Engineering Mathematics – I
(10 MAT11)

LECTURE NOTES
(FOR I SEMESTER B E OF VTU)

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Dr. V. Lokesha
Professor and Head
DEPARTMENT OF MATHEMATICS
ACHARYA INSTITUTE OF TECNOLOGY
Soldevanahalli, Bangalore – 90

ENGNEERING MATHEMATICS – I

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UNIT - I

DIFFERENTIAL CALCULUS – I

Introduction:

The mathematical study of change like motion, growth or decay is calculus. The Rate of change of given function is derivative or differential.

The concept of derivative is essential in day to day life. Also applicable in Engineering, Science, Economics, Medicine etc.

Successive Differentiation:

Let $y = f(x)$ --(1) be a real valued function.

The first order derivative of y denoted by $\dfrac{dy}{dx}$ or y' or y_1 or Δ^1

The Second order derivative of y denoted by $\dfrac{d^2 y}{dx^2}$ or y'' or y_2 or Δ^2

Similarly differentiating the function (1) n-times, successively,

the nth order derivative of y exists denoted by $\dfrac{d^n y}{dx^n}$ or y^n or y_n or Δ^n

The process of finding 2nd and higher order derivatives is known as Successive Differentiation.

n^{th} derivative of some standard functions:

1. $y = e^{ax}$

Sol : $y_1 = a\, e^{ax}$

            $y_2 = a^2\, e^{ax}$

Differentiating Successively

            $y_n = a^n\, e^{ax}$

ie. $D^n[e^{ax}] = a^n\, e^{ax}$

        For, $a = 1$ $D^n[e^x] = e^x$

2. $y = \log(ax + b)$

Solution : $y_1 = \dfrac{a}{ax+b}$

$$y_2 = \dfrac{(-1)a.a}{(ax+b)^2} = \dfrac{(-1)^1 a^2}{(ax+b)^2}$$

$$y_3 = \dfrac{(-1)(-2)a^2.a}{(ax+b)^3} = \dfrac{(-1)^2(1)(2)a^3}{(ax+b)^3} = \dfrac{(-1)^{3-1}(3-1)!a^3}{(ax+b)^3}$$

$$D^n[\log(ax+b)] = y_n = \dfrac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$$

Similarly $D^n[\log x] = y_n = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$

3. $y = (ax+b)^m$

Solution: $y_1 = m(ax+b)^{m-1}\,a$

$$y_2 = m(m-1)(ax+b)^{m-2}\,a^2$$

$$y_3 = m(m-1)(m-2)(ax+b)^{m-3}\,a^3$$

Similarly

$$y_n = m(m-1)(m-2)\ldots\ldots(m-n+1)(ax+b)^{m-n}\,a^n$$

$$\ldots(*)$$

Case (i) :- If $m = n$ in (*)

$$D^n[(ax+b)^n] = n(n-1)(n-2)\ldots\ldots 3\ 2\ 1\ .\ a^n$$

$$= n!\ a^n$$

$$D^n[x^n] = n!$$

Case (ii) :- If $m > n$ in (*)

$$D^n[(ax+b)^m] = \frac{m(m-1)\ldots..(m-n+1)(m-n)(m-n-1)\ldots.321}{(m-n)(m-n-1)\ldots..3.2.1}(ax+b)^{m-n}a^n$$

$$D^n[(ax+b)^m] = \frac{m!}{(m-n)!}(ax+b)^{m-n}a^n$$

$$D^n[x^m] = \frac{m!}{(m-n)!}x^{m-n}a^n$$

Case iii :- If $m < n$ in (*)

$$D^n[(ax+b)^m] = 0$$

Case iv :- If $m = -1$ in (*)

$$D^n\left[\frac{1}{ax+b}\right] = (-1)(-2)\ldots\ldots(-n)(ax+b)^{-1-n}a^n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$$

$$D^n\left[\frac{1}{(ax+b)^p}\right] = \frac{(-1)^n p(p+1)\ldots\ldots(p+n-1)a^n}{(ax+b)^{p+n}}$$

$$D^n\left[\frac{1}{(ax+b)^p}\right] = (-1)^n \frac{(p+n-1)!}{(p-1)!}\frac{a^n}{(ax+b)^{p+n}}$$

If a=1, $$D^n\left[\frac{1}{x^p}\right] = (-1)^n \frac{(p+n-1)!}{(p-1)!}\frac{1}{x^{p+n}}$$

4. $y = Cos(ax + b)$

$y_1 = -Sin(ax + b) \cdot a = a\, Cos(ax + b + \pi/2)$

$y_2 = -Sin(ax + b + \pi/2) \cdot a^2 = a^2\, Cos(ax + b + 2\pi/2)$

$\cdot$

$\cdot$

$y_n = D^n[Cos(ax + b)] = a^n\, Cos(ax + b + n\pi/2)$

If $a = 1,\ b = 0$

$D^n[Cos\, x] = Cos(x + n\pi/2)$

5. $y = Sin(ax + b)$

$y_n = D^n[Sin(ax + b)] = a^n\, Sin(ax + b + n\pi/2)$

If $a = 1,\ b = 0$

$D^n[Sin\, x] = Sin(x + n\pi/2)$

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6. $y = e^{ax} Sin(bx + c)$

$y_1 = a\, e^{ax} Sin(bx + c) + b\, e^{ax} Cos(bx + c)$

$= e^{ax}[a\, Sin(bx + c) + b\, Cos(bx + c)]$

Put $a = r\cos\theta \quad b = r\, Sin\,\theta \quad then \quad r = \sqrt{a^2 + b^2},\ \theta = \tan^{-1}\dfrac{b}{a}$

$y_1 = e^{ax}[r\cos\theta\, Sin(bx + c) + r\, Sin\,\theta\, Cos(bx + c)]$

$y_1 = r\, e^{ax}[Sin(bx + c + \theta)]$

$y_2 = r[e^{ax} a\, Sin(bx + c + \theta) + e^{ax} b\, Cos(bx + c + \theta)]$

Put $a = r\cos\theta \quad b = r\, Sin\,\theta \quad then \quad r = \sqrt{a^2 + b^2},\ \theta = \tan^{-1}\dfrac{b}{a}$

$y_2 = r\, e^{ax}[r\cos\theta\, Sin(bx + c + \theta) + r\, Sin\,\theta\, Cos(bx + c + \theta)]$

$y_2 = r^2\, e^{ax}[Sin(bx + c + 2\theta)]$

Similarly,

$$y_n = r^n e^{ax} [Sin(bx + c + n\theta)]$$

$$y_n = D^n [e^{ax} Sin(bx + c)] = (a^2 + b^2)^{n/2} e^{ax} [Sin(bx + c + n_{tan^{-1}\frac{b}{a}})]$$

For $a = b = 1, c = 0$

$$D^n [e^x Sin\, x] = (2)^{n/2} e^x [Sin(x + n\pi/4)]$$

7. $y = e^{ax} [Cos(bx + c)]$

$$y_n = D^n [e^{ax} Cos(bx + c)] = (a^2 + b^2)^{n/2} e^{ax} [Cos(bx + c + n_{tan^{-1}\frac{b}{a}})]$$

For $a = b = 1, c = 0$

$$D^n [e^x Sin\, x] = (2)^{n/2} e^x [Sin(x + n\pi/4)]$$

8.
$$y = a^{mx}$$
$$y_1 = a^{mx} (log\, a^m) = a^{mx} (mlog\, a)$$
$$y_2 = a^{mx} (mlog\, a)^2$$

Differentiating Successively

$$y_n = a^{mx} (mlog\, a)^n$$

For $m = 1$, $D^n [a^x] = a^x (log\, a)^n$

Leibnitz's Theorem :

It provides a useful formula for computing the nth derivative of a product of two functions.

Statement : If u and v are any two functions of x with u_n and v_n as their n^{th} derivative. Then the nth derivative of uv is

$$(uv)_n = u_0 v_n + {}^nC_1\ u_1 v_{n-1} + {}^nC_2 u_2 v_{n-2} + \ldots + {}^nC_{n-1}u_{n-1}v_1 + u_n v_0$$

Note : We can interchange u & v $(uv)_n = (vu)_n$,

$${}^nC_1 = n\ ,\ \ {}^nC_2 = n(n\text{-}1)\ /2!\ ,\ {}^nC_3 = n(n\text{-}1)(n\text{-}2)\ /3!\ \ldots$$

1. **Find the nth derivations of** $e^{ax}\cos(bx + c)$

    Solution: y_1 = e^{ax} – b sin (bx +c) + a e^{ax} cos (b x + c), by product rule.

    .i.e, y_1 = $e^{ax}\left[a\cos(bx + c) - b\sin(bx + c)\right]$

    Let us put a = r cos θ , and b = r sin θ .

    $\therefore\ \ a^2 + b^2 = r^2$ and tan $\theta = b/a$

    .ie., $r = \sqrt{a^2 + b^2}$ and θ = tan^{-1} (b/a)

    Now, $y_1 = e^{ax}\left[r\cos\theta\cos(bx + c) - r\sin\theta\sin(bx + c)\right]$

    Ie., y_1 = r e^{ax} cos $(\theta + bx + c)$

    where we have used the formula cos A cos B – sin A sin B = cos (A + B)

    Differentiating again and simplifying as before,

    y_2 = r^2 e^{ax} cos $(2\theta + bx + c)$.

    Similarly y_3 = r^3 e ^{ax} cos $(3\theta + bx + c)$.

    ………………………………………

    Thus $y_n = r^n e^{ax}\cos(n\theta + bx + c)$

    Where $r = \sqrt{a^2 + b^2}$ and θ = tan^{-1} (b/a).

    Thus **D^n [e^{ax} cos (b x + c)]**

    $$=\ \left[(\sqrt{a^2 + b^2})^n\, e^{ax}\cos\left[n\tan^{-1}(b/a) + bx + c\right]\right]$$

2. **Find the nth derivative of log** $\sqrt{4x^2 + 8x + 3}$

   Solution : Let y = log $\sqrt{4x^2 + 8x + 3}$ = log $(4x^2 + 8x + 3)^{½}$

   ie., $y = \dfrac{1}{2}$ log $(4x^2 + 8x + 3)$ $\because$ log $x^n = n$ log x

   $y = \dfrac{1}{2}$ log { (2x + 3) (2x+1)}, by factorization.

   $\because \; y = \dfrac{1}{2}$ {log (2x + 3) + log (2x + 1)}

   Now $y_n = \dfrac{1}{2}\left\{\dfrac{(-1)^{n-1}(n-1)!2^n}{(2x+3)^n} + \dfrac{(-1)^{n-1}(n-1)!2^n}{(2x+1)^n}\right\}$

   Ie., $y_n = 2^{n-1}(-1)^{n-1}(n-1)!\left\{\dfrac{1}{(2x+3)^n} + \dfrac{1}{(2x+1)^n}\right\}$

3. **Find the nth derivative of** log $_{10}$ {(1-2x)^3 (8x+1)^5}

   Solution : Let y = log $_{10}$ {1-2x)^3 (8x+1)^5}

   It is important to note that we have to convert the logarithm to the base e by the property:

   $$\log_{10} x = \dfrac{\log_e x}{\log_e 10}$$

   Thus $y = \dfrac{1}{\log_e 10}\log_e\left\{(1-2x)^3(8x+1)^5\right\}$

   Ie., $y = \dfrac{1}{\log_e 10}\left\{3\log(1-2x) + 5\log(8x+1)\right\}$

   $\therefore y_n = \dfrac{1}{\log_e 10}\left\{3.\dfrac{(-1)^{n-1}(n-1)!(-2)^n}{(1-2x)^n} + 5\dfrac{(-1)^{n-1}(n-1)!8^n}{(8x+1)^n}\right\}$

   Ie., $y_n = \dfrac{(-1)^{n-1}(n-1)!2^n}{\log_e 10}\left\{\dfrac{3(-1)^n}{(1-2x)^n} + \dfrac{5(4)^n}{(8x+1)^n}\right\}$

4. **Find the nth derivative of $e^{2x} \cos^2 x \sin x$**

    Solution : $\gg$ let $y = e^{2x} \cos^2 x \sin x = e^{2x} \left[\dfrac{1+\cos 2x}{2} \right] \sin x$

      ie., $y = \dfrac{e^{2x}}{2} (\sin x + \sin x \cos 2x)$

           $= \dfrac{e^{2x}}{2} \left\{ \sin x + \dfrac{1}{2} [\sin 3x + \sin(-x)] \right\}$

           $= \dfrac{e^{2x}}{4} (2\sin x + \sin 3x - \sin x) \quad \because \sin(-x) = -\sin x$

    $\therefore y = \dfrac{e^{2x}}{4} (\sin x + \sin 3x)$

    Now $y_n = \dfrac{1}{4} \left\{ D^n (e^{2x} \sin x) + D^n (e^{2x} \sin 3x) \right\}$

    Thus $y_n = \dfrac{1}{4} \left\{ (\sqrt{5})^n e^{2x} \sin \left[n \tan^{-1}(1/2) + x \right] + (\sqrt{13})^n e^{2x} \sin \left[n \tan^{-1}(3/2) + 3x \right] \right\}$

    $\therefore y_n = \dfrac{e^{2x}}{4} \left\{ (\sqrt{5})^n \sin \left[n \tan^{-1}(1/2) + x \right] + (\sqrt{13})^n \sin \left[n \tan^{-1}(3/2) + 3x \right] \right\}$

5. **Find the nth derivative of $e^{2x} \cos^3 x$**

    Solution : Let $y = e^{2x} \cos^3 x = e^{2x} \cdot \dfrac{1}{4} (3\cos x + \cos 3x)$

    Ie., $y = \dfrac{1}{4} (3 e^{2x} \cos x + e^{2x} \cos 3x)$

    $\therefore y_n = \dfrac{1}{4} \left\{ 3D^n (e^{2x} \cos x) + D^n (e^{2x} \cos 3x) \right\}$

    $y_n = \dfrac{1}{4} \left\{ (3\sqrt{5})^n e^{2x} \cos \left[n \tan^{-1}(1/2) + x \right] + (\sqrt{13})^n e^{2x} \cos \left[n \tan^{-1}(3/2) + 3x \right] \right\}$

    Thus $y_n = \dfrac{e^{2x}}{4} \left\{ 3(\sqrt{5})^n \cos \left[n \tan^{-1}(1/2) + x \right] + (\sqrt{13})^n \cos \left[n \tan^{-1}(3/2) + 3x \right] \right\}$

6. **Find the nth derivative of** $\dfrac{x^2}{(2x+1)(2x+3)}$

Solution : $y = \dfrac{x^2}{(2x+1)(2x+3)}$ is an improper fraction because; the degree of the

numerator being 2 is equal to the degree of the denominator. Hence we must divide and
rewrite the fraction.

$y = \dfrac{x^2}{4x^2+8x+3} = \dfrac{1}{4} \cdot \dfrac{4x^2}{4x^2+8x+3}$ for convenience.

$$4x^2+8x+3 \,\overline{\left)\begin{array}{l} 1 \\ 4x^2 \\ \underline{4x^2+8x+3} \\ -8x-3 \end{array}\right.}$$

$\therefore \quad y = \dfrac{1}{4}\left[1 + \dfrac{-8x-3}{4x^2+8x+3}\right]$

Ie., $y = \dfrac{1}{4} - \dfrac{1}{4}\left[\dfrac{8x+3}{4x^2+8x+3}\right]$

The algebraic fraction involved is a proper fraction.

Now $\quad y_n = 0 - \dfrac{1}{4}D^n\left[\dfrac{8x+3}{4x^2+8x+3}\right]$.

Let $\dfrac{8x+3}{(2x+1)(2x+3)} = \dfrac{A}{2x+1} + \dfrac{B}{2x+3}$

Multiplying by (2x + 1) (2x + 3) we have, 8x + 3 = A (2x + 3) + B (2x + 1)

................(1)

By setting 2x + 1 = 0, 2x + 3 = 0 we get x = -1/2, x = -3/2.

Put x = -1/2 in (1): -1 -1 + A (2) $\Rightarrow$ A = -1/2

Put x = -3/2 in (1): -9 = B (-2) $\Rightarrow$ B = 9/2

$\therefore \quad y_n = -\dfrac{1}{4}\left\{-\dfrac{1}{2}D^n\left[\dfrac{1}{2x+1}\right] + \dfrac{9}{2}D^n\left[\dfrac{1}{2x+3}\right]\right\}$

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$$= -\frac{1}{8}\left\{(-1)\cdot\frac{(-1)^n n!2^n}{(2x+1)^{n+1}} + 9\cdot\frac{(-1)^n n!2^n}{(2x+3)^{n+1}}\right\}$$

ie., $y_n = \frac{(-1)^{n+1} n!2^n}{8}\left\{\frac{1}{(2x+1)^{n+1}} + \frac{9}{(2x+3)^{n+1}}\right\}$

7. **Find the nth derivative of** $\dfrac{x^4}{(x+1)(x+2)}$

Solution : $y = \dfrac{x^4}{(x+1)(x+2)}$ is an improper fraction.

(deg of nr. = 4 > deg. of dr. = 2)

On dividing x^4 by $x^2 + 3x + 2$, We get

$y = (x^2 - 3x + 7) + \left[\dfrac{-15x - 14}{x^2 + 3x + 2}\right]$

$\therefore \quad y_n = D^n (x^2-3x+7) - D^n\left[\dfrac{15x-14}{x^2+3x+2}\right]$

But $D = (x^2 - 3x + 7) = 2x - 3$, $D^2 (x^2 - 3x + 7) = 2$

$D^3(x^2 - 3x + 7) = 0$......... $D^n (x^2 - 3x + 7) = 0$ if $n > 2$

Hence $y_n = -D^n\left[\dfrac{15x+14}{(x+1)(x+2)}\right]$

Now, let $D^n\dfrac{15x+14}{x^2+3x+2} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+2)}$

=> 15x+ 14 = A(x+2) + B(x+ 1)

Put x = - 1 ; - 1 = A (1)   or   A = - 1

Put x = - 2 ; - 16 = B (- 1) or   B = 16

$$Y_n = \left\{-D^n\left[\frac{1}{x+1}\right] + 16D^n\left[\frac{1}{x+2}\right]\right\}$$

$$= \frac{(-1)^n n! \, 1^n}{(x+1)^{n+1}} - 16\frac{(-1)^n n! \, 1^n}{(x+2)^{n+1}}$$

$$y_n = (-1)^n \, n! \left\{ \frac{1}{(x+1)^{n+1}} - \frac{16}{(x+2)^{n+1}} \right\} n > 2$$

8. **Show that**

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}\left\{ \log x - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{n} \right\}$$

Solution :   Let y = $\frac{\log x}{x} = \log x.\frac{1}{x}$ and let u = log x, v = $\frac{1}{x}$

We have Leibnitz theorem,

$$(uv)_n = uv_n + n_{C_1} u_1 v_{n-1} + n_{C_2} u_2 v_{n-2} + ... + u_n v \qquad \qquad ...... (1)$$

Now, u = log x $\therefore u_n = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$

$v = \dfrac{1}{x}$ $\therefore v_n = \dfrac{(-1)^n \, n!}{x^{n+1}}$

Using these in (1) by taking appropriate values for n we get,

$$D_n = \left(\frac{\log x}{x}\right) = \log x..\frac{(-1)^n n!}{x^{n+1}} + n\frac{1}{x}.\frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$+ \frac{n(n-1)}{1. \; 2}\left(-\frac{1}{x^2}\right)\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}$$

$$+ ...... + \frac{(-1)^{n-1}(n-1)!}{x^n}.\frac{1}{x}$$

Ie.. = log x $\dfrac{(-1)^n n!}{x^{n+1}} + \dfrac{(-1)^{n-1} n!}{x^{n+1}}$

$$- \frac{(-1)^{n-2} n!}{2x^{n+1}} + ... + \frac{(-1)^{n-1}(n-1)!}{x^{n+1}}$$

$$- \frac{(-1)^{n-2} n!}{x^{n+1}}\left[\log x(-1)^{-1} - \frac{(-1)^{-2}}{2} + ... + \frac{(-1)^{-1}(n-1)!}{n^1} \right]$$

Note : (-1)$^{-1}$ = $\dfrac{1}{-1} = -1; (-1)^{-2} = \dfrac{1}{(-1)^2} = 1$

Also $\dfrac{(n-1)!}{n!} = \dfrac{(n-1)!}{n(n-1)!} = \dfrac{1}{n}$

$$\therefore \frac{d^n}{dx^n}\left[\frac{\log x}{x}\right] = \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3}... - \frac{1}{n}\right]$$

9. If $y_n = D^n (x^n \log x)$

Prove that $y_n = n\, y_{n-1} + (n-1)!$ and hence deduce that

$$y_n = n\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right)$$

Solution : $y_n = D^n(x^n \log x) = D^{n-1}\{D\,(x^n \log x)\}$

$= D^{n-1}\left\{x^n \cdot \dfrac{1}{x} + nx^{n-1}\log x\right\}$

$= D^{n-1}(x^{n-1}) + nD^{n-1}(x^{n-1}\log x\}$

$\therefore y_n = (n-1)! + ny_{n-1}$. This proves the first part.

Now Putting the values for n = 1, 2, 3...we get

$y_1 = 0! + 1\ \ y_0 = 1 + \log x = 1!\,(\log x + 1)$

$y_2 = 1! + 2y_1 = 1 + 2\,(1 + \log x)$

ie., $y_2 = 2\log x + 3 = 2(\log x + 3/2) = 2!\left(\log x + 1 + \dfrac{1}{2}\right)$

$y_3 = 2! + 3y_2 = 2 + 3(2 \log x + 3)$

ie., $y_3 = 6\log x + 11 = 6\,(\log x + 11/6) = 3!\left(\log x + 1 + \dfrac{1}{2} + \dfrac{1}{3}\right)$

………………………………………………………………………..

$$y_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right)$$

10. If $y = a \cos (\log x) + b \sin (\log x)$, show that

$x^2 y_2 + x y_1 + y = 0$. Then apply Leibnitz theorem to differentiate this result n times.

or

If $y = a \cos (\log x) + b \sin (\log x)$, show that

$x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1)y_n = 0$.   [July-03]

Solution : $y = a \cos (\log x) + b \sin (\log x)$

Differentiate w.r.t x

$\therefore$ $y_1 = -a \sin (\log x) \dfrac{1}{x} + b \cos (\log x). \dfrac{1}{x}$

(we avoid quotient rule to find y_2) .

$\Rightarrow xy_1 = - a \sin (\log x) + b \cos (\log x)$

Differentiating again w.r.t x we have,

$xy_2 + 1 \ y_1 = - a \cos (\log x) + b \sin (\log x) \dfrac{1}{x}$

or $\quad x^2 y_2 + xy_1 = - [a \cos (\log x) + b \sin (\log x)] = -y$

$\therefore$ $x^2 y_2 + xy_1 + y = 0$

Now we have to differentiate this result n times.

ie., $D^n (x^2 y_2) + D^n (xy_1) + D^n (y) = 0$

We have to employ Leibnitz theoreom for the first two terms.

