Math 230 Learning Objectives

These learning objectives are organized in the following hierarchy.

> ## (S) Strand
> This is a broad categorization, and all strands are interconnected by providing necessary technical tools or motivation to others.
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> **S.Thresh Threshold Concept**
> This is a more focused categorization of objectives. In the interest of time, specific learning objectives may be skipped, but all threshold concepts must be covered to create a comprehensive course.
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> S.Thresh.1 Learning Objective. These are specific topics, often presented with examples, and should be used to construct parts of lectures and individual exam questions.

The course should **not** be structured to cover these objectives in the order presented here. Rather, the organization here reflects larger themes of the course. The capstone strand for Math 230 is Optimization, and all other strands should be structured to motivate and provide computational tools for optimization objectives.

(V) Visualizing in $\mathbb{R}^2$ and $\mathbb{R}^3$

V.Coord Coordinate Systems

V.Coord.1 (a) Find the rectangular coordinates of a point specified with a description or a picture. (b) Plot points specified in rectangular coordinates. (c) Extract the x, y, or z *component* from the rectangular coordinates or picture of a point.

V.Coord.2 (a) Plot points in $\mathbb{R}^2$ specified by polar coordinates. (b) Convert between polar and rectangular coordinates. (c) Find polar equations for circles centered at the origin and lines through the origin. (d) Graph and recognize graphs of polar equations.

V.Coord.3 (a) Plot a point in $\mathbb{R}^3$ given in cylindrical coordinates. (b) Convert between cylindrical and rectangular coordinates.

V.Coord.4 (a) Plot a point in $\mathbb{R}^3$ given in spherical coordinates. (b) Convert from spherical coordinates to rectangular coordinates. (c) Convert from rectangular coordinates to spherical coordinates.

V.Coord.5 (a) Identify if two points are equal when given descriptions in a coordinate system (e.g. are the points $(1,\pi)$ and $(-1,\pi)$ equal in polar coordinates? Are they equal in rectangular coordinates?). (b) Draw the coordinate surfaces for rectangular, polar, cylindrical, and spherical coordinates (e.g. plot in $\mathbb{R}^3$ the surface described in spherical coordinates by $\varphi=\frac{\pi}{4}$). (c) When presented with a geometric description of an object with straightforward symmetries, choose an appropriate coordinate system to describe it.

V.ScalarFunc Scalar-valued functions of two (or more) variables

- V.ScalarFunc.1 (a) When given a formula for a function, find the largest domain on which it is defined. (b) Given the formula for a function and its domain, determine its range.
- V.ScalarFunc.2 (a) Define level curves and the traces of a function in a coordinate plane. (b) Draw a particular level curve from a formula for a function or a description or picture of its graph (e.g. draw the level curve $f(x, y) = 1$ for the function $f(x, y) = x^2y$) (e.g. draw the level curve $f(x, y) = 1$ where f is the function whose graph is an upside-down right cone with vertex at $(-2, -3, 0)$).

V.Draw **Visualize an object in $\mathbb{R}^3$ using a 2-d picture by perspective drawing**

- V.Draw.1 (a) Match contour maps of functions with perspective drawings of their graphs. (b) Given plots of the level curves or traces of a function of two variables, plot its graph. (c) Plot the following quadric surfaces: ellipsoids, paraboloids, hyperbolic paraboloids, cones, and cylinders.
- V.Draw.2 (a) Sketch plots of planes and their normal vectors in $\mathbb{R}^3$. (b) Draw informative pictures to assist in problem solving.
- V.Draw.3 Plot a level surface of a function of three variables when the level surface is an implicit function of two variables or a quadric surface (e.g. plot the $f(x, y, z) = 4$ level surface of $f(x, y, z) = x^2 + y^2 - z$) (e.g. plot the $f(x, y, z) = 4$ level surface of $f(x, y, z) = 2x^2 + 3y^2 + 4z^2$).

V.Rel **Relationships between objects in $\mathbb{R}^3$**

- V.Rel.1 Find all points of intersections between two parameterized curves.
- V.Rel.2 Find all points intersections between a parameterized curve and a level surface.
- V.Rel.3 Find a parameterization for the line of intersection between two planes.

