# TD 3 – Wavelets and Approximation

**Notes:** We will be playing with different wavelet bases and Fast Wavelet Transform (FWT).
Multiresolution approximations of $L^2(\mathbb{R})$ are families $(V_j)_{j\in\mathbb{Z}}$ of nested closed subspaces of $L^2(\mathbb{R})$, s.t.

$$L^2(\mathbb{R}) \supset \ldots \supset V_{j-1} \supset V_j \supset V_{j+1} \supset \ldots \supset \{0\}.$$

$$f \in V_j \iff f\left(\frac{\cdot}{2}\right) \in V_{j+1} \quad \text{and} \quad f \in V_j \iff \forall n \in \mathbb{Z}, f(\cdot + n2^j) \in V_j$$

and $\exists \varphi \in L^2(\mathbb{R})$ so that $\{\varphi(\cdot - n)\}_n$ is a Hilbertian orthonormal basis of $V_0$. This implies that $\varphi_{j,n}$ form an orthonormal basis of $L^2(\mathbb{R})$:

$$\left\{\varphi_{j,n}(t) = \frac{1}{\sqrt{2^j}}\varphi\left(\frac{t-2^j n}{2^j}\right)\right\}_{(j,n)\in\mathbb{Z}^2}$$

## Exercises

### Multi-resolution Approximation Spaces

Show that these families $(V_j)_j$ are multiresolution approximations of $L^2(\mathbb{R})$, with scaling functions $\varphi$.

1. (Piecewise constant functions)
   $V_j = \{f \in L^2(\mathbb{R}); \forall n, f \text{ is constant on } [2^j n, 2^j(n+1))\}$ and $\varphi(t) = 1_{[0,1)}$.
2. (Shannon approximations)
   $V_j = \{f \in L^2(\mathbb{R}); \forall n, \hat{f} \text{ is zero outside of } [-2^{-j}\pi, 2^{-j}\pi]\}$ and $\varphi(t) = \frac{\sin \pi t}{\pi t}$.

### Haar Wavelets

For $j \in \mathbb{Z}$, we define the space $V_j \subset L^2(\mathbb{R})$ of functions which are constant on each interval $I_{j,k}$, where $\forall k \in \mathbb{Z}$, $I_{j,k} := [2^j k, 2^j(k+1))$. We also define the functions

$$\forall x \in \mathbb{R}, \varphi(x) := \begin{cases} 1 & \text{if } x \in [0,1) \\ 0 & \text{otherwise} \end{cases} \text{ and } \psi(x) := \begin{cases} 1 & \text{if } x \in [0,1/2) \\ -1 & \text{if } x \in [1/2,1) \\ 0 & \text{otherwise} \end{cases}$$

Their dilated and translated versions are defined as

$$\forall (j,k) \in \mathbb{Z}, \psi_{j,k}(t) := \frac{1}{\sqrt{2^j}}\psi\left(2^{-j}t - k\right) \text{ and } \varphi_{j,k}(t) := \frac{1}{\sqrt{2^j}}\varphi\left(2^{-j}t - k\right)$$

The functions $\psi_{j,k}$ are the so-called "Haar wavelets".

1. Draw the graphs of the functions $\varphi_{0,0}$, $\varphi_{1,0}$ and $\varphi_{2,2}$. Do the same for $\psi_{0,0}$, $\psi_{1,0}$ and $\psi_{2,2}$.
2. Find and show an explicit characterization of $W_j$ where $W_j$ satisfies $W_j \subset L^2(\mathbb{R})$ such that $W_j \perp V_j$ and $V_{j-1} = V_j \oplus W_j$.
3. Show that $\mathcal{B}_{\varphi_j} := \{\varphi_{j,k}; k \in \mathbb{Z}\}$ and $\mathcal{B}_{\psi_j} := \{\psi_{j,k}; k \in \mathbb{Z}\}$ are ortho-bases of respectively $V_j$ and $W_j$.

