Pumping Lemma for CFL

Intuition

- Recall the pumping lemma for regular languages.
- It told us that if there was a string long enough to cause a cycle in the DFA for the language, then we could “pump” the cycle and discover an infinite sequence of strings that had to be in the language.

Intuition

- For CFL’s the situation is a little more complicated.
- We can always find two pieces of any sufficiently long string to “pump” in tandem.
  - That is: if we repeat each of the two pieces the same number of times, we get another string in the language.

The size of Parse Trees

Theorem 7.17: Suppose we have a parse tree according to a Chomsky-Normal-Form grammar $G = (V, T, P, S)$, and suppose that the yield of the tree is a terminal string w. If the length of the longest path is n, then $|w| \leq 2^{n-1}$.

PROOF: The proof is a simple induction on n.

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PROOF: The proof is a simple induction on n.

- If $|w| > 2^{n-1}$, then the longest path is $> n$

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- For $|w| = 2^m$, longest path is $> m + 1$

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- If $|w| > 2^{n-1}$, then the longest path is $> n$
- Let $|V| = m$
- For $|w| = 2^m$, longest path is $> m + 1$
- In that longest path a variable must have been repeated (since we have only m variables).

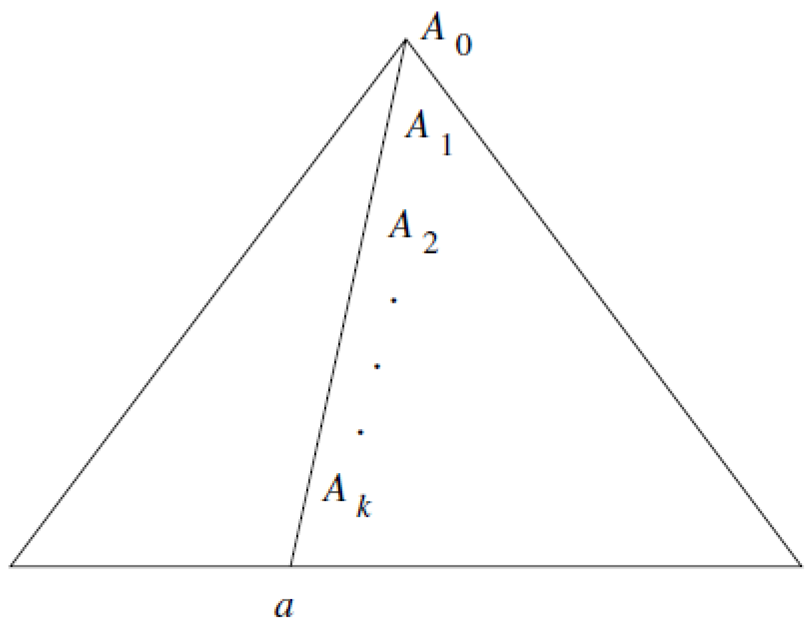


Figure 7.5: Every sufficiently long string in L must have a long path in its parse tree