V.Vect **Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$**

- V.Vect.1 (a) Define a vector as an object with a magnitude and a direction. (b) Plot and write vectors using rectangular coordinates.
- V.Vect.2 (a) Add, subtract, and scalar multiply vectors described in coordinates. (b) Estimate the result of adding, subtracting, and/or scalar multiplying vectors described in a picture (e.g. given a picture of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ with the same initial point, graphically estimate whether $\|\mathbf{a} + \mathbf{b} + \mathbf{c}\| < \|\mathbf{c}\|$).
- V.Vect.3 (a) Compute the norm of a vector given in coordinates or as its magnitude and direction. (b) Determine if a vector is a *unit vector*. (c) Given a vector described in coordinates, produce a unit vector in the same direction. (d) Find the distance between two vectors.
- V.Vect.4 (a) Compute the dot product of two vectors described in coordinates or in terms of their magnitudes and directions. (b) Recognize a formula that looks like the result of a dot product and "factor it" as the dot product of two vectors.
- V.Vect.5 Compute the angle between two vectors described in coordinates.
- V.Vect.6 (a) Compute the projection of a vector onto another vector. (b) Compute the projection of a vector onto a coordinate plane.
- V.Vect.7 (a) In terms of its magnitude and direction, define the cross product of two vec-

tors in $\mathbb{R}^3$. (b) Compute the cross product of two vectors described in coordinates. (c) Use the magnitude of the cross product to compute the area of a parallelogram spanned by two vectors.

V.Vect.8 (a) Define orthogonality of vectors. (b) Use the dot product to determine when two vectors described in coordinates are orthogonal. (c) When given one vector in $\mathbb{R}^2$ or one or two vectors in $\mathbb{R}^3$, produce another vector that is orthogonal to the original vector(s).

V.Vect.9 Use scalar multiplication or the cross product to determine when two vectors are parallel.

V.Vect.10 Determine whether an expression involving operations on vectors is well-defined, and if so, whether the result is a scalar or vector (e.g. for vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^3$ are the expressions $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$, and $(\mathbf{a} \cdot \mathbf{b}) + \mathbf{c}$ scalars, vectors, or nonsense?).

(P) Parameterization

P.Struct Structure of parameterized curves

P.Struct.1 (a) Parametrize lines in 2-d and 3-d when determined by two points, a point and a direction vector, or the intersection of two planes. (b) Parameterize line segments between two points by parameterizing the line and giving bounds on the parameter. (c) Work backwards from a parameterization of a line to recover points on the line and direction vectors.

P.Struct.2 (a) When presented with an expression, determine if it is a parameterized curve (e.g. is $\begin{cases} x+y+z=0 \\ 2x-z=3 \end{cases}$ a parameterization of a line?). (b) Distinguish between relations, functions, and parameterizations of curves (e.g. $y = x^2$ is a relation, $f(x) = x^2$ is a function, and $\mathbf{r}(t) = (t, t^2)$ is a parameterization). (c) State from memory the definition of a parameterized curve in 2-d and 3-d space.

P.Struct.3 (a) Plot parameterized curves in 2-d and 3-d space. (b) Translate between traces/projections and plots of parameterized curves, and vice versa.

P.Struct.4 (a) From a parameterization, identify what space the associated curve is a subset of. (b) Given a parameterization of a 2-d curve with additional information, write a parameterization of the 3-d embedding (e.g. find a parameterization of the curve C in $\mathbb{R}^3$ that lies in the plane $z = 2$ and has x, y-coordinates given by $\mathbf{r}(t) = (t, t^2)$).

P.Prop Properties of parameterized curves

P.Prop.1 Define and compute velocities and accelerations of parameterizations.

P.Prop.2 Use a piecewise functions to parameterize paths with a corners.

P.Reparam Reparameterization

P.Reparam.1 Given a parametric curve, change its speed or time shift by producing a new parameterization.

P.Reparam.2 Identify whether two equations parameterize the same curve.

P.Reparam.3 When appropriate, use the properties of polar/cylindrical/spherical coordinates to find parameterization of curves (e.g. use polar coordinates to obtain a parameterization for a circle; use cylindrical coordinates to obtain a parameterization for a helix).

