

集合论II

# 二元关系: 介绍与例子(1)

对应《问题求解》

二元关系: 简介与简单的运算







ant-hengxin [发消息](#)

为TA充电

请关注蚂蚁老师!

你们之所以会对这样一门硬核的课感到恐惧, 这是因为你们有成年人的思维. 成年人会想这个对我有没有用, 会考虑利益, 会考虑难还是不难; 但是当你们回想你们的时候, 当

你们还是小学生甚至是婴幼儿的时候, 你们面对一个新鲜事物的时候应该想的不是难还是不难, 你们大概想知道它好不好玩.

— 魏恒峰, 南京大学  
《编译原理》课程介绍



# “关系”

父母: 和同学的**关系**怎样

数学课: 3和5的的大小**关系**是什么?

Google

Relation meaning



## Dictionary

Definitions from [Oxford Languages](#) · [Learn more](#)



relation

1. the way in which two or more people or things are **connected**; a thing's effect on or relevance to another.
2. a person who is connected by blood or marriage; a relative.

# “关系”



## examples

connected? 地铁线路应该是connected的





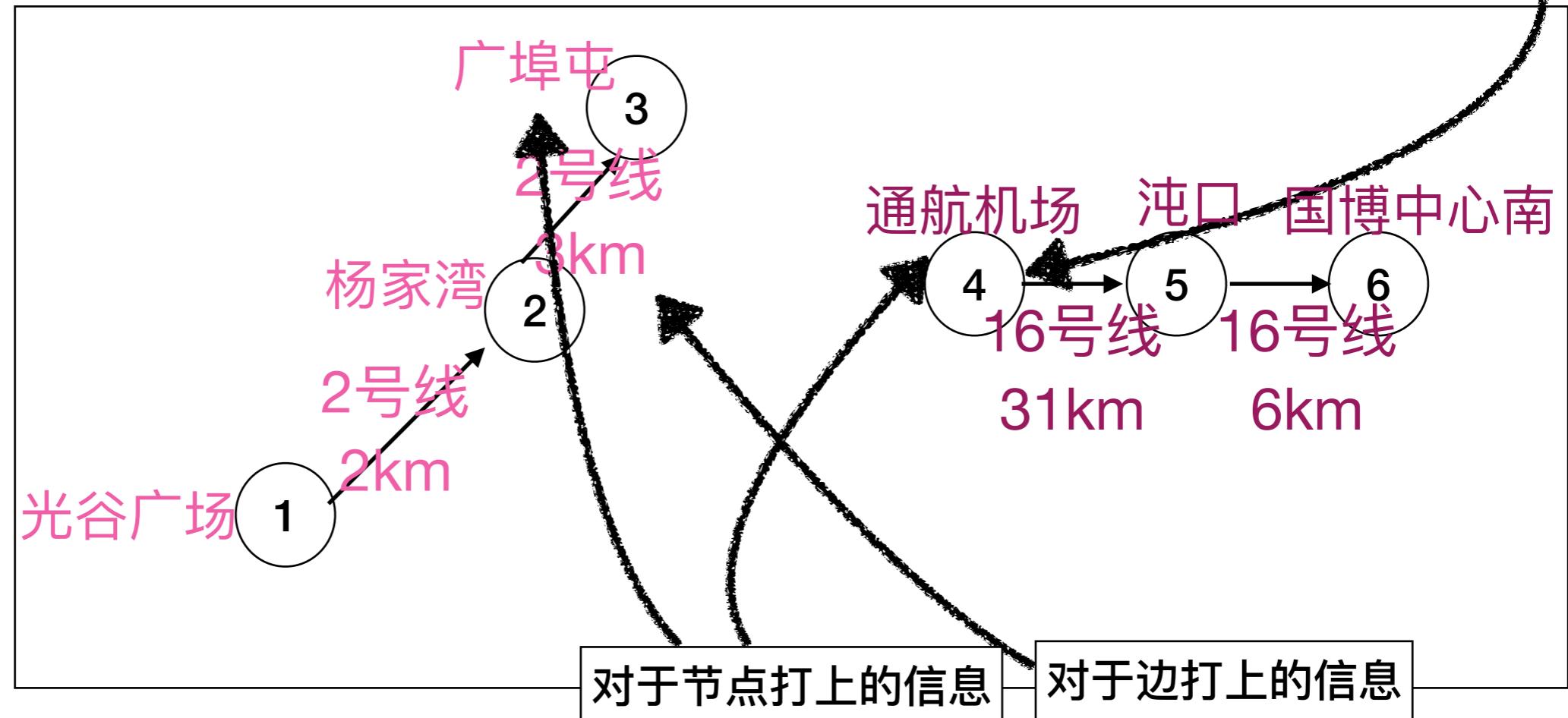
# “关系”

connected? 地铁线路应该是connected的

假设现在只关心2号线, 16号线其中的一些站点...

- 我们不关心把顶点摆在哪里, 形状...

节点的编号



但是我们可以在顶点和边上打上去额外的**信息**

# “关系”

$\mathbb{R}$ 上的Near关系

如果  $|a - b| < 1 (a, b \in \mathbb{R})$ , 则称  $a, b$  具有Near关系.  
满足所有这样的关系记作的集合是  $R$ .



$a$        $b$

我们发现:

这时候  $|1-1|=0<1$ , 在  $R$  里面. 上面的关系可以用  $(a, b)$  表示

- $\forall a \in X. (a, a) \in R$  – **自反性**(reflexivity)
- $\forall a, b \in X. ((a, b) \in R \rightarrow (b, a) \in R)$  – **对称性**(symmetry)
- $\forall a, b, c \in X. ((a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R)$ 
  - **传递性**(transitivity) (这个关系里面不存在)

# “关系”

整除关系

如果  $X = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$ , “关系”是  $X$  上的整除关系.

有关系的个体是:

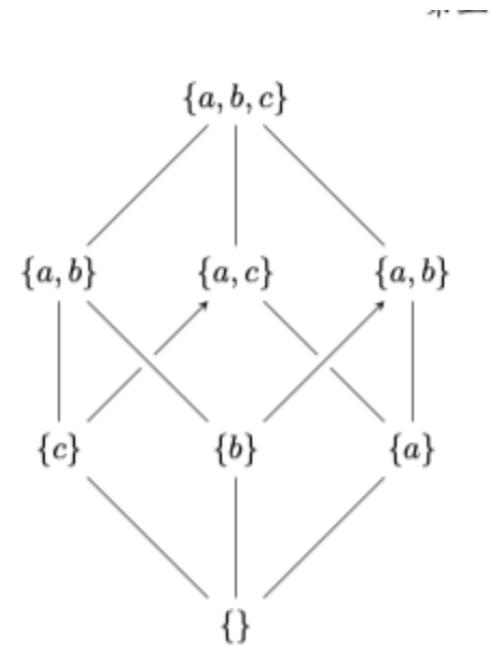
$$R = \{(1, 2), \dots, (4, 12), \dots, (12, 60), \dots, (4, 60), \dots, (60, 60)\}$$

我们发现:

- $\forall a \in X. (a, a) \in R$  – **自反性**(reflexivity)
- $\forall a, b \in X. ((a, b) \in R \rightarrow (b, a) \in R)$  – **对称性**(symmetry)
- $\forall a, b, c \in X. ((a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R)$   
– **传递性**(transitivity) (这个就有了)

# “关系”

集合上的包含关系: 集合 $\{a, b, c\}$



我们发现:

- $\forall a \in X. (a, a) \in R$  – **自反性**(reflexivity)
- $\forall a, b \in X. ((a, b) \in R \rightarrow (b, a) \in R)$  – **对称性**(symmetry)
- $\forall a, b, c \in X. ((a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R)$   
– **传递性**(transitivity) (这个就有了)

# “关系”

整数上的“大于”关系

我们发现：

- $\forall a \in X. (a, a) \in R$  – **自反性**(reflexivity)
- $\forall a, b \in X. ((a, b) \in R \rightarrow (b, a) \in R)$  – **对称性**(symmetry)
- $\forall a, b, c \in X. ((a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R)$   
– **传递性**(transitivity)
- $\forall a, b \in X. ((a, b) \in R \vee (b, a) \in R)$  – **连接性**(connectivity)

# 接下来的工作

如何用(a,b)表示a, b有关系?

定义: 元组(序偶)

定义出来的东西有什么好玩的性质?

# 一. 有序对

问题: 我们怎么刻画有序对?

