TA Session 4: Linear Panel Models

Shuowen Chen[1]

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[1] Contents are based on Wooldridge (2010) and teaching slides by Manuel Arellano.

Outline

- Different Exogeneity Assumptions
- Pooled OLS
- Linear Panel with Time-Invariant Individual Effects
  - Random Effects
  - Fixed Effects
- Hausman Test Revisited

Linear Panel Model Set Up

$$y_{it} = x_{it}\beta + v_{it}, \quad i = 1, \dots, N; t = 1, \dots, T$$

Level of time:

$$y_t = x_t\beta + v_t, \quad t = 1, \dots, T,$$

where $y_t = (y_{1t}, y_{2t}, \dots, y_{Nt})'$

Level of individual:

$$y_i = x_i\beta + v_i, \quad i = 1, \dots, N,$$

where $y_i = (y_{i1}, y_{i2}, \dots, y_{iT})'$

Exogeneity Assumptions of Errors

1. Contemporaneous exogeneity

$$\mathbb{E}(v_t | x_t) = 0, \ \ t = 1, \dots, T$$

2. Sequential exogeneity

$$\mathbb{E}(v_t | x_t, x_{t-1}, \dots, x_1) = 0, \ \ t = 1, \dots, T$$

3. Strict exogeneity

$$\mathbb{E}(v_t | x_T, x_{T-1}, \dots, x_1) = 0, \ \ t = 1, \dots, T$$

Fails if regressor includes lagged dependent variable

Dynamic Panel

Consider having lagged dependent variable as the regressor

$$y_t = \beta_0 + \beta_1 y_{t-1} + v_t,$$

Strict exogeneity would require

$$\mathbb{E}(v_t | y_0, y_1, \ldots, y_{T-1}) = 0$$

When $t = T$,

$$\mathbb{E}(v_T | y_0, y_1, \ldots, y_{T-1}) = 0,$$

But when $t = 1, \ldots, T-1$,

$$\mathbb{E}(v_t | y_0, y_1, \ldots, y_{T-1}) = y_t - \beta_0 - \beta_1 y_{t-1} \neq 0$$

Pooled OLS

$$y_t = x_t\beta + v_t$$

Assume that

1. $\mathbb{E}(x_t' v_t) = 0, \quad t = 1, \ldots, T$
2. $rank(\sum_{t=1}^{T} \mathbb{E}(x_t' x_t)) = K$, no perfect linear dependencies

Ignore the panel structure and treat data as a giant cross section:

$$\widehat{\beta}_{POLS} = (X'X)^{-1}X'y$$

Error Component Model

$$y_{i,t} = x_{i,t}\beta + v_{i,t} \tag{1}$$

$$v_{i,t} = c_i + u_{i,t} \tag{2}$$

- c_i in equation (2) is called individual effects
- Depending on whether we assume c_i is correlated with $x_{i,t}$, either random effects or fixed effects
- Pooled OLS is consistent if $\mathbb{E}(x_t'(c_i\mathbf{1}_T + u_t)) = 0$, $t = 1, \ldots, T$. Since $c_i\mathbf{1}_T$ is in each time period, need to use robust variance estimator.

Random Effects Model

Assumption 1:

- Strict exogeneity: $\mathbb{E}(u_{i,t} \mid x_i, c_i) = 0$
- Orthogonality: $\mathbb{E}(c_i \mid x_i) = \mathbb{E}(c_i) = 0$ (KEY assumption)

Assumption 2: Equicorrelated random effect structure

$$\mathbb{E}(u_i u_i' \mid x_i, c_i) = \sigma_u^2 I_T, \quad \mathbb{E}(c_i^2 \mid x_i) = \sigma_c^2.$$

This implies $\mathbb{E}(v_i' v_i) = \boldsymbol{\Omega} = \sigma_u^2 I_T + \sigma_c^2 \mathbf{1}_T \mathbf{1}_T'$

