PUMA

Design explicit solution of direct and inverse problem for PUMA. The kinematic chain is presented on Fig. 1.

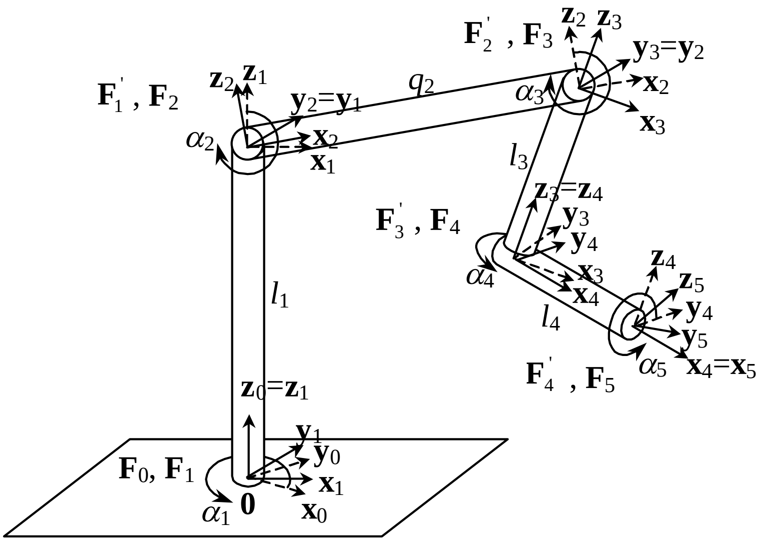

Fig. 1. Kinematic chain of PUMA.

Solution

The kinematic chain PUMA consists of one prismatic and five revolute joints, that imply the 6C configuration space (α_1, α_2, q_2, α_3, α_4, α_5) where $0 \le \alpha_i < 2\pi$ and $q_2 \ge 0$. Given lengths of constant rods are equal to l_1, l_3 and l_4; positions and orientations of joints are denoted by the frames $\mathbf{F}_i = (\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i, \mathbf{p}_i)$ and $\mathbf{F}_i' = (\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i, \mathbf{p}_i')$ (assume the scene frame $\mathbf{F}_0 = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{0})$).

The direct problem is to find the end effector of the chain configuration (as a rigid body defined by coordinates of the frame $\mathbf{F}_5$ in the scene frame $\mathbf{F}_0$) from the given kinematic chain configuration (α_1, α_2, q_2, α_3, α_4, α_5). The solution is the affine transformation matrix $\mathbf{F}_{05}$ converting the co-ordinates from the effector frame $\mathbf{F}_5$ to the scene frame $\mathbf{F}_0$:

$$\mathbf{F}_5 = \mathbf{F}_0\mathbf{F}_{05} \tag{6}$$

defined as

$$\mathbf{F}_{05} = \mathbf{R}_{\mathbf{x}_4}(\alpha_5)\mathbf{T}_{\mathbf{x}_4}(l_4)\mathbf{R}_{\mathbf{z}_3}(\alpha_4)\mathbf{T}_{\mathbf{z}_3}(-l_3)\mathbf{R}_{\mathbf{y}_2}(\alpha_3)\mathbf{T}_{\mathbf{x}_2}(q_2)\mathbf{R}_{\mathbf{y}_1}(\alpha_2)\mathbf{T}_{\mathbf{z}_1}(l_1)\mathbf{R}_{\mathbf{z}_0}(\alpha_1)$$

where $\mathbf{R}_\mathbf{a}(\alpha)$ is the rotation matrix around the axis $\mathbf{a}$ by the angle α and $\mathbf{T}_\mathbf{a}(l)$ – translation along the axis $\mathbf{a}$ by the length l (the axis $\mathbf{a}$ co-ordinates are expressed in the scene frame $\mathbf{F}_0$). Note that matrix $\mathbf{F}_{05}$ can be also defined by transformations in corresponding local frames as:

$$\mathbf{F}_{05} = \underbrace{\mathbf{R}_\mathbf{z}(\alpha_1)}_{\mathbf{F}_{01}}\underbrace{\mathbf{T}_\mathbf{z}(l_1)}_{\mathbf{F}_{11'}}\underbrace{\mathbf{R}_\mathbf{y}(\alpha_2)}_{\mathbf{F}_{1'2}}\underbrace{\mathbf{T}_\mathbf{x}(q_2)}_{\mathbf{F}_{22'}}\underbrace{\mathbf{R}_\mathbf{y}(\alpha_3)}_{\mathbf{F}_{2'3}}\underbrace{\mathbf{T}_\mathbf{z}(-l_3)}_{\mathbf{F}_{33'}}\underbrace{\mathbf{R}_\mathbf{z}(\alpha_4)}_{\mathbf{F}_{3'4}}\underbrace{\mathbf{T}_\mathbf{x}(l_4)}_{\mathbf{F}_{44'}}\underbrace{\mathbf{R}_\mathbf{x}(\alpha_5)}_{\mathbf{F}_{4'5}}$$

Then coordinates of axes are expressed by ($\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$) := ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) and rotations and translations have the simplest form.
The explicit form of (6) in the scene frame:

$$\mathbf{x}_5 = \begin{bmatrix} c_1c_4(c_2c_3 - s_2s_3) - s_1s_4 \\ s_1c_4(c_2c_3 - s_2s_3) + c_1s_4 \\ -c_4(s_2c_3 + c_2s_3) \end{bmatrix}$$

$$\mathbf{y}_5 = \begin{bmatrix} c_1\big(c_2(s_3s_5 - c_3s_4c_5) + s_2(c_3s_5 + s_3s_4c_5)\big) - s_1c_4c_5 \\ s_1\big(c_2(s_3s_5 - c_3s_4c_5) + s_2(c_3s_5 + s_3s_4c_5)\big) + c_1c_4c_5 \\ -s_2(s_3s_5 - c_3s_4c_5) + c_2(c_3s_5 + s_3s_4c_5) \end{bmatrix}$$

$$\mathbf{z}_5 = \begin{bmatrix} c_1\big(c_2(s_3c_5 + c_3s_4s_5) + s_2(c_3c_5 - s_3s_4s_5)\big) + s_1c_4s_5 \\ s_1\big(c_2(s_3c_5 + c_3s_4s_5) + s_2(c_3c_5 - s_3s_4s_5)\big) - c_1c_4s_5 \\ -s_2(s_3c_5 + c_3s_4s_5) + c_2(c_3c_5 - s_3s_4s_5) \end{bmatrix} \qquad (7)$$

$$\mathbf{p}_5 = \begin{bmatrix} c_1\big(c_2(q_2 - s_3l_3 + c_3c_4l_4) - s_2(c_3l_3 + s_3c_4l_4)\big) - s_1s_4l_4 \\ s_1\big(c_2(q_2 - s_3l_3 + c_3c_4l_4) - s_2(c_3l_3 + s_3c_4l_4)\big) + c_1s_4l_4 \\ -s_2(q_2 - s_3l_3 + c_3c_4l_4) - c_2(c_3l_3 + s_3c_4l_4) + l_1 \end{bmatrix}$$

where $s_i = \sin(\alpha_i)$ and $c_i = \cos(\alpha_i)$.
The inverse problem: calculation of the kinematic chain configuration (α_1, α_2, q_1, α_3, α_4, α_5) defined by the manipulator (rigid body) configuration at the end of the kinematic chain (co-ordinates of the frame $\mathbf{F}_5$ in the scene frame $\mathbf{F}_0$), can be solved explicitly from the expression (7) or by using of geometrical properties of the system.

Ver.1.

