

On Enhancing Hierarchical Modulation

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Abstract—Hierarchical modulation offers an important coverage/throughput tradeoff for wireless communication, however it has received relatively little attention to date. Regular hierarchical modulation suffers from the interference between layers. It results in both achievable capacity decrease and bit-error rate increase. In this paper, an enhanced hierarchical modulation technique along with three optimization schemes, in which the enhancement-layer signal constellation is rotated, are presented and analyzed. The first optimization scheme is proposed to maximize the achievable spectral efficiency with rotating the enhancement-layer signal constellation. The second scheme is proposed to lower the demodulation symbol-error rate with maximizing the modulation efficiency and asymptotic modulation efficiency. The parameters, modulation efficiency and asymptotic modulation efficiency, are formulated for quantizing inter-layer interference. The last one is proposed to maximize RF power amplifier efficiency with reducing the peak-to-average-power ratio of modulated symbols. This perspective is especially important for multi-carrier communications. All proposed schemes are simple and efficient. They can help recover the performance loss of regular hierarchical modulation due to inter-layer interference with little complexity increase. Computer simulations are provided to support our conclusions.

I. INTRODUCTION

Broadcast multicast service (BCMCS) has increasingly been popular for delivering multimedia content to mobiles. BCMCS can be implemented through either a dedicated digital broadcast infrastructure like DVB-T/H/S2 and MediaFLO or a 3rd generation and beyond radio access network like UMTS and cdma2000 network. Traditional digital broadcast networks are designed with the tradeoff between maximum achievable capacity and intended coverage in mind. Their throughput is limited by the maximum transmit power and the worst channel condition so that every user in the coverage can reliably receive services. All covered users share the services of same quality. The users under good reception condition and having the capability of advanced signal processing may not have many advantages, even though their achievable throughput can be much higher. In addition, it is known that the reception condition of mobiles frequently changes. Therefore, it is interesting to provide users with differentiated quality of services (QoS's) based on not only the content they requested but also the receiver capability as well as their reception conditions. On the other hand, there are rising interests in upgrading existing digital broadcast systems with more services for new users while be able to keep existing users unchanged, delivering better QoS's to users with upgraded receivers while still be able to guarantee other users' services, and providing unequal protection on digital contents with high spectral efficiency [1],

[2], [3], [4]. Many technologies are under investigation for these goals, e.g., rateless coding, hierarchical modulation, multiple-input multiple-output (MIMO) and selective retransmission. Backward compatibility and implementation complexity are among the major concerns in upgrading existing networks. This is because there are a large number of users already served by existing networks and it is prohibitively expensive for simply replacing existing terminal equipments by upgraded ones. It is expected that existing receivers can still operate in upgraded systems, even though they are not able to receive supplemental services provided by upgraded networks. And there always are power consumption and computation complexity concerns in receiver design. This is very critical for mobile terminals. Therefore, hierarchical modulation is one of the promising technologies for upgrading existing systems while maintaining strictly backward compatibility. One of the key advantages of hierarchical modulation is that the added complexity and cost is low. It has been proven and included in various standards, such as DVB-T [1], MediaFLO [2], UMB (Ultra Mobile Broadband, a new 3.5th generation mobile network standard developed by 3GPP2) [4], etc, and is under study for DVB-H.

Hierarchical modulation, also named layered modulation, is one of the techniques for multiplexing and modulating multiple data streams into one single symbol stream, where the base-layer subsymbols and enhancement-layer subsymbols are synchronously superimposed together before transmission. When hierarchical modulation is employed, users with good reception and advanced receiver can demodulate and decode more than one data streams. For a receiver with conventional receiver or poor reception, only the data streams embedded in low layer(s), e.g. the base layer, can be modulated and decoded. From an information-theoretical perspective, hierarchical modulation is taken as one practical implementation of superposition precoding, which was proposed for achieving the maximum sum rate of Gaussian broadcast channel with successive interference cancellation at receivers. From a network operation perspective, a network operator can seamlessly target users with different services or QoS's when hierarchical modulation is employed. However, traditional hierarchical modulation suffers from inter-layer interference (ILI) so that the achievable rates by low-layer data streams, e.g. the base-layer data stream, are dented by the interference from high-layer signal(s). For example, for the hierarchically modulated two-layer symbols consisting of a 16QAM base layer and a QPSK enhancement layer, the base-layer throughput loss due