Assumption 3: $rank(\mathbb{E}(X_i' \Omega^{-1} X_i)) = K$

RE estimator:

$$\widehat{\beta}_{RE} = \Big(\sum_{i=1}^{N} X_i' \Omega^{-1} X_i\Big)^{-1} \Big(\sum_{i=1}^{N} X_i' \Omega^{-1} y_i\Big)$$

In practice we need to estimate Ω and thus use

$$\widehat{\beta}_{RE} = \Big(\sum_{i=1}^{N} X_i' \widehat{\Omega}^{-1} X_i\Big)^{-1} \Big(\sum_{i=1}^{N} X_i' \widehat{\Omega}^{-1} y_i\Big)$$

Estimating $\widehat{\Omega}$ in Random Effects Estimator

Recall $\mathbf{\Omega} = \sigma_u^2 I_T + \sigma_c^2 \mathbf{1}_T \mathbf{1}_T'$, therefore it suffices to get $\widehat{\sigma}_u^2$ and $\widehat{\sigma}_c^2$.

1. Run POLS, obtain $\widehat{v}_{i,t} = y_{i,t} - x_{i,t}\widehat{\beta}_{POLS}$, calculate

$$\widehat{\sigma}_v^2 = \frac{1}{NT - K} \sum_{t=1}^{T} \sum_{i=1}^{N} \widehat{v}_{i,t}^2.$$

2. Recall $\sigma_c^2 = \mathbb{E}(v_{i,t} v_{i,s})$, $t \neq s$. Use the $T(T-1)/2$ nonredundant off-diagnol terms to calculate

$$\widehat{\sigma}_c^2 = \frac{1}{NT(T-1)/2 - K} \sum_{i=1}^{N} \sum_{t=1}^{T-1} \sum_{s=t+1}^{T} \widehat{v}_{i,t} \widehat{v}_{i,s}.$$

3. Compute $\widehat{\sigma}_u^2 = \widehat{\sigma}_v^2 - \widehat{\sigma}_c^2$.

Fixed Effects Model

Assumption 1:

- Strict exogeneity: $\mathbb{E}(u_{i,t} \mid x_i, c_i) = 0$

FE transformation:

$$y_{i,t} - \overline{y}_i = (x_{i,t} - \overline{x}_i)\beta + u_{i,t} - \overline{u}_i$$

Define demean operator: $Q_T = I_T - \lambda(\mathbf{1}_T(\mathbf{1}_T'\mathbf{1}_T)^{-1}\mathbf{1}_T')$

Assumption 2:

- $rank(\mathbb{E}(X_i' Q_T X_i)) = K$

FE estimator is

$$\widehat{\beta}_{FE} = \Big(\sum_{i=1}^{N} X_i' Q_T X_i\Big)^{-1}\Big(\sum_{i=1}^{N} X_i' Q_T y_i\Big)$$

Between and Within Estimators

- Fixed Effects estimator is also called within estimators as it exploits time variation within each cross section
- Between estimator is OLS applied to the following

$$\overline{y}_i = \overline{x}_i \beta + c_i + \overline{u}_i,$$

  which exploits variation between the cross sections.
- Define $P = I_N \otimes \mathbf{1}_T(\mathbf{1}_T'\mathbf{1}_T)^{-1}\mathbf{1}_T'$ and $Q = I_{NT} - P$, POLS and RE are both linear combinations of between and within estimators

$$\begin{aligned}\widehat{\beta}_{POLS} &= a\widehat{\beta}_{between} + (1-a)\widehat{\beta}_{within}\\ \widehat{\beta}_{RE} &= b\widehat{\beta}_{between} + (1-b)\widehat{\beta}_{within}\end{aligned}$$

  where $a = (X'X)^{-1}(X'PX)$ and $b = \left[X'\left(P + \frac{T\sigma_c^2+\sigma_u^2}{\sigma_u^2}Q\right)X\right]^{-1}X'PX$

