Time Complexity

Objectives

- In this lecture, we focus on problems that are computable, and investigate the amount of time required to solve these problems
  - Later, we will investigate the amount of space, and other resources required to solve a problem
- Before that, we will review the big-O, small-o, big-Ω, and small-ω notations

Big-O and Big-Ω Notations

Definition: Let f and g be functions that maps N to R^+. We say $f(n) = O(g(n))$ if there exists positive integers c and n' such that for every $n \geq n'$, $f(n) \leq cg(n)$.

When $f(n) = O(g(n))$, we say $g(n)$ is an asymptotic upper bound for $f(n)$

Big-O and Big-Ω Notations

We say $g(n) = \Omega(f(n))$ if $f(n) = O(g(n))$

Important: $f(n) = O(g(n))$

is a special notation, so that we will never write $O(g(n)) = f(n)$ instead

Although, we can write something like:

$f(n) = O(g(n)) = O(h(n))$, which means:

$f(n) = O(g(n))$, and $g(n) = O(h(n))$

Small-o and Small-ω Notations

Definition: Let f and g be functions that maps N to R^+. We say $f(n) = o(g(n))$ if

$$\lim_{n \to \infty} f(n)/g(n) = 0$$

We say $g(n) = \omega(f(n))$ if $f(n) = o(g(n))$

Examples

Is the following true?

1. $5n^2 + 1002n + 17 = O(n^2)$
2. $\log_3 n = O(\log n)$
3. $\log n = O(\log_3 n)$
4. $\log n = O(n^{0.00001})$
5. $\log (n^2 \log n) = O(\log n)$
6. $2^n = O(3^n)$
7. $3^n = O(2^n)$
8. $n^{1/(\log n)} = o(\ (n^{1/(\log n)})^2\ )$

the Limit Rule

Suppose $L \leftarrow \lim_{n\to\infty} f(n)/g(n)$ exists (may be ∞)

Then if $L = 0$, **then** f **is** $O(g)$

if $0 < L < \infty$, **then** f **is** $\Theta(g)$

if $L = \infty$, **then** f **is** $\Omega(g)$

To compute the limit, the standard **L'Hopital rule** of calculus is useful: if $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x)$ and f, g are positive differentiable functions for $x > 0$, then $\lim_{x\to\infty} f(x)/g(x) = \lim_{x\to\infty} f'(x)/g'(x)$ where $f'(x)$ is the derivative

EXAMPLE 7.6

The following are easy to check.

1. $\sqrt{n} = o(n)$.
2. $n = o(n \log \log n)$.
3. $n \log \log n = o(n \log n)$.
4. $n \log n = o(n^2)$.
5. $n^2 = o(n^3)$.

However, $f(n)$ is never $o(f(n))$.

Analyzing Algorithms

Let A be the language $\{ 0^k1^k \mid k \geq 0 \}$, and we have seen that A is decidable before. Below is one such TM that decides A:

M_1 = "On input string w,

1. Scan across the tape and reject if 0 appears on the right of a 1
2. Repeat if both 0s and 1s remain in tape
   a. Scan the tape, cross of a 0 and a 1
3. If all 0s and 1s are crossed, accept. Otherwise, reject."

Analyzing Algorithms (2)

How many steps will M_1 need to decide if w is in A or not? Let n be the length of w

- Step 1 takes at most $O(n)$ steps
- Step 2 will repeat at most $n/2$ times, each time taking $O(n)$ steps

  ➔ In total, Step 2 requires $O(n^2)$ steps
- Step 3 takes $O(n)$ steps

Thus, M_1 needs $O(n^2)$ steps to decide if w is in A or not

Running time of a DTM

- Number of steps the machine will take on input whose size is n.
- To formalize …

Running Time

Definition: Let M be a deterministic Turing machine that **halts** on all inputs. The **running time** of M is the function $f:N \rightarrow N$, where $f(n)$ is the maximum number of steps that M uses on any input of length n

If $f(n)$ is the running time of M, we say M runs in time $f(n)$, and M is an $f(n)$-time TM

Time Complexity Class

Definition: Let $t: N \rightarrow R^+$ be a function. We define the time complexity class, TIME(t(n)), to be the collection of all languages that are decidable by an O(t(n))-time Turing machine

In the previous example, M1 is an $O(n^2)$ time TM, so that the language $A = \{0^k1^k \mid k \geq 0\}$ is in $TIME(n^2)$

Analyzing Algorithms (3)

Can we decide $A = \{ 0^k 1^k \mid k \geq 0 \}$ faster?

Below is another TM that decides A:

M_2 = "On input string w,

1. If 0 appears on the right of a 1, reject
2. Repeat if both 0s and 1s remain in tape
   (i) If total # of 0s and 1s is odd, reject
   (ii) Scan the tape, cross off every other 0. Then cross off every other 1.
3. If all 0s and 1s are crossed, accept. Otherwise, reject."

Analyzing Algorithms (4)

Question 1: Why M_2 can decide A correctly?

Question 2: What is running time of M_2?

- Step 1 and Step 3 takes O(n) steps.
- For each time Step 2 is repeated, # of 0s is halved ➔ repeated for log n times
- Each time Step 2 is run, it takes O(n) steps ➔ in total takes O(n log n) steps

Thus, the running time of M_2 is O(n log n)

Analyzing Algorithms (5)

This implies that *A* is in TIME(n log n)

Question 1: Earlier, we show that *A* is in TIME(n^2) … Is there a contradiction??

Question 2: Can we find a TM that decides *A* faster? That is, in o(n log n) time?

- The answer is NO… (if TM just have a single tape)
- In fact, it is shown that if a language can be decided by a single-tape TM in o(n log n) time, the language is regular

Analyzing Algorithms (6)

How about if we have 2 tapes?

M_3 = "On input string w,

1. If 0 appears on the right of a 1, reject
2. Scan across 0s on tape 1 until first 1. At the same time, copy 0s to tape 2
3. Scan tape 1 and tape 2 together. Each time, match a 0 with a 1
4. If all 0s and 1s match, accept. Otherwise, reject."

Analyzing Algorithms (7)

The running time of M3 is O(n)!

What we have learnt before:

Single-tape and Multi-tape TM have the same power (in terms of computability, I.e., whether a problem can be solved)

What we have learnt now:

Single-tape and Multi-tape does not have the same power (in terms of complexity, I.e., how fast a problem can be solved)

Next Time

- Complexity relationship among models
  - Single-Tape versus Multi-Tape
  - Deterministic versus Non-Deterministic
- P and NP
  - Two important classes of problems in time complexity theory