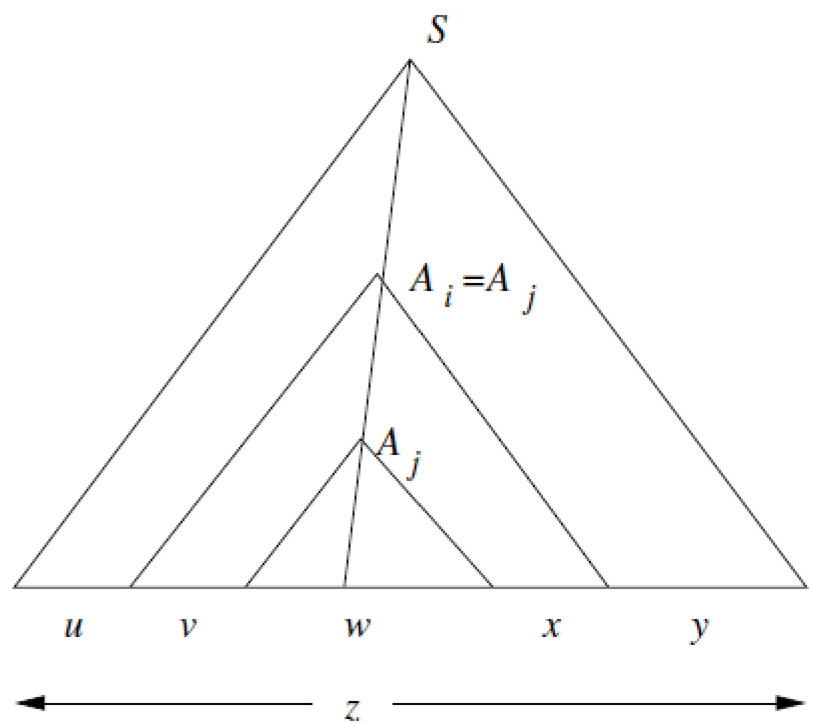
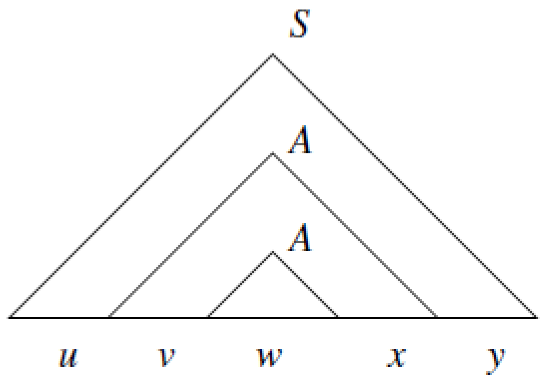


