Inquiry Based Vector Calculus

About the Document

This document was originally designed in the spring of 2016 to guide students through an ten week Linear Algebra course (Math 281-3) at Northwestern University.

A typical class day using the problem-sets:

1. **Introduction by instructor.** This may involve giving a definition, a broader context for the day's topics, or answering questions.

2. **Students work on problems.** Students work individually or in pairs on the prescribed problem. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.

3. **Instructor intervention.** If most students have successfully solved the problem, the instructor regroups the class by providing a concise explanation so that everyone is ready to move to the next concept. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to do some computation while being oblivious to the larger context).

   If students are having trouble, the instructor can give hints to the group, and additional guidance to ensure the students don't get frustrated to the point of giving up.

4. **Repeat step 2.**

Using this format, students are working (and happily so) most of the class. Further, they are especially primed to hear the insights of the instructor, having already invested substantially into each problem.

This problem-set is geared towards concepts instead of computation, though some problems focus on simple computation.

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Linear Combinations, Span, and Linear Independence

Linear Combination

DEFINITION

A ***linear combination*** of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a vector

$$\vec{w} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars.

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1 Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\vec{w} = 2\vec{v}_1 + \vec{v}_2$.

1.1 Write the coordinates of $\vec{w}$.

1.2 Draw a picture with $\vec{w}$, $\vec{v}_1$, and $\vec{v}_2$.

1.3 Is $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

1.4 Is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

1.5 Is $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

1.6 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

1.7 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$?

Span

DEF

The ***span*** of a set of vectors V is the set of all linear combinations of vectors in V.

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2 Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

2.1 Draw $\operatorname{span}\{\vec{v}_1\}$.

2.2 Draw $\operatorname{span}\{\vec{v}_2\}$.

2.3 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.

2.4 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_3\}$.

2.5 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

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3 Give an example of:

3.1 two vectors in $\mathbb{R}^3$ that span a plane;

3.2 two vectors in $\mathbb{R}^3$ that span a line;

3.3 four vectors in $\mathbb{R}^3$ that span a plane;

3.4 a set of 50 vectors in $\mathbb{R}^3$ whose span is the line through the origin and the point $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

In some sets, every vector is essential for computing a span. In others, there are "excess" vectors. This leads us to the concept of linear independence.

Linearly Dependent & Independent

DEFINITION

We say $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is ***linearly dependent*** if for at least one i,

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n\},$$

and a set is ***linearly independent*** otherwise.

Rank-nullity Theorem

THEOREM

The ***nullity*** of a matrix is the dimension of the null space.

The rank-nullity theorem states

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = \#\text{ of columns in } A.$$

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4 The vectors $\vec{u}, \vec{v} \in \mathbb{R}^9$ are linearly independent and $\vec{w} = 2\vec{u} - \vec{v}$. Define $A = [\vec{u}|\vec{v}|\vec{w}]$.

4.1 What is the rank and nullity of A^T?

4.2 What is the rank and nullity of A?

Rank-nullity Lemma

LEMMA

The ***nullity*** of a matrix is the dimension of the null space.

The rank-nullity theorem states

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = \#\text{ of columns in } A.$$

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5 More question

5.1 What is the rank and nullity of A^T?

5.2 What is the rank and nullity of A?

And a break

5.3 And a continuation of parts.