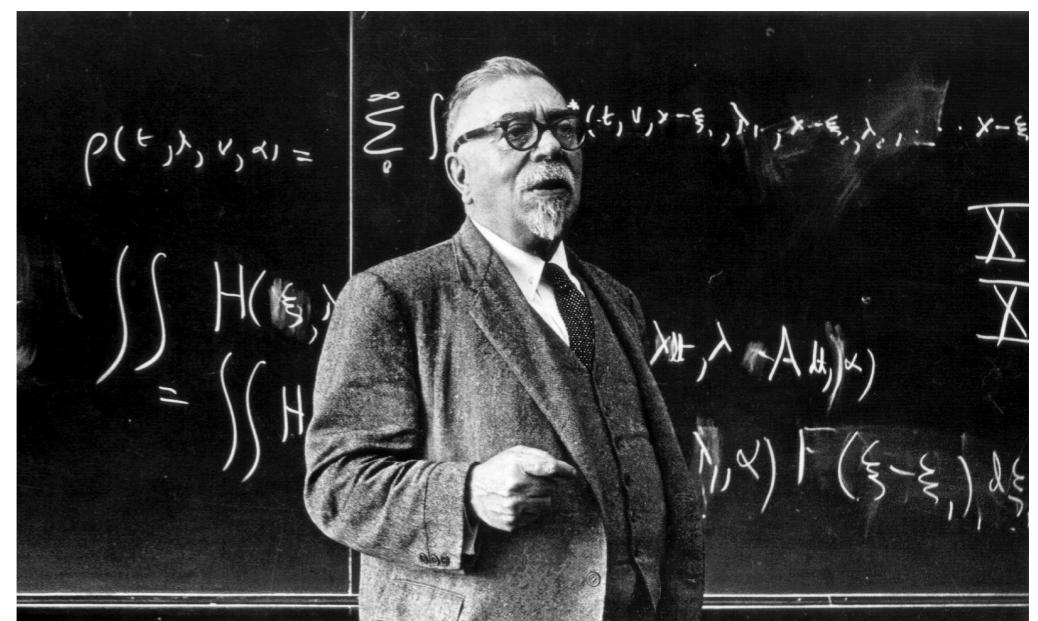
- 我们希望:  $(a, b) = (c, d) \iff a = c \wedge b = d$

## 1. 有序对

Def1. 有序对(Ordered Pairs)

$$(a, b) \triangleq \{\{a\}, \emptyset\}, \{\{b\}\}$$

然后我们就证明看到前面的直觉了...



# 一. 有序对

定理 3.4.1.

$$(a, b) = (c, d) \iff a = c \wedge b = d$$

证明. 也就是证明

1. 有

$$\left( \{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} \right) \iff (a = c \wedge b = d)$$

De

我们有:

$$\begin{aligned} & \{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} \\ \iff & (\{a\} = \{c\} \vee \{a\} = \{c, d\}) \wedge (\{a, b\} = \{c\} \vee \{a, b\} = \{c, d\}) \\ \iff & (\{a\} = \{c\} \wedge \{a, b\} = \{c\}) \vee \\ & (\{a\} = \{c\} \wedge \{a, b\} = \{c, d\}) \vee \\ & (\{a\} = \{c, d\} \wedge \{a, b\} = \{c\}) \vee \\ & (\{a\} = \{c, d\} \wedge \{a, b\} = \{c, d\}) \end{aligned}$$

然



# 一. 有序对

## 1. 有序对

Def2.  $n$ 元组( $n$ -ary tuples)

$$(x, y, z) \triangleq ((x, y), z)$$

## 2. Cartesian积:

Def3. The Cartesian product  $A \times B$  of  $A$  and  $B$  is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \wedge b \in B\}$$

有什么运算律呢

# 一. 有序对

## 2. Cartesian积:

Def3. The Cartesian product  $A \times B$  of  $A$  and  $B$  is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \wedge b \in B\}$$

有什么运算律呢

### (2) 运算规律

例子:

$$X \times \emptyset = \emptyset \times X$$

$$X \times Y \neq Y \times X$$

$$(X \times Y) \times Z \neq X \times (Y \times Z)$$

$$A = \{1\} \quad (A \times A) \times A \neq ;A \times (A \times A)$$

没有交换律, 结合律, 只有分配率.

# 一. 有序对

## 2. Cartesian积:

Def3. The Cartesian product  $A \times B$  of  $A$  and  $B$  is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \wedge b \in B\}$$

有什么运算律呢

## (2) 运算规律: 分配率

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Why?

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C)$$

# 一. 有序对

## 2. Cartesian积:

Def3. The Cartesian product  $A \times B$  of  $A$  and  $B$  is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \wedge b \in B\}$$

有什么运算律呢

Proof.  $A \times (B \cap C) = (A \times B) \cap (A \times C)$

对于任意的有序对 $(a, b)$ :

$$(a, b) \in A \times (B \cap C)$$

起关键作用的就是这个和连接词

$$\iff a \in A \wedge b \in (B \cap C)$$

$$\iff a \in A \wedge b \in B \wedge b \in C$$

$$\iff (a \in A \wedge b \in B) \wedge (a \in A \wedge b \in C)$$

$$\iff (a, b) \in A \times B \wedge (a, b) \in A \times C$$

$$\iff (a, b) \in (A \times B) \cap (A \times C)$$

# 一. 有序对

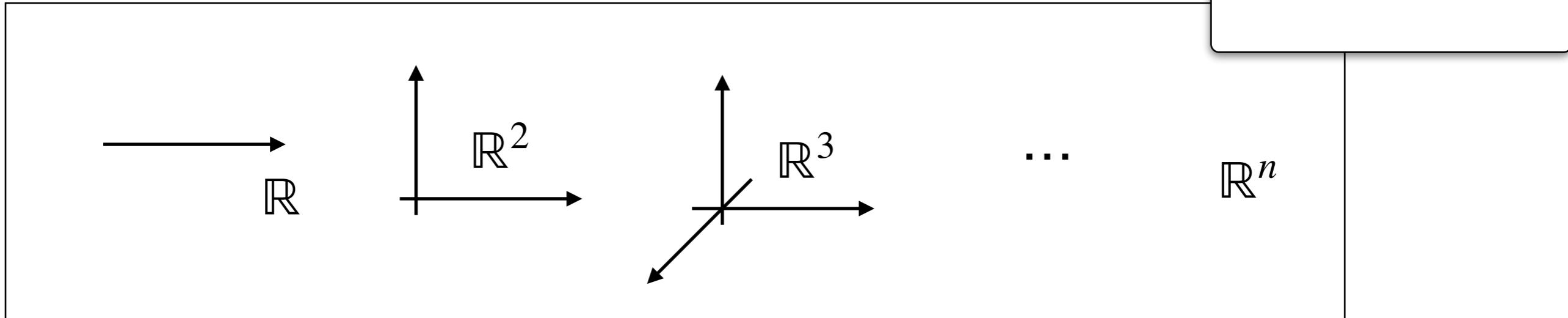
3. n-ary Cartesian Product:

Def4.

$$X_1 \times X_2 \times X_3 \triangleq (X_1 \times X_2) \times X_3$$

$$X_1 \times X_2 \times \dots \times X_n \triangleq (X_1 \times X_2 \times \dots \times X_{n-1}) \times X_n$$

同样是归纳法



## 二. 有序对定义的二元关系

### 1. 关系(relations)

Def5. A *relation* from  $A$  to  $B$  is a subset of  $A \times B$

$$R \subseteq A \times B$$

Sepcially, if  $A = B$ ,  $R$  is called a relation on  $A$ .

记法: 如果  $(a, b) \in R$ , 记作  $aRb$

如果  $(a, b) \notin R$ , 记作  $a\bar{R}b$

## 二. 有序对定义的二元关系

### 1. 关系(relations)

Def5. A *relation* from  $A$  to  $B$  is a subset of  $A \times B$

$$R \subseteq A \times B$$

Specially, if  $A = B$ ,  $R$  is called a relation on  $A$ .

记法: 如果  $(a, b) \in R$ , 记作  $aRb$

如果  $(a, b) \notin R$ , 记作  $a\bar{R}b$

小于关系:  $< = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a \text{ is less than } b\}$

整除关系:  $D = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid \exists q \in \mathbb{N}. a \cdot q = b\}$

妈妈关系:  $M = \{(a, b) \in \text{人} \times \text{人} \mid a \text{ is the mother of } b\}$

爸爸关系:  $F = \{(a, b) \in \text{人} \times \text{人} \mid a \text{ is the father of } b\}$

# 三. 关系的简单运算

(一) 三个定义 1. 定义域(domain) 2. 值域(range) 3. 域(field)

(二) 五种操作 逆, 限制, 像, 逆像, 复合.