Figure 7.6: Dividing the string w so it can be pumped



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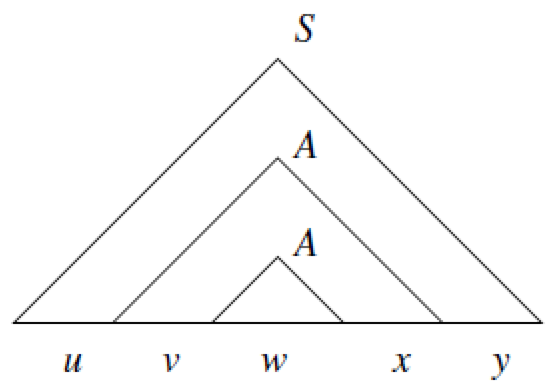
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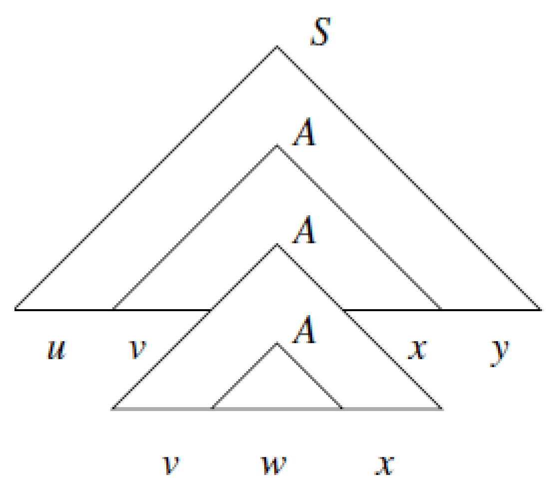
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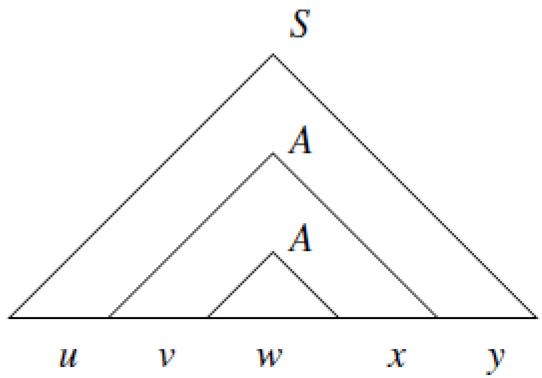
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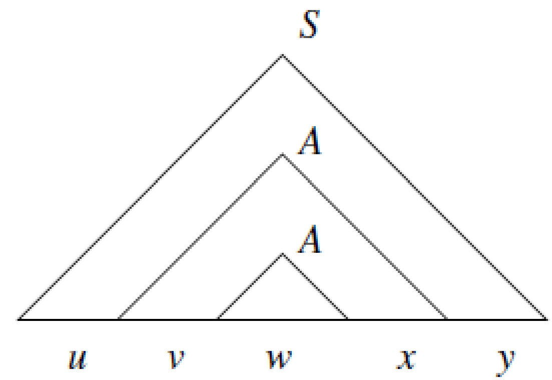
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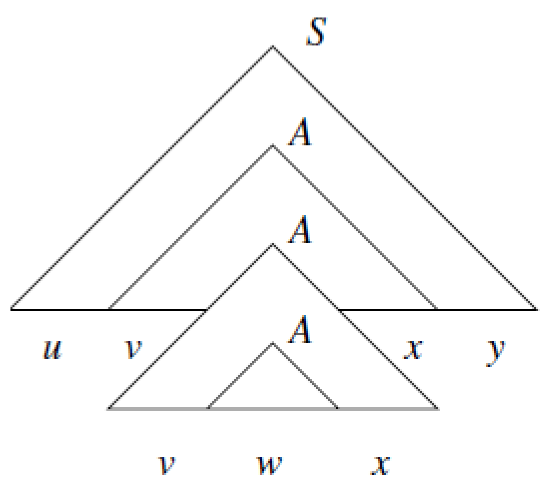