Explicit solution uses substitution of complex sub-expressions of (7) by auxiliary labels a_i.

a) Angle α_1 evaluation:

$$\begin{cases} \mathbf{x}_{5,1} = c_1a_1 - s_1s_4 \\ \mathbf{x}_{5,2} = s_1a_1 + c_1s_4 \end{cases} \Rightarrow a_1 = \frac{\mathbf{x}_{5,1} + s_1s_4}{c_1} = \frac{\mathbf{x}_{5,2} - c_1s_4}{s_1}$$

$$\begin{cases} \mathbf{p}_{5,1} = c_1a_2 - s_1s_4l_4 \\ \mathbf{p}_{5,2} = s_1a_2 + c_1s_4l_4 \end{cases} \Rightarrow a_2 = \frac{\mathbf{p}_{5,1} + s_1s_4l_4}{c_1} = \frac{\mathbf{p}_{5,2} - c_1s_4l_4}{s_1}$$

Hence

$$s_4 = c_1\mathbf{x}_{5,2} - s_1\mathbf{x}_{5,1} = \frac{c_1\mathbf{p}_{5,2} - s_1\mathbf{p}_{5,1}}{l_4} \qquad (8)$$

From the last equality

$$\tan\alpha_1 = \frac{s_1}{c_1} = \frac{\mathbf{p}_{5,2} - l_4\mathbf{x}_{5,2}}{\mathbf{p}_{5,1} - l_4\mathbf{x}_{5,1}}$$

that leads to

$$\alpha_1 = \arctan\frac{\mathbf{p}_{5,2} - l_4\mathbf{x}_{5,2}}{\mathbf{p}_{5,1} - l_4\mathbf{x}_{5,1}} \qquad (9)$$

Note that the function arctan() returns two equivalent solutions.

b) Angle α_4 can be evaluated from equation (8):

$$\alpha_4 = \arcsin(c_1\mathbf{x}_{5,2} - s_1\mathbf{x}_{5,1}) \qquad (10)$$

that gives 1 or 2 solutions for given α_1.

c) Angle α_5 evaluation:

$$\begin{cases} \mathbf{y}_{5,1} = c_1 a_3 - s_1 c_4 c_5 \\ \mathbf{y}_{5,2} = s_1 a_3 + c_1 c_4 c_5 \end{cases} \Rightarrow a_3 = \frac{\mathbf{y}_{5,1} + s_1 c_4 c_5}{c_1} = \frac{\mathbf{y}_{5,2} - c_1 c_4 c_5}{s_1}$$

$$\begin{cases} \mathbf{z}_{5,1} = c_1 a_4 + s_1 c_4 s_5 \\ \mathbf{z}_{5,2} = s_1 a_4 - c_1 c_4 s_5 \end{cases} \Rightarrow a_4 = \frac{\mathbf{z}_{5,1} - s_1 c_4 s_5}{c_1} = \frac{\mathbf{z}_{5,2} + c_1 c_4 s_5}{s_1}$$

leads to two conditions

$$\begin{cases} c_5 = \dfrac{c_1 \mathbf{y}_{5,2} - s_1 \mathbf{y}_{5,1}}{c_4} \\ s_5 = \dfrac{s_1 \mathbf{z}_{5,1} - c_1 \mathbf{z}_{5,2}}{c_4} \end{cases} \tag{11}$$

that identify the angle α_5 uniquely.

d) The angle α_2 evaluation needs two auxiliary labels $a_5 = c_2 c_3 - s_2 s_3$ and $a_6 = s_2 c_3 + c_2 s_3$:

$$\begin{cases} \mathbf{x}_{5,1} = c_1 c_4 a_5 - s_1 s_4 \Rightarrow a_5 = \dfrac{\mathbf{x}_{5,1} + s_1 s_4}{c_1 c_4} \\ \mathbf{x}_{5,3} = -c_4 a_6 \Rightarrow a_6 = -\dfrac{\mathbf{x}_{5,3}}{c_4} \end{cases} \tag{12}$$

Then

$$\begin{cases} \mathbf{p}_{5,1} = c_1\big(c_2(q_2 - s_3 l_3 + c_3 c_4 l_4) - s_2(c_3 l_3 + s_3 c_4 l_4)\big) - s_1 s_4 l_4 \\ \mathbf{p}_{5,3} = -s_2(q_2 - s_3 l_3 + c_3 c_4 l_4) - c_2(c_3 l_3 + s_3 c_4 l_4) + l_1 \end{cases}$$

transforms into

$$\begin{cases} c_2 q_2 = \dfrac{\mathbf{p}_{5,1} + s_1 s_4 l_4}{c_1} + l_3(\underbrace{s_2 c_3 + c_2 s_3}_{a_6}) - c_4 l_4(\underbrace{c_2 c_3 - s_2 s_3}_{a_5}) \\ s_2 q_2 = -\mathbf{p}_{5,3} + l_1 - c_4 l_4(\underbrace{s_2 c_3 + c_2 s_3}_{a_6}) - l_3(\underbrace{c_2 c_3 - s_2 s_3}_{a_5}) \end{cases} \tag{13}$$