**Up and down sampling (from exam 2020)**

We consider $\ell_2(\mathbb{Z})$ the set of sequences $(x_i)_{i\in\mathbb{Z}}$ with $\sum_i x_i^2 < +\infty$ with the inner product $\langle x, y\rangle = \sum_i x_i y_i$. The down and up-sampling operators are $x\downarrow_2 := (x_{2i})_{i\in\mathbb{Z}}$ and $(x\uparrow^2)_i := \begin{cases} x_{i/2} & \text{if } i \text{ is even} \\ 0 & \text{otherwise} \end{cases}$

1. Prove that $\downarrow_2$ and $\uparrow^2$ are adjoints i.e. $\langle x\downarrow_2, y\rangle = \langle x, y\uparrow^2\rangle$.
2. What is the adjoint of the linear operator $x \mapsto x \star h$, where $(x\star h)_i := \sum_{j\in\mathbb{Z}} x_j h_{i-j}$?
3. Prove that $(x\downarrow_2)\star h = (x\star(h\uparrow^2))\downarrow_2$. Then use this recursively to show $f^k(x) = (x\star H^k)\downarrow_{2^k}$ where $H^k := h^0 \star h^1 \star \ldots \star h^{k-1}$ with $h^s = h\uparrow^{2^s}$ ($\uparrow^{2^s}$ is defined similarly to $\uparrow^2$ by inserting $2^s - 1$ zeros), and $f : x \mapsto (x\star h)\downarrow_2$.
4. What is $H^k$ in the case of a box filter $[h_0, h_1] = [1, 1]/2$ (the other entries of the vector being 0)?

**Heisenberg inequality**

A wavelet is a filter with mean 0, localized both in the spatial and Fourier domains. How localized can it be? For a function $f \in \mathrm{L}^2(\mathbb{R})$ such that the following quantities exist, we define:

- Time mean $u = \frac{1}{\|f\|_2^2}\int_{-\infty}^{+\infty} t|f(t)|^2 \mathrm{d}t$
- Time Variance $\sigma_t^2 = \frac{1}{\|f\|_2^2}\int_{-\infty}^{+\infty}(t-u)^2|f(t)|^2\mathrm{d}t$
- Fourier Mean $\xi = \frac{1}{2\pi\|f\|_2^2}\int_{-\infty}^{+\infty}\omega|\hat{f}(\omega)|^2\mathrm{d}\omega$
- Fourier Variance $\sigma_\omega^2 = \frac{1}{2\pi\|f\|_2^2}\int_{-\infty}^{+\infty}(\omega-\xi)^2|\hat{f}(\omega)|^2\mathrm{d}\omega$

1. Let $s \in \mathbb{R}_+^*$ and $f_s(t) = \frac{1}{\sqrt{s}} f(\frac{t}{s})$.
   (a) Show that $\|f_s\|_2^2 = \|f\|_2^2$
   (b) Derive the time mean and variance of $f_s$ as a function of those of $f$.
   (c) Derive the Fourier mean and variance of $f_s$ as a function of those of $f$.
   (d) How does scaling affect time and Fourier resolution ?
2. In the following we assume $u = 0$ and $\xi = 0$. This amounts to replacing $f$ with $g : t \mapsto f(t-u)e^{i\xi t}$. Assuming $\lim_{|t|\to+\infty}\sqrt{t}f(t) = 0$, show that the spatial and Fourier variances verify the Heisenberg inequality:
$$\sigma_t^2\sigma_\omega^2 \geq 1/4$$
3. In which case is this an egality?

**Linear and non-linear approximation (from exam 2019)**

For a fixed $y \in [0, 1]$, we consider $f \stackrel{\text{def.}}{=} 1_{[y,1]} - 1_{[0,y[}$. We denote $\langle f, g\rangle \stackrel{\text{def.}}{=} \int_0^1 fg$ the inner product and $\|f\|^2 \stackrel{\text{def.}}{=} \langle f, f\rangle$.