to inter-layer interference can be up to around 1.5 bits/symbol when the total receive signal-to-noise ratio (SNR) is about 23 dB. This is about $1.5/4 = 37.5\%$ loss of the base-layer achievable throughput at 23dB SNR. Meanwhile, the demodulation error rate of either the base-layer and enhancement-layer symbols increases too. Therefore it is interesting to study the approaches for recovering the performance loss of hierarchical modulations. Furthermore, multicarrier transmission, e.g. orthogonal frequency-division multiplexing (OFDM), is widely used for BCMCS as well as next generation wireless networks, due to its high diversity gain and high spectral efficiency with simple receiver design. However, the advantages of OFDM, specially when it is modulated by high-order signal constellations, are counter-balanced by the high peak-to-average-power ratio (PAPR) issue. High PAPR of modulated signals can significantly reduce the average output power of the high-power amplifier (HPA) at the transmitter due to more back-offs. It also increases the receiver demodulation and decoding errors and therefore limits the throughput of whole transceiver chain. Therefore, it is important to understand how hierarchical modulation affects the PAPR of multicarrier modulations.

In order to recover the capacity loss due to ILI as well as high PAPR, an enhanced hierarchical modulation and three associated optimization schemes are presented and analyzed in this paper. With the proposed hierarchical modulation, the signal constellation of enhancement layer is rotated so that a better performance, such as high throughput, low error rate and high power efficiency, is achievable. For this, three optimization schemes are presented and discussed. The first approach is from an information-theoretical perspective and straightforward to maximize the achievable spectral efficiency. On the other hand, it is known that the performance loss of a hierarchical modulation is also related to the Euclidean distance profile of the signal constellation. For tracking the Euclidean distance profile change, several parameters like effective signal power, effective SNR and modulation efficiency are defined. The second approach then is to minimize demodulation errors with maximizing the modulation efficiency of the hierarchical modulation. The last approach is more from an implementation perspective with considering the power efficiency in multicarrier transmission. With avoiding back-offs and maximizing average output power, high transmit power efficiency is shown to be achievable by optimizing hierarchical modulation. Above all, one of the most important advantages of the proposed enhanced hierarchical modulation is the incurred implementation complexity is low. This makes it attractive and easy to be employed in actual system implementations. The proposed enhancement was adopted in UMB by 3GPP2 [4]. Though the schemes proposed here can straightforwardly be generalized for most hierarchical modulations and multicarrier transmission schemes, the following discussion will be limited to two-layer signal constellations over OFDM with QPSK-modulated enhancement layer and QPSK- or 16QAM-modulated base layer. The reason for this is not only because of the simplicity of QPSK and 16QAM modulations over OFDM but also because QPSK

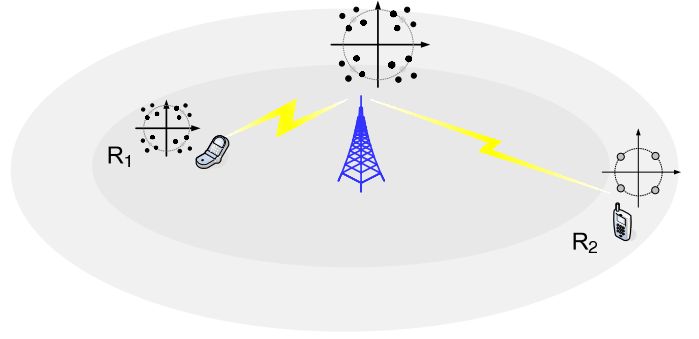


Fig. 1. A broadcast channel model with two coverages and the enhanced QPSK/QPSK hierarchical modulation.

and 16QAM as well as OFDM are among the most popular signal constellations and transmission schemes adopted in various digital communication systems and standards. It is shown that the use of QPSK as enhancement layer can yield significant performance gain with our approaches. On the other hand, many high-order regular or hierarchical signal constellations may be decomposed into multiple QPSK signals adding together and most results for OFDM can be applied to other multicarrier transmission too.