Dividing in (13) the second equation by the first equation and substitution of a_5 and a_6 lead to:

$$\tan\alpha_2 = \frac{s_2}{c_2} = -\frac{c_1 c_4(\mathbf{p}_{5,3} - l_4 \mathbf{x}_{5,3} - l_1) + l_3(\mathbf{x}_{5,1} + s_1 s_4)}{c_4(\mathbf{p}_{5,1} - l_4 \mathbf{x}_{5,1}) - c_1 l_3 \mathbf{x}_{5,3}}$$

and as a consequence two values of angle α_2 may be evaluated:

$$\alpha_2 = \arctan\left(-\frac{c_1 c_4(\mathbf{p}_{5,3} - l_4 \mathbf{x}_{5,3} - l_1) + l_3(\mathbf{x}_{5,1} + s_1 s_4)}{c_4(\mathbf{p}_{5,1} - l_4 \mathbf{x}_{5,1}) - c_1 l_3 \mathbf{x}_{5,3}}\right) \tag{14}$$

e) The transition q_2 can be calculated from the first equation of (13) and substitution of a_5 and a_6 by (12):

$$q_2 = \frac{c_4(\mathbf{p}_{5,1} - l_4 \mathbf{x}_{5,1}) - c_1 l_3 \mathbf{x}_{5,3}}{c_1 c_2 c_4} \tag{15}$$

f) The angle α_3 evaluation is based on a_5 and a_6 definition (12)

$$\begin{cases} a_5 = c_2c_3 - s_2s_3 = \cos(\alpha_2+\alpha_3) \\ a_5 = \dfrac{\mathbf{x}_{5,1} + s_1s_4}{c_1c_4} \end{cases} \Rightarrow \cos(\alpha_2+\alpha_3) = \frac{\mathbf{x}_{5,1} + s_1s_4}{c_1c_4}$$

$$\begin{cases} a_6 = s_2c_3 + c_2s_3 = \sin(\alpha_2+\alpha_3) \\ a_6 = -\dfrac{\mathbf{x}_{5,3}}{c_4} \end{cases} \Rightarrow \sin(\alpha_2+\alpha_3) = -\frac{\mathbf{x}_{5,3}}{c_4}$$

and leads to the unique value $\alpha_{23} := \alpha_2 + \alpha_3$. Then

$$\alpha_3 = \alpha_{23} - \alpha_2 \tag{16}$$

In general expressions (9), (10), (11), (14), (15) and (16) define eight (or less for some specific positions of $\mathbf{F}_5$) different solutions in the kinematic chain configuration that imply two sequences ($\mathbf{p}_0 = \mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$, $\mathbf{p}_5$) that differ in $\mathbf{p}_3$ co-ordinates. However, for some positions of the end effector there is indefinite number of solutions; eg. for $\mathbf{p}_4$ on $\mathbf{z}_0$ axis. Then one of them should be chosen: especially ‘the simplest’ configuration or the configuration that is the nearest to the previous and next position in interpolation. The same strategy should be used in the previous case (selecting one of explicit configurations).

Ver.2.