Hence we have,

$$\left\{ x^2 . D^n (y_2) + n. \ 2x . D^{n-1} (y_2) + \frac{n(n-1)}{1 . 2} . \ 2 . D^{n-2} (y^{2)}) \right\}$$

$$\left\{ x. D^n (y_1) + n. \ 1 . \ D^{n-1} (y_1) \right\} + y_n = 0$$

ie., $\{ x^2 y_{n+2} + 2n x y_{n+1} + n (n-1) y_n \} + \{ xy_{n+1} + ny_n \} + y_n = 0$

ie., $x^2 y_{n+2} + 2n x y_{n+1} + n^2 y_n - ny_n + xy_{n+1} + ny_n + y_n = 0$

ie., $x^2 y_{n+2} + (2n+l) xy_{n+l} + (n^2 + l) y_n = 0$

11. **If $\cos^{-1} (y/b) = \log (x/n)^n$, then show that**

$x^2 y_{n+2} + (2n+l) xy_{n+l} + 2n^2 y_n = 0$

Solution : By data, $\cos^{-1} (y/b) = n \log (x/n)$ $\quad \therefore \log(a^m) = m \log a$

$\Rightarrow \dfrac{y}{b} = \cos [n \log (x/n)]$

or $\quad y = b \ . \ \cos [n \log (x/n)]$

Differentiating w.r.t x we get,

$y_1 = -b \sin [n \log (x/n)] \cdot n \cdot \dfrac{1}{(x/n)} \cdot \dfrac{1}{n}$

or $xy_1 = - n\, b \sin [n \log (x/n)]$

Differentiating w.r.t x again we get,

$xy_2 + 1.\ y_1 = - n \cdot b \cos [n \log (x/n)]\ n\, \dfrac{1}{(x/n)} \cdot \dfrac{1}{n}$

or $x (xy_2+y_1) = n^2 b \cos [n \log (x/n)] = -n^2 y$, by using (1).

or $x^2 y_2 + xy_1 + n^2 y = 0$

Differentiating each term n times we have,

$$D(x^2 y_2) + D^n(xy_1) + n^2 D^n (y) = 0$$

Applying Leibnitz theorem to the product terms we have,

$$\left\{ x^2 y_{n+2} + n.\, 2x.\, y_{n+1} + \frac{n(n-1)}{1.\, 2}.\, 2 .\, y_n \right\}$$
$$+ \left\{ xy_{n+1} + n.\, 1 .y_n \right\} + n^2 y_n = 0$$

ie $x^2 y_{n+2} + 2 x\, y_{n+1} + n^2 y_n + xy_{n+1} + ny_n + n^2 y_n = 0$

or $x^2\, y_{n+2} + (2n + l)\, xy_{n+1} + 2n^2 y_n = 0$

12. If y = sin(log (x² + 2 x + 1)),

or   [Feb-03]

If $\sin^{-1} y = 2 \log (x + 1)$,   show that

$(x+l)^2 y_{n+2} + (2n+1)(x+1)y_{n+l} + (n^2 + 4)y_n = 0$

Solution :   By data $y = \sin \log (x^2 + 2 x + 1)$

$\therefore\ y_1 = \cos \log (x^2 + 2 x + 1)\, \dfrac{1}{(x+1)^2}\, 2x + 2$

ie., $y_1 = \cos \log (x^2 + 2 x + 1)\, \dfrac{1}{x^2 + 2x + 1}\, 2 (x + 1)$

ie., $y_1 = \dfrac{2 \cos \log(x^2 + 2 x + 1)}{(x+1)}$

or $(x + 1) y_1 = 2 \cos \log (x^2 + 2 x + 1)$

Differentiating w.r.t x again we get

$(x+1)y_2 + 1$ $y_1 = -2 \sin \log (x^2 + 2x + 1) \dfrac{1}{(x+1)^2} . 2(x+1)$

or $(x + 1)^2 y_2 + (x+1) y_1 = -4y$

or $(x+l)^2 y_2 + (x+l) y_1 + 4y = 0$,

Differentiating each term n times we have,

$D^n [(x + 1)^2 y_2] + D^n [(x+ 1)y_1] + D^n [y] = 0$

Applying Leibnitz theorem to the product terms we have,

$$\left\{ (x+1)^2 \, y_{n+2} + n. \; 2(x+1). \;\; y_{n+1} + \frac{n(n-1)}{1. \; 2}.2 \; . y_n \right\}$$

$$+ \{(x+l) \, y_{n+1} + n. \;\; 1 \;\; . y_n\} + 4y_n = 0$$

ie., $(x+l)^2 y_{n+2} + 2n (x+1)y_{n+1}$

$\qquad\qquad + n^2 y_n - n y_n + (x+l)y_{n+l} + n y_n + 4 y_n = 0$

ie., $(x+l)^2 y_{n+2} + (2n + l) (x + l) \, y_{n+1} + (n^2 + 4) y_n = 0$

13. **If = log** $\left(x + \sqrt{1 + x^2} \right)$ **prove that**

$(1 + x^2) \, y_{n+2} + (2n + 1) \, xy_{n+1} + n^2 y_n = 0$

$\gg$ By data, $y = \log \left(x + \sqrt{1 + x^2} \right)$

$$\therefore y_1 = \frac{1}{(x + \sqrt{1+x^2})} \left\{ 1 + \frac{1}{2\sqrt{1+x^2}} . \; 2x \right\}$$

Ie., $y_1 \dfrac{1}{(x + \sqrt{1+x^2})} \dfrac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}} = \dfrac{1}{\sqrt{1+x^2}}$

or $\sqrt{1 + x^2} \, y_1 = 1$

Differentiating w.r.t.x again we get

$$\sqrt{1 + x^2} y_2 + \frac{1}{2\sqrt{1+x^2)}} . 2x.y_1 = 0$$

or $(1+x^2)y_2 + xy_1 = 0$

Now $D^n [(l+x^2)y_2] + D^n[xy_1] = 0$

Applying Leibnitz theorem to each term we get,

$$\left\{ (1+x^2)\,y_{n+2} + n.\,2x\,.y_{n+1} + \frac{n(n-1)}{1.2}.2\,.y_n \right\}$$

$$+\,[x\,.\,y_{n+1} + n\,.1\,y_n] = 0$$

Ie., $(1 + x^2)\,y_{n+2} + 2n\,x\,y_{n+1} + n^2 y_n - n y_n + x y_{n+l} + n y_n = 0$

or $(l+x^2)y_{n+2} + (2n + l)xy_{n+1} + n^2 y_n = 0$

14. If x = sin t and y = cos mt, prove that

$(l-x^2)y_{n+2} - (2n+1)xy_{n+l} + (m^2-n^2)y_n = 0.$ [Feb-04]

Solution : By data $x = \sin t$ and $y = \cos mt$

$x = \sin t \Rightarrow t = \sin^{-1} x$ and $y = \cos mt$ becomes

$y = \cos [\,m \sin^{-1}x)$

Differentiating w.r.t.x we get

$y_1 = -\sin (m \sin^{-1}x)\,\dfrac{m}{\sqrt{1-x^2}}$

or $\sqrt{1+x^2}\,y_1 = -m \sin (m \sin^{-1}x)$

Differentiating again w.r.f .x we get,

$$\sqrt{1-x^2}\,y_2 + \frac{1}{2\sqrt{1-x^2}}(-2x)\,y_1 = -m\cos(m\sin^{-1}x).\frac{m}{\sqrt{1-x^2}}$$

or $(1 - x^2)y_2 - xy_l = -m^2 y$

or $(1 - x^2)y_2 - xy_1 + m^2 y = 0$

Thus $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

15. If x = tan (log y), find the value of

$(l+x^2)y_{n+1} + (2nx-l)\,y_n + n(n-1)y_{n-1}$ [July-04]

Solution : By data $x = \tan(\log y) \Rightarrow \tan^{-1} x = \log y$ or $y = e^{\tan^{-1} x}$ Since the desired relation involves

$y_{n+1},\ y_n$ and y_{n-1} we can find y_1 and differentiate n times the result associated with y_1 and y.

Consider $y = e^{\tan^{-1} x} \cdot \quad \therefore y. = e^{\tan^{-1} x} \cdot \dfrac{1}{1+x^2}$

or $(1 + x^2)y_1 = y$

Differentiating n times we have

$D^n[(1+x^2)y_1]=D^n[y]$

Anplying Leibnitz theorem onto L.H.S, we have,

$\{(1+x^2)D^n(y_1) + n .2x .D^{n-1} (y_1)$

$$+ \frac{n(n-1)}{1.2}.2 \ .D^{n-2}(y_1)\} = y_n$$

Ie., $(1+x^2)y_{n+1}+2n x y_n + n (n-1) y_{n-1}-y_n=0$

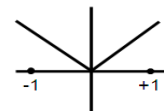Or $(1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0$

Continuity & Differentiability

Some Fundamental Definitions

A function $f(x)$ is defined in the interval I, then it is said to be continuous at a point $x = a$ if $\lim\limits_{x \to a} f(x) = f(a)$

A function $f(x)$ is said to be differentiable at $x = a$ if $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(a)}{h} = f'(a)$ $\;exists$ $\qquad a \in I$

Ex : Consider a function $f(x)$ is defined in the interval [-1,1] by $f(x) =$ $\;|x| = \begin{cases} -x & -1 \le x \le 0 \\ x & 0 \le x \le 1 \end{cases}$
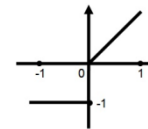

It is continuous at $x = 0$
But not differentiable at $x = 0$

Note : If a function $f(x)$ is differentiable then it is continuous, but converse need not be true. Geometrically :

(1) If $f(x)$ is Continuous at $x = a$ means, $f(x)$ has no breaks or jumps at the point $x = a$

Ex : $\qquad f(x) = \begin{cases} -1 & -1 \le x \le 0 \\ x & 0 < x \le 1 \end{cases}$

Is discontinuous at x=0

(2) If $f(x)$ is differentiable at $x = a$ means, the graph of $f(x)$ has a unique tangent at the point or graph is smooth at $x = a$

1. **Give the definitions of Continuity & Differentiability:**

Solution: A function f (x) is said to be continuous at x = a, if corresponding to an arbitrary positive number ε, however small, their exists another positive number δ such that.

$\big| f(x) - f(a) \big| < \varepsilon$, where $\big| x - a \big| < \delta$

It is clear from the above definition that a function f (x) is continuous at a point 'a'.

If (i) it exists at x = a

(ii) $\underset{x \to a}{Lt}\ f(x) = f(a)$

i.e, limiting value of the function at x = a is to the value of the function at x = a

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Differentiability:

A function f (x) is said to be differentiable in the interval (a, b), if it is differentiable at every point in the interval.

In Case [a,b] the function should posses derivatives at every point and at the end points a & b i.e., Rf^1 (a) and Lf^1 (a) exists.

2. State Rolle's Theorem with Geometric Interpretation.

Statement: Let f (x) be a function is defined on [a,b] & it satisfies the following Conditions.

(i)   f (x) is continuous in [a,b]

(ii)   f (x) is differentiable in (a,b)

(iii)   f (a) = f (b)

Then there exists at least a point $C \in$ (a,b), Here a < b such that f^1 (c) = 0

Proof:

Geometrical Interpretation of Rolle's Theorem:
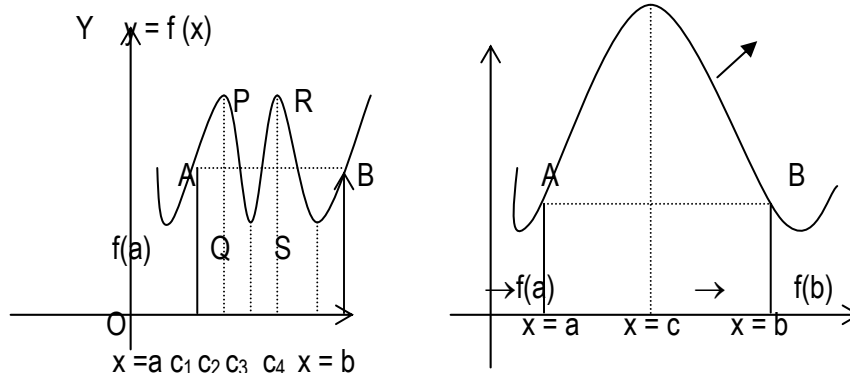
Let us consider the graph of the function y = f (x) in xy – plane. A (a,.f(a)) and B (*b, f(b)*) be the two points in the curve f (x) and a, b are the corresponding end points of A & B respectively. Now, explained the conditions of Rolle's theorem as follows.

(i)   f (x) is continuous function in [a,b], Because from figure without breaks or jumps in between A & B on y = f (x).

(ii)   f (x) is a differentiable in (a,b), that means let us joining the points A & B, we get a line AB.

∴ Slope of the line AB = 0 then ∃ a point C at P and also the tangent at P (or Q or R or S) is Parallel to x –axis.

∴ Slope of the tangent at P (or Q or R or S) to be Zero even the curve y = f (x) decreases or increases, i.e., f (x) is Constant.

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$$f^1 (x) = 0$$

$$\therefore f^1 (c) = 0$$

(iii) The Slope of the line AB is equal to Zero, i.e., the line AB is parallel to x – axis.

$$\therefore f (a) = f (b)$$

3. **Verify Rolle's Theorem for the function f (x) = x² – 4x + 8 in the internal [1,3]**

Solution: We know that every Poly nominal is continuous and differentiable for all points and hence f (x) is continuous and differentiable in the internal [1,3].

Also f (1) = 1 – 4 + 8 = 5, f (3) = 3² – 43 + 8 = 5

Hence f (1) = f (3)

Thus f (x) satisfies all the conditions of the Rolle's Theorem. Now f¹ (x) = 2x – 4 and f¹ (x) = 0
$\Rightarrow$ 2x – 4 = 0 or x = 2. Clearly 1 < 2 < 3. Hence there exists 2t (1,3) such that f¹ (2) = 0. This shows that Rolle's Theorem holds good for the given function f (x) in the given interval.

4. **Verify Rolle's Theorem for the function f (x) = x (x + 3)** $e^{-x/2}$ **in the interval [-3, 0]**

Solution: Differentiating the given function W.r.t 'x' we get

$$f^{1}(x) = (x^2 + 3x)\left(-\frac{1}{2}\right)e^{-x/2} + (2x+3)e^{-x/2}$$

$$= -\frac{1}{2}(x^2 - x - 6)e^{-x/2}$$

$\therefore$ f¹(x) exists (i.e finite) for all x and hence continuous for all x.

Also f (-3) = 0, f (0) = 0 so that f (-3) = f (0) so that f (-3) = f (0). Thus f (x) satisfies all the conditions of the Rolle's Theorem.

Now, f¹ (x) = 0

$$\Rightarrow -\frac{1}{2}(x^2 - x - 6)e^{-x/2} = 0$$

Solving this equation we get x = 3 or x = -2

Clearly –3 < -2 < 0. Hence there exists –2$\in$ (-3,0) such that f¹ (-2) = 0

This proves that Rolle's Theorem is true for the given function.

5.   **Verify the Rolle's Theorem for the function Sin x in [-π, π]**

 Solution: Let f (x) = Sin x

Clearly Sinx is continuous for all x.

Also f^1 (x) = Cos x exists for all x in (-π, π) and f (-π) = Sin (-π) = 0; f (π) = Sin (π) = 0 so that f (-π) = f (π)

Thus f (x) satisfies all the conditions of the Rolle's Theorem .

Now f^1 (x) = 0 $\Rightarrow$ Cos x = 0 so that

$$X = \pm \frac{\pi}{2}$$

Both these values lie in (-π,π). These exists C = $\pm \dfrac{\pi}{2}$

Such that f^1 (c) = 0

Hence Rolle's theorem is vertified.

6.   **Discuss the applicability of Rolle's Theorem for the function f (x) = lxl in [-1,1].**

Solution: Now f (x) = $\left| x \right|$ = $\begin{cases} x \text{ for } 0 \le x \le 1 \\ -x \text{ for } -1 \le x \le 0 \end{cases}$

f (x) being a linear function is continuous for all x in [-1, 1]. f(x) is differentiable for all x in (-1,1) except at x = 0.   Therefore Rolle's Theorem does not hold good for the function f (x) in [-1,1]. Graph of this function is shown in figure. From which we observe that we cannot draw a tangent to the curve at any point in (-1,1) parallel to the x – axis.
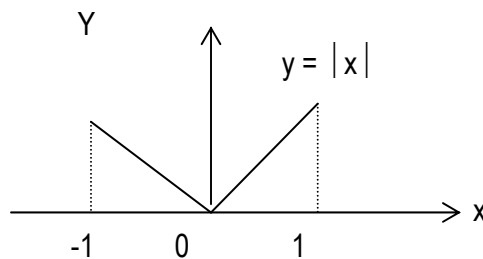
Exercise:

7. **Verify Rolle's Theorem for the following functions in the given intervals.**

   a) $x^2 - 6x + 8$ in [2,4]

   b) $(x - a)^3 (x - b)^3$ in [a,b]

   c) $\log \left\{ \dfrac{x^2 + ab}{(a+b)x} \right\}$ in [a,b]

8. **Find whether Rolle's Theorem is applicable to the following functions. Justify your answer.**

   a) $f(x) = |x - 1|$ in [0,2]

   b) $f(x) = \tan x$ in $[0, \pi]$.

9. **State & prove Lagrange's (1st) Mean Value Theorem with Geometric meaning.**

Statement: Let f (x) be a function of x such that
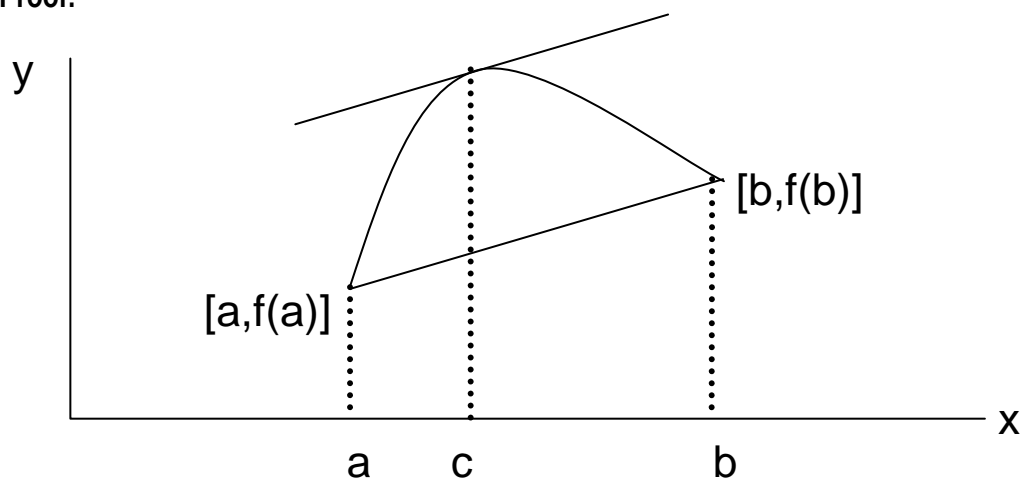
   (i) If is continuous in [a,b]

   (ii) If is differentiable in (a,b)

Then there exists atleast a point (or value) $C \in$ (a,b) such that.

$$f^1(c) = \frac{f(b) - f(a)}{b - a}$$

   i.e., f (b) = f (a) + (b – a) f¹ (c)

Proof:

Define a function g (x) so that g (x) = f (x) – Ax ---------- (1)

Where A is a Constant to be determined.

So that g (a) = g (b)

Now, g (a) = f (a) – Aa

G (b) = f (b) – Ab

$\therefore$ g (a) = g (b) $\Rightarrow$ f (a) – Aa = f (b) – Ab.

i.e., $A = \dfrac{f(b) - f(a)}{b - a}$ ---------------- (2)

Now, g (x) is continuous in [a,b] as rhs of (1) is continuous in [a,b]
G(x) is differentiable in (a,b) as r.h.s of (1) is differentiable in (a,b).

Further g (a) = g (b), because of the choice oif A.

Thus g (x) satisfies the conditions of the Rolle's Theorem.

$\therefore$ These exists a value x = c sothat a < c < b at which g^1 (c) = 0

$\therefore$ Differentiate (1) W.r.t 'x' we get

g^1 (x) = f ^{1} (x) – A

$\therefore$ g^1 (c) = f^1 (c)- A ($\because$ x =c)

$\Rightarrow$ f ^{1} (c) - A = 0 ($\because g^1$(c) = 0)
$\therefore$ f^1 (c) = A   -------------- (3)

From (2) and (3) we get

$f^1(c) = \dfrac{f(b) - f(a)}{b - a}$ (or) f (b) = f (a) + (b – a) f^1 (c) For a < c < b

Corollary: Put b – a = h
      i.e., b = a + h and c = a + θ h
      Where 0 < θ < 1
   Substituting in f (b) = f (a) + (b – a) f^1 (c)

$\therefore$ f (a + h) = f (a) + h f (a + θ h), where 0 < θ < 1.

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Geometrical Interpretation:-

Since y = f (x) is continuous in [a,b], it has a graph as shown in the figure below,

At x = a, y = f (a)

At $x = b, y = f(b)$

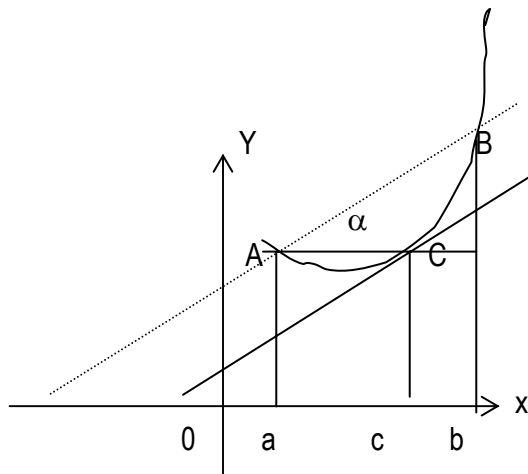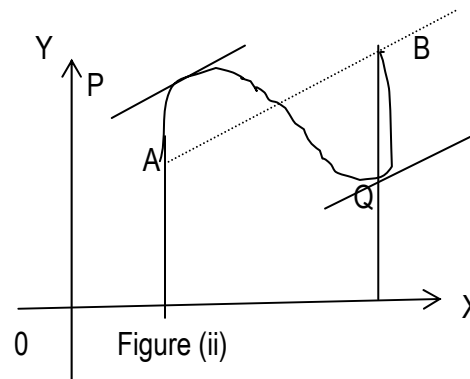Figure (i)

Figure (ii)

Slope of the line joining the points A (a,f(a)) and B ((b,f (b))

Is $\dfrac{f(b) - f(b)}{b - a}$ (∴ Slope = m = tan θ)

= tan α

Where α is the angle mode by the line AB with x – axis

= Slope of the tangent at x = c

= f1 (c), where a < c < b

Geometrically, it means that there exists at least are value of x = c, where a < c < b at which the tangent will be parallel to the line joining the end points at x = a & x = b.

Note: These can be more than are value at which the tangents are parallel to the line joining points A & B (from Fig (ii)).

10. **Verify Lagrange's Mean value theorem for f(x) = (x – 1) (x – 2) (x – 3) in [0,4].**

 Solution: Clearly given function is continuous in [0,4] and differentiable in (0,4), because f (x) is in polynomial.

$$f (x) = (x – 1) (x – 2) (x – 3)$$

$$f (x) = x^3 – 6x^2 + 11x – 6$$

and $f (0) = 0^3 – 6(0)^2 + 11 (0) – 6 = -6$

$$f(4) = 4^3 – 6 (4)^2 + 11 (4) – 6 = 6$$
Differentiate f (x) W.r.t x, we get

$$F^1 (x) = 3x^2 – 6x + 11$$

Let x = c, $f^1 (c) = 3c^2 – 6c + 11$

By Lagrange's Mean value theorem, we have

$$f^1(c) = \frac{f(b) - f(a)}{b-a} = \frac{f(4) - f(0)}{(4-0)}$$

$$= \frac{6 - (-6)}{4} = 3$$

$\therefore 3c^2 – 6c + 11 = 3$

$\Rightarrow 3c^2 – 6c + 8 = 0$

Solving this equation, we get

$$C = 2 \pm \frac{2}{\sqrt{3}} \in (0,4)$$
Hence the function is verified.

11. **Verify the Lagrange's Mean value theorem for f (x) = logx in [1,e].**

Solution: Now Logx is continuous for all x > 0 and hence [1,e].
Also $f^1(x) = \frac{1}{x}$ which exists for all x in (1,e)
Hence f (x) is differentiable in (1,e)

$\therefore$ by Lagrange's Mean Value theorem, we get

$$\frac{Loge - Log1}{e - 1} = \frac{1}{c} \Rightarrow \frac{1}{e-1} = \frac{1}{c}$$

$$\Rightarrow C = e - 1$$

$$\Rightarrow 1 < e - < 2 < e$$

Since $e \in (2,3)$

$\therefore$ So that $c = e - 1$ lies between 1 & e

Hence the Theorem.