(三) 七个性质

自反/反自反; 对称/反对称; 传递性; 连接性; 三分的



Figure 13. A selection of consistency axioms over an execution  $(E, \text{repl}, \text{obj}, \text{oper}, \text{ral}, \text{ro}, \text{vis}, \text{ar})$

#### Auxiliary relations

$$\text{sameobj}(e, f) \iff \text{obj}(e) = \text{obj}(f)$$

Per-object causality (aka happens-before) order:

$$\text{hbo} = ((\text{ro} \cap \text{sameobj}) \cup \text{vis})^+$$

Causality (aka happens-before) order:  $\text{hb} = (\text{ro} \cup \text{vis})^+$

#### Axioms

EVENTUAL:

$$\forall e \in E. \neg(\exists \text{ infinitely many } f \in E. \text{sameobj}(e, f) \wedge \neg(e \xrightarrow{\text{vis}} f))$$

THINAIR:  $\text{ro} \cup \text{vis}$  is acyclic

POCV (Per-Object Causal Visibility):  $\text{hbo} \subseteq \text{vis}$

POCA (Per-Object Causal Arbitration):  $\text{hbo} \subseteq \text{ar}$

COCV (Cross-Object Causal Visibility):  $(\text{hb} \cap \text{sameobj}) \subseteq \text{vis}$

COCA (Cross-Object Causal Arbitration):  $(\text{hb} \cap \text{ar})$  is acyclic

我的工作日常 ...

Figure 17. Optimized state-based multi-value register and its simulation

```

 $\Sigma$ 
 $= \text{ReplicaID} \times \mathcal{P}(\mathbb{Z} \times (\text{ReplicaID} \rightarrow \mathbb{N}_0))$ 
 $\emptyset_0$ 
 $= \{\langle r, V \rangle | r \in \mathcal{P}(\mathbb{Z} \times (\text{ReplicaID} \rightarrow \mathbb{N}_0))\}$ 
 $M$ 
 $= \mathcal{P}(\mathbb{Z} \times (\text{ReplicaID} \rightarrow \mathbb{N}_0))$ 
 $\text{do}(\text{receive}(r, V), t)$ 
 $= \langle \langle r, \lambda s. \text{if } s \neq r \text{ then } \max\{v(s) | (s, v) \in V\} \rangle, \perp \rangle$ 
 $\quad \text{else } \max\{v(s) | (s, v) \in V \wedge s = 1\} \rangle, \perp \rangle$ 
 $\text{do}(\text{ext}, \langle r, V \rangle, t)$ 
 $= \langle \langle r, \lambda s. \{a | (s, a) \in V\} \rangle, \perp \rangle$ 
 $\text{send}(\langle r, V \rangle, V')$ 
 $= \langle r, \lambda s. \{a | (s, a) \in V' \wedge \bigvee_{i=1}^k \{a' | (s, a') \in V' \wedge a' \neq a\}\} \rangle$ 
 $\text{receive}(\langle r, V \rangle, V')$ 
 $= \langle r, \lambda s. \{a' | (s, a') \in V' \wedge \bigvee_{i=1}^k \{a'' | (s, a'') \in V'' \wedge a'' \neq a'\}\} \rangle$ 
 $\text{where } V' = \{\langle a | \{a' | (a, a') \in V \wedge a' \in V'\} \rangle | (a, a) \in V \wedge V'\}$ 
 $\langle a, V \rangle | R_a : I \iff (r = s) \wedge (V \setminus \{M\} : I)$ 
 $V : [M] | (E, \text{rep}, \text{obj}, \text{oper}, \text{ral}, \text{ro}, \text{vis}, \text{ar}) \iff$ 
 $\langle \langle V(a, v), \langle a', v' \rangle | (a = a' \implies v = v') \rangle \rangle \wedge$ 
 $\langle \langle V(a, v), \langle a, v \rangle | V \setminus \{a\} \neq \emptyset \wedge v(a) > 0 \rangle \rangle \wedge$ 
 $\langle \langle V(a, v) | V, \bigvee_{i=1}^k \{a' | (a, a') \in V \wedge a' \neq a\} \rangle \rangle \wedge$ 
 $\exists \text{ distinct } e_{a,k}$ 
 $\langle \langle e \in E | \exists a. \text{oper}(e) = \text{vr}(a) \rangle \rangle = \{e_{a,k} | a \in \text{ReplicaID} \wedge$ 
 $1 \leq k \leq \max\{v(s) | \exists a. (a, v) \in V\}\} \wedge$ 
 $\langle \langle V(a, v), \langle e | \text{oper}''(e_{a,k}) = s \rangle \rangle = \{e_{a,k} \xrightarrow{\text{opr}} j \iff j < k\} \rangle \wedge$ 
 $\langle \langle V(a, v) | V, \forall g. \{j | \text{oper}'(e_{g,j}) = \text{vr}(v)\} \rangle \rangle \wedge$ 
 $\langle \langle V(a, v) | V, \forall g. \{j | \text{oper}'(e_{g,j}) = \text{vr}(a)\} \rangle \rangle =$ 
 $\langle \langle V(a, v) | V, \forall g. \{j | \text{oper}'(e_{g,j}) = \text{vr}(a) \wedge \text{oper}'(e_{g,j}) = \text{vr}(v)\} \rangle \rangle \wedge$ 
 $\langle \langle V(a, v) | V, \forall f. \{j | \text{oper}''(f) = \text{vr}(a) \wedge e \xrightarrow{\text{opr}} f \} \rangle \rangle = \{e_{a,k} | V\}$ 

```

the former. The only non-trivial obligation is to show that if

$$V : [M] | (E, \text{rep}, \text{obj}, \text{oper}, \text{ral}, \text{ro}, \text{vis}, \text{ar}),$$

then

$$\langle \langle a, \langle a, V \rangle | \exists e \in E. \text{oper}(e) = \text{vr}(a) \wedge$$

$$\neg \exists f \in E. \exists a'. \text{oper}(e) = \text{vr}(a') \wedge e \xrightarrow{\text{opr}} f \rangle \rangle \quad (13)$$

the reverse inclusion is straightforwardly implied by  $\mathcal{R}_{\text{ar}}$ .

$$\langle \langle a, \langle a, V \rangle | \forall (a, v) \in V. \exists a. \text{vr}(a) > 0,$$

$$v \sqsubseteq \{a' | \exists a. (a, a') \in V \wedge a \neq a'\}$$

and

$$\langle \langle a, \langle a, V \rangle | \forall g. \{j | \text{oper}(e_{g,j}) = \text{vr}(a)\} \rangle \rangle =$$

$$\langle \langle j | \exists a. k. e_{a,k} \xrightarrow{\text{opr}} e_{a,k} \wedge \text{oper}(e_{a,k}) = \text{vr}(a) \rangle \rangle =$$

From this we get that for some  $e \in E$

$$\text{oper}(e) = \text{vr}(a) \wedge \neg \exists f \in E. \exists a'. a' \neq a \wedge$$

$$\text{oper}(e) = \text{vr}(a') \wedge e \xrightarrow{\text{opr}} f.$$

Since  $\text{vis}$  is acyclic, this implies that for some  $e' \in E$

$$\text{oper}(e') = \text{vr}(a) \wedge \neg \exists f \in E. \text{oper}(e') = \text{vr}(.) \wedge e' \xrightarrow{\text{opr}} f,$$

which establishes (13).

Let us now discharge RECEIVE. Let  $\text{receive}(\langle r, V \rangle, V') =$

$$\langle \langle V, V' | \rangle \rangle,$$

$$V' = \{ (a, \bigcup_{a' \in V} \{a' | (a, a') \in V \wedge a' \in V'\}) | (a, a) \in V \wedge V'\};$$

$$V = \{ (a, \bigcup_{a' \in V} \{a' | (a, a') \in V \wedge a' \in V\}) | (a, a) \in V \wedge a \neq a'\}.$$

Assume  $\langle r, V \rangle | I, V' | [M] | J$  and

$$I = ((E, \text{rep}, \text{obj}, \text{oper}, \text{ral}, \text{ro}, \text{vis}, \text{ar}), \text{info});$$

$$J = ((E', \text{rep}', \text{obj}', \text{oper}', \text{ral}', \text{ro}', \text{vis}', \text{ar}', \text{info}'))$$

$$J \cup J = ((E'' \text{rep}'' \text{obj}'', \text{oper}'', \text{ral}'', \text{ro}'', \text{vis}'', \text{ar}'', \text{info}'').$$

By agree we have  $I \cup J \in \mathbb{E}$ . Then

$$\langle \langle (a, v), \langle a', v' \rangle | V, (a = a' \implies v = v') \rangle \rangle \wedge$$

$$\langle \langle (a, v), \langle a, v \rangle | \exists a. v(a) > 0 \rangle \rangle \wedge$$

$$\langle \langle (a, v) | V, \forall g. \{j | \{a' | (a, a') \in V \wedge a' \neq a\} \rangle \rangle \wedge$$