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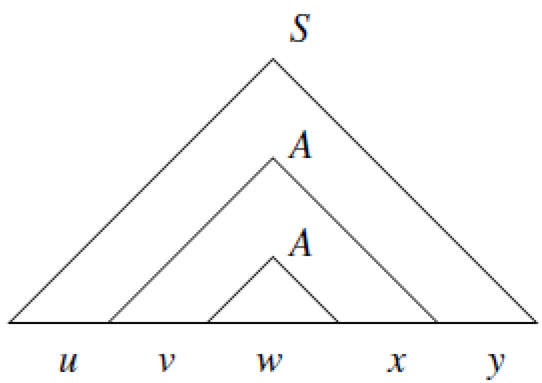
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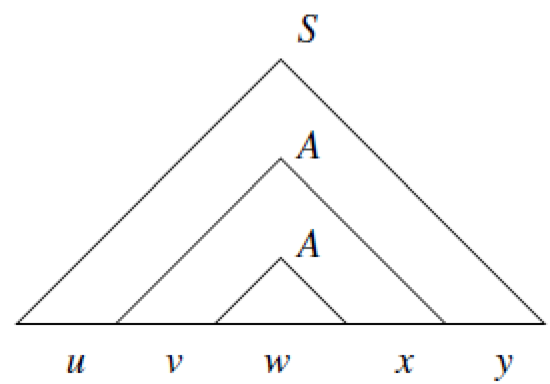
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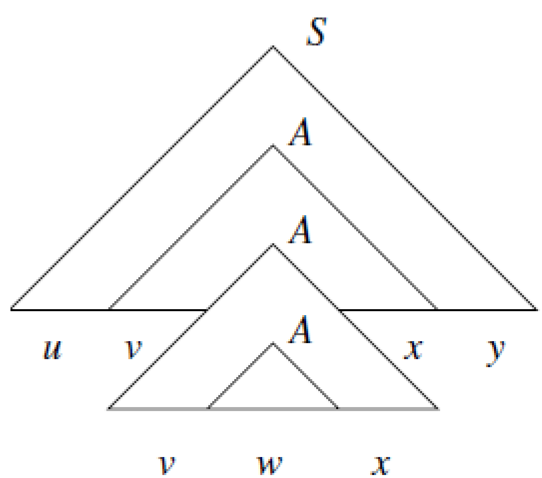
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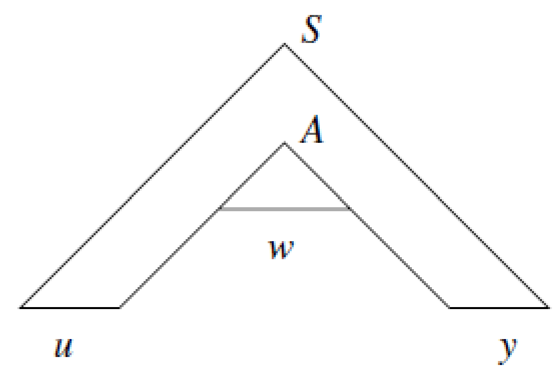
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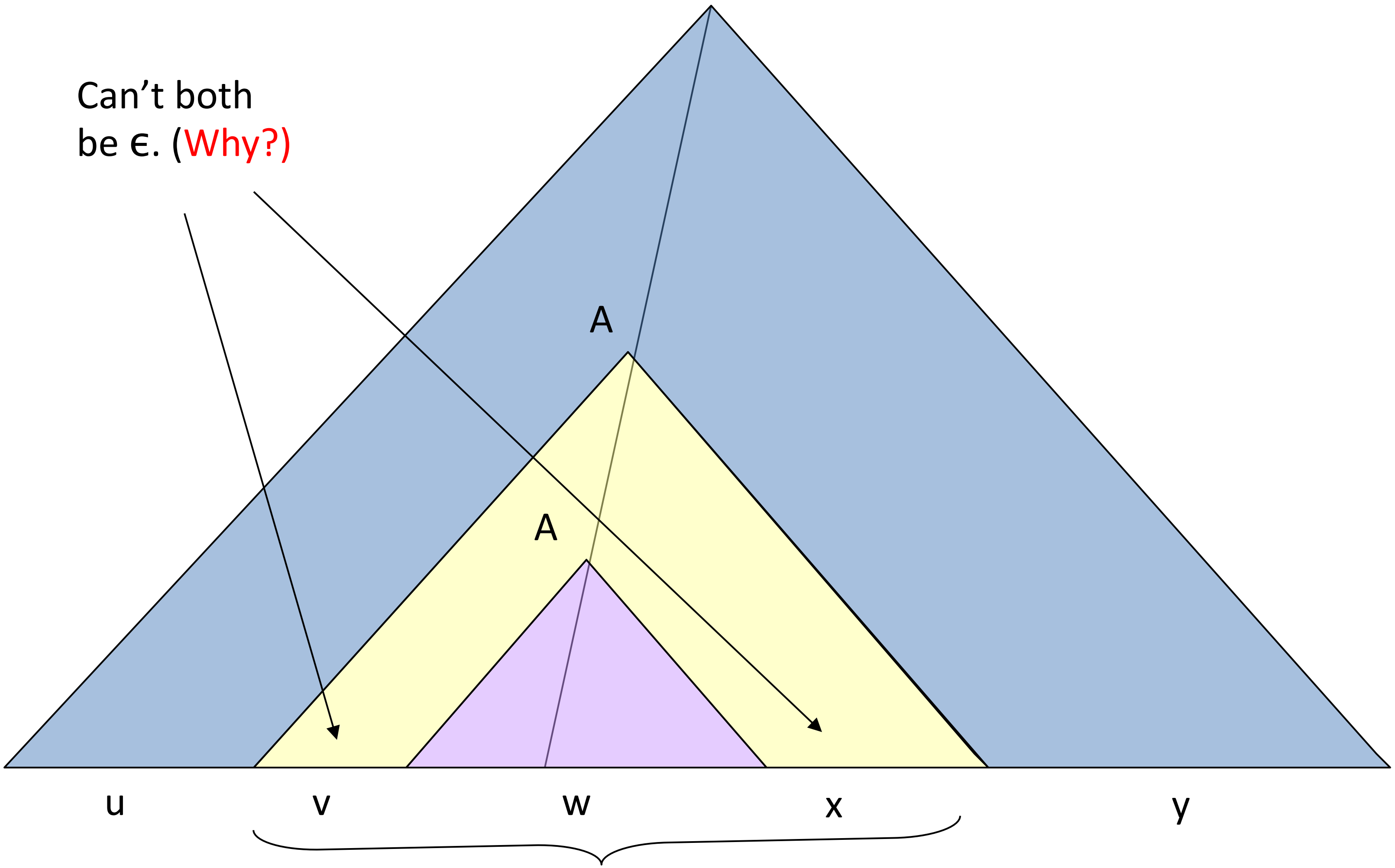
Parse Tree in the Pumping-Lemma