12. Find θ for f (x) = Lx² + mx + n by Lagrange's Mean Value theorem.

Solution: f (x) = Lx² + mx + n

$\therefore f^1 (x) = 2 Lx + m$

We have $f (a + h) = f (a) + hf^1 (a + \theta h)$

Or $f (a + h) – f (a) = hf^1 (a + \theta h)$

i.e., $\{\ell (a + h)^2 + m (a + h) + n\} – \{\ell a^2 + ma + n\} = h \{2\ell (a + \theta h) + m\}$

Comparing the Co-efficient of ℓh^2, we get

$$1 = 2\theta \qquad\qquad \therefore \theta = \frac{1}{2} \in (0,1)$$

Exercise:

13. Verify the Lagrange's Mean Value theorem for $f (x) = Sin^2x$ in $\left[0, \frac{\pi}{2}\right]$

14. Prove that, $\dfrac{b-a}{1+b^2} < tan^{-1} b – tan^{-1} a < \dfrac{b-a}{1+a^2}$ if 0 < a < b and reduce that

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

15. Show that $\dfrac{2}{\pi} < \dfrac{Sinx}{x} < 1$ in $\left(0, \dfrac{\pi}{2}\right)$

16. Prove that $\dfrac{b-a}{\sqrt{1-a^2}} < Sin^{-1}b – Sin^{-1}a < \dfrac{b-a}{\sqrt{1-b^2}}$ Where a < b. Hence reduce

$$\frac{\pi}{6} - \frac{1}{2\sqrt{3}} < Sin^{-1}\frac{1}{4} < \frac{\pi}{6} - \frac{1}{\sqrt{15}} .$$

17. State & prove Cauchy's Mean Value Theorem with Geometric meaning.

Proof: The ratio of the increments of two functions called Cauchy's Theorem.
Statement: Let g (x) and f (x) be two functions of x such that,

(i) Both f (x) and g (x) are continuous in [a,b]

(ii) Both f (x) and g (x) are differentiable in (a,b)

(iii) g^1 (x) $\neq$ 0 for any x ε (a,b)

These three exists at least are value x = c ε (a,b) at which

$$\frac{f^1(c)}{g^1(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof: Define a function,

ϕ (x) = f (x) – A. g (x)   ------------------ (1)

So that ϕ (a) = ϕ (b) and A is a Constant to be determined.

Now, ϕ (a) = f (a) – A g (a)

ϕ (b) = f (b) – A. g (b)

∴ f (a) – A g (a) = f (b) – A. g (b)

$$\Rightarrow A = \frac{f(b) - f(b)}{g(b) - g(a)}$$ -------------------- (2)

Now, ϕ is continuous in [a,b] as r.h.s of (1) is continuous in [a,b] and ϕ (x) is differentiable in (a,b) as r.h.s of (1) is differentiable in [a,b].

Also ϕ (a) = ϕ (b)

Hence all the conditions of Rolle's Theorem are satisfied then there exists a value x = c $\in$ (a,b) such that ϕ1 (c) = 0.

Now, Differentiating (1) W.r.t x, we get

ϕ^1 (x) = f^1 (x) – A.g^1 (x)

at x = c $\in$ (a,b)

$$\therefore \phi^1(c) = f^1(c) - A\,g^1(c)$$

$$0 = f^1(c) - A\,g^1(c) \qquad\qquad (\because g^1(x) \neq 0)$$

$$\Rightarrow A = \frac{f^1(c)}{g^1(c)} \qquad\qquad \text{--------------- (3)}$$

Substituting (3) in (2), we get

$$\frac{f^1(c)}{g^1(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \text{ , where a < c < b}$$

Hence the proof.

18. Verify Cauchy's Mean Value Theorem for the function f (x) = x² + 3, g (x) = x³ + 1 in [1,3]

Solution: Here f (x) = x² + 3, g (x) = x³ + 1

      Both f (x) and g (x) are Polynomial in x. Hence they are continuous and differentiable for all x and in particular in [1,3]

      Now, f¹ (x) = 2x, g¹ (x) = 3x²

      Also g ¹ (x) ≠ 0 for all x ∈ (1,3)

Hence f (x) and g (x) satisfy all the conditions of the cauchy's mean value theorem. Therefore

$$\frac{f(3) - f(1)}{g(3) - g(1)} = \frac{f^1(c)}{g^1(c)} \text{ , for some c : 1 < c < 3}$$

i.e., $\dfrac{12 - 4}{28 - 2} = \dfrac{26}{3c^2}$

i.e., $\dfrac{2}{13} = \dfrac{1}{3c} \Rightarrow C = \dfrac{13}{6} = 2\dfrac{1}{6}$

Clearly C = $2\dfrac{1}{6}$ lies between 1 and 3.

Hence Cauchy's theorem holds good for the given function.

19. Verify Cauchy's Mean Value Theorem for the functions f (x) Sin x, g (x) = Cos x in $\left[0, \dfrac{\pi}{2}\right]$

Solution: Here f (x) = Sin x, g (x) = Cos x so that

f^1 (x) = Cos x ,g^1 (x) = - Sinx

Clearly both f (x) and g (x) are continuous in $\left[0, \dfrac{\pi}{2}\right]$, and differentiable in $\left(0, \pi/2\right)$

Also g^1 (x) = -Sin x $\neq$ 0 for all x $\in$ $\left(0, \pi/2\right)$

$\therefore$ From cauchy's mean value theorem we obtain

$$\frac{f\left(\dfrac{\pi}{2}\right) - f(0)}{g\left(\dfrac{\pi}{2}\right) - g(0)} = \frac{f^1(c)}{g^1(c)} \quad \text{for some C} : 0 < C < \frac{\pi}{2}$$

i.e., $\dfrac{1-0}{0-1} = \dfrac{Cosc}{-Sinc}$ i.e., -1 = - Cot c (or) Cot c = 1

$\therefore$ C = $\dfrac{\pi}{4}$, clearly C = $\dfrac{\pi}{4}$ lies between 0 and $\dfrac{\pi}{2}$

Thus Cauchy's Theorem is verified.

Exercises:

20. Find C by Cauchy's Mean Value Theorem for

a)   f (x) = e^x, g (x) = e^{-x} in [0,1]

b)   f (x) = x^2, g (x) = x in [2,3]

21. Verify Cauchy's Mean Value theorem for

a)   f (x) = tan^{-1} x, g (x) = x in $\left[\dfrac{1}{\sqrt{3}}, 1\right]$

b)   f(x) = log x, g (x) = $\dfrac{1}{x}$ in [1,e]

Generalized Mean Value Theorem:

22. State Taylor's Theorem and hence obtain Maclaurin's expansion (series)

Statement: If f (x) and its first (n – 1) derivatives are continuous in [a,b] and its nth derivative exists in (a,b) then

$$f(b) = f(a) + (b - a) f^1 (a) + \frac{(b-a)^2}{2!} f^{11} (a) + \text{---------} + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n (c)$$

Where a < c < b

Remainder in Taylor's Theorem:

We have

$$f (x) = f (a) + (x - a) f^1 (a) + \frac{(x-a)^2}{2!} f^{11} (a) + \text{---------} + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1} (a) +$$

$$\frac{(x-a)^n}{\angle n} f^n [a + (x - a) \theta\}$$

$$f (x) = S_n (x) + R_n (x)$$

Where $R_n (x) = \frac{(x-a)^n}{n!} f^n [a + (x - a) \theta]$ is called the Largrange's form of the Remainder.

Where a = 0, $R_n (x) = \frac{x^n}{n!} f^n (\theta x), 0 < \theta < 1$

Taylor's and Maclaurin's Series:

We have f (x) = $S_n (x) + R_n (x)$

$$\therefore \underset{n\to\infty}{Lim}[f(x) - S_n(x)] = \underset{n\to\infty}{Lim} R_n(x)$$

If $\underset{n\to\infty}{Lim} R_n(x) = 0 then f(x) \underset{n\to\infty}{Lim} S_n(x)$

Thus $\underset{n\to\infty}{Lim} S_n(x)$ converges and its sum is f (x).

This implies that f (x) can be expressed as an infinite series.

i.e., f (x) = f (a) + (x – a) f ^{1} (a) + $\frac{(x-a)^2}{2!}$ f ^{11} (a) + ---------- to ∞

This is called Taylor's Series.

Putting a = 0, in the above series, we get

$$F(x) = f(0) + x f^1(0) + \frac{(x)^2}{2!} f^{11}(0) + \text{--------- to } \infty$$

This is called Maclaurin's Series. This can also denoted as

$$Y = y(0) + x y_1(0) + \frac{(x)^2}{2!} y_2(0) + \text{-------- } + \frac{(x)^n}{n!} y_n(0) \text{ ---------- to } \infty$$

Where $y = f(x)$, $y_1 = f_1(x)$, -------------- $y_n = f_n(x)$

23. By using Taylor's Theorem expand the function e^x in ascending powers of (x – 1)

Solution: The Taylor's Theorem for the function f (x) is ascending powers of (x – a) is

$$f(x) = f(a) + (x - a) f^1(a) + \frac{(x-a)^2}{\angle 2} f^{11}(a) + \text{----------- (1)}$$

Here $f(x) = e^x$ and a = 1

$f^1(x) = e^x \Rightarrow f^1(a) = e$

$f^{11}(x) = e^x \Rightarrow f^{11}(a) = e$

∴ (1) becomes

$$e^x = e + (x - 1) e + \frac{(x-1)^2}{2} e + \text{-------------}$$

$$= e \left\{ 1 + (x - 1) + \frac{(x-1)^2}{2} + \text{--------} \right\}$$

24. By using Taylor's Theorem expand log sinx in ascending powers of (x – 3)

Solution: $f(x) = \text{Log Sin } x$, a = 3 and f (3) = log sin3

$$\text{Now } f^1(x) = \frac{Cosx}{Sinx} = \text{Cotx}, \quad f^1(3) = \text{Cot3}$$

$f^{11}(x) = -\text{Cosec}^2 x$, $f^{11}(3) = -\text{Cosec}^2 3$

$f^{111}(x) = -2\text{Cosecx}(-\text{Cosecx Cotx}) = 2\text{Cosec}^3 x \text{ Cotx}$

∴ $f^{111}(3) = 2\text{Cosec}^3 3 \text{ Cot3}$

$$\therefore f(x) = f(a) + (x - a) f^1(a) + \frac{(x-a)^2}{2!} f^{11}(a) + \frac{(x-a)^3}{3!} f^{111}(a) + \text{-----------}$$

$$\therefore \text{Log Sinx} = f(3) + (x - 3) f^1(3) + \frac{(x-3)^2}{2!} f^{11}(3) + \frac{(x-3)^3}{3!} f^{111}(3) + \text{-----------}$$

$$= \text{logsin3} + (x - 3) \text{Cot3} + \frac{(x-3)^2}{2!} (-\text{Cosec}^2 3) + \frac{(x-3)^3}{3!} 2\text{Cosec}^3 3\text{Cot3} + \text{---}$$

Exercise:

25.   **Expand Sinx is ascending powers of** $\left(x - \dfrac{\pi}{2} \right)$

26.   **Express tan $^{-1}$ x in powers of (x – 1) up to the term containing (x – 1) ^{3}**

27.   **Apply Taylor's Theorem to prove**

$$\mathbf{e}^{\,x+h} = \mathbf{e}^{\,x} \left[1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + - - - - - - - - - \right]$$

Problems on Maclaurin's Expansion:

28.   **Expand the log (1 + x) as a power series by using Maclaurin's theorem.**

Solution: Here f (x) = log (1 + x), Hence f (0) = log 1 = 0

We know that

$$f^{\,n}(x) = \frac{d^n}{dx^n}\{\log(1+x)\} = \frac{d^{n-1}}{dx^{n-1}}\left\{ \frac{1}{1+x} \right\}$$

$$= \frac{(-1)^{n-1}(n-1)!}{(1+n)^n} , \qquad n = 1,2,\text{---------}$$

Hence f ^{n} (0) = (-1) $^{n-1}$ (n – 1) !

f ^{1} (0) = 1, f ^{11} (0) = -1, f ^{111} (0) = 3!, f ^{1v} (0) = - 3!

Substituting these values in

$$f(x) = f(0) + x\,f^{\,1}(0) + \frac{x^2}{2!}\,f^{\,11}(0) + \text{--------} + \frac{x^n}{n!}\,f^{\,n}(0) + \text{------------}$$

$$\therefore \log(1+x) = 0 + x \cdot 1 + \frac{x^2}{2!}\,(-1) + \frac{x^3}{3!}\,2! + \frac{x^4}{4!}\,- 3! + \text{---------------}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{---------}$$

This series is called Logarithmic Series.

29.   Expand tan $^{-1}$ x by using Macluarin's Theorem up to the term containing x ^{5}

Solution: let y = tan $^{-1}$ x, Hence y (0) = 0

We find that y $_1$ = $\dfrac{1}{1+x^2}$ which gives y $_1$ (0) = 1

Further y $_1$ (1 + x ^{2}) = 1, Differentiating we get

Y $_1$. 2x + (1 + x ^{2}) y $_2$ = 0 (or) (1 + x^2) y $_2$ + 2xy $_1$ = 0
Hence y $_2$ (0) = 0

Taking n th derivative an both sides by using Leibniz's Theorem, we get

(1 + x ^{2}) y $_{n+2}$ + n . 2xy $_{n+1}$ + $\dfrac{n(n-1)}{1.2}$. 2. y $_n$ + 2xy $_{n-1}$ + n.2.y $_n$ = 0

i.e., (1 + x ^{2}) y $_{n+2}$ + 2 (n +1) x y $_{n+1}$ + n (n + 1) y $_n$ = 0

Substituting x = 0, we get, y $_{n+2}$ (0) = -n (n + 1) y $_n$ (0)

For n = 1, we get y $_3$ (0) = - 2y $_1$ (0) = - 2

For n = 2, we get y $_4$ (0) = - 2 .3.y $_2$ (0) = 0

For n = 3, we get y $_5$ (0) = - 3.4.y $_3$ (0) = 24

Using the formula

Y = y (0) + x y $_1$ (0) + $\dfrac{x^2}{2!}$ y $_2$ (0) + $\dfrac{x^3}{3!}$ y $_3$ (0) + ---------

We get tan $^{-1}$ x = x - $\dfrac{x^3}{3}$ + $\dfrac{x^5}{5}$ - -------------

Exercise:

30.   Using Maclaurin's Theorem prove the following:

a) Secx = 1 + $\dfrac{x^2}{2!}$ **+** $\dfrac{5x^4}{4!}$ **+ --------**

b) Sin $^{-1}$ x = x + $\dfrac{x^3}{6}$ **+** $\dfrac{3x^5}{40}$ **+ -------------**

c) e ^{x} Cos x = 1 + x - $\dfrac{x^3}{3}$ **+ ---------------**

d) Expand e ^{ax} Cos bx by Maclaurin's Theorem as far as the term containing x ^{3}

Exercise : Verify Rolle's Theorem for

(i) $f(x) = e^x(\sin x - \cos x)$ in $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$,

(ii) $f(x) = x(x-2)e^{x/2}$ in $[0,2]$

(iii) $f(x) = \dfrac{\sin 2x}{e^{2x}}$ in $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$.

Exercise : Verify the Lagrange's Mean Value Theorem for

(i) $f(x) = x(x-1)(x-2)$ in $\left[0, \dfrac{1}{2}\right]$

(ii) $f(x) = Tan^{-1}x$ in $[0,1]$

Exercise : Verify the Cauchy's Mean Value Theorem for

(i) $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{\sqrt{x}}$ in $\left[\dfrac{1}{4}, 1\right]$

(ii) $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x}$ in $[a,b]$

(iii) $f(x) = Sin\ x$ and $g(x) = Cos\ x$ in $[a,b]$

UNIT – II

DIFFERENTIAL CALCULUS-II

Give different types of Indeterminate Forms.

If f (x) and g (x) be two functions such that $\underset{x \to a}{Lim}\, f(x)$ and $\underset{x \to a}{Lim}\, g(x)$ both exists, then

$$\underset{x \to a}{Lim}\, \frac{f(x)}{g(x)} = \frac{\underset{x \to a}{Lim}\, f(x)}{\underset{x \to a}{Lim}\, g(x)}$$

If $\underset{x \to a}{Lim}\, f(x) = 0$ and $\underset{x \to a}{Lim}\, g(x) = 0$ then

$$\underset{x \to a}{Lim}\, \frac{f(x)}{g(x)} = \frac{0}{0}$$ Which do not have any definite value, such an expression is called

indeterminate form. The other indeterminate forms are $\dfrac{\infty}{\infty}, 0 \times \infty, \infty - \infty$, 0^0, ∞^0 and 1^{∞}

1. State & prove L' Hospital's Theorem (rule) for Indeterminate Forms.

L'Hospital rule is applicable when the given expression is of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$

Statement: Let f (x) and g (x) be two functions such that

(1) $\underset{x \to a}{Lim}\, f(x) = 0$ and $\underset{x \to a}{Lim}\, g(x) = 0$

(2) f¹ (a) and g¹ (a) exist and g¹ (a) ≠ 0

Then $\underset{x \to a}{Lim}\, \dfrac{f(x)}{g(x)} = \dfrac{\underset{x \to a}{Lim}\, f^1(x)}{\underset{x \to a}{Lim}\, g^1(x)}$

Proof: Now $\underset{x \to a}{Lim}\, \dfrac{f(x)}{g(x)} = \underset{x \to a}{Lim} \left[\dfrac{1/g \, x}{1/f(x)} \right]$, which takes the indeterminate form $\dfrac{0}{0}$. Hence applying the

L' Hospitals theorem, we get

$$Lim_{x \to a} \frac{f(x)}{g(x)} = Lim_{x \to a} \frac{-g^1(x) \big/ [g(x)]^2}{-f^1(x) \big/ [f(x)]^2} = Lim_{x \to a} \frac{g^1(x)}{f^1(x)} \left[\frac{f(x)}{g(x)} \right]^2$$

$$= \left[Lim_{x \to a} \frac{g^1(x)}{f^1(x)} \right] \left[Lim_{x \to a} \frac{f(x)}{g(x)} \right]^2$$

If $Lim_{x \to a} \frac{f(x)}{g(x)} \neq 0 \, and \neq \infty$ then

$$1 = \left[Lim_{x \to a} \frac{g^1(x)}{f^1(x)} \right] \left[Lim_{x \to a} \frac{f(x)}{g(x)} \right]$$

i.e $Lim_{x \to a} \frac{f(x)}{g(x)} = Lim_{x \to a} \frac{f^1(x)}{g^1(x)}$

If $Lim_{x \to a} \frac{f(x)}{g(x)} = 0$ or ∞ the above theorem still holds good.

2. Evaluate $Lim_{x \to a} \dfrac{Sinx}{x} = \dfrac{0}{0}$ **form**

Solution: Apply L'Hospital rule, we get

$$Lim_{x \to a} \frac{Cosx}{1} = \frac{Cos\theta}{1} = \frac{1}{1} = 1$$

$$\therefore Lim_{x \to a} \frac{Sinx}{x} = 1$$

3. Evaluate $Lim_{x \to a} \dfrac{\log Sinx}{Cotx}$

Solution: $Lim_{x \to a} \dfrac{\log Sinx}{Cotx} = \dfrac{\log Sin0}{Cot0} = \dfrac{\log 0}{\infty} = \dfrac{-\infty}{\infty}$ form

Apply L' Hospital rule

$$= Lim_{x \to a} \frac{-Cosce^2 x}{-2Co\sec xCo\sec xCotx}$$

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$$= \underset{x \to a}{Lim} \frac{1}{2Cotx} = 0$$

$$\therefore \underset{x \to a}{Lim} \frac{\log Sinx}{Cotx} = 0$$

Exercise: 1

Evaluate

a) $\underset{x \to 0}{Lim} \dfrac{\tan x}{x}$

b) $\underset{x \to 0}{Lim}(1+x)^{\frac{1}{x}}$

c) $\underset{x \to \infty}{Lim} \dfrac{a^x - 1}{x}$

d) $\underset{x \to 0}{Lim} \dfrac{x^n - a^n}{x - a}$

4.   **Explain ∞ - ∞ and $0 \times \infty$ Forms:**

Solution: Suppose $\underset{x \to a}{Lim} f(x) = 0$ and $\underset{x \to a}{Lim} g(x) = \infty$ in this case

$\underset{x \to a}{Lim} f(x)$ - g(x) = $0 \times \infty$, reduce this to $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ form

Let $\underset{x \to a}{Lim}[f(x).g(x)] = \underset{x \to a}{Lim}\left\{ \dfrac{f(x)}{1/g(x)} \right\} = \dfrac{0}{0}$ form

Or $\underset{x \to a}{Lim}[f(x).g(x)] = \underset{x \to a}{Lim}\left\{ \dfrac{g(x)}{1/f(x)} \right\} = \dfrac{\infty}{\infty}$ form

L' Hospitals rule can be applied in either case to get the limit.

Suppose $\underset{x \to a}{Lim} f(x) = \infty$ and $\underset{x \to a}{Lim} g(x) = \infty$ in this case $\underset{x \to a}{Lim}[f(x).g(x)] = \infty - \infty$ form, reduce

this $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ form and then apply L'Hospitals rule to get the limit

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5. **Evaluate** $\displaystyle\lim_{x\to 0}\left\{\frac{1}{x}-\frac{\log(1+x)}{x^2}\right\}$

Solution: Given $\displaystyle\lim_{x\to 0}\left\{\frac{1}{x}-\frac{\log(1+x)}{x^2}\right\}=\infty-\infty$ form

$\therefore$ Required limit = $\displaystyle\lim_{x\to 0}\left\{\frac{x-\log(1+x)}{x^2}\right\}=\frac{0}{0}$ form

Apply L'Hospital rule.

$$=\lim_{x\to 0}\frac{1-\dfrac{1}{1+x}}{2x}$$

$$=\lim_{x\to 0}\frac{\dfrac{x}{1+x}}{2x}=\lim_{x\to 0}\frac{1}{2(1+x)}=\frac{1}{2}$$

6. **Evaluate** $\displaystyle\lim_{x\to 0}\left\{\frac{1}{x}-Cotx\right\}$

Solution: Given limit is $\infty-\infty$ form at x = 0. Hence we have

Required limit = $\displaystyle\lim_{x\to 0}\left\{\frac{1}{x}-\frac{Cosx}{Sinx}\right\}$

$$=\lim_{x\to 0}\left(\frac{Sinx-xCosx}{xSinx}\right)=\left(\frac{0}{0}\right)\text{ form}$$

Apply L' Hospital's rule

$$=\lim_{x\to 0}\frac{Cosx-Cosx+xSinx}{xCosx+Sinx}$$

$$=\lim_{x\to 0}\frac{xSinx}{xCosx+Sinx}=\left(\frac{0}{0}\right)\text{ form}$$

Apply L' Hospitals rule

$$= \lim_{x \to 0} \frac{x Cosx + Sinx}{Cosx - xSinx + Cosx}$$

$$= \frac{0}{2-0} = \frac{0}{2} = 0$$

7. Evaluate $\lim_{x \to 0}$ **tan x log x**

Solution: Given limit is $(0 \times -\infty)$ form at x = 0

$\therefore$ Required limit = $\lim_{x \to 0} \frac{\log x}{Cotx} = \left[\frac{-\infty}{\infty} \right]$ form

Apply L' Hospitals rule

$$= \lim_{x \to 0} \frac{1/x}{-Co\sec^2 x}$$

$$= \lim_{x \to 0} \frac{-Sin^2 x}{x} \left(\frac{0}{0} \right) \text{ form}$$

Apply L Hospitals rule

$$= \lim_{x \to 0} \frac{-2SinxCosx}{1} = 0$$

8. Evaluate $\lim_{x \to 1} Sec\left(\frac{\pi x}{2} \right). \log x$

Solution: Given limit is $(\infty \times 0)$ form at x = 1

$\therefore$ Required limit = $\lim_{x \to 1} \frac{\log x}{Cos \dfrac{\pi x}{2}} \dfrac{0}{0}$ form

Apply L' Hospitals rule

$$= \lim_{x \to 1} \frac{1/x}{-Sin\dfrac{\pi x}{2} \cdot \dfrac{\pi}{2}} = -\frac{2}{\pi}$$

Exercise: 2

Evaluate

a) $\underset{x \to 1}{Lim} \left(\dfrac{x}{x-1} - \dfrac{1}{\log x} \right)$

b) $\underset{x \to 0}{Lim} \left(\dfrac{a}{x} - Cot\left(\dfrac{x}{a} \right) \right)$

c) $\underset{x \to \frac{\pi}{2}}{Lim} \left(Secx - \dfrac{1}{1 - Sinx} \right)$

d) $\underset{x \to \infty}{Lim} \left(a^{1/x} - 1 \right) x$

9.   **Explain Indeterminate Forms** 0^0 , 1^∞ , ∞^0 , 0^∞

Solution: At x = a, $\left[f(x) \right]^{g(x)}$ takes the indeterminate form

(i) 0^0 if $\underset{x \to a}{Lim} \ f(x) = 0$ and $\underset{x \to a}{Lim} \ g(x) = 0$