Variances under Equicorrelated Random Structure

Suppose that $E[c_i|x_i] = 0$, $E[c_i^2|x_i] = \sigma_c^2$ and $E[u_i u_i'|x_i, c_i] = \sigma_u^2 I_T$, then

$$Var(\widehat{\beta}_{FE}|X) = \sigma_u^2\Big(\sum_{i=1}^N X_i' Q_T X_i\Big)^{-1}, \quad Var(\widehat{\beta}_{RE}|X) = \Big(\sum_{i=1}^N X_i' \widehat{\Omega}^{-1} X_i\Big)^{-1}$$

To estimate the two variances consistently, it suffices to estimate σ_u^2 and σ_c^2 consistently. See previous slide for details.

Quasi Demeaning Transformation of Random Effects

Denote $\lambda = 1 - \sqrt{1/\left[1 + T\left(\sigma_\alpha/\sigma_u\right)^2\right]}$ and observe that

$$\Omega^{-1/2} = \frac{1}{\sigma_u}\left[I_T - \lambda(\mathbf{1}_T(\mathbf{1}_T'\mathbf{1}_T)^{-1}\mathbf{1}_T')\right]$$

Then the RE estimator is obtained by estimating the following

$$C_T y_i = C_T x_i \beta + C_T v_i,$$

where $C_T = I_T - \lambda(\mathbf{1}_T(\mathbf{1}_T'\mathbf{1}_T)^{-1}\mathbf{1}_T')$. Note $\mathbb{E}[(C_T u_i)(C_T u_i)'] = \sigma_u^2 I_T$. The t−th element of the previous regression is

$$y_{i,t} - \lambda\overline{y}_i = (x_{i,t} - \lambda\overline{x}_i)\beta + (v_{i,t} - \lambda\overline{v}_i),$$

where $\overline{y}_i = \frac{1}{T}\sum_{t=1}^{T} y_{i,t}$. Denote $\widetilde{y}_{i,t} \equiv y_{i,t} - \lambda\overline{y}_i$, then RE estimator is

$$\widehat{\beta}_{RE} = (\widetilde{X}'\widetilde{X})^{-1}\widetilde{X}'\widetilde{y}$$

Relationship Between FE and RE

- As $T \to \infty$ or $\frac{\sigma_u}{\sigma_c} \to 0$, $\lambda \to 1$ and RE coincides with *FE*.
- Under equicorrelated random effect structure,

$$Var(\widehat{\beta}_{FE}|X) = \sigma_u^2\Big(\sum_{i=1}^{N} X_i' Q_T X_i\Big)^{-1} = \sigma_u^2\Big(X'QX\Big)^{-1}$$

$$Var(\widehat{\beta}_{RE}|X) = \Big(\frac{1}{\sigma_u^2}X'QX + \frac{1}{T\sigma_c^2 + \sigma_u^2}X'PX\Big)^{-1}$$

RE is more efficient than FE

Hausman Test: FE vs RE

Under H_0 RE is efficient. Under H_1 only FE is consistent

$$H_0 : \mathbb{E}(c_i|x_i) = 0, \quad H_1 : \mathbb{E}(c_i|x_i) \geq 0$$

Hausman statistic:

$$H = (\widehat{\beta}_{RE} - \widehat{\beta}_{FE})' \left[Var(\widehat{\beta}_{FE}) - Var(\widehat{\beta}_{RE}) \right]^{-1} (\widehat{\beta}_{RE} - \widehat{\beta}_{FE})$$

Under the null: $H \xrightarrow{d} \chi^2_K$. **Caveats**[2]:

- Strict exogeneity is maintained under the null and alternative
- Failure of equicorrelated random effect structure leads to non-standard limiting distributions
- Cannot compare time–invariant coefficients as they are unidentified in FE
- Hausman pretest for selecting RE or FE leads to second stage (post model selection) size distortion

[2] Refer to Wooldridge (2010 p.329) and Guggenberger (2010, JoE) for details.