$\exists \text{ distinct } e_{a,k}$

$$\langle \langle e \in E | \exists a. \text{oper}(e) = s \rangle \rangle = \{e_{a,k} | a \in \text{ReplicaID} \wedge$$

$$1 \leq k \leq \max\{v(s) | \exists a. (a, v) \in V\}\} \wedge$$

$$\langle \langle V(a, v), \langle e | \text{oper}''(e_{a,k}) = s \rangle \rangle = \{e_{a,k} \xrightarrow{\text{opr}} j \iff j < k\} \rangle \wedge$$

$$\langle \langle V(a, v) | V, \forall g. \{j | \text{oper}'(e_{g,j}) = \text{vr}(v)\} \rangle \rangle \wedge$$

$$\langle \langle V(a, v) | V, \forall g. \{j | \text{oper}'(e_{g,j}) = \text{vr}(a)\} \rangle \rangle =$$

$$\langle \langle V(a, v) | V, \forall f. \{j | \text{oper}''(f) = \text{vr}(a) \wedge e \xrightarrow{\text{opr}} f \} \rangle \rangle = \{e_{a,k} | V\}$$

and

$$\langle \langle (a, v), \langle a', v' \rangle | V', (a = a' \implies v = v') \rangle \rangle \wedge$$

$$\langle \langle (a, v), \langle a, v \rangle | \exists a. v(a) > 0 \rangle \rangle \wedge$$

$$\langle \langle (a, v) | V', \forall g. \{j | \{a' | (a, a') \in V' \wedge a' \neq a\} \rangle \rangle \wedge$$

$\exists \text{ distinct } e_{a,k}'$

$$\langle \langle e \in E' | \exists a. \text{oper}(e) = s \rangle \rangle = \{e'_{a,k} | a \in \text{ReplicaID} \wedge$$

$$1 \leq k \leq \max\{v(s) | \exists a. (a, v) \in V'\}\} \wedge$$

$$\langle \langle V(a, v), \langle e | \text{oper}''(e'_{a,k}) = s \rangle \rangle = \{e'_{a,k} \xrightarrow{\text{opr}} j \iff j < k\} \rangle \wedge$$

$$\langle \langle V(a, v) | V', \forall g. \{j | \text{oper}'(e'_{g,j}) = \text{vr}(v)\} \rangle \rangle \wedge$$

$$\langle \langle V(a, v) | V', \forall g. \{j | \text{oper}'(e'_{g,j}) = \text{vr}(a)\} \rangle \rangle =$$

$$\langle \langle V(a, v), \langle e | \text{oper}''(e) = \text{vr}(a) \wedge e \xrightarrow{\text{opr}} f \} \rangle \rangle = \{e'_{a,k} | V\}$$

and

$$\langle \langle (a, v), \langle a', v' \rangle | V'', (a = a' \implies v = v') \rangle \rangle \wedge$$

$$\langle \langle (a, v), \langle a, v \rangle | \exists a. v(a) > 0 \rangle \rangle \wedge$$

$$\langle \langle (a, v) | V'', \forall g. \{j | \{a' | (a, a') \in V'' \wedge a' \neq a\} \rangle \rangle \wedge$$

$\exists \text{ distinct } e'_{a,k}$

$$\langle \langle e' \in E'' | \exists a. \text{oper}(e') = s \rangle \rangle = \{e'_{a,k} | a \in \text{ReplicaID} \wedge$$

$$1 \leq k \leq \max\{v(s) | \exists a. (a, v) \in V''\}\} \wedge$$

$$\langle \langle V(a, v), \langle e' | \text{oper}''(e'_{a,k}) = s \rangle \rangle = \{e'_{a,k} \xrightarrow{\text{opr}} j \iff j < k\} \rangle \wedge$$

$$\langle \langle V(a, v) | V'', \forall g. \{j | \text{oper}'(e'_{g,j}) = \text{vr}(v)\} \rangle \rangle \wedge$$

$$\langle \langle V(a, v) | V'', \forall g. \{j | \text{oper}'(e'_{g,j}) = \text{vr}(a)\} \rangle \rangle =$$

$$\langle \langle V(a, v), \langle e' | \text{oper}''(e') = \text{vr}(a) \wedge e' \xrightarrow{\text{opr}} f \} \rangle \rangle = \{e'_{a,k} | V\}$$

离散数学学得好不好, 一个重要的标准在于是否完成了这样的转变

— 魏恒峰, 南京大学  
在2021年《离散数学》

# 三. 关系的简单运算

## (一) 三个定义

### 1. 定义域(domain)

Def6.

$$\text{dom}(R) = \{a \mid \exists b . (a, b) \in R\}$$

“所有有定义的地方 第一个元素的集合”

### 2. 值域(range)

Def7.

$$\text{ran}(R) = \{b \mid \exists a . (a, b) \in R\}$$

“所有有定义的地方 第二个元素的集合”

### 3. 域(field)

Def8.

$$\text{fld}(R) = \text{dom}(R) \cup \text{ran}(R)$$

提示: 和中学的函数类比

# 三. 关系的简单运算

## (一) 三个定义

提示: 和中学的函数类比

$$R = \{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R} \times \mathbb{R}$$

$$\text{dom}(R) = [1,1], \text{ran}(R) = [-1,1], \text{fld}(R) = [-1,1]$$

就是用集合的眼光来理解这些内容.

$$\text{dom}(R) \subseteq \bigcup R$$



I'm so excited.



# 三. 关系的简单运算

## (一) 三个定义

$$\text{dom}(R) \subseteq \bigcup \bigcup R$$

任何的定义域, 值域都会在二元组的某一个元素中“出现”.

证明. 对任意  $a$ ,

$$\begin{aligned} a &\in \text{dom}(R) \\ \implies \exists b. (a, b) &\in R \\ \implies \exists b. \{\{a\}, \{a, b\}\} &\in R \\ \implies \exists b. \{a, b\} &\in \bigcup R \\ \implies \exists b. a &\in \bigcup \bigcup R \\ \implies a &\in \bigcup \bigcup R \end{aligned}$$

□

# 三. 关系的简单运算

## (二) 五种操作

### 1. 逆变换(inverse)

Def9. The *inverse* of  $R$  is the **relation**

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

例子：

- 如果  $R = \{(x, y) \mid x = y\} \subseteq \mathbb{R} \times \mathbb{R}$ ,  $R^{-1} = R$
- $R = \{(x, y) \mid y = \sqrt{x}\} \subseteq \mathbb{R} \times \mathbb{R}$ ,  $R^{-1} = \{(x, y) \mid y = x^2 \wedge x > 0\}$
- $\leq = \{(x, y) \mid x \leq y\} \subseteq \mathbb{R} \times \mathbb{R}$  ,  $\leq^{-1} = \geq \triangleq \{(x, y) \mid x \geq y\}$

# 三. 关系的简单运算

## (二) 五种操作

### 1. 逆变换(inverse)

Def9. The *inverse* of  $R$  is the **relation**

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

Th1. 关系的逆的逆还是原来的关系

$$(R^{-1})^{-1} = R$$

Proof. 对任意 $(a, b)$

$$\begin{aligned}(a, b) \in (R^{-1})^{-1} \\ \iff (b, a) \in R^{-1} \\ \iff (a, b) \in R\end{aligned}$$