II. SYSTEM MODEL AND PROBLEM DESCRIPTION

Instead of the traditional single-coverage broadcast model, a broadcast channel model with two-layer coverages is discussed in this paper. It can be illustrated in Fig. 1. In this model, a broadcast station (BS) transmits signals to all mobile stations (MS) in the covered area. The MS's far from the BS may only be able to reliably decode the data stream of a low rate R_1 while the MS's close to the BS can possibly decode multiple data streams with a high sum rate R_2 . For achieving this, hierarchical modulation is employed by the BS so that the modulated symbol stream comprises two synchronously superimposed substreams. The rationale behind it is the finding from the study of broadcast channel capacity, which states that with superposition precoding (SPC) a little throughput sacrifice on the users near to the coverage edge may lead to a big increase for the users with good reception [5]. This concept is illustrated in Fig. 2, where the achievable rates of two multiplexing schemes, time-division multiplexing (TDM) and SPC, are plotted and compared. It shows that SPC can outperform TDM most of the time. With moving up the network operation point ($R_1, R_2 - R_1$) a little bit along the SPC curve a much higher sum throughput R_2 is achievable. And hierarchical modulation can be taken as a practical implementation of SPC.

The signal constellation of regular square-shaped QPSK/QPSK and 16QAM/QPSK hierarchical modulation are shown in Fig. 3¹. The minimum Euclidean distance (MED)

¹In this paper, a hierarchical modulation is denoted by *layer 0 (or base layer) constellation / layer 1 constellation / ...*, where the signal constellation of different layers are separated by backslash from the lowest layer (also called base layer) to the highest layer.

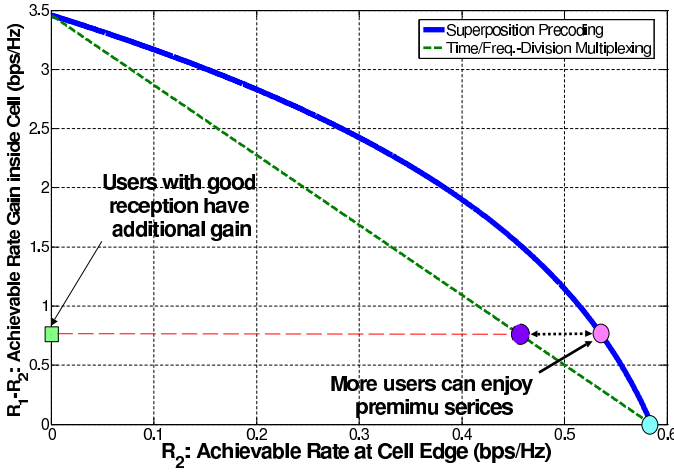


Fig. 2. Achievable capacity region of broadcast channel.

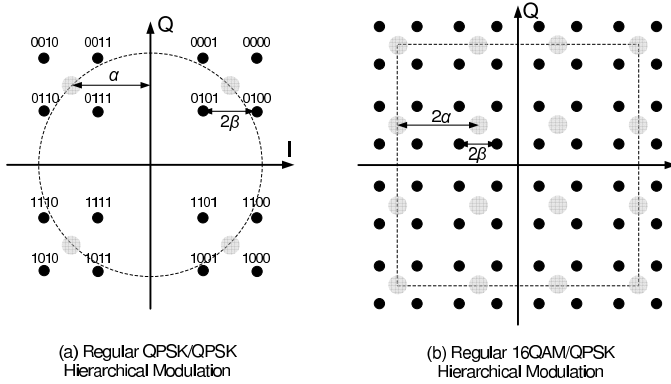


Fig. 3. Regular hierarchical modulation examples: the base layer is QPSK/16QAM and the enhancement layer is QPSK.

