Condition for apparent jet separation

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1 Definitions

The coordinate system used here places the emitting object at the origin, with the observer at a large distance along the positive z-axis looking towards the origin. θ represents an angle measured from the positive z-axis in the xz-plane and ϕ represents an angle above the xz-plane. The unit vector in the direction of the observer is therefore described by $\theta_{obs} = 0, \phi_{obs} = 0$.

The rotation axis $\vec{r}(t)$ is the axis along which matter is ejected to form the jets. It is characterized the angles $\theta_r(t)$ and $\phi_r(t)$, and two initial conditions $\theta_{r,0}$ and $\phi_{r,0}$. Due to the boundary condition given by the current state of the system, where $\vec{r}$ directly points at the observer, we can set $\theta_{r,0} = \phi_{r,0} = 0$. In rectangular coordinates, this initial condition $\vec{r_0}$ is simply the unit vector along the z-axis, $\hat{k}$.

The precession axis $\hat{p}$ is the axis around which rotation axis precesses. It is assumed to be constant in time and is given by the two angles θ_p and ϕ_p. In rectangular coordinates, $\hat{p} = \langle \sin\theta_p \cos\phi_p, \sin\phi_p, \cos\theta_p \cos\phi_p \rangle$.

The period of precession of the rotation axis is denoted by ω, and the emission velocity of matter along the jet is denoted by v_e. The radius of jet termination is denoted by R.

2 Expression for $\vec{r}(t)$

Applying the vector rotation formula[1] gives:

$$\vec{r}(t) = \vec{r}_0 \cos\omega t + (\hat{p} \times \vec{r}_0)\sin\omega t + \hat{p}(\hat{p}\cdot\vec{r}_0)(1-\cos\omega t)$$

Substituting $\vec{r}_0 = \hat{k}$ and $\hat{p} = \langle \sin\theta_p \cos\phi_p, \sin\phi_p, \cos\theta_p\cos\phi_p\rangle$, and evaluating the products yields:

$$\vec{r}(t) = \langle 0, 0, \cos\omega t\rangle + \langle \sin\phi_p, -\sin\theta_p\cos\phi_p, 0\rangle \sin\omega t + \hat{p}(\cos\theta_p\cos\phi_p)(1-\cos\omega t)$$

Further simplification produces:

$$\vec{r}(t) = \begin{bmatrix} \sin\phi_p \sin\omega t + (1-\cos\omega t)\sin\theta_p\cos\theta_p\cos^2\phi_p \\ -\sin\theta_p\cos\phi_p\sin\omega t + (1-\cos\omega t)\sin\phi_p\cos\theta_p\cos\phi_p \\ \cos\omega t + (1-\cos\omega t)\cos^2\theta_p\cos^2\phi_p \end{bmatrix}$$

The expression $\vec{r}(t)\cdot\hat{k}$ is useful later:

$$\vec{r}(t)\cdot\hat{k} = \cos\omega t + (1-\cos\omega t)\cos^2\theta_p\cos^2\phi_p \tag{1}$$

3 Apparent velocity scaling due to time delay

Due to light travel time, projected velocities of objects moving with velocity $|\vec{v}| \ll c$ appear to be decreased for objects moving away from the observer and increased for objects moving toward the observer. For an object with constant radial speed v_0, the time for an object starting at the observer's position to travel an apparent distance x_a, and have the light propagate back to the observer is given by:

$$t = \frac{x_a}{v_0} + \frac{x_a}{c}$$

Solving for x_a:

$$x_a = \frac{v_0 t c}{v_0 + c}$$

But $v_0 t$ is the absolute distance x traveled by the object:

$$x_a = \frac{xc}{v_0 + c}$$

[1] `https://en.wikipedia.org/wiki/Rodrigues%27_rotation_formula`

Taking the derivative with respect to time produces the relation between apparent and absolute velocities:

$$v_a = \frac{v_0 c}{v_0 + c} \tag{2}$$

4 Velocity of ejecta as a function of time

By projecting the velocities onto the z-axis and inverting the z-axis to match our coordinate system, we can use (2) to see that the apparent speed of matter emitted at time t is:

$$s(t) := \frac{v_e c}{c - v_e(\vec{r}(t) \cdot \hat{k})} \tag{3}$$

So the apparent velocity of matter emitted at time t is simply $s(t)\vec{r}(t)$.

5 Condition for jet separation

The jet will appear to separate if matter emitted at some time appears to be overtaken by matter emitted at a later time; the matter emitted later will appear to strike the termination sphere first.
Quantitatively, define $T(t)$ as the apparent travel time of matter from the origin to the termination sphere at radius R. This is given by:

$$T(t) := \frac{R}{|s(t)\vec{r}(t)|} = \frac{R}{s(t)} \tag{4}$$

Then matter emitted at time t arrives at the termination sphere at time $t+T(t)$. The jet separates if there exists a time t such that $\frac{d}{dt}(t+T(t)) < 0$; i.e. the arrival time of matter at the termination sphere decreases. Substituting (1) and (3) into (4):

$$T(t) = \frac{R}{v_e} - \frac{R}{c}(\cos\omega t + (1 - \cos\omega t)\cos^2\theta_p \cos^2\phi_p)$$

Computing the derivative:

$$\frac{d}{dt}(t + T(t)) = 1 - \frac{R}{c}\omega \sin\omega t(\cos^2\theta_p \cos^2\phi_p - 1)$$

The emission velocity v_e has been differentiated away. Jet separation is therefore independent of v_e. Finally, we set up the inequality:

$$1 - \frac{R}{c}\omega \sin \omega t(\cos^2 \theta_p \cos^2 \phi_p - 1) < 0$$

Note that $-1 \leq \sin \omega t \leq 1$ and $\cos^2 \theta_p \cos^2 \phi_p - 1 \leq 0$, so the minimum sufficient condition occurs when $\omega t = \frac{3\pi}{2}$. Then $\sin \omega t = -1$, and we can write:

$$\frac{R\omega}{c}(1 - \cos^2 \theta_p \cos^2 \phi_p) > 1 \tag{5}$$

Taking R in light-seconds and c as unity, we get the sufficient condition for jet separation:

$$R\omega(1 - \cos^2 \theta_p \cos^2 \phi_p) > 1 \tag{6}$$

Also observe that as $\phi_p \to 0$, this expression simplifies to $R\omega \sin^2 \theta_p > 1$.