(ii) 1^∞ if $\underset{x \to a}{Lim} \ f(x) = 1$ and $\underset{x \to a}{Lim} \ g(x) = \infty$

(iii) ∞^0 if $\underset{x \to a}{Lim} \ f(x) = \infty$ and $\underset{x \to a}{Lim} \ g(x) = 0$ and

(iv) 0^∞ if $\underset{x \to a}{Lim} \ f(x) = 0$ and $\underset{x \to a}{Lim} \ f(x) = \infty$

In all these cases the following method is adopted to evaluate $\underset{x \to a}{Lim} \ \left[f(x) \right]^{g(x)}$

Let L = $\underset{x \to a}{Lim} \ \left[f(x) \right]^{g(x)}$ so that

Log L = $\underset{x \to a}{Lim} \ [g(x) \log f(x)] = 0 \times \infty$

Reducing this to $\dfrac{0}{0} \, or \, \dfrac{\infty}{\infty}$ and applying L' Hospitals rule, we get Log L = a Or

L = e^a

10. Evaluate $\underset{x \to 0}{Lim}$ **x** ^{Sinx}

Solution: let L = $\underset{x \to 0}{Lim}$ x ^{Sinx} $\Rightarrow 0^0$ form at x = 0

Hence Log L = $\underset{x \to 0}{Lim}$ Sinx log x $\Rightarrow 0 \times \infty$ form

$$\therefore LogL = \underset{x \to 0}{Lim} \frac{\log x}{1/Sinx} = \underset{x \to 0}{Lim} \frac{\log x}{Coscex}\left(\frac{\infty}{\infty}\right) \text{ form}$$

Apply L' Hospital rule,

$$= \underset{x \to 0}{Lim} \frac{1/x}{-CoscexCotx} = \underset{x \to 0}{Lim} \frac{-Sinx.\tan x}{x}\left(\frac{0}{0}\right) \text{ form}$$

Apply L' Hospitals rule we get

$$= \underset{x \to 0}{Lim} \frac{\sin xSec^2 x - Cosx \tan x}{1} = -\infty$$

$$\therefore LogL = -\infty \Rightarrow L = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

$$\therefore \text{ L = 0}$$

11. Evaluate $\underset{x \to 1}{Lim}(x)^{1/1-x}$

Solution: let L = $\underset{x \to 1}{Lim}(x)^{1/1-x}$ is 1^{∞} form

$$\therefore LogL = \underset{x \to 1}{Lim}\left(\frac{1}{1-x} \log x\right) \equiv \left(\frac{0}{0}\right) \text{ form}$$

Apply L' Hospitals rule

$$= \underset{x \to 1}{Lim} \frac{1/x}{-1} = \underset{x \to 1}{Lim} \frac{1}{-x} = -1$$

$\therefore$ Log L = -1

$\Rightarrow$ L = $e^{-1} = \dfrac{1}{e}$

12. Evaluate $\displaystyle \lim_{x \to 0} \left(\dfrac{\tan x}{x} \right)^{1/x^2}$

Solution: let L = $\displaystyle \lim_{x \to 0} \left(\dfrac{\tan x}{x} \right)^{1/x^2} \equiv 1^\infty$ form

$\therefore LogL = \displaystyle \lim_{x \to 0} \left\{ \dfrac{1}{x^2} \log\left(\dfrac{\tan x}{x} \right) \right\} \equiv (\infty \times 0)$ form

$\therefore LogL = \displaystyle \lim_{x \to 0} \left\{ \dfrac{\log\left(\tan x / x\right)}{x^2} \right\} \equiv \left(\dfrac{0}{0} \right)$ form

Apply L' Hospitals rule

$= \displaystyle \lim_{x \to 0} \left\{ \dfrac{1}{\dfrac{\tan x}{x}} \cdot \dfrac{\dfrac{xSec^2 x - \tan x}{x^2}}{2x} \right\}$

$LogL = \dfrac{1}{2} \displaystyle \lim_{x \to 0} \left[\dfrac{xSec^2 x - \tan x}{x^3} \right] = \left(\dfrac{0}{0} \right)$ form

Apply L' Hospital rule, we get

$= \dfrac{1}{2} \displaystyle \lim_{x \to 0} \dfrac{Sec^2 x + 2xSec^2 x \tan x - Sec^2 x}{3x^2}$

$= \dfrac{1}{3} \displaystyle \lim_{x \to 0} (Sec^2 x)\left(\dfrac{\tan x}{x} \right)$

Log L = $\dfrac{1}{3}$

$\therefore L = e^{1/3}$

Exercise: 3

Evaluate the following limits.

a) $\lim_{x \to 0} (Secx)^{Cotx}$

b) $\lim_{x \to a} \left(2 - \dfrac{x}{a} \right)^{\tan\left(\frac{\pi x}{2a} \right)}$

c) $\lim_{x \to 0} \dfrac{(1+x)^{1/x} - e}{x}$

d) $\lim_{x \to 0} (Cosax)^{\infty/x^2}$

(i) $\lim_{x \to 0} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$

(ii) $\lim_{x \to 0} \dfrac{\log(1+x^3)}{\sin^3 x}$

(iii) $\lim_{x \to 0} \dfrac{1 + \sin x - \cos x + \log(1-x)}{x \tan^2 x}$

(iv) $\lim_{x \to \frac{\pi}{2}} \dfrac{\log \sin x}{(x - \frac{\pi}{2})^2}$

(v) $\lim_{x \to 0} \dfrac{\cosh x + \log(1-x) - 1 + x}{x^2}$

(vi) $\lim_{x \to \frac{\pi}{2}} \dfrac{\sin x \sin^{-1} x}{x^2}$

(vii) $\lim_{x \to 0} \dfrac{e^{2x} - (1+x)^2}{x \log(1+x)}$

Evaluate the following limits.

(i) $\lim_{x \to a} \left(2 - \dfrac{x}{a} \right) \cot(x - a)$

(ii) $\lim_{x \to 0} (\cos ecx - \cot x)$

(iii) $\lim_{x \to \frac{\pi}{2}} \left[x \tan x - \dfrac{\pi}{2} \sec x \right]$

(iv) $\lim_{x \to 0} \left(\cot^2 x - \dfrac{1}{x^2} \right)$

(v) $\lim_{x \to 0} \left[\dfrac{1}{x^2} - \dfrac{1}{x \tan x} \right]$

(vi) $\lim_{x \to \frac{\pi}{2}} [2x \tan x - \pi \sec x]$

(i) $\lim_{x \to 0} (\cos ax)^{\frac{b}{x^2}}$

(ii) $\lim_{x \to 0} \left(\dfrac{1 + \cos x}{2} \right)^{\frac{1}{x^2}}$

(iii) $\lim\limits_{x \to 1}(1 - x^2)^{\frac{1}{\log(1-x)}}$ (iv) $\lim\limits_{x \to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$

(v) $\lim\limits_{x \to 0}(\sin x)^{\tan x}$ (iv) $\lim\limits_{x \to 0}\left(1 + \sin x\right)^{\cot x}$

(vii) $\lim\limits_{x \to 0}(\cos x)^{\cos ec^2 x}$ $(viii)$ $\lim\limits_{x \to \frac{\pi}{4}}\left(\tan x\right)^{\tan 2x}$

(ix) $\lim\limits_{x \to \infty}(\dfrac{ax+1}{ax-1})^x$ (x) $\lim\limits_{x \to 0}\left(\dfrac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}}$

Evaluate the following limits.

(i) $\lim\limits_{x \to 0}(\cos ax)^{\frac{b}{x^2}}$ (ii) $\lim\limits_{x \to 0}\left(\dfrac{1 + \cos x}{2}\right)^{\frac{1}{x^2}}$

(iii) $\lim\limits_{x \to 1}(1 - x^2)^{\frac{1}{\log(1-x)}}$ (iv) $\lim\limits_{x \to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$

(v) $\lim\limits_{x \to 0}(\sin x)^{\tan x}$ (iv) $\lim\limits_{x \to 0}\left(1 + \sin x\right)^{\cot x}$

(vii) $\lim\limits_{x \to 0}(\cos x)^{\cos ec^2 x}$ $(viii)$ $\lim\limits_{x \to \frac{\pi}{4}}\left(\tan x\right)^{\tan 2x}$

(ix) $\lim\limits_{x \to \infty}(\dfrac{ax+1}{ax-1})^x$ (x) $\lim\limits_{x \to 0}\left(\dfrac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}}$

Polar Curves

If we traverse in a hill section where the road is not straight, we often see caution boards hairpin bend ahead, sharp bend ahead etc. This gives an indication of the difference in the amount of bending of a road at various points which is the curvature at various points. In this chapter we discuss about the curvature, radius of curvature etc.
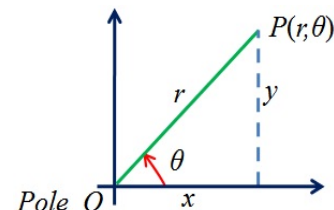
Consider a point P in the *xy*-Plane.

    r = length of OP= radial distance

    θ = Polar angle

    (*r, θ*)→ Polar co-ordinates

Let *r = f* (θ) be the polar curve

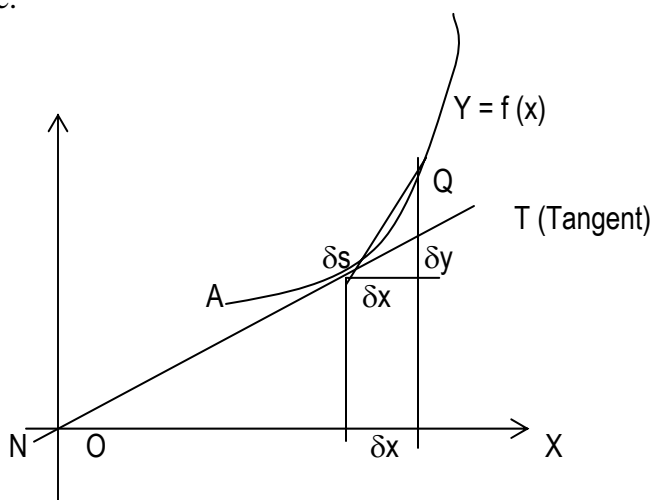$$r=\sqrt{x^2+y^2}, \quad \theta=\tan^{-1}\left(\frac{y}{x}\right) \quad ------(1)$$

    x = r Cos θ   y = r Sin θ

Relation (1) enables us to find the polar co-ordinates (*r*, θ) when the Cartesian co-ordinates (*x, y*) are known.

Expression for arc length in Cartesian form.

Proof: Let P (x,y) and Q (x + δx, Y + δy) be two neighboring points on the graph of the function y = f (x). So that they are at length S and S + δs measured from a fixed point A on the curve.

From figure, $\overset{\frown}{PQ} = \delta S$,

        $\overset{\frown}{AP} = S$

    ∠TPR = ψ   and PR =δx, RQ = δy

∴ Arc PQ = δS

From Δ^{le} PQR, we have

[Chord PQ]^2 = PR^2 + QR^2

[Chord PQ]^2 = (δx)^2 + (δy)^2 (∵ from figure)

When Q is very close to point P, the length of arc PQ is equal to the length of Chord PQ.

i.e arc PQ = Chord PQ = δs

∴ (δs)^2 = (δx)^2 + (δy)^2 -------- (1)

÷ (δx)^2, we get

$$\left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

When Q → P along the curve, δx → 0, δs → 0

$$\therefore \underset{\delta x \to 0}{Lim}\left(\frac{\delta s}{\delta x}\right)^2 = 1 + \underset{\delta x \to 0}{Lim}\left(\frac{\delta y}{\delta x}\right)^2$$

i.e., $$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad\text{--------------(2)}$$

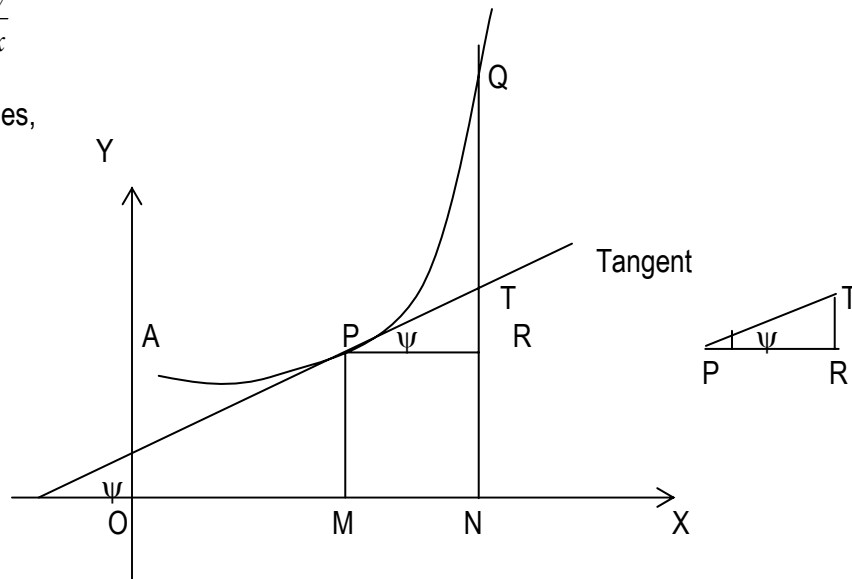Similarly, dividing (1) by δy and taking the limit as δy → 0, we get

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \quad\text{------------(3)}$$

Expressions for $\dfrac{ds}{dx}$ **&** $\dfrac{ds}{dy}$

Trace a tangent to the curve at the point P, it makes an angle ψ with the x – axis. From Δ^{le} PRT, we have

$$\tan \psi = \frac{dy}{dx}$$

Equation (2) becomes,

$$\frac{ds}{dx} = \sqrt{1 + \tan^2 \psi}$$

$$= \sqrt{Sec^2 \psi} = \sec \psi$$

$$\therefore \ \frac{ds}{dx} = \text{Sec } \psi \ (\text{Or}) \ \frac{dx}{ds} = \text{Cos } \psi$$

and equation (3) becomes

$$\frac{ds}{dy} = \sqrt{1 + Cot^2 \psi} = \sqrt{Co\sec^2 \psi} = \text{Cosec } \psi$$

$$\therefore \ \frac{ds}{dy} = \text{Cosec}\psi \ \text{ or } \ \frac{dy}{ds} = \text{Sin } \psi$$

Derive an expression for arc length in parametric form.

Solution: Let the equation of the curve in Parametric from be x = f (t) and y = g (t).

We have,

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$$

$\div$ by $(\delta t)^2$, we get

$$\left(\frac{\delta s}{\delta t}\right)^2 = \left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2$$

Taking the limit as $\delta t \to 0$ on both sides, we get

$$\underset{\delta t \to 0}{Lim}\left(\frac{\delta s}{\delta t}\right)^2 = \underset{\delta t \to 0}{Lim}\left(\frac{\delta x}{\delta t}\right)^2 + \underset{\delta t \to 0}{Lim}\left(\frac{\delta y}{\delta t}\right)^2$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

(Or) $\dfrac{ds}{dt} = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$ ------------ (4)
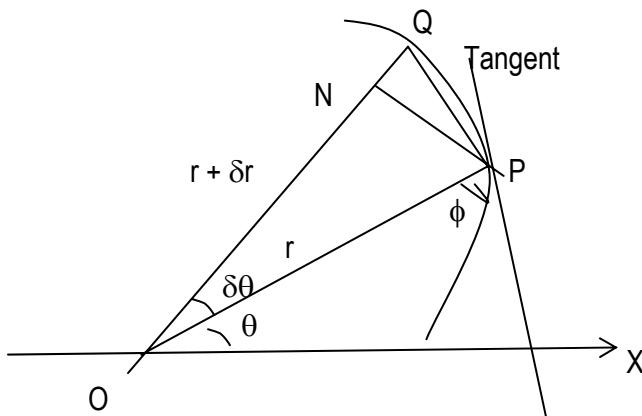
Derive an expression for arc length in Cartesian form.

Solution: Let P (r,θ) and Q $(r + \delta r, \theta + \delta\theta)$ be two neighboring points on the graph of the function $r = f(\theta)$. So that they are at lengths S and $s + \delta s$ from a fixed point A on the curve.

$\therefore$ PQ = $(S + \delta s) - s = \delta s$

Draw PN $\perp$ OQ

From Δ^{le} OPN,

$$\frac{PN}{OP} = Sin\,\delta\theta$$

i.e., $\dfrac{PN}{r}$ = Sin $\delta\theta$ (or) PN = r sin$\delta\theta$

and $\dfrac{ON}{OP}$ = Cos $\delta\theta$

i.e $\dfrac{ON}{r}$ = Cos $\delta\theta$ (or) ON = r Cos $\delta\theta$

When Q is very close to P, the length of arc PQ as equal to δs, where δs as the length of chord PQ.

In Δ^{le} PQN,

$(PQ)^2 = (PN)^2 + (QN)^2$

but PN = r sin $\delta\theta$ ($\because$ Sin $\delta\theta \approx \delta\theta$)

And QN = OQ – ON

$\qquad\qquad$ = (r + δr) – r Cos $\delta\theta$

$\qquad\qquad$ = r + δr – r   ($\because$ Cos $\delta\theta \approx 1$)

$\therefore$ QN = δr

And $(PQ)^2 = (PN)^2 + (QN)^{201}$

$\qquad$ $(\delta S)^2 + (r\delta\theta)^2 + (\delta r)^2$ ---------------- (5)

$\div$ by $(\delta\theta)^2$ we get

$$\left(\frac{\delta S}{\delta\theta}\right)^2 = r^2 + \left(\frac{\delta r}{\delta\theta}\right)^2$$

When Q $\rightarrow$ P along the curve $\delta\theta \rightarrow 0$ as δS $\rightarrow 0$ and δr $\rightarrow 0$

$$\underset{\delta\theta\to0}{Lim}\left(\frac{\delta S}{\delta\theta}\right)^2 = r^2 + \underset{\delta\theta\to0}{Lim}\left(\frac{\delta r}{\delta\theta}\right)^2$$

i.e $\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$

i.e., $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$ -------------- (6)

Similarly equation (5) ÷ by (δr)² and taking the limits as δr → 0 we get

$$\underset{\delta r\to0}{Lim}\left(\frac{\delta S}{\delta r}\right)^2 = \underset{\delta r\to0}{Lim}\, r^2\left(\frac{\delta\theta}{\delta r}\right)^2 + 1$$

i.e $\left(\frac{ds}{dr}\right)^2 = r^2\left(\frac{d\theta}{dr}\right)^2 + 1$

$\therefore \frac{ds}{dr} = \sqrt{r^2\left(\frac{d\theta}{dr}\right)^2 + 1}$ -------------- (7)

Note: Angle between Tangent and Radius Vector:-

We have,

Tan ϕ = r $\dfrac{d\theta}{dr}$

i.e., $\dfrac{Sin\phi}{Cos\phi}$ = r $\dfrac{d\theta}{dr}$

= r $\dfrac{d\theta}{ds} \cdot \dfrac{ds}{dr}$

$\dfrac{Sin\phi}{Cos\phi} = \dfrac{r\,d\theta/ds}{dr/ds}$

$\therefore Sin\phi$ = r $\dfrac{d\theta}{ds}$ and Cos ϕ = $\dfrac{dr}{ds}$

Find $\dfrac{ds}{dx}$ and $\dfrac{ds}{dy}$ for the following curves:-

1) y = C Cos h $\left(\dfrac{x}{c}\right)$

Solution: y = C Cos h $\left(\dfrac{x}{c}\right)$

Differentiating y w.r.t x. we get

$$\frac{dy}{dx} = \text{Sin h}\left(\frac{x}{c}\right)$$

$$\therefore \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\equiv \sqrt{1+\sinh^2\left(x/c\right)} = \sqrt{Cosh^2\left(x/c\right)}$$

$$\frac{ds}{dx} = \text{Cosh}\left(\frac{x}{c}\right)$$

Again Differentiating y w.r.t y we get

$$1 = \text{C Sin h}\left(\frac{x}{c}\right)\ \frac{1}{C}\ \frac{dx}{dy}$$

i.e $\dfrac{dx}{dy}$ = Cosech $\left(\dfrac{x}{c}\right)$

$$\therefore \frac{ds}{dy} = \sqrt{1 + Co\sec h^2\left(x/c\right)}$$

2) $x^3 = ay^2$

Solution : $x^3 = ay^2$

Differentiating w.r.t y and x separately we get

$$3x^2 \; \frac{dx}{dy} = 2 \, ay \text{ and } 3 \, x^2 = 2 \, ay \; \frac{dy}{dx}$$

i.e., $\dfrac{dx}{dy} = \dfrac{2ay}{3x^2}$ and $\dfrac{dy}{dx} = \dfrac{3x^2}{2ay}$

We know that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= \sqrt{1 + \left(\frac{3x^2}{2ay}\right)^2}$$

$$= \sqrt{1 + \frac{9x^3 x}{4a^2 y^2}} = \sqrt{1 + \frac{9ay^2 x}{4a^2 y^2}}$$

$$\frac{ds}{dx} = \sqrt{1 + \frac{9x}{4a}} \quad \text{and}$$

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \left(\frac{2ay}{3x^2}\right)^2}$$

$$\frac{ds}{dy} = \left(1 + \frac{4a^2 y^2}{9x^4}\right)^{\frac{1}{2}} = \left(1 + \frac{4a}{9x}\right)^{\frac{1}{2}} \qquad (\because x^3 = ay^2)$$

3. y = log cos x
Solution . y= log cos x

Differentiating w.r.t x and y separately, we get

$$\frac{dy}{dx} = \frac{1}{Cosx} \; (-\, Sin \; x) = -\tan x$$

i.e., $\dfrac{dy}{dx} = -\tan x$

and $1 = \dfrac{1}{Cosx}$ (- Sin x) $\dfrac{dx}{dy}$

i.e., $1 = -\tan x \ \dfrac{dx}{dy}$ (Or) $\dfrac{dx}{dy} = -\cot x$

We have

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad \text{and} \quad \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \tan^2 x} \quad \& \quad \frac{ds}{dy} = \sqrt{1 + Cot^2 x}$$

$$= \sqrt{Sec^2 x}$$

$$\therefore \frac{ds}{dx} = Sec\ x \text{ and} \qquad \frac{ds}{dy} = (1 + Cot^2 x)^{\frac{1}{2}}$$

Find $\dfrac{ds}{dt}$ *for the following Curves:-*

1. x = a (Cos t + t Sin t), y = a (Sin t = t Cos t)

2. x = a Sec t , y = b tan t

3. x = a $\left(Cost + \log\tan {}^{t}\!\!\diagup\!\!_{2}\right)$**, y = a Sin t**

Solution of 1

Given x = a (Cos t + t Sin t), y = a (Sin t – t Cos t)

Differentiating x & y W.r.t 't', we get

$$\frac{dx}{dt} = a \ (\text{-Sin t + Sin t + t Cos t})$$

$$\frac{dx}{dt} = a \ t \cos t$$

and $\dfrac{dy}{dt} = a$ (Cos t – Cos t + t Sin t)

$$\frac{dy}{dt} = at \ Sin \ t$$

$$\therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{a^2 t^2 Cos^2 t + a^2 t^2 Sin^2 t}$$

$$\frac{ds}{dt} = at$$

Solution of 2

$$x = a \, Sec \, t, \; x = b \, tan \, t$$

$$\therefore \; \frac{dx}{dt} = a \, Sec \, t \, tan \, t, \quad \frac{dy}{dt} = b \, Sec^2 t$$