# 三. 关系的简单运算

## (二) 五种操作

### 1. 逆变换(inverse)

Def9. The *inverse* of  $R$  is the **relation**

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

Th2. 关系的逆与交, 并, 补: 如果 $R, S$ 均为关系, 那么有

$$(R \cup S)^{-1} = R^{-1} \cup S^{-1}$$

$$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$$

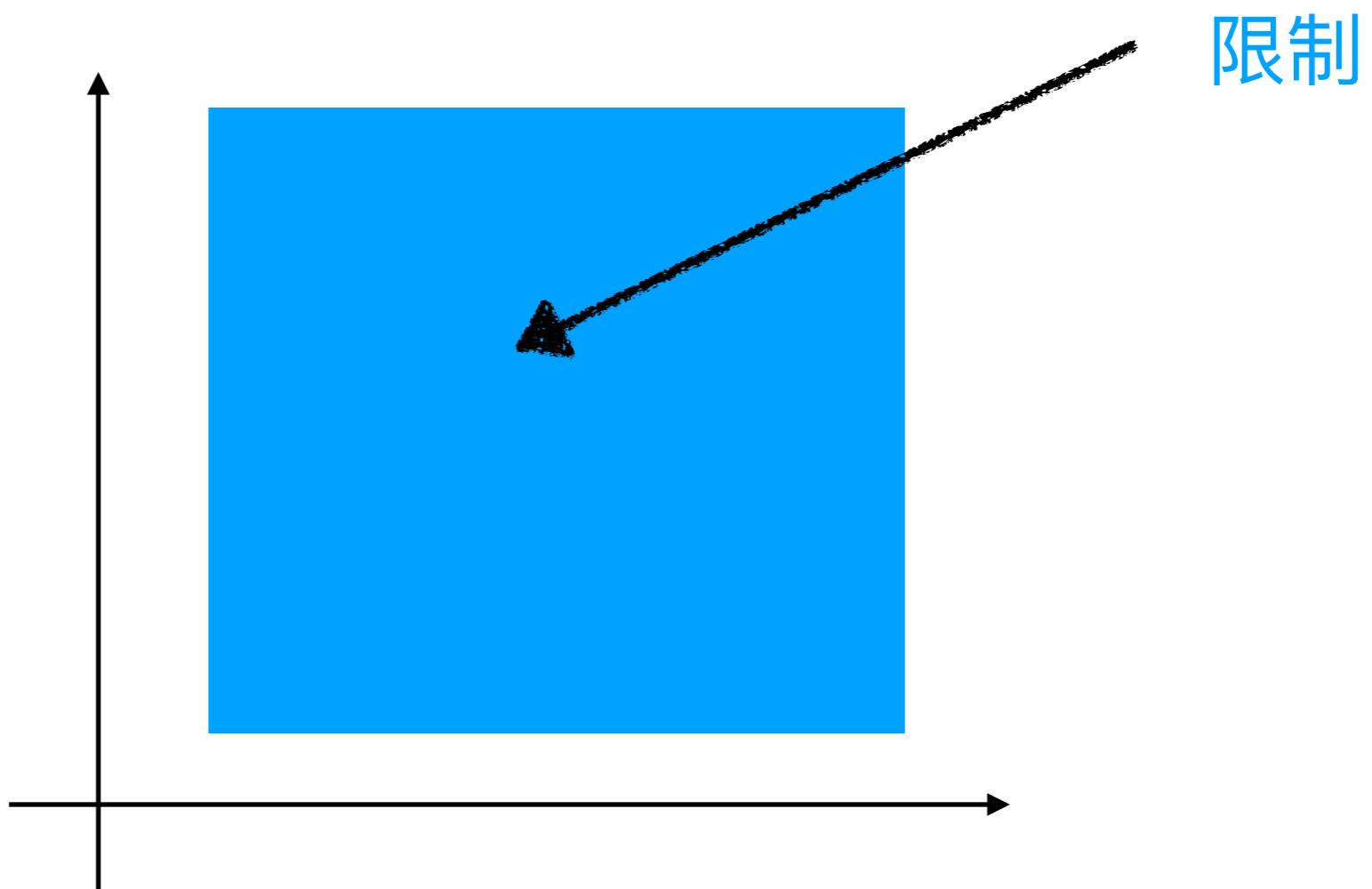
$$(R \setminus S)^{-1} = R^{-1} \setminus S^{-1}$$

Proof. 提示: 把集合写出来就行了.

# 三. 关系的简单运算

## (二) 五种操作

### 2. 限制



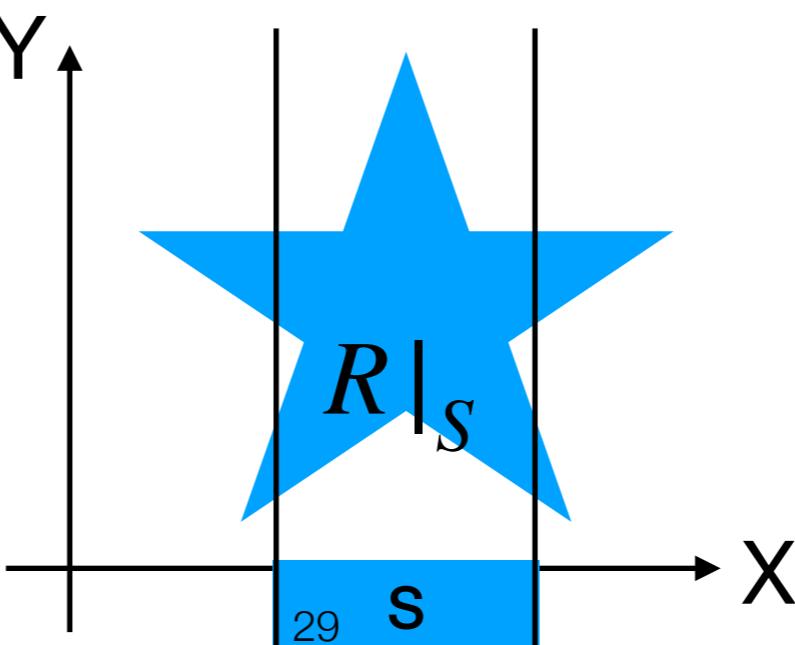
# 三. 关系的简单运算

## (二) 五种操作

### 2. 限制

Def10. 左限制 (Left-Restriction) Suppose  $R \subseteq X \times Y$  and  $S \subseteq X$ . The *left restriction* relation of  $R$  to  $S$  over  $X$  and  $Y$  is

$$R|_S = \{(x, y) \in R \mid x \in S\}$$



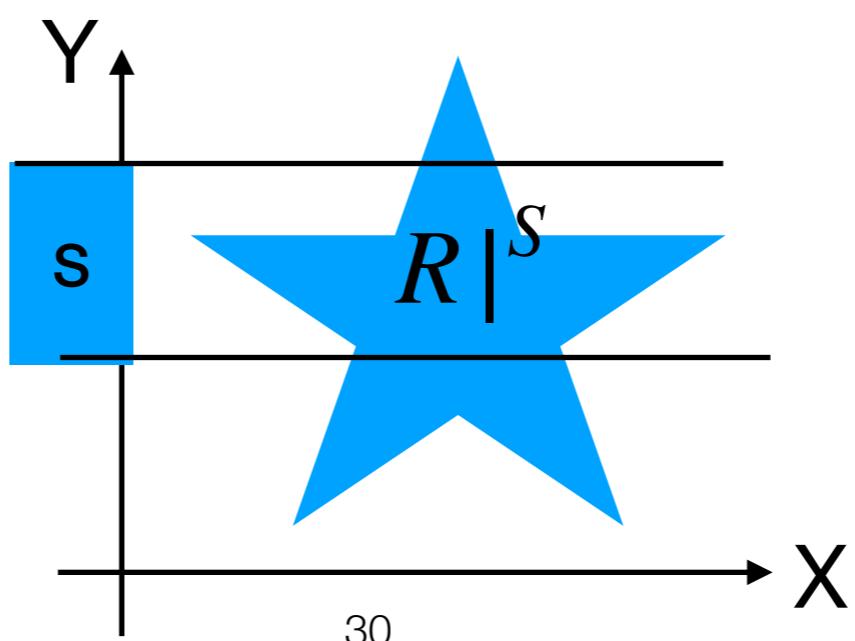
# 三. 关系的简单运算

## (二) 五种操作

### 2. 限制

Def11. 右限制 (Right-Restriction) Suppose  $R \subseteq X \times Y$  and  $S \subseteq Y$ . The *left restriction* relation of  $R$  to  $S$  over  $X$  and  $Y$  is

$$R|_S = \{(x, y) \in R \mid y \in S\}$$



# 三. 关系的简单运算

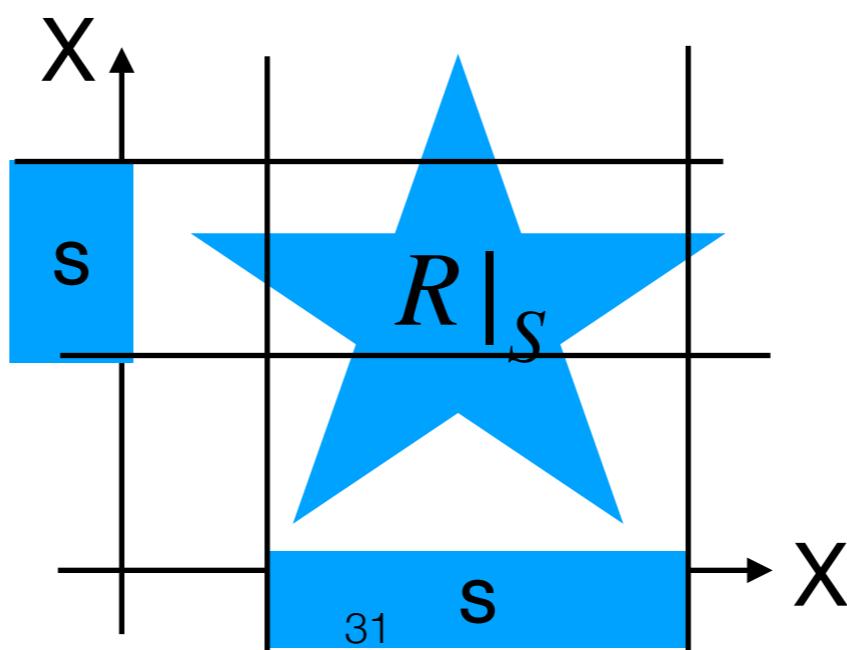
## (二) 五种操作

### 2. 限制

很多时候关系是从自己到自己的, 所以

Def12. 限制 (Restriction) Suppose  $R \subseteq X \times X$  and  $S \subseteq X$ . The *restriction* relation of  $R$  to  $S$  over  $X$  is