The geometric construction of kinematic chain configuration leads to two explicit solutions (besides the cases for which it is impossible to find definite number of solutions). It is based on calculation of frames points $\mathbf{p}_i$ from given frames $\mathbf{F}_0$ and $\mathbf{F}_5$:

$$\begin{aligned} \mathbf{p}_1 &= \mathbf{p}_0 \\ \mathbf{p}_2 &= \mathbf{p}_0 + l_1\mathbf{z}_0 \\ \mathbf{p}_4 &= \mathbf{p}_5 - l_4\mathbf{x}_5 \end{aligned} \tag{17}$$

Note that points $\mathbf{p}_0 = \mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and $\mathbf{p}_4$ lie on the common plane P, fixed by i.e. normal vector $\mathbf{n}_{024}$:

$$\mathbf{n}_{024} := (\mathbf{p}_4 - \mathbf{p}_0) \bullet (\mathbf{p}_2 - \mathbf{p}_0) / \left| (\mathbf{p}_4 - \mathbf{p}_0) \bullet (\mathbf{p}_2 - \mathbf{p}_0) \right|$$

(iff it is defined uniquely). Besides the point $\mathbf{p}_3$ belongs to the plane going through the point $\mathbf{p}_4$ and perpendicular to the vector $\mathbf{x}_5$ and it is in distance l_4 from the point $\mathbf{p}_4$:

$$\begin{cases} \langle \mathbf{p}_3 - \mathbf{p}_4 \,;\, \mathbf{n}_{024} \rangle = 0 \\ \langle \mathbf{p}_3 - \mathbf{p}_4 \,;\, \mathbf{x}_5 \rangle = 0 \\ \langle \mathbf{p}_3 - \mathbf{p}_4 \,;\, \mathbf{p}_3 - \mathbf{p}_4 \rangle = l_4^2 \end{cases} \tag{18}$$

Conditions (18) are equivalent to vector $\mathbf{z}_4$ calculation ($|\mathbf{z}_4| = 1$).

$$\begin{cases} \langle \mathbf{z}_4 \,;\, \mathbf{n}_{024} \rangle = 0 \\ \langle \mathbf{z}_4 \,;\, \mathbf{x}_5 \rangle = 0 \\ \langle \mathbf{z}_4 \,;\, \mathbf{z}_4 \rangle = 1 \end{cases} \tag{19}$$

Then two possible sets of co-ordinates of point $\mathbf{p}_3$ are equal to:

$$\mathbf{p}_3 = \mathbf{p}_4 \pm l_3\mathbf{z}_4$$

In both cases the linear set of equations with additional length condition (18) or normalization (19) needs to be solved. If the vector $\mathbf{n}_{024}$ is parallel to the vector $\mathbf{x}_5$ (both planes are parallel) there exists indefinite number of solutions and co-ordinates of $\mathbf{p}_3$ may be identified by the minimal value of parameter q_2 (the length of the rod $\mathbf{p}_2\mathbf{p}_3$).
Then from points' coordinates (17) and (18) the corresponding chain configuration is evaluated:

$$\begin{aligned}
\alpha_1 &= \arctan_2(\mathbf{p}_{4,2}, \mathbf{p}_{4,1}) \\
\alpha_2 &= \mathrm{angle}(\mathbf{p}_2 - \mathbf{p}_0, \mathbf{p}_3 - \mathbf{p}_2) - \pi/2 \\
q_2 &= |\mathbf{p}_3 - \mathbf{p}_2| \\
\alpha_3 &= \mathrm{angle}(\mathbf{p}_3 - \mathbf{p}_2, \mathbf{p}_4 - \mathbf{p}_3) - \pi/2 \\
\alpha_4 &= \mathrm{angle}(\mathbf{n}_{024}, \mathbf{x}_5) + \pi/2 \\
\alpha_5 &= \mathrm{angle}(\mathbf{p}_3 - \mathbf{p}_4, \mathbf{z}_5)
\end{aligned}$$

Functions $\arctan_2(.)$ and angle(.) return one unique value. The function angle(.) is defined as:

$$\alpha := \mathrm{angle}(\mathbf{v}, \mathbf{w}) \Leftrightarrow \begin{cases} \sin\alpha = \dfrac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}| \cdot |\mathbf{w}|} \\ \cos\alpha = \dfrac{\langle \mathbf{v};\ \mathbf{w} \rangle}{|\mathbf{v}| \cdot |\mathbf{w}|} \end{cases}$$

As the result two of eight chain configurations (because of two equivalent sets of coordinates of $\mathbf{p}_3$) are found. It is sufficient, unless there are some limitations on values of chain configuration angles α_i.