of base layer and enhancement layer are denoted by 2α and 2β , respectively². With superimposing base-layer signal constellation and enhancement layer signal constellation together, the MED of resulted two-layer hierarchical modulation becomes

$$d_{\min} = \min\{2(\alpha - \beta), 2\alpha, 2\beta\} < 2\alpha. \quad (1)$$

When the MED of the base-layer and enhancement-layer signal constellations satisfies $\alpha = 2\beta$, the regular QPSK/QPSK hierarchical modulation becomes the regular 16QAM. And smaller MED usually results in more ambiguity and more demodulation errors especially when ζ becomes large. The power-splitting ratio ζ between the enhancement layer and the base layer is defined by

$$\zeta = \frac{P_E}{P_B} \quad (2)$$

with $\zeta < 1$ in most cases. For QPSK/QPSK hierarchical modulation, the power-splitting ratio is $\zeta_{\text{QPSK/QPSK}} = \frac{\beta^2}{\alpha^2}$. For 16QAM/QPSK modulation, $\zeta_{\text{16QAM/QPSK}} = \frac{\beta^2}{4\alpha^2}$. When $\zeta_{\text{QPSK/QPSK}} = \frac{1}{4}$, the QPSK/QPSK modulation in Fig. 3(a) becomes regular square-shaped 16QAM. In general, the

²Without loss of generality, it is assumed that $\alpha \geq \beta$ in most cases.

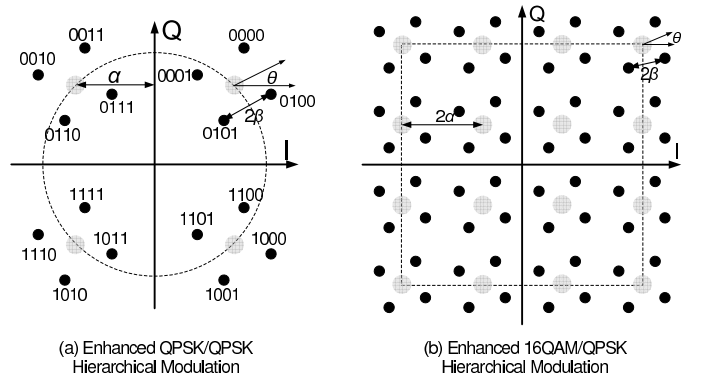


Fig. 4. Enhancing hierarchical modulation by rotation.

enhancement-layer signal can be taken as additional noise by base layer. This means most existing conventional receivers can continue to demodulate base-layer signals with no additional change but at a lower signal-to-noise-and-interference ratio (SINR) $\hat{\gamma}$ defined by

$$\hat{\gamma} = \frac{P_B}{P_E + \sigma^2} < \gamma = \frac{P_B}{\sigma^2} \quad (3)$$

with the background additive Gaussian white noise (AWGN) power σ^2 .

III. THE ENHANCED HIERARCHICAL MODULATION

Regular hierarchical modulation may seriously suffer from ILI, which not only decreases the base-layer SINR from γ to $\hat{\gamma}$ but also lowers the achievable spectral efficiency. It is well-known that the achievable throughput of a hierarchical modulated signal essentially depends on the power distribution profile of the signal [6] in signal space. This is somehow similar to channel coding. From a channel coding point of view, higher throughput is achievable by the *i.i.d. Gaussian code* defined in coding space, even though it may not be implementable from an engineering standpoint. How to transmit a signal close to Shannon channel capacity and implementable in a relatively easy way is not only critical for the signal constellation design but also every other component in a communication system, including modulating signals.

In order to increase channel capacity with minimum complexity increase, we proposed to optimize the signal constellation of hierarchical modulation with optimally rotating the enhancement-layer signal constellation for improving the spectral efficiency, decreasing demodulation error rate and increasing transmit power efficiency. For the QPSK/QPSK hierarchical modulation shown in Fig. 3(a), the QPSK signal constellation of the enhancement layer is rotated in counter-clockwise by θ , $0 \leq \theta \leq \frac{1}{4}\pi$, and resulted signal constellation is shown in Fig. 4(a). For the 16QAM/QPSK signal constellation, the regular and enhanced hierarchical modulations are shown in Fig. 3(b) and Fig. 4(b), respectively. Now the question becomes how to choose θ for each modulation. In the following, three schemes are proposed from different perspectives of signal modulation and demodulation.