$$R|_S = \{(x, y) \in R \mid x \in S \wedge y \in S\}$$

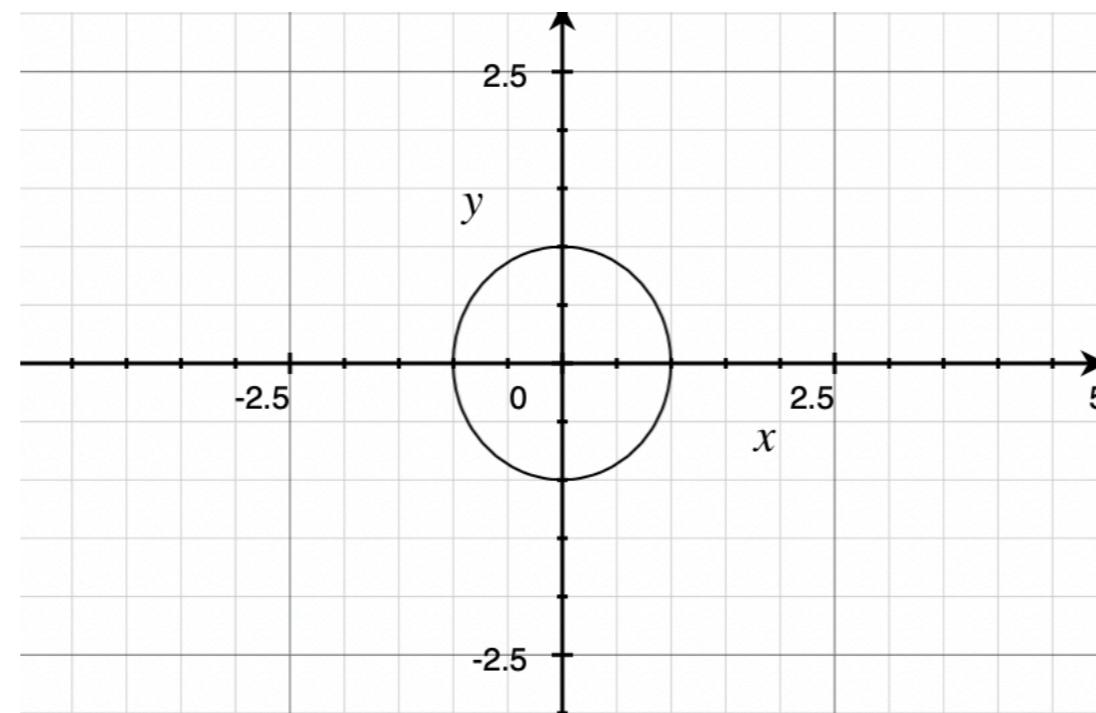


# 三. 关系的简单运算

## (二) 五种操作

### 2. 限制

$$R = \{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R} \times \mathbb{R}$$

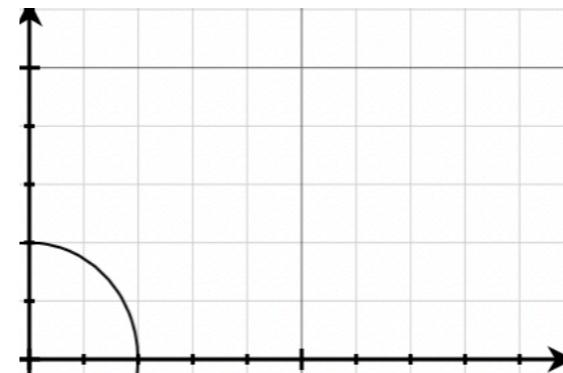


# 三. 关系的简单运算

## (二) 五种操作

### 2. 限制

$$R|_{\mathbb{R}^+} = \{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R} \times \mathbb{R}$$



# 三. 关系的简单运算

## (二) 五种操作

### 3. 像

Def 13. The *image* of  $X$  under relation  $R$  is the set

$$R[X] = \{b \in \text{ran}(R) \mid \exists a \in X. (a, b) \in R\}$$

为了简化记号, 我们一般写作

$$R[a] \triangleq R[\{a\}] = \{b \mid (a, b) \in R\}$$

### 4. 逆像

Def 14. The *inverse image* of  $Y$  under relation  $R$  is the set

$$R^{-1}[Y] = \{a \in \text{dom}(R) \mid \exists b \in Y. (a, b) \in R\}$$

为了简化记号, 我们一般写作

$$R^{-1}[b] \triangleq R^{-1}[\{b\}] = \{a \mid (a, b) \in R\}$$

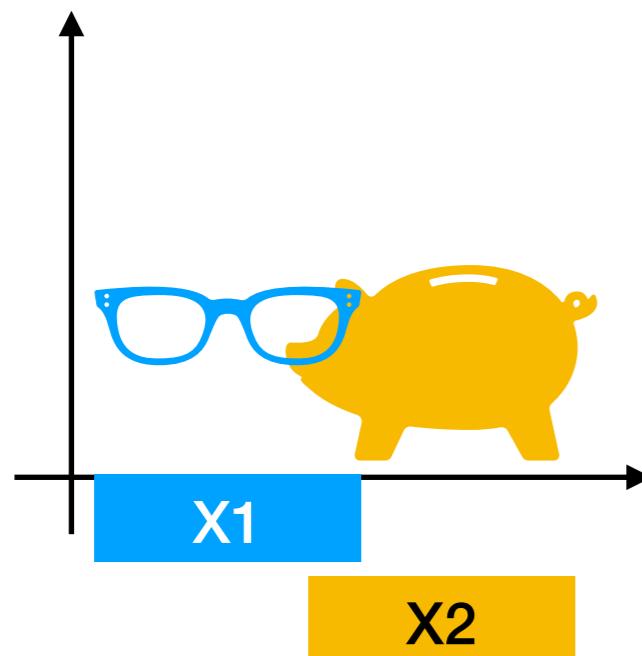
# 三. 关系的简单运算

## (二) 五种操作

[像和逆像的一些性质]

$$(1) R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$

Th3



Proof. 对于任意的  $b$

$$b \in R[X_1 \cup X_2]$$

Definition is always your friend!

$$\iff \exists a \in X_1 \cup X_2. (a, b) \in R$$

$$\iff \exists a \in X_1. (a, b) \in R \vee \exists a \in X_2. (a, b) \in R$$

$$\iff b \in R[X_1] \cup b \in R[X_2]$$

$$\iff b \in R[X_1] \cup R[X_2]$$

# 三. 关系的简单运算

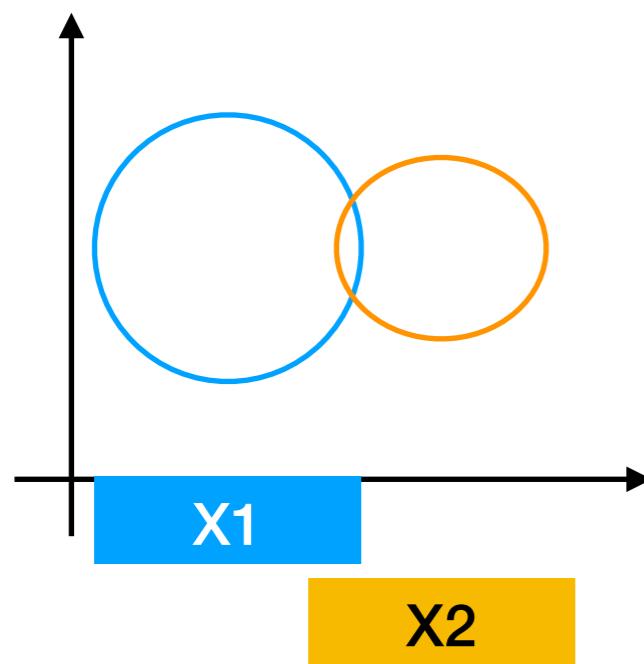
## (二) 五种操作

[像和逆像的一些性质]



$$(2) R[X_1 \cap X_2] \subsetneq R[X_1] \cap R[X_2]$$

Th4



左边: 只有两个点(交点)