We have

$$\therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= (a^2 Sec^2 \, t \, tan^2 t + b^2 \, Sec^4 t)^{\frac{1}{2}}$$

$$= [a^2 \, Sec^2 \, t \, (Sec^2 t - 1) + b^2 \, Sec^4 \, t]^{\frac{1}{2}}$$

$$= [a^2 \, Sec^4 \, t - a^2 \, Sec^2 \, t + b^2 \, Sec^4 \, t]^{\frac{1}{2}}$$

$$\frac{ds}{dt} = [(a^2 + b^2) \, Sec^4 \, t - a^2 \, Sec^2 t]^{\frac{1}{2}}$$

Solution of 3

$$x = a \, (Cos \, t + log \, tan \, \tfrac{t}{2}), \; y = a \, Sin \, t$$

Differentiating x and y w.r.t 't' we get

$$\frac{dx}{dt} = a \left(- S \, int + \frac{1}{\tan \tfrac{t}{2}} Sec^2 \, \tfrac{t}{2} \cdot \frac{1}{2} \right), \; \frac{dy}{dt} = a \, Cos \, t$$

$$\therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \left[a^2 \left[- S \, int + \frac{Sec^2 \, \tfrac{t}{2}}{2 \tan \tfrac{t}{2}} \right]^2 + a^2 Cos^2 t \right]^{\frac{1}{2}}, \quad = \left[a^2 - \frac{a^2 S \, int \, Sec^2 \, \tfrac{t}{2}}{\tan \tfrac{t}{2}} + \frac{a^2 Sec^4 \, \tfrac{t}{2}}{4 \tan^2 \tfrac{t}{2}} \right]$$

$$= a \, Cot \, t$$

Find $\dfrac{ds}{d\theta}$ *and* $\dfrac{dr}{d\theta}$ *for the following curves*

1. r = a (1 – Cos θ)

2 .r² = a² Cos 2θ

3. r = a e^{θ Cot α}

Solution of 1

r = a (1 – Cos θ)

       Differentiating r w.r.t θ we get

$$\frac{dr}{d\theta} = a \, Sin \, \theta$$

Hence

$$\frac{ds}{d\theta} = \left\{ r^2 + \left(\frac{dr}{d\theta} \right)^2 \right\}^{\frac{1}{2}}$$

$$= \{a^2 \, (1 - Cos \, \theta)^2 + a^2 \, Sin^2 \, \theta\}^{\frac{1}{2}}$$

$$= \{a^2 \, (1 - 2 \, Cos \, \theta + Cos^2\theta) + a^2 \, Sin^2 \, \theta\}^{\frac{1}{2}}$$

$$= \{a^2 - 2a^2 \, Cos \, \theta + a^2\}^{\frac{1}{2}}$$

$$= \{2a^2 - 2a^2 \, Cos \, \theta\}^{\frac{1}{2}}$$

$$= a\{ 2(1 - Cos \, \theta)\}^{\frac{1}{2}}$$

$$= a \{2 (2 \, Sin^2 \theta/2 \,)\}^{\frac{1}{2}}$$

$$\frac{ds}{d\theta} = 2 \, a \, Sin \, \theta/2$$

And $\dfrac{ds}{dr} = \left\{ 1 + r^2 \left(\dfrac{d\theta}{dr} \right)^2 \right\}^{\frac{1}{2}}$

$$= \left\{ 1 + \frac{a^2 (1 - Cos\theta)^2}{a^2 Sin^2\theta} \right\}^{\frac{1}{2}}$$

$$= \left\{ \frac{Sin^2\theta + (1 - Cos\theta)^2}{Sin^2\theta} \right\}^{\frac{1}{2}}$$

$$= \frac{\{2(1 - Cos\theta)\}^{\frac{1}{2}}}{Sin\theta} = \frac{2Sin\left(\frac{\theta}{2}\right)}{Sin\theta} = \frac{2Sin\frac{\theta}{2}}{2Sin\frac{\theta}{2}Cos\frac{\theta}{2}}$$

$$\frac{ds}{dr} = \frac{1}{Cos\frac{\theta}{2}}$$

Solution of 2

$$r^2 = a^2 \, Cos \, 2\theta$$

Differentiating W.r.t 'θ' we get

$$2r \, \frac{dr}{d\theta} = -a^2 \, Sin \, 2\theta \, . \, 2$$

$$r \, \frac{dr}{d\theta} = -a^2 \, Sin \, 2\theta$$

$$\frac{dr}{d\theta} = \frac{a^2 Sin2\beta\theta}{r}$$

Hence

$$\frac{ds}{d\theta} = \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{1}{2}}$$

$$= \left\{ r^2 + \left(\frac{-a^2 Sin2\theta}{r}\right)^2 \right\}^{\frac{1}{2}}$$

$$= \left\{ r^2 + \frac{a^4 Sin^2 2\theta}{r^2} \right\}^{\frac{1}{2}}$$

$$= \left\{ a^2 Cos2\theta + \frac{a^4 Sin^2\theta}{a^2 Cos2\theta} \right\}^{\frac{1}{2}}$$

$$= \left\{ \frac{a^4 Cos^2 2\theta + a^4 Sin^2 2\theta}{a^2 Cos2\theta} \right\}^{\frac{1}{2}}$$

$$= \left\{ \frac{a^4}{a^2 Cos2\theta} \right\}^{\frac{1}{2}} = \frac{a}{\sqrt{Cos2\theta}}$$

Or $\frac{ds}{d\theta} = \left\{ \frac{a^4}{r^2} \right\}^{\frac{1}{2}} = \frac{a^2}{r}$ ($\because r^2 = a^2 Cos 2\theta$)

And $\frac{ds}{dr} = \left\{ 1 + r^2 \left(\frac{d\theta}{dr} \right)^2 \right\}^{\frac{1}{2}}$

$$= \left\{ 1 + r^2 \left(\frac{r}{-a^2 Sin2\theta} \right)^2 \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 + \frac{r^4}{a^4 Sin^2 2\theta} \right\}^{\frac{1}{2}}$$

$$= \left\{ \frac{a^4 Sin^2 2\theta + a^4 Cos^2 2\theta}{a^4 sin^2 2\theta} \right\}^{\frac{1}{2}}$$

$$= \left\{ \frac{a^4}{a^4 sin^2 2\theta} \right\}^{\frac{1}{2}} = \frac{1}{Sin2\theta} = Cosec\ 2\theta$$

Solution 3

　　　　$r = ae^{\theta Cot\ \alpha}$, here α is constant

Differentiating w.r.t 'θ' we get

$\frac{dr}{d\theta}$ = a $e^{\theta Cot\ \alpha}$. Cot α

Hence

$$\frac{ds}{d\theta} = \left\{ r^2 + \left(\frac{dr}{d\theta} \right)^2 \right\}^{\frac{1}{2}}$$

$$= \{a^2\ e^{2\theta Cot\alpha} + a^2\ e^{2\theta Cot\ \alpha}\ cot^2\ \alpha\}^{\frac{1}{2}}$$

$$= a\ e^{\theta Cot\ \alpha}\ \{1 + Cot^2\alpha\}^{\frac{1}{2}}$$

$$= a\ e^{\theta Cot\ \alpha}\ \{Cosce^2\alpha\}^{\frac{1}{2}}$$

$$\frac{ds}{d\theta} = a\, e^{\theta \cot \alpha}\, \text{Cosce}\, \alpha$$

and $\frac{ds}{dr} = \left\{1 + r^2 \left(\frac{d\theta}{dr}\right)^2\right\}^{\frac{1}{2}}$

$$= \left\{1 + a^2 e^{2\theta Cot\alpha}\, \frac{1}{a^2 e^{2\theta Cot\alpha Cot^2 \alpha}}\right\}^{\frac{1}{2}}$$

$$= \{1 + \tan^2 \alpha\}^{\frac{1}{2}}$$

$$= \{Sec^2 \alpha\}^{\frac{1}{2}} = Sec\, \alpha$$

Exercises:

Find $\frac{ds}{dr}$ and $\frac{ds}{d\theta}$ to the following curves.

1. $r^n = a^n \cos n\theta$

2. $r(1 + \cos \theta) = a$

3. $r\theta = a$
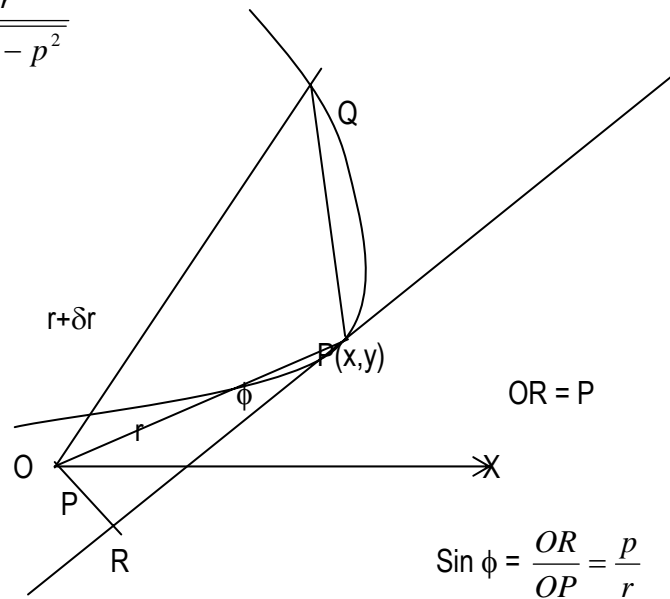
Note:

We have $\sin \phi = r\, \frac{d\theta}{ds}$ and

$\cos \phi = \frac{dr}{ds}$

$\therefore \frac{dr}{ds} = \cos\phi = (1 - \sin^2\phi)^{\frac{1}{2}}$

$$= \left[1 - \frac{p^2}{r^2}\right]^{\frac{1}{2}} \quad \text{Since } P = r \sin \phi.$$

$$\frac{dr}{ds} = \frac{\sqrt{r^2 - p^2}}{r}$$

$$\therefore \frac{ds}{dr} = \frac{r}{\sqrt{r^2 - p^2}}$$

OR = P

$$\text{Sin } \phi = \frac{OR}{OP} = \frac{p}{r}$$

$$\therefore \text{ P = r Sin } \phi$$

Prove that with usual notations $\tan \phi = r\left(\dfrac{d\theta}{dr}\right)$
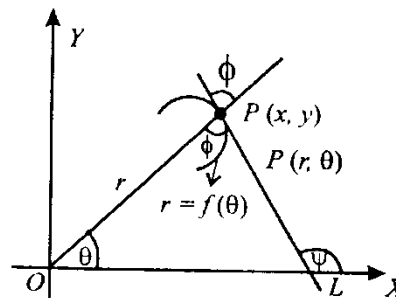
Let P (r, θ) be any point on the curve r = f (θ)

$$\therefore \ X\hat{O}P = \theta \ \text{ and } OP = r$$

Let PL be the tangent to the curve at P subtending an angle ψ with the positive direction of the initial line (x – axis) and ϕ be the angle between the radius vector OP and the tangent PL.

That is $\ O\hat{P}L = \phi$

From the figure we have

$$\psi = \phi + \theta$$

(Recall from geometry that an exterior angle is equal to the sum of the interior opposite angles)

$$\Rightarrow \ \tan \psi = \tan (\phi + \theta)$$

or $\ \tan \psi \ \dfrac{\tan \phi + \tan \phi}{1 - \tan \phi \ \tan \theta}$ ...(1)

Let (x, y) be the Cartesian coordinates of P so that we have,

X = r cos θ , y = r sin θ

Since r is a function of θ , we can as well regard these as parametric equations in terms of θ .

We also know from the geometrical meaning of the derivative that

$$\tan \psi = \frac{dy}{dx} = \text{slope of the tangent } PL$$

ie., $\ \tan \psi \ = \ \dfrac{dy}{d\theta} \Big/ \dfrac{dx}{d\theta} \quad$ since x and y are function of θ

ie., $\ \tan \psi = \dfrac{\dfrac{d}{d\theta}(r \sin \theta)}{\dfrac{d}{d\theta}(r \cos \theta)} = \dfrac{r \cos \theta + r' \sin \theta}{-r \sin \theta + r' \cos \theta} \text{ where } r' = \dfrac{dr}{d\theta}$

Dividing both the numerator and denominator by $r' \cos \theta$ we have,

$$\tan \psi \ = \ \frac{\dfrac{r \cos \theta}{r' \cos \theta} + \dfrac{r' \sin \theta}{r' \cos \theta}}{\dfrac{-r \sin \theta}{r' \cos \theta} + \dfrac{r' \cos \theta}{r' \cos \theta}}$$

Or $\qquad \tan \psi \ = \ \dfrac{\dfrac{r}{r'} + \tan \theta}{1 - \dfrac{r}{r'} . \ \tan \theta}$ ...(2)

Comparing equations (1) and (2) we get

$$\tan \phi = \frac{r}{r'} = \frac{r}{\left(\dfrac{dr}{d\theta}\right)} \quad \text{or} \ \ \tan\phi = r\left(\frac{d\theta}{dr}\right)$$

Prove with usual notations $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$ **or**

$$\dfrac{1}{p^2} = u^2 + \left(\dfrac{du}{d\theta}\right)^2 \text{ where } u = \dfrac{1}{r}$$
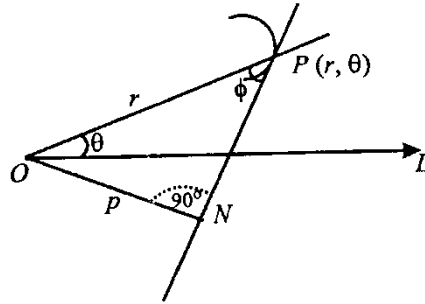
Proof :

Let O be the pole and OL be the initial line. Let P (r, θ) be any point on the curve and hence

we have OP = r and $L\hat{O}P = \theta$

Draw ON = p (say) perpendicular from the pole on the tangent at P and let ϕ be the angle

made by the radius vector with the tangent.

From the figure $O\hat{N}P = 90° \quad L\hat{O}P = \theta$

Now from the right angled triangle ONP

$$\sin\phi = \dfrac{ON}{OP}$$

ie., $\sin\phi = \dfrac{P}{r}$ or $p = r\sin\phi$

we have p = r sin ϕ ...(1)

and $\cot\phi = \dfrac{1}{r}\dfrac{dr}{d\theta}$ ...(2)

Squaring equation (1) and taking the reciprocal we get,

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$$\frac{1}{p^2} = \frac{1}{r^2} \cdot \frac{1}{\sin^2 \phi} \quad \text{ie.,} \quad \frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 \phi$$

Or $\qquad \frac{1}{p^2} = \frac{1}{r^2} \left(1 + \cot^2 \phi \right)$

Now using (2) we get,

$$\frac{1}{p^2} = \frac{1}{r^2} \left[1 + \frac{1}{r^2} \left(\frac{dr}{d\theta} \right)^2 \right]$$

Or $\qquad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$ $\qquad\qquad\qquad$ ...(3)

Further, let $\dfrac{1}{r} = u$

Differentiating w.r.t. θ we get,

$$-\frac{1}{r^2} \left(\frac{dr}{d\theta} \right) = \frac{du}{d\theta} \implies \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 = \left(\frac{du}{d\theta} \right)^2 \text{, by squaring}$$

Thus (3) now becomes

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta} \right)^2 \qquad\qquad\qquad \text{...(4)}$$

1. **Find the angle of intersection of the curves:**

$r = a \left(1 + \cos \theta \right) \& r = b \left(1 - \cos\theta \right)$

Solution : $r = a \left(1 + \cos \theta \right)$ $\qquad : \qquad$ $r = b(1 - \cos \theta)$

$\implies \ \log r = \log a + \log \left(1 + \cos \theta \right)$ $\quad : \qquad$ $\log r = \log b + \log \left(1 - \cos \theta \right)$

Differentiating these w.r.t. θ we get

$\dfrac{1}{r} \ \dfrac{dr}{d\theta} = 0 + \dfrac{-\sin \theta}{1 + \cos \theta}$ $\qquad\qquad : \qquad$ $\dfrac{1}{r} \ \dfrac{dr}{d\theta} = 0 + \dfrac{\sin \theta}{1 - \cos \theta}$

$\cot \phi_1 = \dfrac{-2 \sin \left(\theta/2 \right) \cos \left(\theta/2 \right)}{2 \cos^2 \left(\theta/2 \right)}$ $\qquad : \qquad$ $\cot \phi_2 = \dfrac{2 \sin \left(\theta/2 \right) \cos \left(\theta/2 \right)}{2 \sin^2 \left(\theta/2 \right)}$

ie., $\cot \phi_1 = - \tan \left(\theta/2 \right) = \cot \left(\pi/2 + \theta/2 \right)$ $\qquad : \qquad$ $\cot \phi_2 = \cot \left(\theta/2 \right)$

$\implies \ \phi_1 = \pi/2 + \theta/2$ $\qquad\qquad\qquad : \qquad$ $\phi_2 = \theta/2$

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$\therefore$ angle of intersection = $\left|\phi_1 - \phi_2\right| = \left|\pi/2 + \theta/2 - \theta/2\right| = \pi/2$

Hence the curves intersect orthogonally.

2. S.T. the curves $r = a\left(1 + \sin\theta\right) \& r = a\left(1 - \sin\theta\right)$ **cut each other orthogonally**

Solution : log r = log a + log (1 + sin θ) : log r = log a + log (1 – sin θ)

Differentiating these w.r.t θ we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{1 + \sin\theta} \qquad : \qquad \frac{1}{r}\frac{dr}{d\theta} = \frac{-\cos\theta}{1 - \sin\theta}$$

ie., $\cot\phi_1 = \dfrac{\cos\theta}{1 + \sin\theta} \qquad : \qquad \cot\phi_2 = \dfrac{-\cos\theta}{1 - \sin\theta}$

We have $\tan\phi_1 = \dfrac{1 + \sin\theta}{\cos\theta}$ and $\tan\phi_2 = \dfrac{1 - \sin\theta}{-\cos\theta}$

$\therefore \qquad \tan\phi_1 . \tan\phi_2 = \dfrac{1 - \sin^2\theta}{-\cos^2\theta} = \dfrac{\cos^2\theta}{-\cos^2\theta} = -1$

Hence the curves intersect orthogonally.

3.   Find the angle of intersection of the curves:

r = sin θ + cos θ, r = 2 sin θ

Solution : log r = log (sin θ + cos θ)   :   r = 2 sin θ

$\Rightarrow$ log r = log $\left(\sin\theta + \cos\theta\right)$:   log r = log 2 + log (sin θ)

Differentiating these w.r.t θ we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta} \qquad : \qquad \frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{\sin\theta}$$

ie., $\cot\phi_1 = \dfrac{\cos\theta\left(1 - \tan\theta\right)}{\cos\theta\left(1 + \tan\theta\right)} \qquad : \qquad \cot\phi_2 = \cot\theta \Rightarrow \phi_2 = \theta$

ie., $\cot\phi_1 = \cot\left(\pi/4 + \theta\right) \Rightarrow \phi_1 = \pi/4 + \theta$

$\therefore \left|\phi_1 - \phi_2\right| = \left|\pi/4 + \theta - \theta\right| = \pi/4$

The angle of intersection is $\pi/4$

4. Find the angle of the curves: r = a log θ and r = a/ log θ

Solution : r = a log θ :   r = a/ log θ

$$\Rightarrow \ \log r = \log a + \log (\log \theta) \qquad : \log r = \log a - \log (\log \theta)$$

Differentiating these w.r.t q, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\log \theta . \theta} \qquad : \qquad \frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\log \theta . \theta}$$

ie., $\cot \phi_1 = \dfrac{1}{\theta \log \theta}$: $\cot \phi_2 = -\dfrac{1}{\theta \log \theta}$

Note : We can not find ϕ_1 and ϕ_2 explicitly.

$\therefore$ $\tan \phi_1 = \theta \log \theta$: $\tan \phi_2 = \theta \log \theta$

Now consider, $\tan \phi_1 = \theta \log \theta$: $\tan \phi_2 = - \theta \log \theta$

Now consider, $\tan (\phi_1 - \phi_2) = \dfrac{2\theta \log \theta}{1 + \tan \phi_1 \tan \phi_2} = \dfrac{2\theta \log \theta}{1 - (\theta \log \theta)^2}$ ....(1)

We have to find θ by solving the given pair of equations :

$r = a \log \theta$ and $r = a/\log \theta$

Equating the R.H.S we have $a \log \theta = \dfrac{a}{\log \theta}$

ie., $(\log \theta = 1)^2 = 1$ or $\log \theta \Rightarrow \theta = e$

Substituting $\theta = e$ in (1) we get

$\tan (\phi_1 - \phi_2) = \dfrac{2e}{1 - e^2}$ $(\because \log e = 1)$

$\therefore$ angle of intersection $\phi_1 - \phi_2 = \tan^{-1}\left(\dfrac{2e}{1 - e^2}\right) = 2 \tan^{-1} e$

5. Find the angle of intersection of the curves:

r = a (1 – cos θ) and   r = 2a cos θ

Solution : r = a (1 – cos θ)   :   r = 2a cos θ

Taking logarithms we have,

Log r = log a + log (1 – cos θ)   :   log r = log 2a + log (cos θ)

Differentiating these w.r.t θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} \qquad : \qquad \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{\cos\theta}$$

ie., $\cot\phi_1 = \dfrac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} : \cot\phi_2 = -\tan\theta$

ie., $\cot\phi_1 = \cot(\theta/2) \qquad : \qquad \cot\phi_2 = \cot(\pi/2 + \theta)$

$\Rightarrow \phi_1 = \theta/2 \qquad\qquad : \qquad \phi_2 = \pi/2 + \theta$

$\therefore \left|\phi_1 - \phi_2\right| = \left|\theta/2 - \pi/2 - \theta\right| = \left|\pi/2 + \theta 2\right|$ 　　　　　　...(1)

Now consider $r = a(1-\cos\theta) = 2a\cos\theta$

Or $\quad 3\cos\theta = 1$ or $\theta = \cos^{-1}(1/3)$

Substituting this value in (1) we get,

The angle of intersection $\pi/2 + 1/2 \cdot \cos^{-1}(1/3)$

6. Find the angle of intersection of the curves :

$r = a\theta$ and $r = a/\theta$

Solution : $r = a\theta \qquad\qquad : \qquad r = a/\theta$

$\Rightarrow \log r = \log a + \log\theta \qquad : \qquad \log r = \log a - \log\theta$

Differentiating these w.r.t θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\theta} \qquad : \qquad \frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\theta}$$

ie., $\qquad \cot\phi_1 = \dfrac{1}{\theta} \qquad : \qquad \cot\phi_2 = -\dfrac{1}{\theta}$

or $\qquad \tan\phi_1 = \theta \qquad : \qquad \tan\phi_2 = -\theta$

Also by equating the R.H.S of the given equations we have

$a\theta = a/\theta$ or $\theta^2 = 1 \Rightarrow \theta = \pm 1$

When $\theta = 1$, $\tan\phi_1 = 1$, $\tan\phi_2 = -1$ and

When $\theta = -1$, $\tan\phi_1 = -1$, $\tan\phi_2 = 1$.

$\therefore \tan\phi_1 \cdot \tan\phi_2 = -1 \Rightarrow \phi_1 - \phi_2 = \pi/2$

The curves intersect at right angles.

7. Find the pedal equation of the curve: $r\left(1-\cos\theta\right)=2a$

Solution : $r\left(1-\cos\theta\right)=2a$

$$\Rightarrow\ \log r+\log\left(1-\cos\theta\right)=\log 2a$$

Differentiating w.r.t θ, we get

$$\frac{1}{r}\frac{dr}{d\theta}+\frac{\sin\theta}{1-\cos\theta}=0\ \text{ or }\frac{1}{r}\frac{dr}{d\theta}=\frac{-\sin\theta}{1-\cos\theta}$$

$$\therefore\ \cot\phi\ =\ \frac{-2\sin\left(\theta/2\right)\cos\left(\theta/2\right)}{2\sin^2\left(\theta/2\right)}=-\cot\left(\theta/2\right)$$

ie., $\therefore\ \cot\phi\ =\ \cot\left(-\theta/2\right)\Rightarrow\phi=-\left(\theta/2\right)$

Consider $p=r\sin\phi$

$$\therefore\ p=r\sin\left(-\theta/2\right)\ \text{ or } p=-r\sin\left(\theta/2\right)$$

Now we have, $r\left(1-\cos\theta\right)=2a$ ...(1)

$$p=\ -r\sin\left(\theta/2\right)$$ ...(2)

We have to eliminate θ from (1) and (2)

(1) can be put in form $r\ .\ 2\sin^2\left(\theta/2\right)=2a$

ie., $r\sin^2\left(\theta/2\right)=a$

But p/-r = sin (θ/2), from (2)

$$\therefore\ r\left(\frac{p^2}{r^2}\right)=a\ \text{ or } p^2=ar$$

Thus p² = ar is the required pedal equation.