IV. OPTIMIZING HIERARCHICAL MODULATION

A. Maximizing Achievable Rates

In general, the achievable rate of a N -ary modulated signal, either regular or hierarchical signal constellation, through AWGN channel is given by [6]

$$R = \log_2(N) - \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E} \left\{ \log_2 \left[\sum_{j=0}^{N-1} e^{-\frac{|s_j + n - s_i|^2 - |n|^2}{2\sigma^2}} \right] \right\}. \quad (4)$$

This is the achievable rate when a receiver tries to decode the whole hierarchically modulated symbol. With (4), the AWGN capacity of regular QPSK and 16QAM can be plotted in Fig 5. Though the rate in (4) is achievable for a user capable to decode the whole constellation, it is more than achievable for a user with a conventional receiver and detecting the base-layer signals only. The achievable rate of either base layer or enhancement layer along is lower than the total rate in (4). Following the concept of the successive interference cancellation, the achievable rate, also termed *equivalent capacity*, for a receiver decoding up to l layers of a hierarchical modulated symbol is [7]

$$\tilde{R}_l = \sum_{i=0}^{l-1} R_i = R - \sum_{j=l}^L R_j. \quad (5)$$

To illustrate this, let's take the regular 16QAM as an example since a regular 16QAM can be take as a special case of QPSK/QPSK hierarchical modulation with $\zeta_{\text{QPSK/QPSK}} = \frac{1}{4}$. This means the achievable rate of the enhancement layer is the same as the regular QPSK capacity but the achievable rate of the QPSK base layer becomes

$$\begin{aligned} R_{\text{QPSK/QPSK}}^B(\gamma, \frac{4}{5}) &= R_{\text{QPSK/QPSK}}(\gamma, \frac{1}{4}) - R_{\text{QPSK}}(\frac{1}{5}\gamma) \\ &= R_{16\text{QAM}}(\gamma) - R_{\text{QPSK}}(\frac{1}{5}\gamma), \end{aligned} \quad (6)$$

where $R_{\text{QPSK/QPSK}}^B(\gamma, \frac{4}{5})$ denotes the achievable rate of the base-layer of a QPSK/QPSK hierarchical modulation with total SNR γ and the power fraction $\frac{4}{5}$. It is plotted in Fig 5 too. Due to the ILI from the QPSK-modulated enhancement layer, it shows that the actual throughput of the QPSK base layer $R_{\text{QPSK/QPSK}}^B(\gamma, \frac{4}{5})$ is lower than the corresponding QPSK rate $R_{\text{QPSK}}(\frac{4}{5}\gamma)$, i.e.,

$$R_{\text{QPSK/QPSK}}^B(\gamma, \frac{4}{5}) \leq R_{\text{QPSK}}(\frac{4}{5}\gamma). \quad (7)$$

This kind of degradation can be further illustrated in Fig. 6, where the hierarchical modulation is 16QAM/QPSK-modulated with the 16QAM-modulated base layer. In Fig. 6, the achievable rates of each layer and the whole signal constellation are plotted with the total SNR fixed at $\frac{P}{\sigma^2} = 20\text{dB}$ and the power of the 16QAM sublayer changing from 50% to 100% of the total power P . One of the interesting things shown in Fig. 6 is the equivalent capacity of the 16QAM base layer changes periodically instead of monotonically with increase the power ratio of the base layer. One of the good things in Fig. 6 is this kind of capacity loss can be recovered by optimally rotating the enhancement layer. This is one of the advantages of the proposed enhanced hierarchical modulations.

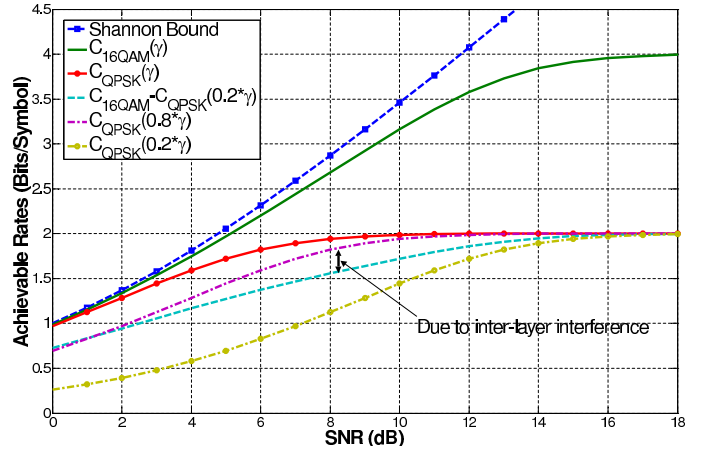


Fig. 5. Achievable rates of regular 16QAM modulation: a hierarchical modulation perspective.