右边: 是介于 $x_1, x_2$ 之间的图形区域

证明可以仿照上述, 但是注意哪一步变为了  
单方向的推演

# 三. 关系的简单运算

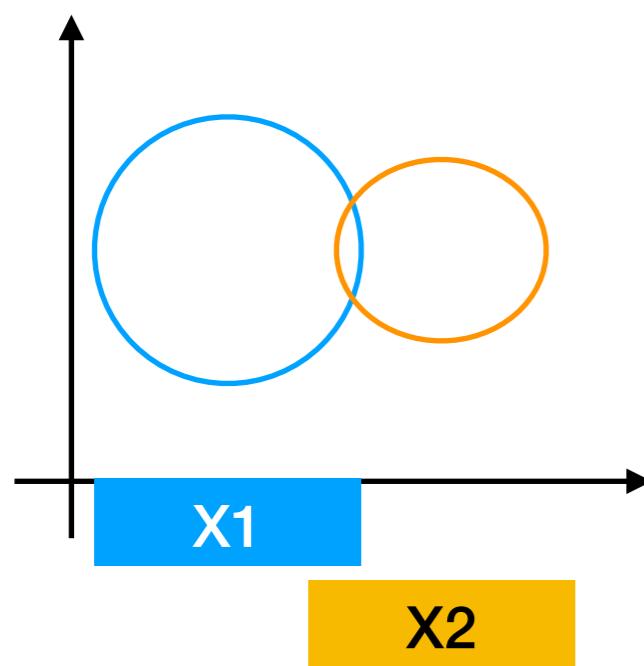
## (二) 五种操作

[像和逆像的一些性质]



$$(3) R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2]$$

Th5



左边: 蓝色圆环拿掉两个交点  
右边: 是从 $x_1$ , 到 $x_2$ 做端点的蓝色区域

证明可以仿照上述, 但是注意哪一步变为了  
单方向的推演

# 三. 关系的简单运算

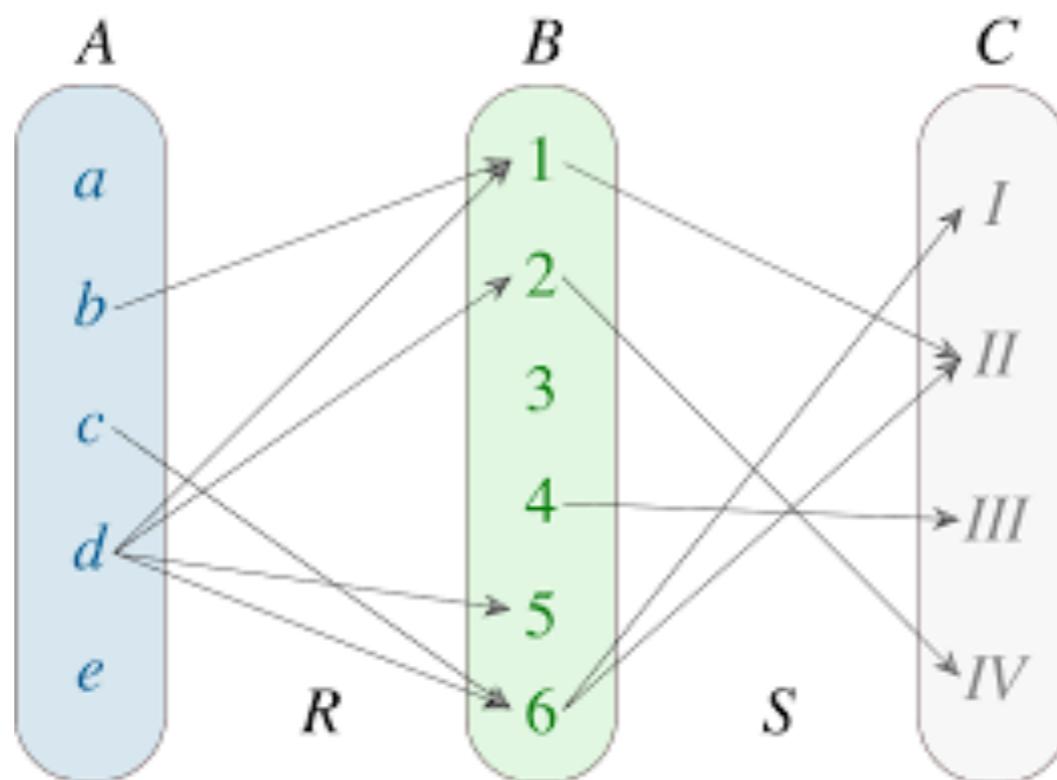
## (二) 五种操作

### 5. 复合

Def 15. The *composition of relation*

$R \subseteq X \times Y$  and  $S \subseteq Y \times Z$  is the *relation*

$$R \circ S = \{(a, c) \mid \exists b . (a, b) \in S \wedge (b, c) \in R\}$$



# 三. 关系的简单运算

## (二) 五种操作

这是以前排版了大部分的文稿... 实在懒得改了...



### ⚠ 警告: 记号的解释不一的问题

关于 $R \circ S$ 的记号

- 南京大学《离散数学》课程
  - 从右往左计算
- 中国地质大学的《离散数学》课程
  - 定义是从左往右的, 也就是定义如
$$R \circ S = \{(a, c) \mid \exists b. (a, b) \in R \wedge (b, c) \in S\}.$$
- 个人感觉文献中从右往左的情况多一些
- 余下的文档中, 仍然保持从右往左的顺序, 以便于书写.

# 三. 关系的简单运算

## (二) 五种操作

### 5. 复合

Def 15. The *composition* of *relation*

$R \subseteq X \times Y$  and  $S \subseteq Y \times Z$  is the *relation*

$$R \circ S = \{(a, c) \mid \exists b . (a, b) \in S \wedge (b, c) \in R\}$$

问:  $(A \circ B) \circ C \stackrel{?}{=} A \circ (B \circ C)$

# 三. 关系的简单运算

## (二) 五种操作

### 5. 复合

Def 15. The *composition* of *relation*

$R \subseteq X \times Y$  and  $S \subseteq Y \times Z$  is the *relation*

$$R \circ S = \{(a, c) \mid \exists b . (a, b) \in S \wedge (b, c) \in R\}$$

Th6. 满足结合律  $(R \circ S) \circ T = R \circ (S \circ T)$

如果这个是这样的话就太好了! (矩阵乘法)

# 三. 关系的简单运算

## (二) 五种操作

5. 复合 Th6.  $(R \circ S) \circ T = R \circ (S \circ T)$

Proof. 对任意  $(a, b)$   $(a, b) \in (R \circ S) \circ T$

$$\iff \exists c. \left( (a, c) \in T \wedge (c, b) \in R \circ S \right)$$

$$\iff \exists c. \left( (a, c) \in T \wedge \left( \exists d. (c, d) \in S \wedge (d, b) \in R \right) \right)$$

$$\iff \exists d. \exists c. \left( (a, c) \in T \wedge (c, d) \in S \wedge (d, b) \in R \right)$$

$$\iff \exists d. \left( \left( \exists c. (a, c) \in T \wedge (c, d) \in S \right) \wedge (d, b) \in R \right)$$

$$\iff \exists d. \left( (a, d) \in S \circ T \wedge (d, b) \in R \right)$$

$$\iff (a, b) \in R \circ_4 (S \circ T)$$

# 三. 关系的简单运算

## (二) 五种操作

5. 复合 Th7. 和矩阵的逆类似  $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$

提示:

$$\begin{aligned}(a, b) &\in (R \circ S)^{-1} \\ \iff (b, a) &\in R \circ S \iff \exists c . (c, b) \in S^{-1} \wedge (a, c) \in R^{-1} \\ \iff \exists c . (b, \textcolor{blue}{c}) &\in S \wedge (\textcolor{blue}{c}, a) \in R \\ \iff (a, b) &\in S^{-1} \circ R^{-1}\end{aligned}$$

# 三. 关系的简单运算

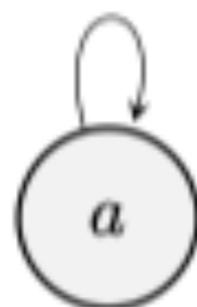
## (三) 七个性质

### 1. 自反的:

Def16.  $R \subseteq X \times X$  is **reflexive** if

$$\forall a \in X. (a, a) \in R$$

$$\forall a \in X. (a, a) \in R$$



举几个例子:

- $\leq \subseteq \mathbb{R} \times \mathbb{R}$  is reflexive
- 三角形上的**全等关系**是自反的

# 三. 关系的简单运算

## (三) 七个性质

1. 自反的:

Def16.  $R \subseteq X \times X$  is **reflexive** if

$$\forall a \in X. (a, a) \in R$$

Th8. 所有自反的关系都是这个关系的一个子集

$R$  is reflexive  $\iff I \subseteq R$ , 其中

$$I = \{(a, a) \in A \times A \mid a \in A\}.$$

Th9.  $R$  is reflexive  $\iff R^{-1} = R$

# 三. 关系的简单运算

## (三) 七个性质

### 2. 反自反的(Irreflexive):

Def17.  $R \subseteq X \times X$  is **irreflexive** if

$$\forall a \in X. (a, a) \notin R$$

例子:

- $< \subseteq \mathbb{R} \times \mathbb{R}$  is irreflexive
- $> \subseteq \mathbb{R} \times \mathbb{R}$  is irreflexive

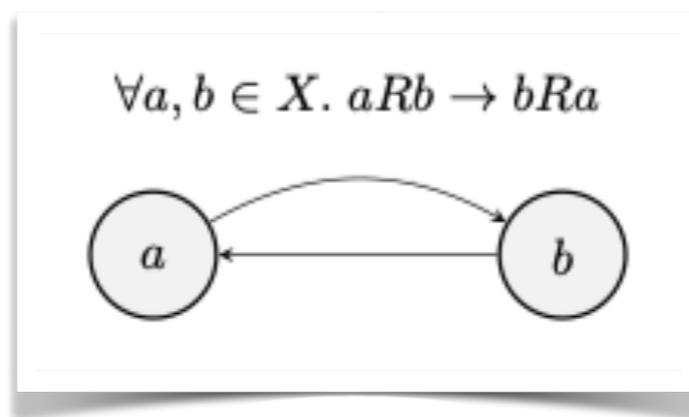
# 三. 关系的简单运算

## (三) 七个性质

### 3. 对称(symmetric)

Def18.  $R \subseteq X \times X$  is **irreflexive** if

$$\forall a, b \in X. aRb \rightarrow bRa$$



对于求逆运算很友好!

Th10.  $R$  is symmetric  $\iff R^{-1} = R$

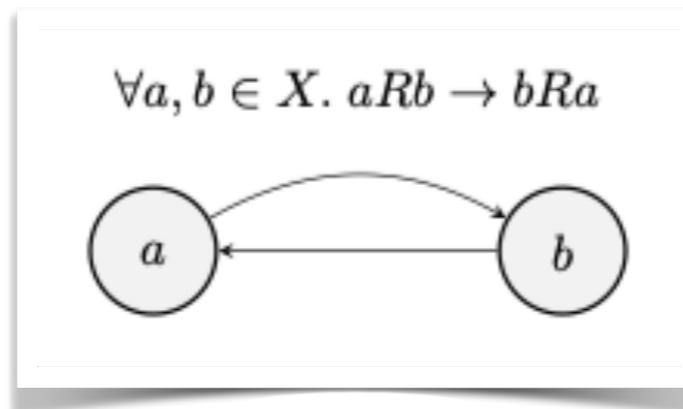
# 三. 关系的简单运算

## (三) 七个性质

### 4. 反对称(antisymmetric)

Def19.  $R \subseteq X \times X$  is **antisymmetric** if

$$\forall a, b \in X. ((aRb \wedge bRa) \rightarrow a = b)$$



例如:  $>$ (大于),  $|$ (整除)就是反对称的.

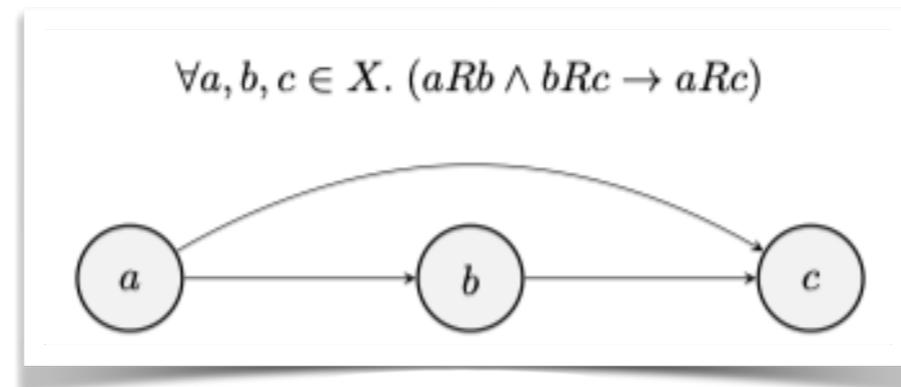
# 三. 关系的简单运算

## (三) 七个性质

### 5. 传递性(transitive)

Def20.  $R \subseteq X \times X$  is **transitive** if

$$\forall a, b, c \in X. (aRb \wedge bRc \rightarrow aRc)$$



具有这样的关系的就意味着经过反复作用, 它不会跑出去

# 三. 关系的简单运算

## (三) 七个性质

### 5. 传递性(transitive)

这样就构成了偏序关系

Def20.  $R \subseteq X \times X$  is **transitive** if

$$\forall a, b, c \in X. (aRb \wedge bRc \rightarrow aRc)$$

Th11. 封闭性:  $R$  is transitive  $\iff R \circ R \subseteq R$

Proof. 对任意( $a, b$ )

$$(a, b) \in R \circ R$$

$$\implies \exists c. (a, c) \in R \wedge (b, c) \in R$$

$$\implies (a, b) \in R$$

对任意( $a, b, c$ )

$$(a, b) \in R \wedge (b, c) \in R \implies (a, c) \in R \circ R \implies (a, c) \in R$$

# 三. 关系的简单运算

## (三) 七个性质

### 6. 连接性(Connex)

有了这个就有了  
全序关系

Def21.  $R \subseteq X \times X$  is **connex** if

$$\forall a, b \in X. (aRb \vee bRa)$$

### 7. 三分的(Trichotomous)

Def22.  $R \subseteq X \times X$  is **trichotomous** if

$$\forall a, b \in X. (\text{exactly one of } aRb, bRa, \text{ or } a = b \text{ holds})$$

比如实数上的大于关系就满足这一个.

# 三. 关系的简单运算

## (三) 七个性质

Th12. 求逆的可行性:

$$R \text{ is symmetric and transitive} \iff R = R^{-1} \circ R$$

Proof. 对任意  $(a, b)$

$$(a, b) \in R \circ R$$

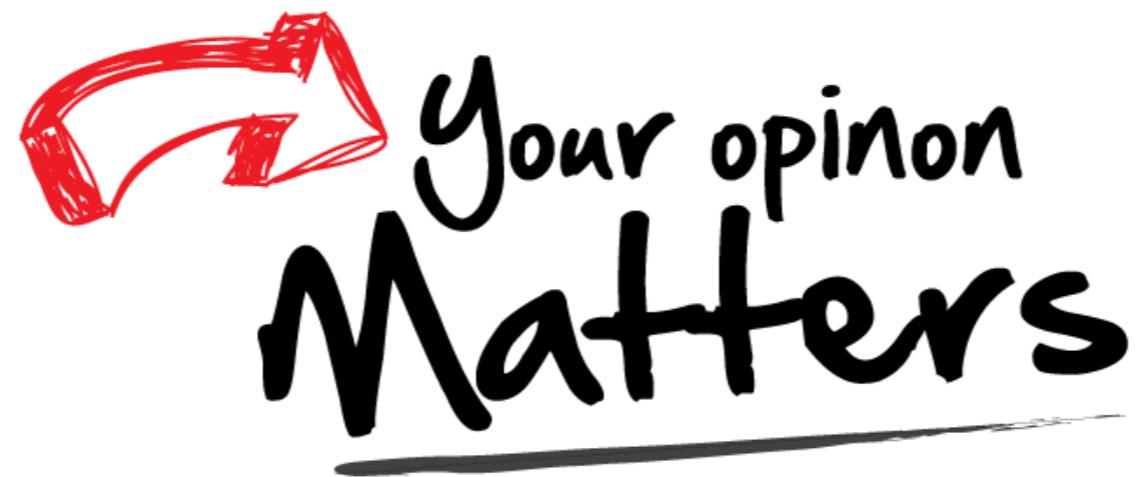
$$\implies \exists c. (a, c) \in R \wedge (b, c) \in R$$

$$\implies (a, b) \in R$$

# 参考文章和课件

1. 魏恒峰《离散数学2020》集合论II. 关系
2. 南京大学《计算机问题求解2021》集合论II. 关系

# Thank You!



QQ: 2095728218

Email: [micoael@qq.com](mailto:micoael@qq.com)

(学校) [gwzhang@cug.edu.cn](mailto:gwzhang@cug.edu.cn)