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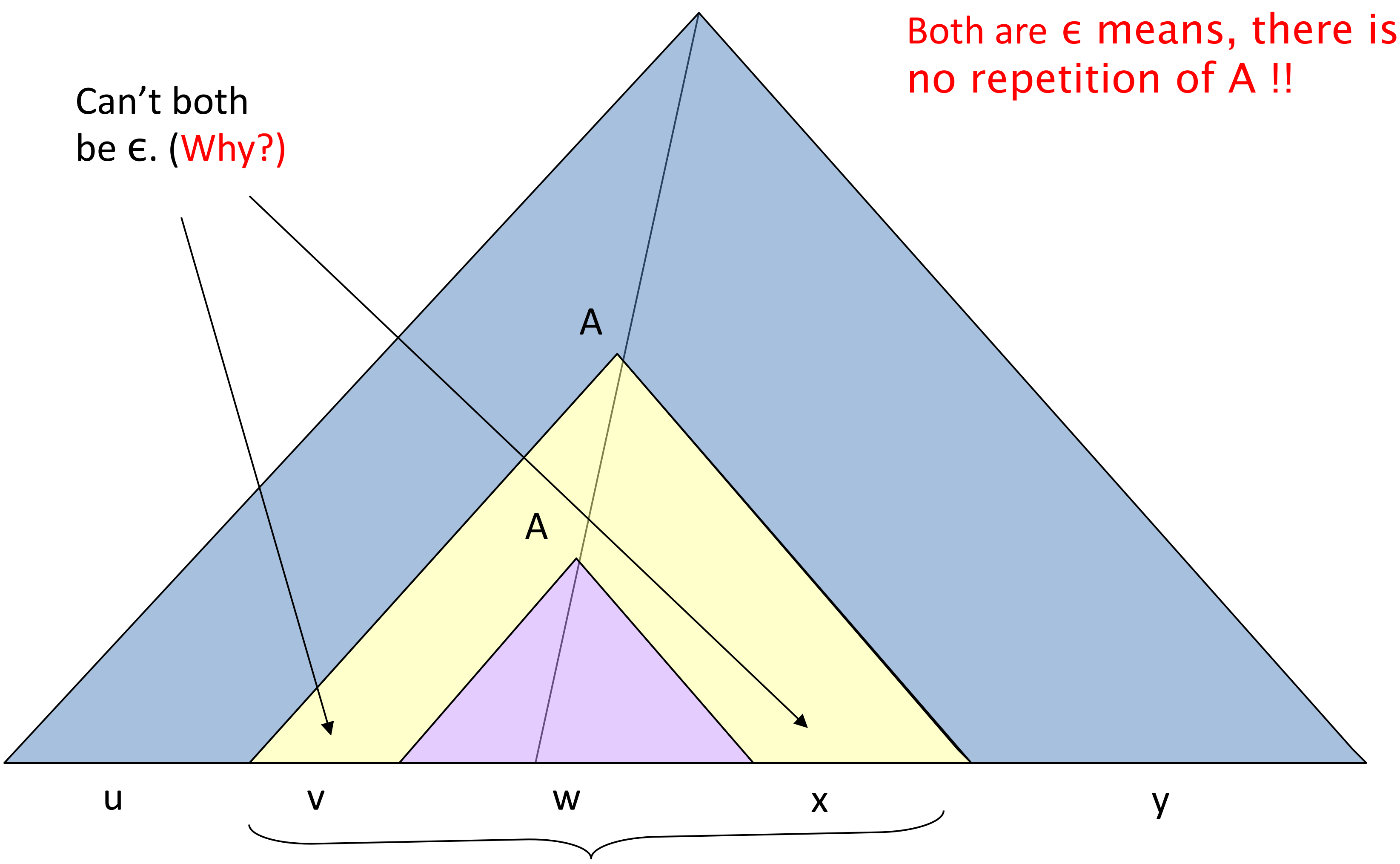