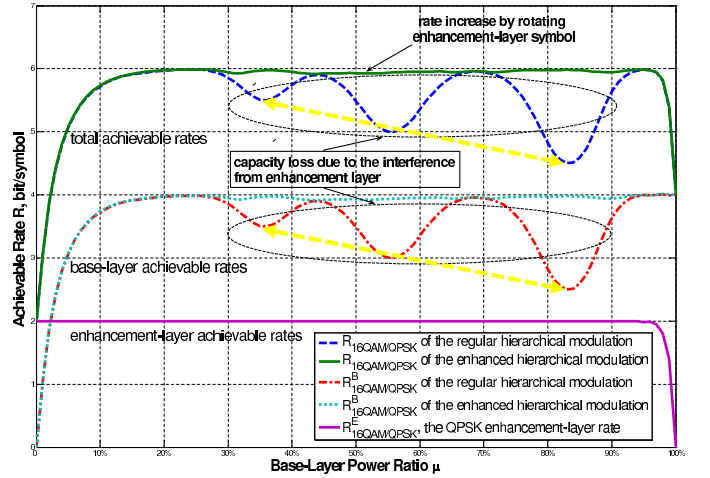


Fig. 6. Achievable rates of 16QAM/QPSK hierarchical modulation with different power splitting and $\frac{P}{\sigma^2} = 20\text{dB}$.

B. Maximizing Modulation Efficiency

Besides the above information-theoretical point of view on hierarchical modulation, it is also interesting to understand hierarchical modulation from a practical signal-processing perspective. At this time, the performance of hierarchical modulation will be evaluated through an actual implementation, where demodulation error is one of the major concerns. In general, it is difficult to give a simple closed-form BER or symbol-error rate (SER) expression for hierarchical signal constellation, which also depends on receiver design and bits-to-symbol mapping. The BER of square-shaped M -QAM constellation and a hierarchical QAM constellation for maximum likelihood (ML) demodulator can be computed by using recursive algorithms [8]. For a simple QPSK modulation, it is known that the BER is

$$P_{e,\text{QPSK}}(\gamma) = Q\left(\sqrt{\frac{\gamma}{2}}\right), \quad (8)$$

where $Q(x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)$ denotes the Q-function.

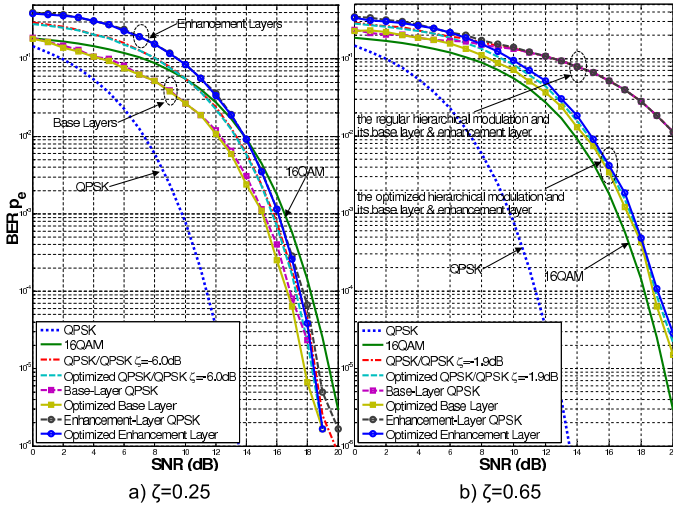


Fig. 7. Bit-error rate of uncoded QPSK/QPSK hierarchical modulations using maximum likelihood demodulation.

