atleast one solution $\mathbf{X} \in \mathbb{R}^n$.. i.e $dim(Col(\mathbf{A})) = m$ or $Rank(\mathbf{A}) = m$ If $m > n$, then $\mathbf{AX} = \mathbf{b}$ has no solution because rank-nullity theorem is not satisfied.

3 Proof

Let $\mathbf{A}$ be an $m \times n$ matrix with entries in F and let T be the linear transformation from $F^{n\times 1}$ into $F^{m\times l}$ defined by $T(\mathbf{X}) = \mathbf{AX}$. If,		
	$m < n$	$m > n$
singular	Since $rank(\mathbf{A}) < n$, by definition T is singular	Consider an non-singular T such that $rank(\mathbf{A}) > n$
onto	Since $m < n$, by definition T can be onto	Since $m > n$, by definition T is not onto.

4 Examples

4.1 $m < n$

$$\text{Let, } T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \tag{4.1.1}$$

$$T(\mathbf{X}) = \mathbf{AX} = \mathbf{b} \tag{4.1.2}$$

$$\text{Let, } \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{4.1.3}$$

$$\text{Consider, } \mathbf{X} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \tag{4.1.4}$$

$$\implies \mathbf{AX} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \tag{4.1.5}$$

$$= \begin{pmatrix} 6 \\ 5 \end{pmatrix} \tag{4.1.6}$$

Hence T is onto.

$$\text{Consider, } \mathbf{X} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{4.1.7}$$

$$\implies \mathbf{AX} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{4.1.8}$$

$$= \mathbf{0} \tag{4.1.9}$$

Since $\exists \quad \mathbf{X} \neq \mathbf{0}$ such that $\mathbf{AX} = \mathbf{0}$, T is singular.

$\therefore$ T is both onto and singular.

4.2 $m > n$

$$\text{Let, } T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \tag{4.2.1}$$

$$T(\mathbf{X}) = \mathbf{AX} = \mathbf{b} \tag{4.2.2}$$

$$\text{Let, } \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.2.3}$$

$$\text{Consider, } \mathbf{X} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{4.2.4}$$

$$\implies \mathbf{AX} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{4.2.5}$$

$$= \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \tag{4.2.6}$$

$$\tag{4.2.7}$$

$\therefore$ T is not onto, and is also non-singular.