TD 2 – Signal processing: Fourier, filtering, FFT

Exercises

- We denote as $\Sigma_n \overset{\text{def.}}{=} \{a \in \mathbb{R}_+^n ; \sum_i a_i = 1\}$ the probability simplex.
- We identify the indices $\{1, \ldots, n\}$ with the elements of $\mathbb{Z}/n\mathbb{Z}$ (i.e. we consider periodic indices).
- We define the convolution between $f \in \mathbb{C}^n$ and $g \in \mathbb{C}^n$ by $(f \star g)_i = \sum_j f_{i-j} g_j$
- For all $f \in \mathbb{C}^n$, we recall that the discrete Fourier transform is defined as

$$\forall k \in \mathbb{Z}/n\mathbb{Z}, \hat{f}_k \overset{\text{def.}}{=} \sum_{p=0}^{n-1} f_p e^{-2i\pi kp/n}.$$

Common Fourier Transforms

Compute the discrete Fourier transforms of the following signals $f \in \mathbb{C}^n$.

1. *Frequency shift.* $f_p = x_p e^{2i\pi p\tau/n}$ for $\tau \in \mathbb{Z}/n\mathbb{Z}$ (use $\hat{x}_k$ and τ to express $\hat{f}_k$).
2. *Geometric progression.* $f_p = a^p$.
3. *Window.* f is a window of size $W = 2K + 1$ centered around 0:

$$\begin{aligned} f_p &= 1/W \text{ if } \min(p, n-p) \leq K \\ &= 0 \text{ otherwise.} \end{aligned}$$

Total Variation (from exam 2020)

1. Let $f \in \mathbb{C}^n$. Bound $|\hat{f}_k|$ using $\|f\|_1$ (the l^1 norm of f).
2. We define the total variation of f as

$$\|f\|_V \overset{\text{def.}}{=} \sum_{i=0}^{n-1} |f_{i+1} - f_i|.$$

   Determine the set of f such that $\|f\|_V = 0$. Show that $\|\cdot\|_V$ statisfies the triangular inequality (it is a semi-norm).
3. Write $\|f\|_V$ as $\|f \star h\|_1$, where $h \in \mathbb{C}^n$.
4. Compute $\hat{h}$ the discrete Fourier transform of h, and compute $|\hat{h}_k|$.
5. Bound $|\hat{f}_k|$ using $\|f\|_V$, n, and k.

Markov Chains (from exam 2018)

1. Show that a matrix $P \in \mathbb{R}_+^{n \times n}$ such that $P^\top \mathbb{1}_n = \mathbb{1}_n$ defines a map $a \in \Sigma_n \mapsto Pa \in \Sigma_n$ from the simplex to itself.
2. We denote $T_\tau : a \mapsto (a_{i-\tau})_i$ the translation operator for $\tau \in \mathbb{Z}/n\mathbb{Z}$. Show that P commutes with T_τ for all τ if and only if there exists $h \in \Sigma_n$ such that $Pa = h \star a$.
3. Starting from some $a^{(0)} \in \Sigma_n$, we define $a^{(l+1)} \overset{\text{def.}}{=} Pa^{(l)}$. Show that if $\forall k \neq 0, |\hat{h}_k| < 1$, then $a^{(l)} \to a^\star$ as $l \to +\infty$. What is $a^\star$?

Solutions

Common Fourier Transforms

1. $\hat{f}_k = \hat{x}_{k-\tau}$
2. Using the geometric progression formula, we get: $\hat{f}_k = n$ if $a = e^{2ik\pi/n}$, else
$$\hat{f}_k = \frac{1-a^n}{1-ae^{-2ik\pi/n}}$$
3.
$$\begin{aligned}\hat{f}_k = \frac{1}{W}\sum_{p=n-K}^{K} e^{-2ik\pi p/n} &= 1/W\frac{e^{2ik\pi K/n} - e^{-2ik\pi(K+1)/n}}{1-e^{-2ik\pi/n}} \\ &= \frac{1}{W2i\sin(k\pi/n)}e^{i\pi k/n}\left(e^{2ik\pi K/n} - e^{-2ik\pi(K+1)/n}\right) \\ &= \frac{1}{W2i\sin(k\pi/n)}(e^{ik\pi(2K+1)/n} - e^{-ik\pi(2K+1)/n}) \\ &= \frac{\sin(k\pi W/n)}{W\sin(k\pi/n)}.\end{aligned}$$