8. Find the pedal equation of the curve: r² = a² sec 2θ

Solution : r² = a² sec 2θ

$$\Rightarrow\ 2\log r=2\log a+\log\left(\sec 2\theta\right)$$

Differentiating w.r.t θ, we get,

$$\frac{2}{r}\frac{dr}{d\theta}=\frac{2\sec 2\theta\tan 2\theta}{\sec 2\theta}\ \ \text{ie., }\frac{1}{r}\frac{dr}{d\theta}=\tan 2\theta$$

ie., $\cot\phi=\cot\left(\pi/2-2\theta\right)\Rightarrow\phi=\pi/2-2\theta$

Consider $p = r \sin \phi$ $\therefore p = r \sin (\pi/2 - 2\theta)$ ie., $p = r \cos 2\theta$

Now we have, $r^2 = a^2 \sec 2\theta$ ...(1)

$\qquad\qquad\qquad p = r \cos 2\theta$ ...(2)

From (2) p/r = cos 2θ or r/p = sec 2 θ

Substituting in (1) we get, $\qquad r^2 = a^2 (r/p)$ or pr = a²

Thus pr = a² is the required pedal equation.

9. Find the pedal equation of the curve: rⁿ = aⁿ cos nθ

Solution : rⁿ = aⁿ cos nθ

$\Rightarrow$ n log r = n log a + log (cos nθ)

Differentiating w.r.t q we get

$\dfrac{n}{r} \dfrac{dr}{d\theta} = \dfrac{-n \sin n\theta}{\cos n\theta}$ ie., $\dfrac{1}{r} \dfrac{dr}{d\theta} = -\tan n\theta$

$\therefore$ cot ϕ = cot $(\pi/2 + n\theta)$ $\Rightarrow$ $\phi = \pi/2 + n\theta$

Consider $p = r \sin \phi$

$\therefore p = r \sin (\pi/2 = n\theta)$ ie., $p = r \cos n\theta$

Now we have, $r^n = a^n \cos n\theta$ ...(1)

$\qquad\qquad\qquad p = r \cos n\theta$ ...(2)

$\therefore$ (1) as a consequence of (2) is $r^n = a^n (p/r)$

Thus rⁿ⁺¹ = paⁿ is the required pedal equation.

10. Find the pedal equation of the curve: rᵐ = aᵐ (cos mθ + sin mθ)

Solution : rᵐ = aᵐ (cos mθ + sin mθ)

Differentiating w.r.t θ, we get,

$\dfrac{m}{r} \dfrac{dr}{d\theta} = \dfrac{-m \sin m\theta + m \cos m\theta}{\cos m\theta + \sin m\theta}$

ie., $\dfrac{1}{r} \dfrac{dr}{d\theta} = \dfrac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta} = \dfrac{\cos m\theta (1 - \tan m\theta)}{\cos m\theta (1 + \tan m\theta)}$

$\therefore$ cot ϕ = cot $(\pi/4 + m\theta)$ $\Rightarrow$ $\phi = \pi/4 + m\theta$

Consider $p = r \sin \phi$

$\therefore\ p = r \sin (\pi/4 + m\theta)$

ie., $p = r \left[\sin (\pi/4) \cos m\theta + \cos (\pi/4) \sin m\theta\right]$

ie., $p = \dfrac{r}{\sqrt{2}} \left(\cos m\theta + \sin m\theta\right)$

(we have used the formula of sin (A + B) and also the values $\sin (\pi/4) = \cos (\pi/4) = 1/\sqrt{2}$)

Now we have, $r^m = a^m \left(\cos m\,\theta + \sin m\,\theta\right)$ ...(1)

$$p = \dfrac{r}{\sqrt{2}} \left(\cos m\,\theta + \sin m\theta\right) \qquad\qquad ...(2)$$

Using (2) in (1) we get,

$$r^m = a^m \ . \ \dfrac{p\sqrt{2}}{r} \quad \text{or}\ \ r^{m+1} = \sqrt{2}\ a^m p$$

Thus $r^{m+1} = \sqrt{2}\ a^m p$ is the required pedal equation.

11. Establish the pedal equation of the curve:

$r^n = a^n \sin n\theta + b^n \cos n\theta$ **in the form** $p^2 \left(a^{2n} + b^{2n}\right) = r^{2n+2}$

Solution : We have $r^n = a^n \sin n\theta + b^n \cos n\theta$

$\Rightarrow\ n \log r = \log \left(a^n \sin n\theta + b^n \cos n\theta\right)$

Differentiating w.r.t θ we get

$$\dfrac{n}{r} \dfrac{dr}{d\theta} = \dfrac{na^n \cos n\theta - nb^n \sin n\theta}{a^n \sin n\theta + b^n \cos n\theta}$$

Dividing by n, $\cot \phi = \dfrac{a^n \cos n\theta - b^n \sin n\theta}{a^n \sin n\theta + b^n \cos n\theta}$

Consider $p = r \sin \phi$

Since ϕ cannot be found, squaring and taking the reciprocal we get,

$$\dfrac{1}{p^2} = \dfrac{1}{r^2} \csc^2 \phi \ \text{ or }\ \dfrac{1}{p^2} = \dfrac{1}{r^2} \left(1 + \cot^2 \phi\right)$$

$$\therefore\ \dfrac{1}{p^2} = \dfrac{1}{r^2} \left\{ 1 + \dfrac{\left(a^n \cos n\theta - b^n \sin n\theta\right)^2}{\left(a^n \sin n\theta + b^n \cos n\theta\right)^2} \right\}$$

ie., $\dfrac{1}{p^2} = \dfrac{1}{r^2} \left\{ \dfrac{\left(a^n \sin n\theta + b^n \cos n\theta\right)^2 + \left(a^n \cos n\theta - b^n \sin n\theta\right)^2}{\left(a^n \sin n\theta + b^n \cos n\theta\right)^2} \right\}$

ie., $\dfrac{1}{p^2} = \dfrac{1}{r^2} \left\{ \dfrac{a^{2n}\left(\sin^2 n\theta + \cos^2 n\theta\right) + b^{2n}\left(\cos^2 n\theta + \sin^2 n\theta\right)}{\left(a^n \sin n\theta + b^n \cos n\theta\right)^2} \right\}$
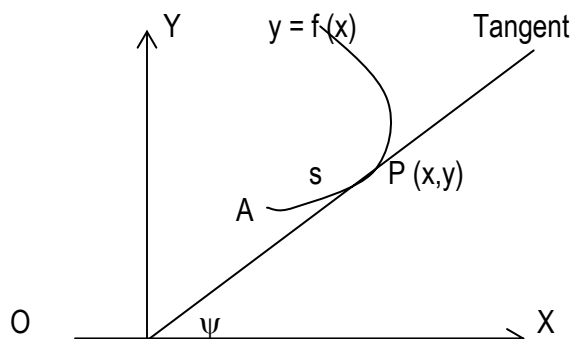
(product terms cancels out in the numerator)

ie., $\dfrac{1}{p^2} = \dfrac{1}{r^2} \cdot \dfrac{a^{2n} + b^{2n}}{\left(a^n \sin n\theta + b^n \cos n\theta\right)^2}$

or $\dfrac{1}{p^2} = \dfrac{1}{r^2} \cdot \dfrac{a^{2n} + b^{2n}}{\left(r^n\right)^2}$, by using the given equation.

$\therefore p^2\left(a^{2n} + b^{2n}\right) = r^{2n+2}$ is the required pedal equation.

12. Define Curvature and Radius of curvature

Solution: A Curve Cuts at every point on it. Which is determined by the tangent drawn.

Let P be a point on the curve y= f (x) at the length 's' from a fixed point A on it. Let the tangent at 'P' makes are angle ψ with positive direction of x – axis. As the point 'P' moves along curve, both s and ψ vary.

The rate of change ψ w.r.t s, i.e., $\dfrac{d\psi}{ds}$ as called the Curvature of the curve at 'P'.

The reciprocal of the Curvature at P is called the radius of curvature at P and is denoted by ρ.

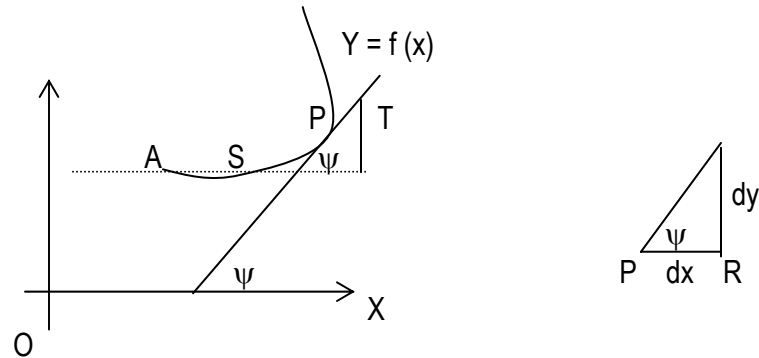$\therefore \rho = \dfrac{1}{\dfrac{d\psi}{ds}} = \dfrac{ds}{d\psi}$

$$\therefore \rho = \frac{ds}{d\psi} \quad \text{(or)} \quad \frac{1}{\rho} = \frac{d\psi}{ds}$$

Also denoted $\rho = \frac{1}{K}$ $\qquad \therefore K = \frac{d\psi}{ds}$

K is read it as Kappa.

13. Derive an expression for radius of curvature in Cartesian form.

Solution :(1) Cartesian Form:

Let y = f (x) be the curve in Cartesian form.

We know that, $\tan \psi = \frac{dy}{dx}$ (From Figure) ----------- (1)

Where ψ is the angle made by the tangent at P with x – axis. Differentiating (1) W.r.t x, we get

$$\text{Sec}^2\psi \cdot \frac{d\psi}{dx} = \frac{d^2y}{dx^2}$$

i.e., $\frac{d^2y}{dx^2} = \text{Sec}^2 \psi \frac{d\psi}{dx}$

$= (1 + \tan^2 \psi) \frac{d\psi}{dx}$

$= (1 + \tan^2 \psi) \frac{d\psi}{dx} \cdot \frac{ds}{dx}$

$= (1 + \tan^2 \psi) \frac{1}{\rho} \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{1}{2}}$

$= \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\} \cdot \frac{1}{\rho} \cdot \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{1}{2}}$ from eqn (1)

$$= \frac{1}{\rho} \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{3}{2}}$$

$$\therefore \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2 \right]^{3/2}}{d^2 y \Big/ dx^2}$$

$$\therefore \rho = \frac{(1 + y_1^2)^{3/2}}{y_2} \quad \dots\dots\dots(1)$$

Where $y_1 = \dfrac{dy}{dx}$, $y_2 = \dfrac{d^2 y}{dx^2}$

This is the formula for Radius of Curvature in Cartesian Form.

14. Show that the Curvature of a Circle at any point on it, is a Constant
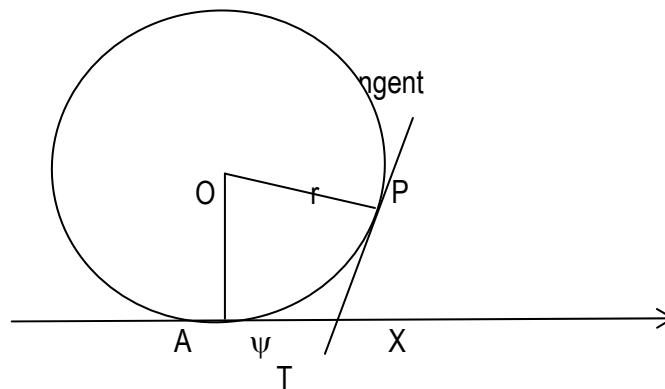
Solution: Consider a Circle of radius r. Let A be a fixed point and 'P' be a given point on the circle such that arc AP = S.

Let the angle between the tangent to the Circle at A and P be ψ. Then clearly AOP = ψ.

$$\therefore AP = r\psi$$
$$\text{i.e., } S = r\psi$$

This is the intrinsic equation of the circle.
Differentiating w.r.t S we get

$$1 = r \frac{d\psi}{ds} \text{ Or K} = \frac{d\psi}{ds} = \frac{1}{r}$$

$$\therefore K = \frac{1}{r} \text{ which is Constant}$$

Hence the Curvature of the Circle at any point on it is constant.

15. Derive an expression for radius of curvature in parametric form.

Solution: We have $\rho = \dfrac{(1 + y_1{}^2)^{3/2}}{y_2}$

Let x = f (t), y = g (t) be the curve in Parametric Form.

Then $y_1 = \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\dot{Y}}{\dot{X}}$

$Y_2 = \dfrac{d^2 y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left(\dfrac{\dot{Y}}{\dot{X}}\right) = \dfrac{d}{dt}\left(\dfrac{\dot{Y}}{\dot{X}}\right)\dfrac{dt}{dx}$

$= \dfrac{d}{dt}\left(\dfrac{\dot{Y}}{\dot{X}}\right)\cdot\left(\dfrac{1}{\dfrac{dx}{dt}}\right) \quad \dfrac{d}{dt}\left(\dfrac{\dot{Y}}{\dot{X}}\right)\cdot\dfrac{1}{\dot{X}}$

$= \dfrac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x})^2}\ \dfrac{1}{(\dot{X})}$

$\therefore Y_2 = \dfrac{\dot{X}\ddot{y} - \dot{y}\ddot{x}}{\left(\dot{x}\right)^3}$

Substituting Y_1 and Y_2 in equation 1.

$\rho = \dfrac{\left(1 + Y_1{}^2\right)^{3/2}}{y_2}$, we get

$\rho = \dfrac{\left[\left(1 + \left(\dot{Y}/\dot{x}\right)\right)^2\right]^{3/2}}{\dfrac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x})^3}}$

$\rho = \dfrac{\left[(\dot{x})^2 + (\dot{y})^2\right]^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$ -------------------- (2)

Equation (2) is called the Radius of Curvature in Parametric Form.

16. Derive an expression for radius of curvature in polar form.

Solution: Let r = f (θ) be the curve in the Polar Form. We know that, Angle between the tangent and radius vector,

Tan ϕ = r $\dfrac{d\theta}{dr}$

= r. $\dfrac{1}{dr/d\theta}$

i.e., tan ϕ = $\dfrac{r}{\left(dr/d\theta\right)}$

Differentiate w.r.t 'θ' we get

$$\text{Sec}^2\phi \cdot \frac{d\phi}{d\theta} = \frac{\dfrac{dr}{d\theta}\cdot\dfrac{dr}{d\theta} - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2}$$

$$\therefore \frac{d\phi}{d\theta} = \frac{1}{Sec^2\phi}\left\{\frac{\left(dr/d\theta\right)^2 \, r\, d^2r/d\theta^2}{\left(dr/d\theta\right)^2}\right\}$$

$$= \frac{1}{1+\tan^2\phi}\left\{\frac{\left(dr/d\theta\right)^2 - r\, d^2r/d\theta^2}{\left(dr/d\theta\right)^2}\right\}$$

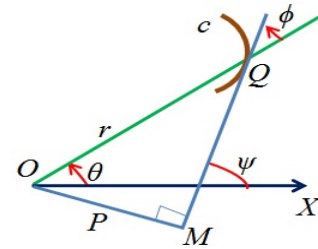$$= \frac{1}{1+\dfrac{r^2}{\left(\dfrac{dr}{d\theta}\right)^2}}\left\{\frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2}\right\}$$

$$\frac{d\phi}{d\theta} = \frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2 + r^2}$$

From figure ψ = θ + ϕ

Differentiating w.r.t θ, we get

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$$\frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta}$$

$$\therefore \quad \frac{d\psi}{d\theta} = 1 + \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

$$\frac{d\psi}{d\theta} = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

Also we know that $\quad \frac{ds}{d\theta} = \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{1}{2}}$

Now, $\rho = \dfrac{ds}{d\psi} = \dfrac{ds}{d\theta} \cdot \dfrac{d\theta}{d\psi}$

$$= \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{1}{2}} \cdot \frac{\left(\frac{dr}{d\theta}\right)^2 + r^2}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$\rho = \frac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 + 2r_1^2 - rr_2} \quad \text{----------- (3)}$$

Where $r_1 = \dfrac{dr}{d\theta}$, $r_2 = \dfrac{d^2r}{d\theta^2}$

Equation (3) as called the radius of curvature in Polar form.

17. Derive an expression for radius of curvature in pedal form.

Solution: Let $p = r \sin \phi$ be the curve in Polar Form.

We have $p = r \sin \phi$

Differentiating p W.r.t r, we get

$$\frac{dp}{dr} = \sin \phi + r \cos \phi \, \frac{d\phi}{dr}$$

But Sin ϕ = r $\dfrac{d\theta}{ds}$, Cos ϕ= $\dfrac{dr}{ds}$

$\therefore$ $\dfrac{dp}{dr}$ = r $\dfrac{d\theta}{ds}$ + r $\dfrac{dr}{ds}$ $\dfrac{d\phi}{dr}$

= r $\dfrac{d\theta}{ds}$ + $\dfrac{d\phi}{dr}$. $\dfrac{dr}{ds}$

= r $\dfrac{d\theta}{ds}$ + r $\dfrac{d\phi}{ds}$

= r $\dfrac{d}{ds}$ (θ + ϕ)

= r $\dfrac{d\psi}{ds}$

$\therefore$ $\dfrac{dp}{dr}$ = r $\dfrac{d\psi}{ds}$

$\dfrac{dp}{dr}$ = r. $\dfrac{1}{\rho}$ $\because \dfrac{d\psi}{ds}$ = $\dfrac{1}{\rho}$

$\therefore$ ρ= r $\dfrac{dr}{dp}$ --------------- (4)

Equation (4) is called Radius of Curvature of the Curve in Polar Form.

18. Find the radius of curvature at (x, y) for the curve $ay^2 = x^3$.

Solution: Given $ay^2 = x^3$ -------- (1) is in Cartesian form.

We have, Radius of curvature in Cartesian form.

$$\rho = \dfrac{\left(1 + y_1^2\right)^{3/2}}{y_2}$$ ----------- (2)

Differentiating (1) w.r.t x we get,

$$a\ 2y\ \frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx} = y_1 = \frac{3}{2}\frac{x^2}{ay} = \frac{3x^2}{2a\left(\frac{x^3}{a}\right)^{\frac{1}{2}}} = \frac{3}{2}\frac{\sqrt{x}}{\sqrt{a}}$$

$$\therefore y_1 = \frac{3}{2}\frac{\sqrt{x}}{\sqrt{a}}$$

Differentiating y_1 w.r.t 'x' we get

$$y_2 = \frac{d^2y}{dx^2} = \frac{3}{2\sqrt{a}}\frac{1}{2\sqrt{x}} = \frac{3}{4\sqrt{a}\sqrt{x}}$$

Substitute Y_1 and Y_2 in (2), we get.

$$\rho = \frac{\sqrt{x}(4a+9x)^{\frac{3}{2}}}{6a}$$

19. Find the radius of curvature at (x,y) for the curve y = c log Sec $\left(\frac{x}{c}\right)$

Solution: Given $y = c\ \log\ Sec\ \left(\frac{x}{c}\right)$ -------- (1)

Differentiating (1) w.r.t x, we get

$$y_1 = c.\frac{1}{Sec\left(\frac{x}{c}\right)}.Sec\left(\frac{x}{c}\right)\tan\left(\frac{x}{c}\right) = c\ \tan\left(\frac{x}{c}\right).\frac{1}{c} = \tan\left(\frac{x}{c}\right)$$

Differentiating y_1 W.r.t x we get

$$y_2 = Sec^2\left(\frac{x}{c}\right)\frac{1}{c} = \frac{1}{c}\ Sec^2\left(\frac{x}{c}\right)$$

Substitute y_1 and y_2 in $\delta = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}$

$$\therefore \delta = \frac{\left[1+\tan^2\left(x/c\right)\right]^{\frac{3}{2}}}{\frac{1}{c}Sec^2\left(x/c\right)} = \frac{\left[Sec^2\left(x/c\right)\right]^{\frac{3}{2}}}{\frac{1}{c}Sec^2\left(x/c\right)}$$

$$\rho = cSec\left(x/c\right)$$

20. Find the radius of curvature at the point 't' on the curve x = a (t + Sint), y = a (1 – Cost).

Solution: Given Curves are in Parametric Form

∴ Radius of Curvature, $\rho = \dfrac{\left((\dot{x})^2 + (\dot{y})^2\right)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$ ------ (1)

Differentiating the given Curves W.r.t t, we get

$\dot{x} = \dfrac{dx}{dt}$ = a (1 + Cost) $\dot{y} = \dfrac{dy}{dt}$ = a Sint

Differentiating W.r.t 't' we get

$\ddot{x}$ = - a Sint, $\ddot{y}$ = a Cost

Substitute $\dot{x}$, $\ddot{y}$, $\dot{y}$ and $\ddot{x}$ in (1), we get

$$\rho = \dfrac{\left(a^2(1+Cost)^2 + a^2 S\text{int}\right)^{3/2}}{a(1+Cost)aCost - aS\text{int}(-aS\text{int})}$$

$$= \dfrac{a^3\left\{1 + 2Cost + Cos^2 t + Sin^2 t\right\}^{3/2}}{a^2\left\{Cost + Cos^2 t + Sin^2 t\right\}}$$

$$= \dfrac{a\left\{2(1+Cost)^{3/2}\right\}}{(1+Cost)} = \dfrac{a\left\{2.2Cos^2\,t/2\right\}^{3/2}}{2Cos^2\,t/2} = \dfrac{8a.Cos^3\,t/2}{2Cos^2\,t/2}$$

$$\rho = 4aCos\,t/2$$

21. Find the Radius of Curvature to x = $a\left\{Cost + \log\tan\left(t/2\right)\right\}$, y = a Sint at t.

Solution: Here x = $a\left\{Cost + \log\tan\left(t/2\right)\right\}$, y = a Sin t

$$\dfrac{dx}{dt} = a\left\{-S\text{int} + \dfrac{1}{\tan t/2}.Sec^2\,t/2.\dfrac{1}{2}\right\}$$

=a { -sint + $\dfrac{1}{2\sin t/2\cos t/2}$ }

=a { -sint + $\dfrac{1}{2\sin t/2}$ }

= a { (1-sin²t) / sint }

$\dfrac{dx}{dt}$ = a Cos²t / sint

$$and \; \frac{dy}{dx} = aCost$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{aCost}{a\cos t\cos t/\sin t} = \text{tant}$$

$$\therefore \frac{dy}{dx} = \tan t$$

Differentiating W.r.t 'x' we get

$$\frac{d^2y}{dx^2} = Sec^2t \frac{dt}{dx} = Sec^2t \frac{1}{\frac{dx}{dt}} = Sec^2t. \frac{1}{aCos^2\;t\Big/_{S\,\text{int}}}$$

$$\frac{d^2y}{dx^2} = \frac{1}{a} = Sec^4tS\,\text{int}$$

Substitute $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in $\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}$, we get

i.e., $\rho = \dfrac{\left(1+\tan^2 t\right)^{3/2}}{\frac{1}{a}Sec^4tS\,\text{int}} = \dfrac{aSec^3t}{Sec^4tS\,\text{int}} = a\dfrac{Cost}{S\,\text{int}} = aCott$

$\therefore \rho = a$ Cot t.

22. Find the Radius of Curvature to $\sqrt{x}+\sqrt{y}=\sqrt{a}$ **at** $\left(\dfrac{a}{4},\dfrac{a}{4}\right)$

Solution: Given $\sqrt{x}+\sqrt{y}=\sqrt{a}$ -------------- (1)

Differentiating (1) w.r.t to '*x*' we get

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

i.e., $y_1 = \dfrac{dy}{dx} = -\dfrac{\sqrt{y}}{\sqrt{x}} = -\dfrac{\left(\sqrt{a}-\sqrt{x}\right)}{\sqrt{x}}$ (From (1))

$$y_1 = 1 - \frac{\sqrt{a}}{\sqrt{x}} \qquad \text{---------- (2)}$$

Also $y_2 = \dfrac{d^2 y}{dx^2} = -\sqrt{a}\left(-\dfrac{1}{2}\right)x^{-3/2} = \dfrac{\sqrt{a}}{2x^{3/2}}$ -------------- (3)

At the given point $\left(\dfrac{a}{4}, \dfrac{a}{4}\right)$

Then $y_1 = 1 - \dfrac{\sqrt{a}}{\sqrt{a/2}} = -1$ & $y_2 = 1 - \dfrac{\sqrt{a}}{2(a/4)^{3/2}} = \dfrac{4}{a}$

$\therefore$ Substitute y_1 and y_2 in $\rho = \dfrac{(1 + y_1^2)^{3/2}}{y_2}$

$\rho = \dfrac{(1 + (-1)^2)^{3/2}}{4/a} = \dfrac{2^{3/2} a}{4} = \dfrac{2\sqrt{2} a}{4} = \dfrac{a}{\sqrt{2}}$

$\rho = \dfrac{a}{\sqrt{2}}$

23. Show that for the Cardioids r = a (1 + Cosθ), ρ² / r² is a constant

Solution: r = a (1 + Cosθ)

$\dfrac{dr}{d\theta}$ = - a Sinθ

We have,

$\dfrac{1}{P^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$, is Pedal Equation.