P.Reparam.4 (a) State the definition of the arclength function and parameterization with respect to arclength. (b) Recognize in context the notation $s(t)$ as the arclength function. (c) Find arclength parameterizations for lines and circles.

(D) Derivatives

D.Partial Partial Derivatives

D.Partial.1 Given a formula for a function of several variables, compute partial derivatives of any order with specified notation (e.g. $\frac{\partial}{\partial x}$, f_{xy}, etc.).

D.Partial.2 For a function $f : \mathbb{R}^n \to \mathbb{R}$ with $n > 1$, explain why the notation f' does not make sense by appealing to the fact that there are multiple "rates of change" for f.

D.Partial.3 (a) Answer true/false questions about the equality of mixed partial derivatives of a function of several variables (e.g. is f_{xy} always equal to f_{yx}?). (b) Use the equality of mixed partial derivatives to simplify the computation of a table of mixed partial derivatives (e.g. if asked to compute f_{xyy}, f_{yxy}, and f_{yyx}, only compute one and use that answer appropriately).

D.Chain Chain Rule

D.Chain.1 (a) Given two formulas for functions, evaluate partial derivatives of their composition using the chain rule. (b) Given a table of values for functions and their partial derivatives, evaluate partial derivatives of their composition or determine if there is not enough information to do so. (c) Factor the chain rule formula as a dot product and explain what the two vectors represent geometrically.

D.Chain.2 Evaluate the derivative of a composition of functions expressed in a story problem using the chain rule.

D.Grad Gradients

D.Grad.1 (a) Given the formula for a function, evaluate the gradient. (b) Draw the gradient vector at a point for a function specified with level curves.

D.Grad.2 (a) Given a formula for a function, identify directions of maximal change and zero change. (b) Given a formula for a function, identify the maximal rate of change as $\|\nabla f\|$.

D.Grad.3 Given a formula for a function of two or three variables, use the gradient to find normal and tangent lines to level curves and normal lines and tangent planes to level surfaces.

D.Direct Directional derivatives

D.Direct.1 (a) Given a formula for a function, evaluate the directional derivative as a dot product with the gradient. (b) Given a formula for a function, calculate a directional derivative as an application of the chain rule by composition with a parameterized line.

D.Direct.2 (a) Interpret the directional derivative as a rate of change of the function. (b) Identify partial derivatives as special cases of directional derivatives.

D.Param **Derivatives of parameterized curves**

D.Param.1 Evaluate the derivate of a parametric function in terms of derivatives of its component functions.

D.Param.2 Define and compute velocities and accelerations of parameterizations.

(A) Approximation

A.Lim **Limits of functions of several variables**

A.Lim.1 given a graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ or a plot of its level curves, state where limits do and do not exist.

A.Lim.2 Given constraints on a function $f : \mathbb{R}^2 \to \mathbb{R}$, draw level curves of a function that satisfies the constraints (e.g. draw level curves of a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous everywhere except for $(0, 0)$) (e.g. draw level curves of a bounded function $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous everywhere except for $(0, 0)$, where it has a non-removable discontinuity).

A.Lim.3 (a) Recognize that checking the limit of a multivariate function along some paths is insufficient evidence for existence of a limit, while checking all paths is sufficient evidence for existence of a limit (e.g. for $f : \mathbb{R}^2 \to \mathbb{R}$, if the limits along every line through $(0, 0)$ exists and are equal, can you conclude $\lim_{(x,y)\to(0,0)} f(x, y)$ exists?). (b) Recognize that if the limits of a multivariate function along two different paths do not agree, then the limit of the function does not exist (e.g. for $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, 0) = \sin x$ and $f(0, y) = \cos y$. Does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist? Why or why not?). (c) Recognize that the Squeeze Theorem can be used to conclude existence of a limit, but not the non-existence (e.g. for $f : \mathbb{R}^2 \to \mathbb{R}$, you know $-x^2 - y^2 \leq f(x, y) \leq x^2 + y^2$. Does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist? Why or why not?). (d) The value of a function close to a limit point need not affect the limit of the function at the limit point (e.g. for $f : \mathbb{R}^2 \to \mathbb{R}$, you know $\lim_{(x,y)\to(0,0)} f(x, y)$ exists and that $f(0.01, 0.01) = 3$ and for $g : \mathbb{R}^2 \to \mathbb{R}$, you know $\lim_{(x,y)\to(0,0)} g(x, y)$ exists and that $g(0.01, 0.01) = -4$. Is it possible that $\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} g(x, y)$? Why or why not?).