From a signal processing standpoint, demodulation error and capacity degradation may happen when there is a change of interference distribution, even though the received SNR γ is the same. The BER performance of regular QPSK/QPSK becomes deteriorated in Fig. 7 when $\zeta_{\text{QPSK/QPSK}}$ increases. However, with optimally rotating the enhancement-layer signal constellation, the performance loss can be recovered. This kind of recovery is more significant with large ζ . In order to quantify and understand this kind of BER performance loss due to interference and receiver design, one approach we propose for capturing this kind of degradation is to calculate the effective signal-to-noise ratio (ESNR) of the whole transceiver chain, which is defined by

$$\tilde{\gamma}(\gamma) \equiv \Psi^{-1}(p_e(\gamma)), \quad (9)$$

where $p_e(\gamma)$ is the demodulation BER of the signal with SNR γ , and $\Psi^{-1}(\cdot)$ denotes the inverse function of $\Psi(\cdot)$, the demodulation error probability function with no ILI. The ESNR for the QPSK-modulated base layer or enhancement layer of any hierarchical modulation can be calculated by

$$\tilde{\gamma}_{\text{QPSK/QPSK}} = 2 [Q^{-1}(p_e(\gamma))]^2. \quad (10)$$

More specifically, the ESNR for the base layer of regular QPSK/QPSK hierarchical modulation with ML demodulator is given by

$$\tilde{\gamma}_{\text{QPSK/QPSK}}^B(\gamma) = 2 \left[Q^{-1} \left(\frac{Q((1-\sqrt{\zeta})\gamma) + Q((1+\sqrt{\zeta})\gamma)}{2} \right) \right]^2. \quad (11)$$

By normalizing ESNR with γ , we can obtain hierarchical modulation efficiency (ME) η by

$$\eta(\gamma) = \frac{\tilde{\gamma}}{\gamma} = \frac{1}{\gamma} \Psi^{-1}(p_e(\gamma)). \quad (12)$$

With no interference, $\eta(\gamma) = 1$; otherwise, $\eta(\gamma) < 1$. η is also the measure of inter-layer resistance for hierarchical modulation. The high ME value means stronger interference-resistance the modulated signal has. The ME of QPSK/QPSK hierarchical modulation are plotted in Fig. 8. We can see

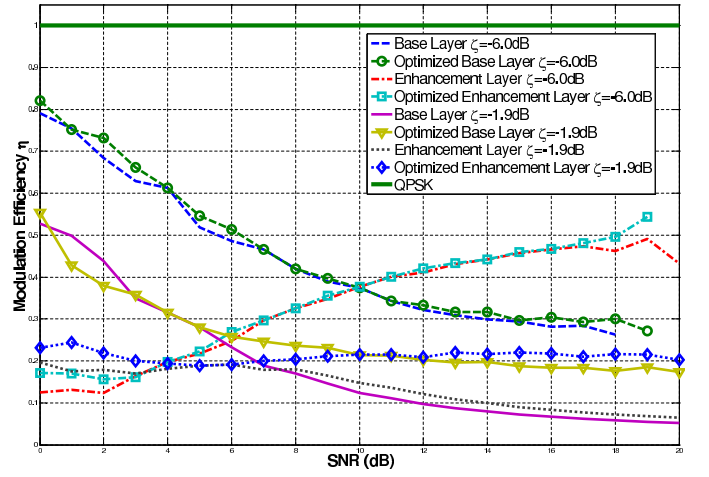


Fig. 8. Hierarchical modulation efficiency of QPSK/QPSK hierarchical modulation using maximum likelihood demodulation.

the enhanced hierarchical modulation has higher ME than the regular modulation. The difference is more obvious when ζ becomes large. This means the enhanced hierarchical modulation has the stronger inter-layer interference resistance than that of regular hierarchical modulation.

Furthermore, in order to isolate the noise effect from demodulation/decoding, the asymptotic modulation efficiency (AME) η_∞ is given by

$$\eta_\infty = \lim_{\gamma \rightarrow \infty} \eta(\gamma) = \lim_{\sigma \rightarrow 0} \frac{\sigma^2}{P} \Psi^{-1}(p_e), \quad (13)$$

which is the ME in high SNR region. For the ME in (11), the AME can be calculated by

$$\eta_\infty = \lim_{\gamma \rightarrow \infty} \frac{2 \left[Q^{-1} \left(\frac{Q((1-\sqrt{\zeta})\gamma) + Q((1+\sqrt{\zeta})\gamma)}{2} \right) \right]^2}{\gamma}. \quad (14)$$

With (13) and (14), it shows that AME indicates how fast ESNR is approaching SNR when $\gamma \rightarrow \infty$. This can be expressed by

$$\eta_\infty = \left. \frac{\partial \eta(\gamma)}{\partial \gamma} \right|_{\gamma=\infty}. \quad (15)$$

The AME for QPSK/QPSK hierarchical modulation can also be found in Fig. 8, where they are the points when SNR becomes larger and larger. The AME of a hierarchical modulation only depends on its signal constellation design and is independent to the background noise.