1. For $M \in \mathbb{N}^*$, we denote $\theta_k \stackrel{\text{def.}}{=} \sqrt{M}1_{[\frac{k}{M},\frac{k+1}{M}[}$ for $0 \leq k < M$. Show that $(\theta_k)_k$ is an orthonormal family and give the expression for the linear approximation
$$f_M \stackrel{\text{def.}}{=} \sum_k \langle f, \theta_k\rangle\theta_k.$$
2. Bound $\|f - f_M\|$ as a function of $M$, independently of $y$, using a bound which is as sharp as possible.

3. We denote $\theta \overset{\text{def.}}{=} 1_{[0,1]}$ and $\psi = 1_{[0,1/2[} - 1_{[1/2,1[}$ the Haar wavelet. We denote, for $j \leq 0$, and $0 \leq n < 2^{-j}$ the wavelet functions as $\psi_{j,n} \overset{\text{def.}}{=} \frac{1}{2^{j/2}}\psi(2^{-j}x - n)$. Draw the wavelets $\psi_{-1,1}$ and $\psi_{-2,3}$.

4. For some $j_{\min} < 0$, show that
$$\{\theta\} \cup \{\psi_{j,n} \ ; \ 0 \geq j \geq j_{\min} \quad \text{and} \quad 0 \leq n < 2^{-j}\}$$
is an orthogonal family. What is the space spanned by this family?

5. For each $j$, what is the set $\Sigma_j$ of index $n$ where $\langle f, \psi_{j,n}\rangle$ is non-zero? For these $n \in \Sigma_j$, bound $|\langle f, \psi_{j,n}\rangle|$ as a function of $j$.

6. For $T > 0$ we define the non-linear approximation of $f$ as
$$\hat{f} \overset{\text{def.}}{=} \langle f, \theta\rangle\theta + \sum_{|\langle f, \psi_{j,n}\rangle| > T} \langle f, \psi_{j,n}\rangle\psi_{j,n}.$$
Bound as a function of $T$ the number $M$ of non-zero coefficients
$$M \overset{\text{def.}}{=} |\{(j,n) \ ; \ |\langle f, \psi_{j,n}\rangle| > T\}|.$$

7. Defining $j_0 \overset{\text{def.}}{=} \lfloor \log_2(T^2) \rfloor$ a cutoff scale, we define an approximation using
$$\tilde{f}_T \overset{\text{def.}}{=} \langle f, \theta\rangle\theta + \sum_{j \geq j_0, n \in \Sigma_j} \langle f, \psi_{j,n}\rangle\psi_{j,n}.$$
Show that $\|f - \hat{f}_T\| \leq \|f - \tilde{f}_T\|$.

8. Bound $\|f - \tilde{f}_T\|$ as a function of $j_0$ and then as a function of $M$ . Compare the decay with $M$ of $\|f - f_M\|$ and $\|f - \hat{f}_T\|$.

# Solutions

### Multi-resolution Approximation Spaces

1. The $(V_j)_{j\in\mathbb{Z}}$ are clearly nested closed subspaces of $L^2(\mathbb{R})$ that satisfy

$$f \in V_j \iff f\left(\frac{\cdot}{2}\right) \in V_{j+1} \iff \forall n \in \mathbb{Z}, f(\cdot + n2^j) \in V_j.$$

Let us show that $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$. Let $f \in L^2(\mathbb{R})$. Let $\epsilon > 0$. As $\mathcal{C}_c(\mathbb{R})$ is dense in $(L^2(\mathbb{R}), \|\cdot\|_2)$, there exists $g \in \mathcal{C}_c(\mathbb{R})$ such that $\|g - f\|_2 \leq \epsilon$. Finally using Heine theorem, there exists $j \in \mathbb{Z}$ and $h \in V_j \subseteq \bigcup_k V_k$ such that $\|g - h\|_2 \leq \epsilon$.