A.Con **Continuity**

A.Con.1 Recall the definition of continuity of a function of several variables. Recognize that multivariate polynomials are continuous everywhere. Recognize that sums, differences, products, quotients, and compositions of continuous functions are continuous on their domains.

A.Con.2 Given a formula for a piecewise function, whose parts are elementary functions, identify which points in the domain need to be tested to determine continuity and which do not (e.g. let $f(x, y) = \begin{cases} \frac{x^2-y^2}{x+y} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ and suppose you know $\lim_{(x,y)=(0,0)} f(x, y) = 0$. Can you conclude that $f(x, y)$ is continuous at $(0, 0)$?)

(e.g. let $f(x,y) = \begin{cases} 2x^2+y^2 & \text{if } x^2+y^2>1 \\ h(x,y) & \text{if } x^2+y^2\leq 1 \end{cases}$ where $h(x,y)$ is continuous. If you know $\lim_{(x,y)\to(1,0)} f(x,y)$ exists, can you conclude that $f(x,y)$ is continuous at $(1,0)$? At what points must you verify the continuity of $f(x,y)$?).

A.Con.3 (a) Given a graph of a function of two variables, conclude where the function is continuous and where it is discontinuous. (b) Given a plot of the level curves of a function of two variables that features a discontinuity, conclude where the function is discontinuous.

A.Der **Derivatives**

A.Der.1 Approximate the value of a partial derivative or directional derivative from a table of values of the function.

A.Der.2 For a function of two variables, approximate the value of a partial derivative or directional derivative from a plot of level curves of the function.

A.TanL **Tangent lines to parameterized curves**

A.TanL.1 Find a formula for the tangent line at a point given a formula for a parameterization.

A.TanL.2 Use a tangent line to approximate values of a parameterization near the point of tangency.

A.TanL.3 Identify where tangent lines exists for a piecewise smooth path.

A.TanP **Tangent planes to surfaces**

A.TanP.1 Find the formula for the tangent plane to the graph of a function of two variables or a level surface of a function of three variables.

A.TanP.2 (a) Use a tangent plane to approximate values of a function of two variables near the point of tangency. (b) Be familiar with the vocabulary *linear approximation*, *tangent plane*, and *first-order approximation*.

A.TanP.3 (a) Define differentiability of a function of two variables as having a "good" tangent plane/linear approximation at a point, where "good" is left to intuition. (b) Identify that corners/cusps admit no "good" linear approximations.

A.Tay **Taylor polynomials of degree two**

A.Tay.1 Find the formula for the Taylor polynomial of degree two given the formula for a function of two variables.

A.Tay.2 Use the second-order Taylor polynomial to approximate values of the function near the point of tangency.

A.Tay.3 Identify whether first-order or second-order approximations given desired information about a function and/or its partial derivatives (e.g. for an unknown function $f:\mathbb{R}^2\to\mathbb{R}$, the linear approximation to f at $(1,1)$ is given by $L(x,y)=2x+3y+4$. Let $g(x,y)=(x^2,y^3)$. If possible, compute $\frac{\partial}{\partial x}(f\circ g)(1,1)$ or explain why you cannot. If possible, compute $\frac{\partial^2}{\partial x^2}(f\circ g)(1,1)$ or explain why you cannot.) (e.g. the second-order Taylor approximation of $f:\mathbb{R}^2\to\mathbb{R}$ is $T(x,y)=2x-y$. Can you conclude f is a plane? Why or why not?).

(O) Optimization

O.Dist Linear distance optimization

O.Dist.1 (a) Using projections, calculate closest distance between parallel planes given formulas for both. (b) Given distances, find missing information in formulas. (e.g. find a plane of distance 5 from the plane $2x - y + 3z = 4$.)