C. Minimizing Peak-to-Average-Power Ratio

Since OFDM has been widely adopted and intensively studied for next-generation wireless systems, it is important to understand how the enhanced hierarchical modulation works with OFDM. Here we investigated the PAPR reduction performance when it is transmitted over OFDM. An OFDM signal is the sum of L independent modulated symbols $\{s_l : 1 \leq l \leq L\}$ mapped onto L different subcarriers with the frequency separation $\frac{1}{T}$, where T is the symbol period with no additional overhead such as cyclic prefix (CP) and zero prefix (ZP). At the transmitter side, the discrete time-domain samples

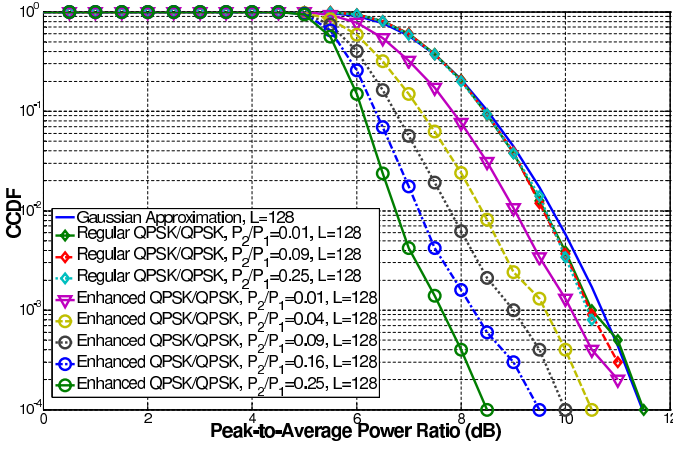


Fig. 9. PAPRs of hierarchical modulations over OFDM with $L = 128$.

$\{x_m : 1 \leq m \leq L\}$ are the inverse fast Fourier transform (IFFT) of the complex symbols $\{s_l : 1 \leq l \leq L\}$:

$$x_m = \frac{1}{\sqrt{L}} \sum_{l=0}^{L-1} s_l e^{j2\pi \frac{ml}{L}}. \quad (16)$$

Before transmission, x_m is usually extended by attaching a CP or ZP. When CP is applied, the extended OFDM symbol

$$\tilde{x}_n = \begin{cases} x_n & 1 \leq n \leq L \\ x_{n+L} & -L_{cp} + 1 \leq n \leq 0 \end{cases}. \quad (17)$$

The extended OFDM symbol then passes through digital-to-analog converter and pulse-shaping filter before up-converted to the carrier frequency. The PAPR of OFDM signal is usually defined by

$$\xi = \frac{1}{\mathbb{E}\{|x_m|^2\}} \max_m |x_m|^2, \quad (18)$$

though in practice the PAPR of the analog signal equivalent of $\{\tilde{x}_n : -L_{cp} + 1 \leq n \leq L\}$ is more of interest.

The PAPR values of the regular QPSK/QPSK hierarchical modulation and its enhancement are plotted in Fig. 9. It shows that the PAPR of regular hierarchical modulation can be reduced with optimally rotating enhancement-layer symbols. And when the power splitting ratio η becomes larger and larger, the PAPR reduction performance will be more significant.

V. CONCLUSIONS

In this paper, one enhanced hierarchical modulation scheme and three optimization approaches are proposed for higher throughput, less error rate and higher power efficiency. One approach is to maximize the spectral efficiency, one approach is to maximize modulation efficiency and the last one is to minimize the PAPR when it is used with OFDM. The rationales as well as the performance of the proposed approaches are analyzed. They can be used for helping upgrade and design BCMCS systems with minimum complexity increase. It is adopted in the 3.5G standard UMB by 3GPP2.

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