= $\dfrac{1}{r^2} + \dfrac{1}{r^4}$ a²Sin²θ

= $\dfrac{r^2 + a^2 Sin^2\theta}{r^4} = \dfrac{a^2(1 + Cos\theta)^2 + a^2 Sin^2\theta}{r^4}$

= $\dfrac{2a^2(1 + Cos\theta)}{r^4}$

$$\therefore \frac{1}{P^2} = \frac{2a^2}{r^4}\left(\frac{r}{a}\right) \qquad\qquad (\because r = a\ (1 + \text{Cos}\theta)$$

$$\therefore \frac{r}{a} = 1 + Cos\,\theta$$

$$\frac{1}{P^2} = \frac{2a}{r^3}$$

$$\therefore P^2 = \frac{r^3}{2a}$$

Differentiating w.r.t 'P' we get

$$2P = \frac{1}{2a} 3r^2 \frac{dr}{dp}$$

$$\frac{dr}{dp} = \frac{4ap}{3r^2}$$

Now, $\quad \rho = r\dfrac{dr}{dp} = \dfrac{4ap}{3r}$

And $\quad \dfrac{\rho^2}{r^2} = \dfrac{1}{r}\left[\dfrac{16a^2 p^2}{9r^2}\right] = \dfrac{16a^2}{9r^3}\cdot\dfrac{r^3}{2a} = \dfrac{8a}{9}$

24. Find the Radius of Curvature of the Curve $\dfrac{1}{P^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2 b^2}$

Solution: Given $\dfrac{1}{P^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2 b^2}$

Differentiating w.r.t to P, we get

$$\frac{-2}{P^3} = \frac{-1}{a^2 b^2} 2r \frac{dr}{dp}$$

$$\therefore \frac{dr}{dp} = \frac{a^2 b^2}{p^3 r}$$

$$\therefore \rho = r.\frac{dr}{dp} = r.\frac{a^2 b^2}{p^3 r} = \frac{a^2 b^2}{p^3}$$

25. Find the Radius of Curvature at (r,θ) on r = $\dfrac{a}{\theta}$

Solution: Given r = $\dfrac{a}{\theta}$

Differentiating w.r.t to 'θ' we get

$$\frac{dr}{d\theta} = \frac{-a}{\theta^2} = \frac{-a}{\theta}\cdot\frac{1}{\theta} = \frac{-r}{\theta}$$

$$\frac{dr}{d\theta} = \frac{-r}{\theta}$$

We have $\dfrac{1}{P^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

$$= \frac{1}{r^2} + \frac{1}{r^4}\frac{r^2}{\theta^2} = \frac{1}{r^2} + \frac{1}{r^2\theta^2}$$

$$\frac{1}{P^2} = \frac{1}{r^2}\left(1 + \frac{1}{\theta^2}\right) = \frac{1}{r^2}\left(\frac{\theta^2+1}{\theta^2}\right)$$

$$\therefore P = \frac{r\theta}{\sqrt{\theta^2+1}}$$

$$P = \frac{r\theta}{\sqrt{\theta^2+1}} = \frac{r.a/r}{\sqrt{\frac{a^2}{r^2}+1}} = \frac{a.r}{\sqrt{a^2+r^2}}$$

Differentiating above result w.r.t to'P' we get

$$1 = \frac{\sqrt{a^2+r^2}.a\dfrac{dr}{dp} - r.a\dfrac{1}{2\sqrt{a^2+r^2}}2r\dfrac{dr}{dp}}{\left(a^2+r^2\right)}$$

$$a^2+r^2 = \left(\sqrt{a^2+r^2}.a - \frac{r^2a}{\sqrt{a^2+r^2}}\right)\frac{dr}{dp}$$

$$a^2+r^2 = \frac{(a^2+r^2).a - r^2a}{\sqrt{a^2+r^2}}\frac{dr}{dp}$$

$$\left(a^2 + r^2\right)\sqrt{a^2 + r^2} = a^3 \frac{dr}{dp}$$

$$\frac{\left(a^2 + r^2\right)^{3/2}}{a^3} = \frac{dr}{dp}$$

Thus, $\rho = r.\dfrac{dr}{dp} = \dfrac{r.\left(a^2 + r^2\right)^{3/2}}{a^3}$

$$\therefore \rho = \frac{r}{a^3}\left(a^2 + r^2\right)^{3/2}$$

Exercises:

(1) Find the Radius of the Curvature at the point (s, ψ) on S = a log tan $\left(\dfrac{\pi}{4} + \dfrac{\psi}{2}\right)$

(2) Find the Radius of the Curvature of $xy = C^2$ at (x,y)

(3) Find the Radius of the Curvature of $xy^3 = a^4$ at (a,a)

(4) Find the Radius of Curvature at the point θ on x = C Sin 2θ (1 + Cos 2θ),
       y = C Cos 2θ (1 – Cos 2θ)

(5) If ρ_1 and ρ_2 are the radii of curvature at the extremities of any chord of the cardiode

   r = a (1 + Cosθ) which posses through the Pole prove that $\rho_1^2 + \rho_2^2 = \dfrac{16a^2}{9}$

DIFFERENTIAL CALCULUS – III

PARTIAL DIFFERENTIATION

Introduction :-

Partial differential equations abound in all branches of science and engineering and many areas of business. The number of applications is endless.

Partial derivatives have many important uses in math and science. We shall see that a partial derivative is not much more or less than a particular sort of directional derivative. The only trick is to have a reliable way of specifying directions ... so most of this note is concerned with formalizing the idea of direction

So far, we had been dealing with functions of a single independent variable. We will now consider functions which depend on more than one independent variable; Such functions are called functions of several variables.

<u>Geometrical Meaning</u>

Suppose the graph of $z = f(x,y)$ is the surface shown. Consider the partial derivative of f with respect to x at a point (x_0, y_0). Holding y constant and varying x, we trace out a curve that is the intersection of the surface with the vertical plane $y = y_0$. The partial derivative $f_x(x_0, y_0)$. measures the change in z per unit increase in x along this curve. That is, $f_x(x_0, y_0)$ is just the slope of the curve at (x_0, y_0). The geometrical interpretation of $f_y(x_0, y_0)$. is analogous.
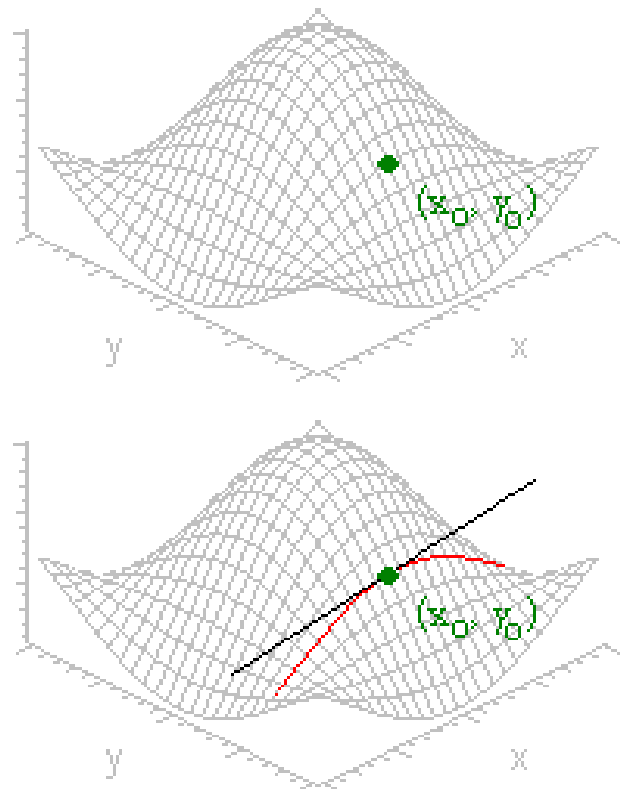
Real-World Applications:

Rates of Change:

In the Java applet we saw how the concept of partial derivative could be applied geometrically to find the slope of the surface in the x and y directions. In the following two examples we present partial derivatives as rates of change. Specifically we explore an application to a temperature function (this example does have a geometric aspect in terms of the physical model itself) and a second application to electrical circuits, where no geometry is involved.

I. Temperature on a Metal Plate

The screen capture below shows a current website illustrating thermal flow for chemical engineering. Our first application will deal with a similar flat plate where temperature varies with position.

* The example following the picture below is taken from the current text in SM221,223:

Multivariable Calculus by James Stewart.
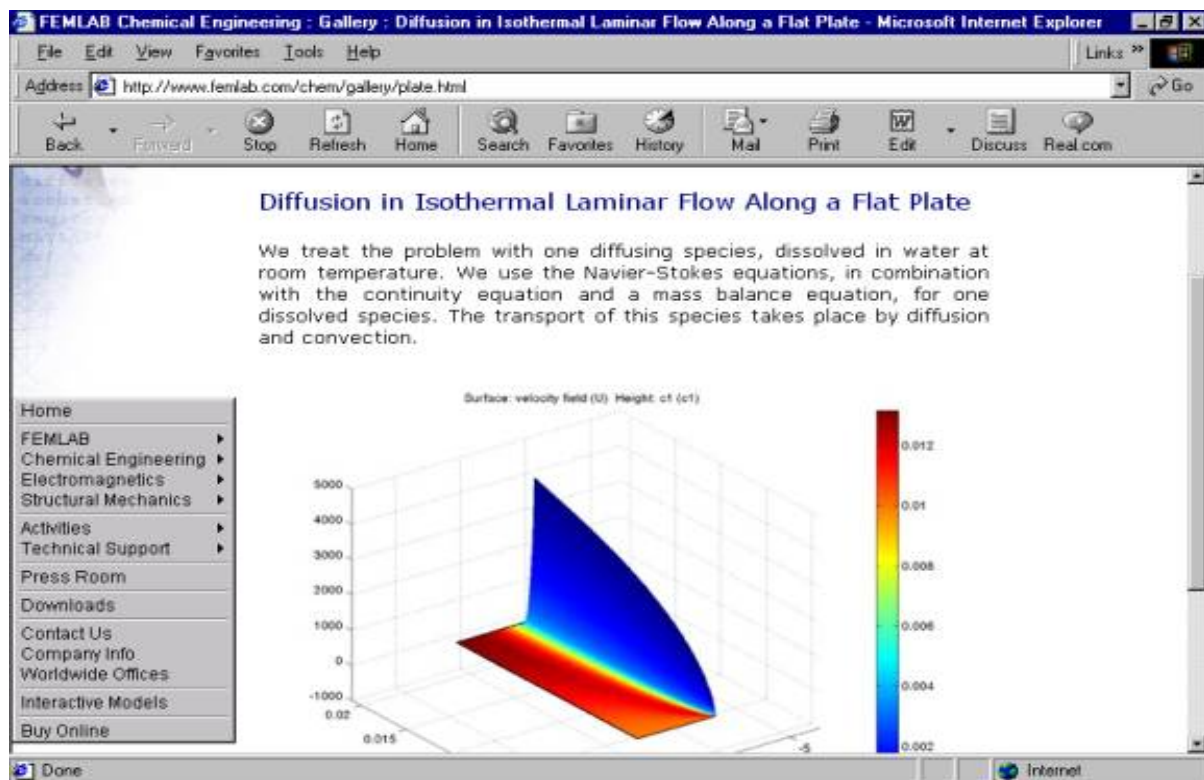
Suppose we have a flat metal plate where the temperature at a point (x,y) varies according to position. In particular, let the temperature at a point (x,y) be given by,

$$T(x, y) = 60 / 1 + x^2 + y^2$$

where T is measured in ℃ and x and y in meters.

Question: what is the rate of change of temperature with respect to distance at the point (2,1) in (a) the x-direction? and (b) in the y-direction ?

Let's take (a) first.

What is the rate of change of temperature with respect to distance at the point (2,1) in (a) the x-direction?

What observations and translations can we make here?

Rate of change of temperature indicates that we will be computing a type of derivative. Since the temperature function is defined on two variables we will be computing a partial derivative. Since the question asks for the rate of change in the x-direction, we will be holding y constant. Thus, our question now becomes:

What is dT/dx at the point (2,1)?

$$T(x, y) = 60 / 1 + x^2 + y^2 = 60(1 + x^2 + y^2)^{-1}$$
$$\partial T/\partial x = -60(2x)(1 + x^2 + y^2)^{-2}$$
$$\partial T/\partial x (2,1) = -60(4)(1 + 4 + 1)^{-2} = -20/3$$

Conclusion :

The rate of change of temperature in the x-direction at (2,1) is $-20/3$ degrees per meter;

note this means that the temperature is decreasing **!**

Part (b):

The rate of change of temperature in the y-direction at (2,1) is computed in a similar manner.

$$T(x, y) = 60/1 + x^2 + y^2 = 60(1 + x^2 + y^2)^{-1}$$
$$\frac{\partial T}{\partial x} = -60(2y)(1 + x^2 + y^2)^{-2}$$
$$\frac{\partial T}{\partial x}(2,1) = -60(2)(1 + 4 + 1)^{-2} = \frac{-10}{3}$$

Conclusion :

The rate of change of temperature in the y-direction at (2,1) is $\frac{-10}{3}$ degrees per meter;

note this means that the temperature is decreasing **!**

II. Electrical Circuits: Changes in Current

The following is adapted from an example in a former text for SM221,223 *Multivariable Calculus* by Bradley and Smith.

* In an electrical circuit with electromotive force (EMF) of E volts and resistance R ohms, the current, I, is

I=E/R amperes.

Question: (a) At the instant when E=120 and R=15 , what is the rate of change of current with respect to voltage.

(b) What is the rate of change of current with respect to resistance?

(a)   Even though no geometry is involved in this example, the rate of change questions can be answered with **partial derivatives.**

we first note that I is a function of E and R, namely,

I(E,R) = ER⁻¹

The rate of change of current with respect to voltage =

 the partial derivative of I with respect to voltage, holding resistance constant is

$$\partial I/\partial E = R^{-1}$$

when **E=120 and R=15** , we have $\partial I/\partial E = 15^{-1} \approx 0.0667$

verbal conclusion : If the resistance is fixed at 15 ohms, the current is increasing with

respect to voltage at the rate of 0.0667 amperes per volt when the EMF is 120 volts.

Part (b):

What is the rate of change of current with respect to   resistance?

Using similar observations to part (a) we conclude:

The partial derivative of I with respect to resistance, holding voltage constant = $\partial I/\partial E = ER^{-1}$

when **E=120 and R=15** , we have $\partial I/\partial E (120,15) = -120(15)^{-1} \approx -0.5333$

Conclusion : If the EMF is fixed at 120 volts, the current is decreasing with respect to

resistance at the rate of 0.5333 amperes per ohm when the resistance is 15 ohms.

Key Words :-

Then the partial derivative of z *w.r.t x* is given by

$$z_x = \frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

The partial derivative of z w.r.t *y* is given by

$$z_y = \frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}$$

1. **If** $u = e^{ax-by} \sin(ax + by)$ **show that** $b\dfrac{\partial u}{\partial x} - a\dfrac{\partial u}{\partial y} = 2\,ab\,u$

Solution : $u = e^{ax-by} \sin(ax + by)$

$$\therefore \frac{\partial u}{\partial x} = e^{ax-by} \cos(ax + by) \cdot a + a.e^{ax-by} \sin(ax + by)$$

ie., $\dfrac{\partial u}{\partial x} = a\, e^{ax-by} \cos(ax + by) + au$ ...(1)

Also $\dfrac{\partial u}{\partial x} = e^{ax-by} \cos(ax + by) \cdot b + (-b)e^{ax-by} \sin(ax + by)$

ie., $\dfrac{\partial u}{\partial y} = b\, e^{ax-by} \cos(ax + by) - bu$ ...(2)

Now $b\,\dfrac{\partial u}{\partial x} = a\,\dfrac{\partial u}{\partial y}$ by using (1) and (2) becomes

$$= abe^{ax-by} \cos(ax + by) + abu - abe^{ax-by} \cos(ax + by) + abu$$

$$= 2\,abu$$

Thus $b\dfrac{\partial u}{\partial x} - a\dfrac{\partial u}{\partial y} = 2\,abu$

2. **If** $u = e^{ax+by} f(ax - by)$, prove that

$$b\frac{\partial u}{\partial x} = a\frac{\partial u}{\partial y} = 2abu$$

Solution : $u = e^{ax+by} f(ax - by)$, by data

$$\frac{\partial u}{\partial x} = e^{ax+by} \cdot f'(ax - by)a + ae^{ax+by}f(ax - by)$$

Or $\dfrac{\partial u}{\partial x} = a\, e^{ax+by} \cdot f'(ax - by) + a\,u$ ...(1)

Next, $\dfrac{\partial u}{\partial y} = e^{ax+by} f'(ax - by) \cdot (-b) + b\, e^{ax+by} f(ax - by)$

Or $\dfrac{\partial u}{\partial y} = -b\,e^{ax+by} f'(ax - by) + ba$ ...(2)

Now consider L.H.S $= b\dfrac{\partial u}{\partial x} + a\dfrac{\partial u}{\partial y}$

$$= b\left\{ae^{ax+by}\,f'\left(ax - by\right)+ au\right\}+ a\left\{- be^{ax+by}\,f'\left(ax - by\right)+ bu\right\}$$

$$= ab\,e^{ax+by}\,f'\left(ax - by\right)+ abu - ab\,e^{ax + by}\,f'\left(ax - by\right)+ abu$$

$$= 2abu = R.H.S$$

Thus $b\dfrac{\partial u}{\partial x}+ a\dfrac{\partial u}{\partial y}= 2\,a\,b\,u$

3. If $u = \log\sqrt{x^2 + y^2 + z^2}$, **show that** $\left(x^2 + y^2 + z^2\right)\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}\right)=1$

Solution : By data $u = \log\sqrt{x^2 + y^2 + z^2}=\dfrac{1}{2}\log\left(x^2 + y^2 + z^2\right)$

The given u is a symmetric function of x, y, z,

(It is enough if we compute only one of the required partial derivative)

$$\frac{\partial u}{\partial x}=\frac{1}{2}\cdot\frac{1}{x^2 + y^2 + z^2}\cdot 2x=\frac{x}{x^2 + y^2 + z^2}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2 + z^2}\right)$$

ie., $\dfrac{\left(x^2 + y^2 + z^2\right)1 - x\cdot 2x}{\left(x^2 + y^2 + z^2\right)^2}=\dfrac{y^2 + z^2 - x^2}{\left(x^2 + y^2 + z^2\right)^2}$

$$\therefore\ \frac{\partial^2 u}{\partial x^2}=\frac{y^2 + z^2 - x^2}{\left(x^2 + y^2 + z^2\right)^2}\qquad\qquad\qquad ...(1)$$

Similarly $\therefore\ \dfrac{\partial^2 u}{\partial y^2}=\dfrac{z^2 + x^2 - y^2}{\left(x^2 + y^2 + z^2\right)^2}\qquad\qquad ...(2)$

$$\frac{\partial^2 u}{\partial z^2}=\frac{x^2 + y^2 - z^2}{\left(x^2 + y^2 + z^2\right)^2}\qquad\qquad\qquad ...(3)$$

Adding (1), (2) and (3) we get,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{x^2 + y^2 + z^2}{\left(x^2 + y^2 + z^2\right)^2}=\frac{1}{x^2 + y^2 + z^2}$$

Thus $\left(x^2 + y^2 + z^2\right)\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}\right)=1$

4.   **If u = log (tan x + tan y + tan z), show that,**

sin 2x u_x + sin 2y u_y + sin 2z u_z = 2

Solution : u = log (tan x + tan y + tan z) is a symmetric function.

$$u_x = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\sin 2x\, u_x = \frac{(2\sin x \cos x)\sec^2 . x}{\tan x + \tan y + \tan z}$$

Or $\sin 2x\, u_x = \dfrac{2\tan x}{\tan x + \tan y + \tan z}$ ...(1)

Similarly $\sin 2y\, u_y = \dfrac{2\tan y}{\tan x + \tan y + \tan z}$ ...(2)

$$\sin 2z\, u_y = \frac{2\tan z}{\tan x + \tan y + \tan z}$$ ...(3)

Adding (1), (2) and (3) we get,

$$\sin 2x\, u_x + \sin 2y\, u_y + \sin 2z\, u_z = \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)} = 2$$

Thus $\sin 2x\, u_x + \sin 2y\, u_y + \sin 2z\ u_z = 2$

5.   **If u = log (x^3 + y^3 + z^3 – 3xyz) then prove that** $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = \dfrac{3}{x + y + z}$ **and hence show**

that $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z} \right)^2 u = \dfrac{-9}{(x + y + z)^2}$

Solution : u = log (x^3 + y^3 + z^3 – 3xyz) is a symmetric function

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$ ...(1)

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz}$$ ...(2)

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$ ...(3)

Adding (1), (2) and (3) we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3\left(x^2 + y^2 + z^2 - xy - yz - zx\right)}{\left(x^3 + y^3 + z^3 - 3xyz\right)}$$

Recalling a standard elementary result,

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)\left(a^2 + b^2 + c^2 - ab - bc - ca\right)$$

We have,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3\left(x^2 + y^2 + z^2 - xy - yz - zx\right)}{(x + y + z)\left(x^2 + y^2 + z^2 - xy - yz - zx\right)}$$

Thus $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = \dfrac{3}{x + y + z}$

Further $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x + y + z}\right), \text{ by using the earlier result.}$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x + y + z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x + y + z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x + y + z}\right)$$

$$= \frac{-3}{(x + y + z)^2} + \frac{-3}{(x + y + z)^2} + \frac{-3}{(x + y + z)^2} = \frac{-9}{(x + y + z)^2}$$

Thus $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u = \dfrac{-9}{(x + y + z)^2}$

6.   If $u = f(r)$ and $x = r \cos \theta$, $y = r \sin \theta$,

        prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r} f'(r)$

Solution : Observing the required partial derivative we conclude that u must be a function of x, y. But

          u = f(r) by data and hence we need to have r as a function of x, y. Since x = r cos θ, y = r

          sin θ we have x^2 + y^2 = r^2.

$$\therefore \ \text{we have } u = f(r) \text{ where } r = \sqrt{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r^3} \left(r^2 - x^2\right) + \frac{f''(r)}{r^2} \cdot x^2 \text{ and}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r^3} \left(r^2 - y^2\right) + \frac{f''(r)}{r^2} \cdot y^2$$

Adding these results we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r^3} \left\{2x^2 - \left(x^2 + y^2\right)\right\} + \frac{f''(r)}{r^2} \left(x^2 + y^2\right)$$

$$= \frac{f'(r)}{r^3} \cdot r^2 + \frac{f''(r)}{r^2} \cdot r^2 = \frac{1}{r} f'(r) + f''(r)$$

Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r} f'(r)$

7.   **Prove that** $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = nu$

   Proof : Since u = f (x, y) is a homogeneous function of degree n we have by the definition,

       $u = x^n \, g\left(y/x\right)$ ...(1)

       Let us differentiate this w.r.t x and also w.r.t.y

$$\therefore \ \frac{\partial u}{\partial x} = x^n \ \cdot g'\left(y/x\right) \cdot \left(-\frac{y}{x^2}\right) + nx^{n-1} g\left(y/x\right)$$

       ie., $\dfrac{\partial u}{\partial x} = x^{n-2} \, y \, g'\left(y/x\right) + nx^{n-1} \, g\left(y / x\right)$ ...(2)

       Also $\dfrac{\partial u}{\partial y} = x^n \cdot g'\left(y/x\right) \cdot \left(\dfrac{1}{x}\right)$

ie., $\dfrac{\partial u}{\partial y} = x^{n-1} \cdot g'(y/x)$ ...(3)