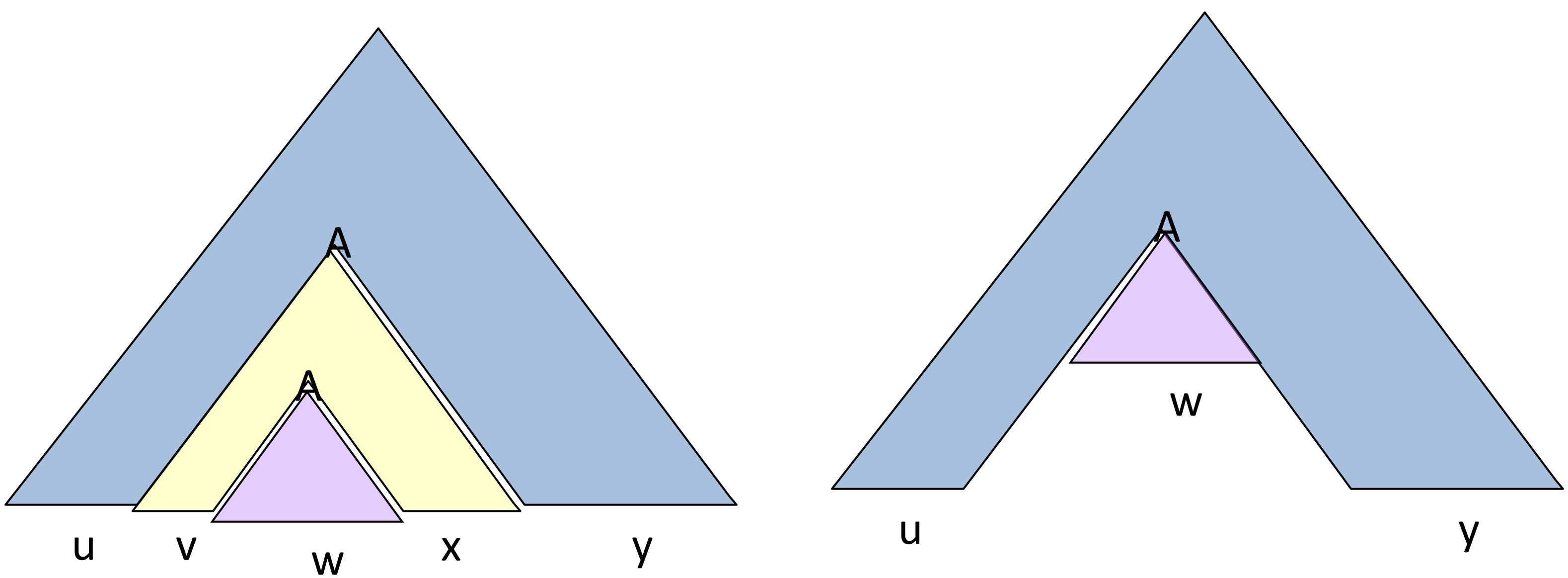
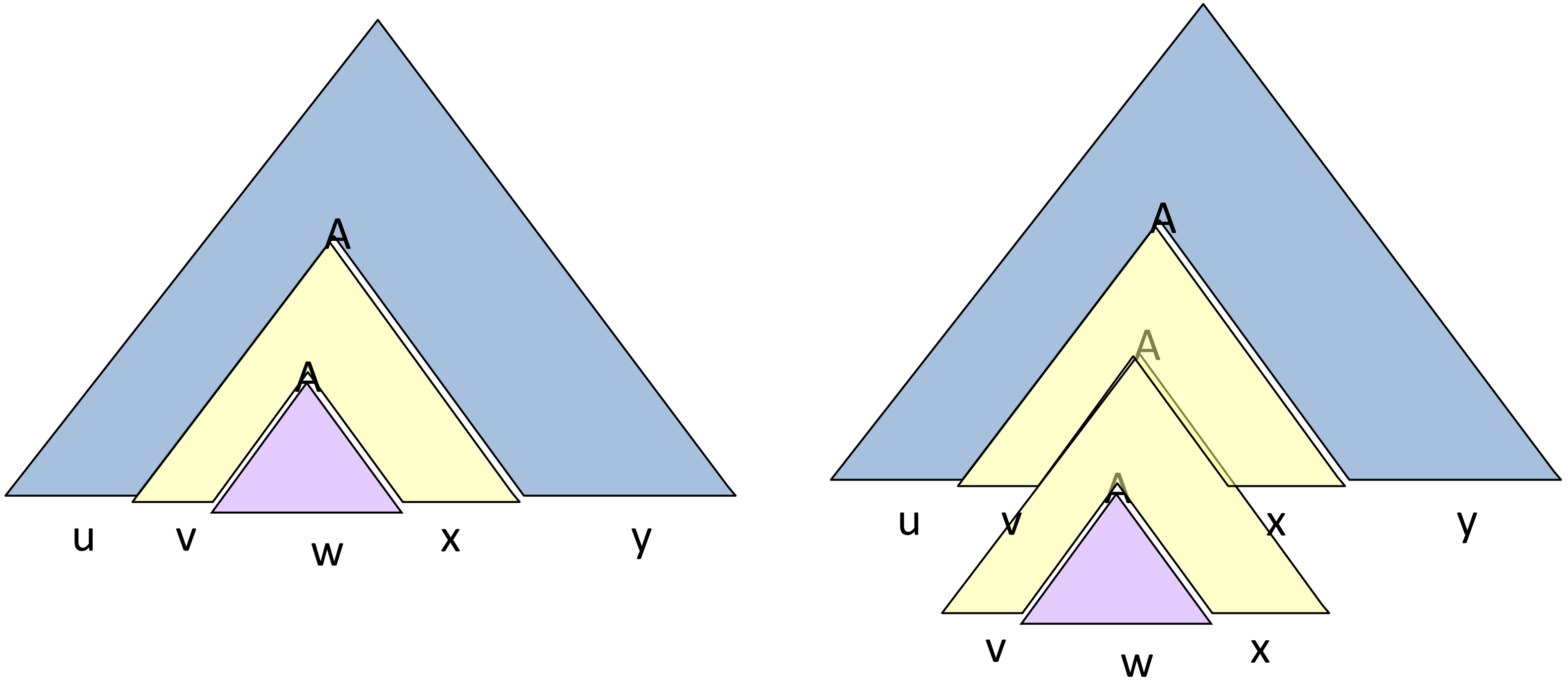