One has $\bigcap_j V_j = \{0\}$ because non-zero constant functions of $\mathbb{R}$ are not in $L^2(\mathbb{R})$.

Finally, $\left\{\varphi(\cdot - n) = 1_{[n,n+1(}\right\}_n$ is clearly an orthonormal family of $V_0$. Moreover let $f \in V_0$ then

$$\|f - \sum_{k=-N}^{N} f(k)1_{[k,k+1[}\|_2 \underset{N\to+\infty}{\longrightarrow} 0.$$

2. The $(V_j)_{j\in\mathbb{Z}}$ are clearly nested closed subspaces of $L^2(\mathbb{R})$ that satisfy

$$f \in V_j \iff f\left(\frac{\cdot}{2}\right) \in V_{j+1} \iff \forall n \in \mathbb{Z}, f(\cdot + n2^j) \in V_j.$$

Let us show that $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$. Let $f \in L^2(\mathbb{R})$. Let $\epsilon > 0$. Again as $\mathcal{C}_c(\mathbb{R})$ is dense in $(L^2(\mathbb{R}), \|\cdot\|_2)$, there exists $g \in \mathcal{C}_c(\mathbb{R})$ such that $\|g - \hat{f}\|_2 \leq \epsilon$. In particular there exists $j \in \mathbb{Z}$ such that supp $g \subseteq [-2^{-j}\pi, 2^{-j}\pi]$. Considering $h = \mathcal{F}^{-1}(g) \in V_j$ and using Plancherel equality one has $2\pi\|h - f\|_2^2 = \|g - \hat{f}\|_2^2 \leq \epsilon^2$. Thus $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$.

One has $\bigcap_j V_j = \{0\}$ as $\|\hat{f}\|^2 = 0$ implies $\|f\|^2 = 0$ by Plancherel equality.

Finally, Nyquist-Shannon theorem yields that $\{\varphi(\cdot - n)\}_n$ is an orthonormal basis of $V_0$.

### Haar wavelets

1. Drawing.

2. Let $j \in \mathbb{Z}$. As $V_{j-1}$ is a Hilbert space (closed subspace in a Hilbert space) and as $V_j \subset V_{j-1}$ is a closed subspace of $V_{j-1}$, one has $V_{j-1} = V_j \oplus^\perp V_j^\perp$. Denote

$$W_j := \left\{f \in L^2(\mathbb{R}), \forall k, f \text{ is constant on } I_{j-1,k} \text{ and has zero mean on the interval } I_{j,k}\right\}.$$

Let us show that $V_j^\perp = W_j$. Let $f \in V_j$, $g \in W_j$, then:

$$\langle f, g\rangle = \sum_k \int_{I_{j,k}} fg \overset{f\in V_j}{=} \sum_k f_k \underbrace{\int_{I_{j,k}} g}_{=0 \text{ as } g\in W_j} = 0.$$

Thus $V_j \perp W_j$. Moreover $V_j \oplus^\perp W_j$ as $\langle\cdot,\cdot\rangle$ is positive-definite. Finally let $h$ be in $V_{j-1}$. Denote $h_k$ the value of $h$ on $I_{j-1,k}$. Consider $f \in V_j$ defined by $f(x) = (h_{2k} + h_{2k+1})/2$ for $x \in I_{j,k}$. Then $h - f =: g \in W_j$.

3. cf Q1 from the previous exercise.