O.Dist.2 (a) Using projections, calculate closest distance between a *point* and a *line* given formulas for both. (b) Given distances, find missing information in formulas (e.g. find a number a so that the point $\mathbf{x} = (1, a)$ has distance 5 from the line $\mathbf{r}(t) = (3t - 1, 2t + 3)$)

O.Dist.3 (a) Using projections, calculate closest distance between a *point* and a *plane* given formulas for both. (b) Given distances, find missing information in formulas (e.g. find a number a so that the point $\mathbf{x} = (1, 2, a)$ has distance 5 from the line $2x - y + 3z = 4$).

O.Param Parameterized curves

O.Param.1 (a) Given a parameterized curve, set up an integral to find the arclength of the curve. (b) Given a formula/table for arclength of a curve with respect to time, find/approximate the speed of the parameterization. (c) Use that arclength is independent of parameterization to make substitutions/simplifications to evaluate arclength (e.g. Find the arclength of $\mathbf{r}(t) = (\cos(t^3), \sin(t^3))$ from $t = 0$ to $t = 1$). (d) Find local linear approximation to arclength via geometric understanding and relate the velocity vector to local arclength (e.g. estimate the arclength from $t = 0$ to $t = \epsilon$ by $\epsilon|\mathbf{v}(0)|$; given a parameterization $\mathbf{r}(t)$ and $\mathbf{r}'(0) = (3, 7)$, find the approximate arclength between $t = 0$ and $t = 0.1$). (e) Be able to answer a true/false question about whether a "typical" parameterization has an elementary arclength formula.

O.Param.2 (a) Visually identify points on a curve with higher/lower curvature. (b) Find the curvature of a circle. (c) Draw an approximate (osculating) circle of best fit. (d) Be able to recite a formula for curvature, $\kappa(t) = \left|\frac{d\mathbf{T}}{ds}\right|$ or $\kappa(t) = |\mathbf{r}''(t)|$ where $\mathbf{r}(t)$ is an arclength parameterization. (e) Apply this formula when given an arclength parameterization or the first or second derivative of an arclength parameterization. (f) There will be **no** expectation to compute the curvature of a non-arclength parameterized curve.

O.ScalarFunc Scalar-valued functions of several variables

O.ScalarFunc.1 State the definition of local extrema using an inequality.

O.ScalarFunc.2 (a) Compute critical points from formulas of a function or its partial derivatives. (b) Estimate critical points using a table of values of a function or its partial derivatives.

O.ScalarFunc.3 Interpret critical points geometrically in terms of horizontal tangent planes (e.g. if $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and has a critical point at $(3, 1)$, find the tangent plane to $z = f(x, y)$ at $(3, 1, f(3, 1))$).

O.ScalarFunc.4 (a) Given a formula or a table of values of partial second-order partial derivatives, classify critical points as local extrema or saddle points using the Second Derivative Test. (b) Given a simple formula that fails the Second Derivative

Test, use arguments about positivity/first derivatives to classify the critical point (e.g. classify the critical point of $f(x, y) = x^2 + y^4$).

O.ScalarFunc.5 (a) Identify and classify critical points based on plots of surfaces or level curves. (b) Draw level curves of a surface with a saddle point or local max/min without reference to a formula.

O.ScalarFunc.6 (a) Given a formula that describes a surface, use the gradient to find a unit vector that maximizes the directional derivative at a given point. (b) Given a formula for a surface and a description in words (e.g. "steepest increase," "steepest decrease," "no change"), use the gradient to find the relevant direction.

O.ScalarFunc.7 Given overlaid plots of level curves for a function and a constraint curve, estimate the location and values of global extrema, or explain why none exist.

O.ScalarFunc.8 (a) Given a formula for a function $f(x_1, \ldots, x_n)$ and a formula/description of a constraint level set $g(x_1, \ldots, x_n) = c$, use Lagrange multipliers to find extrema. (b) Given an optimization problem as a description of a function and/or constraint in a story problem, produce appropriate formulas and apply the method of Lagrange multipliers.