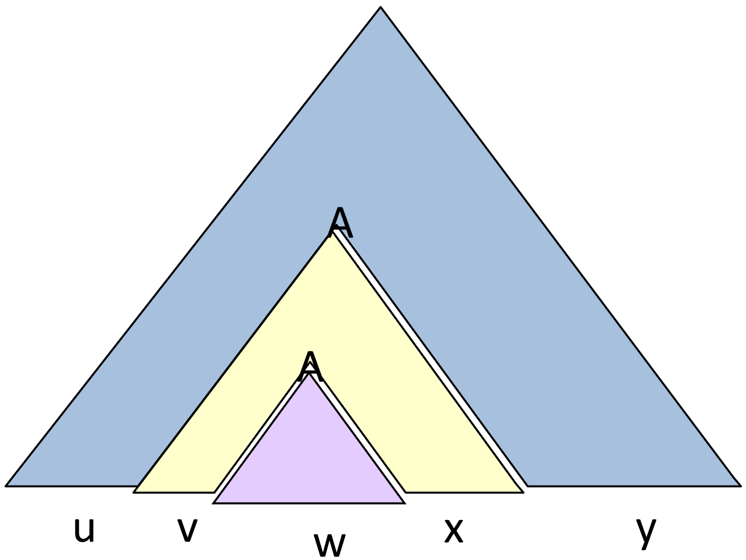
Pump Zero Times

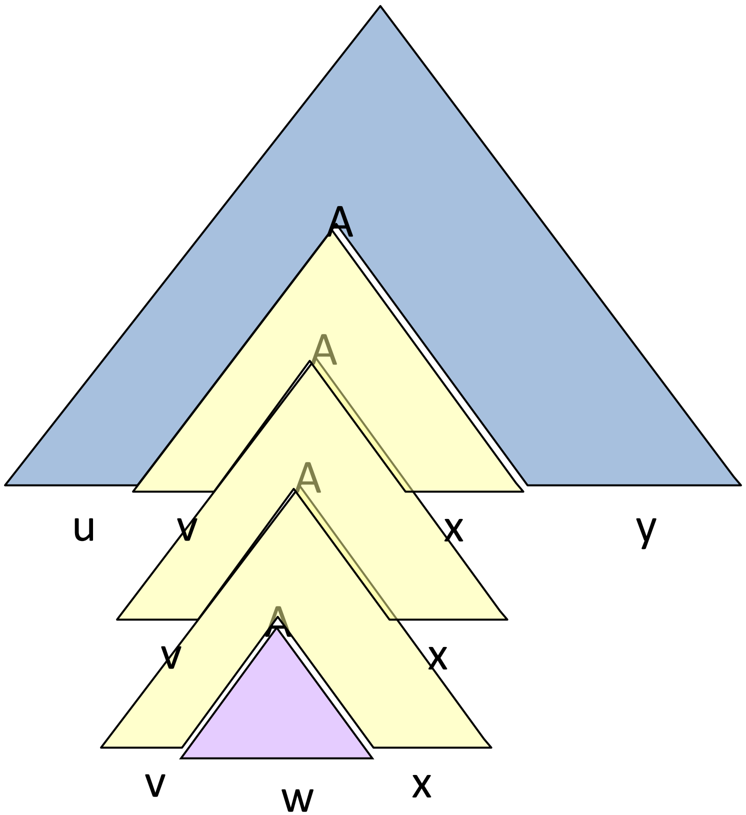


Pump Twice



Pump Thrice   Etc., Etc.