**Up and down sampling**

1. One has

$$\langle x\downarrow_2, y\rangle = \sum_i x_{2i}y_i = \sum_i \left( x_{2i}\underbrace{(y\uparrow^2)_{2i}}_{=y_i} + x_{2i+1}\underbrace{(y\uparrow^2)_{2i+1}}_{=0}\right) = \sum_i x_i y\uparrow^2{}_i = \langle x, y\uparrow^2\rangle.$$

2. By Fubini, it is the convolution against $\tilde{h}$ where $\tilde{h}_n = h_{-n}$.

3. One has

$$[(x\star h\uparrow^2)\downarrow_2]_i = [(x\star h\uparrow^2)]_{2i} = \sum_k \left( x_{2k}\underbrace{(h\uparrow^2)_{2(i-k)}}_{=h_{i-k}} + x_{2k+1}\underbrace{(h\uparrow^2)_{2(i-k)-1}}_{=0}\right) = [(x\downarrow_2)\star h]_i.$$

One thus has

$$f^{k+1}(x) = f(f^k(x)) = \left(f^k(x)\star h\right)\downarrow_2 = [(x\star H^k)\downarrow_{2^k}\star h]\downarrow_2 = [x\star H^k\star(h\uparrow^{2^k})]\underbrace{\downarrow_{2^k}\downarrow_2}_{\downarrow_{2^{k+1}}} = [x\star H^{k+1}]\downarrow_{2^{k+1}}.$$

4. One has $H^2 = [1,1]/2\star[1,0,1,0]/2 = [1,1,1,1]/4$ and more generally $H^s = 1_{[0,\dots,2^s-1]}/2^s$.

**Heisenberg inequality**

1. (a) Change of variables
   (b) By change of variables, $u(s) = s\times u$ and $\sigma_t(s) = s\times\sigma_t$
   (c) Since $\hat{f}_s(\omega) = \sqrt{s}\hat{f}(\omega s)$, by change of variables, $\xi(s) = \xi/s$ and $\sigma_\omega(s) = \sigma_\omega/s$
   (d) Reducing variance in one domain makes it bigger in the other.

2. The product reads

$$\sigma_t^2\sigma_\omega^2 = \frac{1}{2\pi\|f\|^4}\int_{-\infty}^{+\infty}|tf(t)|^2\mathrm{d}t\int_{-\infty}^{+\infty}|\omega\hat{f}(\omega)|^2\mathrm{d}\omega.$$

We notice that since $i\omega\hat{f}(\omega)$ is the Fourier transform of $f'(\omega)$, the Plancherel equality gives

$$\sigma_t^2\sigma_\omega^2 = \frac{1}{\|f\|^4}\int_{-\infty}^{+\infty}|tf(t)|^2\mathrm{d}t\int_{-\infty}^{+\infty}|f'(t)|^2\mathrm{d}t.$$

By appyling Cauchy-Schwarz,

$$\sigma_t^2\sigma_\omega^2 \geq \frac{1}{\|f\|^4}\left(\int_{-\infty}^{+\infty}tf'(t)f^*(t)\mathrm{d}t\right)^2.$$

We can rewrite this as

$$\sigma_t^2\sigma_\omega^2 \geq \frac{1}{\|f\|^4}\left(\int_{-\infty}^{+\infty}\frac{t}{2}(f'(t)f^*(t)+f'^*(t)f(t))\mathrm{d}t\right)^2$$

then

$$\sigma_t^2\sigma_\omega^2 \geq \frac{1}{4\|f\|^4}\left(\int_{-\infty}^{+\infty}t(|f(t)^2|)'\mathrm{d}t\right)^2$$

Finally, integration by parts (since $\lim_{|t|\to+\infty}\sqrt{t}f(t) = 0$) gives

$$\left(\int_{-\infty}^{+\infty}t(|f(t)^2|)'\mathrm{d}t\right)^2 = \left(\int_{-\infty}^{+\infty}|f(t)|^2\mathrm{d}t\right)^2,$$

hence

$$\sigma_t^2\sigma_\omega^2 \geq \frac{1}{4}.$$

3. Equality case of Cauchy-Schwarz, $f(t) = -2btf(t)$. So $f(t) = ae^{-bt^2}$

**Linear and non-linear approximation** See Exam 2019