Total Variation

1. $|\hat{f}_k| = |\sum_p f_p e^{-2i\pi kp/n}| \leq \sum_p |f_p e^{-2i\pi kp/n}| = \sum_p |f_p| = \|f\|_1$.
2. $\|f\|_V = 0 \iff f$ constant. And
$$\|f+g\|_V = \sum_i |f_{i+1} + g_{i+1} - f_i - g_i| \leq \sum_i |f_{i+1} - f_i| + \sum_i |g_{i+1} - g_i| = \|f\|_V + \|g\|_V .$$
3. $h_0 = -1, h_1 = 1$ and 0 elsewhere.
4. $\hat{h}_k = -1 + e^{-2i\pi k/n}$. And $|\hat{h}_k| = 2 \times |\sin(k\pi/n)|$
5. Replacing f by $f \star h$ in the first question, we get $|(\widehat{f \star h})_k| = |\hat{f}_k\hat{h}_k| \leq \|f \star h\|_1 = \|f\|_V$. Hence
$$|\hat{f}_k| \leq \frac{\|f\|_V}{2 \times |\sin(k\pi/n)|}.$$

Markov Chains

1. $\sum_i (Pa)_i = \sum_i \sum_j P_{ij}a_j = \sum_j a_j \underbrace{\sum_i P_{ij}}_{=1 \text{ as } P^\top \mathbb{1}_n = \mathbb{1}_n} = \sum_j a_i = 1.$
2. $(i) \implies (ii)$: We denote $e_i = (0, \ldots, 0, \underset{(i+1)}{1}, 0, \ldots, 0)^\top$, and P_j the $j+1$-th column of P, i.e. $P = (P_0, \cdots, P_{n-1})$.
$$T_\tau(Pe_i) = T_\tau(P_i) \quad \text{and} \quad PT_\tau(e_i) = Pe_{\tau+i} = P_{\tau+i},$$
Thus $\forall \tau, \forall i, T_\tau(P_i) = P_{\tau+i}$, i.e $\forall j, P_{j-\tau,i} = P_{j,\tau+i}$. In particular $(i=0)$, $\forall \tau, \forall j, P_{j-\tau,0} = P_{j,\tau}$. Let $h \overset{\text{def.}}{=} P_0 \in \Sigma_n$.
$$(Pa)_i = \sum_j \underbrace{P_{i,j}}_{=P_{i-j,0}} a_j = \sum_j h(i-j)a_j = (h \star a)_i$$
$(ii) \implies (i)$:
$$\begin{aligned}T_\tau(Pa)_i = T_\tau(h \star a)_i = (h \star a)_{i-\tau} &= \sum_j h_{i-\tau-j}a_j \\ &= \sum_{j'} h_{i-j'}a_{j'-\tau} = (h \star T_\tau(a))_i = (PT_\tau(a))_i\end{aligned}$$

3. We have for $k > 0$,

$$\hat{a}_k^{(l+1)} = \hat{h}_k \times \hat{a}_k^{(l)} = \ldots = (\hat{h}_k)^{(l+1)} \times \hat{a}_k^{(0)},$$

which tends to 0 as $|\hat{h}_k| < 1$. On the other hand, $\hat{a}_0^{(l+1)} = \hat{a}_0^{(0)}$ because $h \in \Sigma_n$.
By continuity of the inverse Fourier transform,

$$a^{(l+1)} \to a^{(\infty)}$$

where $\hat{a}^{(\infty)} = (1, 0, \ldots, 0)^\top$, hence $a^{(\infty)} = \mathbb{1}_n / n$.