Statement

Theorem 7.18: (The pumping lemma for context-free languages) Let L be a CFL. Then there exists a constant n such that if z is any string in L such that $|z|$ is at least n, then we can write $z = uvwxy$, subject to the following conditions:

1. $|vwx| \leq n$. That is, the middle portion is not too long.
2. $vx \neq \epsilon$. Since v and x are the pieces to be "pumped," this condition says that at least one of the strings we pump must not be empty.
3. For all $i \geq 0$, uv^iwx^iy is in L. That is, the two strings v and x may be "pumped" any number of times, including 0, and the resulting string will still be a member of L.

Statement

For every context-free language L

There is an integer n, such that

For every string z in L of length $\geq$ n

There exists $z = uvwxy$ such that:

1. $|vwx| \leq$ n.
2. $|vx| > 0$.
3. For all $i \geq 0$, uv^iwx^iy is in L.

Example 7.19 : Let L be the language $\{0^n 1^n 2^n \mid n \geq 1\}$. That is, L consists of all strings in $\mathbf{0}^+\mathbf{1}^+\mathbf{2}^+$ with an equal number of each symbol, e.g., 012, 001122, and so on. Suppose L were context-free. Then there is an integer n given to us by the pumping lemma. Let us pick $z = 0^n 1^n 2^n$.

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We can write $z = uvwxy$, where $|vwx| \leq n$ and v and x are not both ϵ.

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Then we know that vwx cannot involve both 0's and 2's, since the last 0 and the first 2 are separated by $n + 1$ positions.

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We can write $z = uvwxy$, where $|vwx| \leq n$ and v and x are not both ϵ.

- There can be 5 cases where vwx is having
  - Only 0s
  - Some 0s and some 1s
  - Only 1s
  - Some 1s and some 2s
  - Only 2s.
- In all these 5 cases, $uv^2wx^2y \notin L$.

Example 7.21: Let $L = \{ww \mid w \text{ is in } \{0,1\}^*\}$.

- Show L is not a CFL.

- Note, $\{ww^R | w \in \{0,1\}^* \}$ is a CFL.
- How can you prove this??

Example 7.21: Let $L = \{ww \mid w \text{ is in } \{0,1\}^*\}$.

- Let pumping length is n.
- Let the string be $z = 0^n1^n0^n1^n$
- z can be written as $uvwxy$, such that $|vwx| \leq n$ and $vx \neq \epsilon$
- There are 7 cases, based on where vwx can occur in z.
- In all these cases it can be shown that uwy is not in L.

Using the Pumping Lemma

- $\{0^i10^i \mid i \geq 1\}$ is a CFL.
  - We can match one pair of counts.
  - Can you give CFG??

Using the Pumping Lemma

- $\{0^i10^i \mid i \geq 1\}$ is a CFL.

- But $L = \{0^i10^i10^i \mid i \geq 1\}$ is not.
  - We can't match two pairs, or three counts as a group.
- Proof using the pumping lemma.
- Suppose L were a CFL.
- Let n be L's pumping-lemma constant.

Using the Pumping Lemma

- Consider $z = 0^n10^n10^n$.
- We can write $z = uvwxy$, where $|vwx| \leq n$, and $|vx| \geq 1$.
- Case 1: vx has no 0's.
  - Then at least one of them is a 1, and uwy has at most one 1, which no string in L does.

Using the Pumping Lemma

- Still considering $z = 0^n10^n10^n$.
- Case 2: vx has at least one 0.
  - vwx is too short (length $\leq$ n) to extend to all three blocks of 0's in $0^n10^n10^n$.
  - Thus, uwy has at least one block of n 0's, and at least one block with fewer than n 0's.
  - Thus, uwy is not in L.