Now consider $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ as a consequence of (2) and (3)

$= x\left[- x^{n-2}\, y\, g'(y/x) + n\, x^{n-1}\, g(y/x)\right] + y\left[x^{n-1}\, g'(y/x)\right]$

$= -x^{n-1}\, y\, g'(y/x) + n\, x^{n}\, g(y/x) + x^{n-1}\, y\, g'(y/x)$

$= n \cdot x^{n}\, g(y/x)$

= n u, by using (1)

Thus we have proved Euler's theorem

$$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n\, u \; ; \; x\, u_x + y u_y = n\, u$$

8.   **Prove that** $x^2\dfrac{\partial^2 u}{\partial x^2} + 2\, x\, y\dfrac{\partial^2 u}{\partial x\,\partial y} = n(n-1)u$

Proof : Since u = f (x, y) is a homogeneous function of degree n, we have Euler's theorem

$$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n\, u \qquad\qquad ...(1)$$

Differentiating (1) partially w.r.t. x and also w.r.t y we get,

$$\left(x\dfrac{\partial^2 u}{\partial x^2} + 1 \cdot \dfrac{\partial u}{\partial x}\right) + y\dfrac{\partial^2 u}{\partial x\,\partial y} = n\dfrac{\partial u}{\partial x} \qquad\qquad ...(2)$$

Also, $x\dfrac{\partial^2 u}{\partial y\,\partial x} + \left(y\dfrac{\partial^2 u}{\partial y^2} + 1 \cdot \dfrac{\partial u}{\partial y}\right) = n\dfrac{\partial u}{\partial y} \qquad\qquad ...(3)$

We shall now multiply (2) by x and (3) by y.

$$x^2\dfrac{\partial^2 u}{\partial x^2} + x\dfrac{\partial u}{\partial x} + x\, y\dfrac{\partial^2 u}{\partial x\partial y} = n\, x\dfrac{\partial u}{\partial x} \text{ and}$$

$$x\, y\dfrac{\partial^2 u}{\partial y\,\partial x} + y^2\dfrac{\partial^2 u}{\partial y^2} + y\dfrac{\partial u}{\partial y} = ny\dfrac{\partial u}{\partial y}$$

Adding these using the fact that $\dfrac{\partial^2 u}{\partial y\,\partial x} = \dfrac{\partial^2 u}{\partial y\,\partial x}$ we get,

$$\left(x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} \right) + \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = n \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right)$$

ie., $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2 x y \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} + n\, u = n\,(n\,u)$, by using (1)

or $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2 x y \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} + n\,(nu) - nu = n\,(n-1)u$

Thus $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2 x y \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} + n\,(n-1)u$

ie., $x^2 u_{xx} + 2 x y\, u_{xy} + y^2 u_{yy} = n\,(n-1)u$

9. **If** $u = \dfrac{x}{y+z} + \dfrac{y}{z+x} + \dfrac{z}{x+y}$ **show that** $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} + z \dfrac{\partial u}{\partial z} = 0$

Solution : (Observe that the degree is 0 in every term)

$$u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$$

We shall divide both numerator and denominator of every term by x.

$$u = \frac{1}{y/x + z/x} + \frac{y/x}{z/x + 1} + \frac{z}{1 + y/x} = x^{\circ}\,\{g\,(y/x,\, z/x)\}$$

$\Rightarrow$ u is homogeneous of degree 0. $\therefore$ n = 0

We have Euler's theorem, $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} + z \dfrac{\partial u}{\partial z} = n\, u$

Putting n = 0 we get, $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} + z \dfrac{\partial u}{\partial z} = 0$

10. **If** $u = \log\left(\dfrac{x^4 + y^4}{x + y} \right)$ show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 3$

Solution : we cannot put the given u in the form x^n g (y/x)

$$\therefore\ e^u = \frac{x^4 + y^4}{x + y} = \frac{x^4\left(1 + y^4/x^4\right)}{x\left(1 + y/x\right)} = x^3 \left\{ \frac{1 + (y/x)^4}{1 + (y/x)} \right\}$$

ie., $e^u = x^3$ g (y/x) $\Rightarrow$ e^u is homogeneous of degree 3 $\therefore$ n = 3

Now applying Euler's theorem for the homogeneous function e^u

We have $x\dfrac{\partial(e^u)}{\partial x} + y\dfrac{\partial(e^u)}{\partial y} = n\,e^u$

ie., $x\,e^u\dfrac{\partial u}{\partial x} + y\,e^u\dfrac{\partial u}{\partial y} = 3e^u$

Dividing by e^u we get $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$

11.   If $u = \tan^{-1}\left(\dfrac{x^3 + y^3}{x - y}\right)$ **show that**

(i) $x\,u_x + y\,u_y = \sin 2\,u$

(ii) $x^2 u_{xx} + 2\,x\,y\,u_{xy} + y^2 u_{yy} = \sin 4\,u - \sin 2\,u$

Solution : (i) $u = \tan^{-1}\left(\dfrac{x^3 + y^3}{x - y}\right)$ by data

$$\Rightarrow\ \tan u = \frac{x^3 + y^3}{x - y} = \frac{x^3\left(1 + y^3/x^3\right)}{x\left(1 - y/x\right)} = x^2\left\{\frac{1 + (y/x)^3}{1 - (y/x)}\right\}$$

ie., $\tan u = x^2/g\,(y/x) \Rightarrow \tan u$ is homogeneous of degree 2.

Applying Euler's theorem for the function $\tan u$ we have,

$$x\frac{\partial(\tan u)}{\partial x} + y\frac{\partial(\tan u)}{\partial y} = n\,.\,\tan u\,;\, n = 2$$

ie., $x\sec^2 u\dfrac{\partial u}{\partial x} + y\sec^2 u\dfrac{\partial u}{\partial y} = 2\tan u$

or $\qquad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{2\tan u}{\sec^2 u} = 2\cos^2 u\,\dfrac{\sin u}{\cos u} = 2\cos u\,\sin u = \sin 2\,u$

$\therefore\ x u_x + y u_y = \sin 2u$

(ii)   We have $x u_x + y\,u_y = \sin 2\,u$ …(1)

Differentiating (1) w.r.t x and also w.r.t y partially we get

$x\,u_{xx} + 1\,.\,u_x + y u_{xy} = 2\cos 2u\,.\,u_x$ …(2)

And $x\,u_{yx} + y u_{yy} + 1\,.\,u_y = 2\cos 2u\,.\,u_y$ …(3)

Multiplying (2) by x and (3) by y we get,

$x^2 u_{xx} + x\,u_x + xy\,u_{xy} = 2\cos 2\,u.\,x u_x$

$$xy\, u_{yx} + y^2\, u_{yy} + y\, u_y = 2\cos 2u \cdot yu_y$$

Adding these by using the fact that $u_{yx} = u_{xy}$, we get

$$x^2 u_{xx} + 2\, x\, y\, u_{xy} + y^2 u_{yy} + \left(xy_x + yu_y\right) = 2\cos 2u\left(xu_x + y\, u_y\right)$$

By using (1) we have,

$$x^2 u_{xx} + 2\, x\, y\, u_{xy} + y^2 u_{yy} = 2\cos 2u\ \sin 2u - \sin 2u$$

(since $\sin 2\theta = 2\cos\theta\sin\theta$, first term in the R.H.S becomes $\sin 4u$)

Thus $x^2 u_{xx} + 2\, x\, y\, u_{xy} + y^2\, u_{yy} = \sin 4u - \sin 2u$

12.   **If** $u = f\left(\dfrac{x}{y}, \dfrac{y}{z}, \dfrac{z}{x}\right)$ **Prove that** $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$

>> here we need to convert the given function u into a composite function.

Let $u = f\left(p, q, r\right)$ where $p = \dfrac{x}{y}, q = \dfrac{y}{z}, r = \dfrac{z}{x}$

ie., $\{u \to (p, q, r) \to (x, y, z)\} \Rightarrow u \to x,\, y,\, z$

$$\therefore\ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

ie., $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial p}\ \cdot\ \dfrac{1}{y} + \dfrac{\partial u}{\partial q}\ \cdot\ 0 + \dfrac{\partial u}{\partial r}\ \cdot\left(-\dfrac{z}{x^2}\right)$

$$\therefore\quad x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial p} - \frac{z}{x}\frac{\partial u}{\partial r} \qquad\qquad\qquad \text{...(1)}$$

Similarly by symmetry we can write,

$$y\frac{\partial u}{\partial y} = \frac{y}{z}\frac{\partial u}{\partial q} - \frac{x}{y}\frac{\partial u}{\partial p} \qquad\qquad\qquad \text{...(2)}$$

$$z\frac{\partial u}{\partial z} = \frac{z}{x}\frac{\partial u}{\partial r} - \frac{y}{z}\frac{\partial u}{\partial q} \qquad\qquad\qquad \text{...(3)}$$

Adding (1), (2) and (3) we get $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$

13. If u = f(x – y, y – z, z – x) show that $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0$

>> Let u = f(p, q, r) where p = x – y, q = y – z, r = z – x

$$\therefore \frac{\partial u}{\partial x}=\frac{\partial u}{\partial p}\,\frac{\partial p}{\partial x}+\frac{\partial u}{\partial q}\,\frac{\partial q}{\partial x}+\frac{\partial u}{\partial r}\,\frac{\partial r}{\partial x}$$

ie., $\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial p}\,.\,1+\dfrac{\partial u}{\partial q}\,.0+\dfrac{\partial u}{\partial r}\,(-1)$

$$\therefore \frac{\partial u}{\partial x}=\frac{\partial u}{\partial p}-\frac{\partial u}{\partial r} \qquad\qquad ...(1)$$

Similarly we have by symmetry

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial q}-\frac{\partial u}{\partial p} \qquad\qquad ...(2)$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial q} \qquad\qquad ...(3)$$

Adding (1), (2) and (3) we get, $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0$

14. If z = f(x, y) where x = r cos θ and y = r sin θ

Show that $\left(\dfrac{\partial z}{\partial x}\right)^{2}+\left(\dfrac{\partial z}{\partial y}\right)^{2}=\left(\dfrac{\partial z}{\partial r}\right)^{2}+\dfrac{1}{r^{2}}\left(\dfrac{\partial z}{\partial \theta}\right)^{2}$

Solution : $\{z\rightarrow(x,\,y)\rightarrow(r,\theta)\}\Rightarrow z\rightarrow(r,\theta)$

$$\therefore \frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\,\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\,\frac{\partial y}{\partial r}\,;\,\frac{\partial z}{\partial x}\,\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\,\frac{\partial y}{\partial \theta}$$

ie., $\dfrac{\partial z}{\partial r}=\dfrac{\partial z}{\partial x}\cos\theta+\dfrac{\partial z}{\partial y}\,\sin\theta$ \qquad\qquad ...(1)

and $\dfrac{\partial z}{\partial \theta}=\dfrac{\partial z}{\partial x}(-r\sin\theta)+\dfrac{\partial z}{\partial y}\,(r\cos\theta)=r\left[\dfrac{-\partial z}{\partial x}\sin\theta+\dfrac{\partial z}{\partial y}\cos\theta\right]$

or $\dfrac{1}{r}=\dfrac{\partial z}{\partial \theta}=\dfrac{-\partial z}{\partial x}\,\sin\theta+\dfrac{\partial z}{\partial y}\,\cos\theta$

squaring and adding (1), (2) and collecting suitable terms have,

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\left[\cos^2\theta + \sin^2\theta\right]$$

$$+ \left(\frac{\partial z}{\partial y}\right)^2\left[\sin^2\theta + \cos^2\theta\right] + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta$$

$$\therefore \quad \left(\frac{\partial z}{\partial r}\right)^2 \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \quad \text{ie., R.H.S = L.H.S}$$

15. If $z = f(x, y)$ where $\mathrm{x} = e^u + e^{-v}, \ \mathrm{y} = e^{-u} - e^v$

Prove that $\mathrm{x}\dfrac{\partial z}{\partial x} - \mathrm{y}\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}$

Solution : $\{z \to (\mathrm{x},\,\mathrm{y}) \to (\mathrm{u},\,\mathrm{v})\} \Rightarrow z \to (\mathrm{u},\,\mathrm{v})$

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} ;\ \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$ie., \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot e^u + \frac{\partial z}{\partial y}\left(-e^{-u}\right) \qquad\qquad \text{...(1)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(-e^{-v}\right) + \frac{\partial z}{\partial y}\left(-e^{-v}\right) \qquad\qquad \text{...(2)}$$

Consider R.H.S = $\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}$ and (1) – (2) yields

$$\frac{\partial z}{\partial x}\left(e^u + e^{-v}\right) - \frac{\partial z}{\partial y}\left(e^{-u} - e^v\right) = \frac{\partial z}{\partial x}\cdot x - \frac{\partial z}{\partial y}\cdot y$$

Thus $\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial y} = x\dfrac{\partial z}{\partial x} - \mathrm{y}\dfrac{\partial z}{\partial y}$ ie., R.H.S = L.H.S

16. Find $\dfrac{\partial(\mathrm{u},\mathrm{v},\mathrm{w})}{\partial(\mathrm{x},\mathrm{y},\mathrm{z})}$ where u = x² + y² + z², v = xy+yz+zx, w=x+y+z

Solution :   The definition of J = $\dfrac{\partial(\mathrm{u},\mathrm{v},\mathrm{w})}{\partial(\mathrm{x},\mathrm{y},\mathrm{z})} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\[2mm] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\[2mm] \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$

But   u = x² + y² + z², v = xy+yz+zx, w=x+y+z

Substituting for the partial derivatives we get

$$J = \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

Expanding by the first row,

J = 2x {(x + z) – (y + x)} -2y {(y + z) – (y + x)}

　　　+2z {(y + z) – (x + z)}

= 2x (z-y) – 2y(z-x) + 2z(y-x)

= 2(xz – xy – yz + xy + yz – xz) = 0 Thus **J = 0**

17.　　If u = $\dfrac{yz}{x}$, v = $\dfrac{zx}{y}$, w = $\dfrac{xy}{z}$, **show that** $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$ = 4

Solution :　　by data u = $\dfrac{yz}{x}$, v = $\dfrac{zx}{y}$, w = $\dfrac{xy}{z}$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \dfrac{-yz}{x^2} & \dfrac{z}{x} & \dfrac{y}{x} \\ \dfrac{z}{y} & \dfrac{-zx}{y^2} & \dfrac{x}{y} \\ \dfrac{y}{z} & \dfrac{x}{z} & \dfrac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{-yz}{x^2}\left\{\left(\frac{-zx}{y^2}\right)\left(\frac{-xy}{z^2}\right) - \left(\frac{x}{z}\right)\left(\frac{x}{y}\right)\right\}$$

$$- \frac{z}{x}\left\{\frac{z}{y}\left(\frac{-zx}{y^2}\right) - \frac{y}{z}\cdot\frac{x}{y}\right\} + \frac{y}{x}\left\{\frac{z}{y}\cdot\frac{x}{z} - \frac{y}{z}\left(\frac{-zx}{y^2}\right)\right\}$$

$$= \frac{-yz}{x^2}\left\{\frac{x^2}{yz} - \frac{x^2}{yz}\right\} - \frac{z}{x}\left\{\frac{-x}{z} - \frac{x}{z}\right\} + \frac{y}{x}\left\{\frac{x}{y} + \frac{x}{y}\right\}$$

= 0+1+1+1+1=4

Thus $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$ = 4

18.   If u + v = e^x cos y and u – v = e^x sin y find the jacobian of the functions u and v w.r.t x and y.

Solution :   we have to find $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

Using the given data we have to solve for u and v in terms of x and y.

By data u + v = e^x cos y   ......(1)

  u – v = e^x sin y   ......(2)

(1) + (2) gives : 2 u = e^x (cos y + sin y)

(2) – (2) gives : 2 v = e^x (cos y – sin y)

le., u = $\dfrac{e^x}{2}$ (cos y + sin y) ; v = $\dfrac{e^x}{2}$ (cos y – sin y)

$\therefore \dfrac{\partial u}{\partial x}$ = $\dfrac{e^x}{2}$ (cos y + sin y), $\dfrac{\partial v}{\partial x}$ = $\dfrac{e^x}{2}$ (- sin y - cos y)

Now $\dfrac{\partial(u,v)}{\partial(x,y)}$ = $\begin{vmatrix} \dfrac{e^x}{2}(\cos y + \sin y) & \dfrac{e^x}{2}(-\sin y + \cos y) \\ \dfrac{e^x}{2}(\cos y - \sin y) & \dfrac{-e^x}{2}(\sin y + \cos y) \end{vmatrix}$

  = $\dfrac{e^x}{2} \cdot \dfrac{e^x}{2}$ { - (cos y + sin y)^2 – (cos y – sin y)^2}

  = $\dfrac{-e^{2x}}{4}$ {1+sin 2y) + (1 – sin 2y)} = $\dfrac{-e^{2x}}{2}$

Thus $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{-e^{2x}}{2}$

19.   **(a) If x = r cos θ, y = r sin θ find the value of** $\dfrac{\partial(r,\theta)}{\partial(x,y)}$

   (b)   Further verify that $\dfrac{\partial(x,y)}{\partial(r,\theta)} \cdot \dfrac{\partial(r,\theta)}{\partial(x,y)} = 1$

(a)   **Solution :**   We shall first express r, θ in terms of x and y.

   We have x = r cos θ, y = r sin θ by data.

$\therefore$ x^2 + y^2 = r^2 and $\dfrac{y}{x}$ = tan θ or θ = tan^{-1} (y/x)

Consider r^2 = x^2 + y^2

Differentiating partially w.r.t x and also w.r.t y we get,

$\qquad$ 2r $\dfrac{\partial r}{\partial x}$ = 2x $\qquad$ and $\qquad$ 2r $\dfrac{\partial r}{\partial y}$ = 2y

$\therefore \qquad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$ $\qquad$ and $\qquad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

Also consider $\theta = \tan^{-1}(y/x)$

$\therefore \qquad \dfrac{\partial \theta}{\partial x} = \dfrac{1}{1+(y/x)^2} \cdot \dfrac{-y}{x^2}$ $\qquad$ and $\qquad \dfrac{\partial \theta}{\partial y} = \dfrac{1}{1+(y/x)^2} \cdot \dfrac{1}{x}$

i.e., $\qquad \dfrac{\partial \theta}{\partial x} = \dfrac{-y}{x^2+y^2}$ $\qquad$ and $\qquad \dfrac{\partial \theta}{\partial y} = \dfrac{x}{x^2+y^2}$

Now $\dfrac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial r}{\partial r} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{x}{r} & \dfrac{y}{r} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{vmatrix}$

i.e., $\qquad = \dfrac{x^2}{r(x^2+y^2)} + \dfrac{y^2}{r(x^2+y^2)} = \dfrac{(x^2+y^2)}{r(x^2+y^2)} = \dfrac{1}{r}$

$\therefore \dfrac{\partial(r,\theta)}{\partial(x,y)} = \dfrac{1}{r}$

Solution (b) : Consider $x = r\cos\theta,\ y = r\sin\theta$

$$\frac{\partial(x, y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{d\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$\therefore \frac{\partial(x, y)}{\partial(r,\theta)} = r$$

From (1) and (2) : $\dfrac{\partial(x, y)}{\partial(x,\theta)} \cdot \dfrac{\partial(r,\theta)}{\partial(x, y)} = r \cdot \dfrac{1}{r} = 1$

20. If $x = u(1-v),\ y = uv$ **then show that** $JJ^{/} = 1$

Solution : $x = u(1-v);\ y = uv$

$$\frac{\partial x}{\partial u} = (1-v),\ \frac{\partial y}{\partial u} = v \qquad \frac{\partial x}{\partial v} = -u,\ \frac{\partial y}{\partial v} = u$$

$$J = \frac{\partial(x, y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} (1-v) & -u \\ v & u \end{vmatrix}$$

$$= (1-v)u + uv = u \quad \therefore\ J = u$$

Next we shall express u and v in terms of x and y.

By data x = u - uv and y = uv

Hence $x + y = u$. Also $v = \dfrac{y}{u} = \dfrac{y}{x+y}$

Now we have, $u = x + y;\ v = \dfrac{y}{x+y} \qquad \therefore\quad \dfrac{\partial u}{\partial x} = 1, \dfrac{\partial u}{\partial y} = 1,$

$$\frac{\partial v}{\partial x} = \frac{(x+y)\cdot 0 - y\cdot 1}{(x+y)2} = \frac{x}{(x+y)^2}$$

$$\therefore J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \dfrac{-y}{(x+y)^2} & \dfrac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \frac{x+y}{(x+y)^2} = \frac{1}{(x+y)} = \frac{1}{u}$$

Thus $J' = \dfrac{1}{u}$ Hence $J \cdot J' = u \cdot \dfrac{1}{u}$ Thus $JJ' = 1$

21.   State Taylor's Theorem for Functions of Two Variables.

Statement: Considering f (x + h, y + k) as a function of a single variable x, we have by Taylor's Theorem

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial f(x,y+k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x,y+k)}{\partial x^2} + \text{---------(1)}$$

Now expanding f (x, y + k) as function of y only,

$$f(x, y+k) = f(x, y) + k\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + \text{--------}$$

$\therefore$ (i) takes the form $f(x+h, y+k) = f(x,y) + k\dfrac{\partial f(x,y)}{\partial y} + \dfrac{k^2}{2!}\dfrac{\partial^2 f(x,y)}{\partial y^2}$ +----------------------------- +

$$h\frac{\partial}{\partial x}\left\{ f(x,y) + k\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + -----\right\}$$

$$+ \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{ f(x,y) + k\frac{\partial(x,y)}{\partial y} + --------\right\}$$

Hence $f(x+h, y+k) = f(x,y) + h\dfrac{\partial f}{\partial x} + k\dfrac{\partial f}{\partial y} + \dfrac{1}{2!}\left(h^2\dfrac{\partial^2 f}{\partial x^2} + 2hk\dfrac{\partial^2 f}{\partial x\partial y} + k^2\dfrac{\partial^2 f}{\partial y^2}\right) + --(1)$

In symbol we write it as

$$F(x+h, y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + ----$$

Taking $x = a$ and $y = b$, (1) becomes

$$f(a+h, b+k) = f(a,b) + [h\, f_x(a,b) + k f_y(a,b)] + \frac{1}{2!}\, [h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b)$$

$$+ k^2 f_{yy}(a,b)] + --------$$

Putting $a + h = x$ and $b + k = y$ so that $h = x - a$, $k = y - b$, we get

$$F(x,y) = f(a,b) + [(x - a)\, f_x(a,b) + (y - b)\, f_y(a,b)]$$

$$= \frac{1}{2!}\, [(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)\, f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + ------ \quad (2)$$

This is Taylor's expansion of $f(x,y)$ in powers of $(x - a)$ and $(y - b)$. It is used to expand $f(x,y)$ in the neighborhood of (a,b)

corollary, putting $a = 0$, $b = 0$ in (2), we get

$$f(x,y) = f(0,0) + [x\, f_x(0,0) + y\, f_y(0,0)] + \frac{1}{2!}\, [x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0)$$

$$+ y^2 f_{yy}(0,0)] + --------- \quad (3)$$

This is Maclaurin's Expansion of $f(x,y)$

22.　　Expand $e^x \log(1 + y)$ in powers of x and y up to terms of third degree.

Solution: Here

$$f(x,y) = e^x \log(1 + y) \quad \therefore f(0,0) = 0$$

$$f_x(x,y) = e^x \log(1 + y) \quad \therefore f_x(0,0) = 0$$

$$f_y(x,y) = e^x \frac{1}{1+y} \qquad\qquad \therefore f_y(0,0) = 1$$

$$f_{xx}(x,y) = e^x \log(1 + y) \quad \therefore f_{xx}(0,0) = 0$$

$$f_{xy}(x,y) = e^x \frac{1}{1+y} \qquad\qquad \therefore f_{xy}(0,0) = 1$$

$f_{yy}(x,y) = -e^x(1+y)^{-2}$ $\therefore f_{yy}(0,0) = -1$

$f_{xxx}(x,y) = e^x \log(1+y)$ $\therefore f_{xxx}(0,0) = 0$

$f_{xxy}(x,y) = e^x \dfrac{1}{1+y}$ $\therefore f_{xxy}(0,0) = 1$

$f_{xyy}(x,y) = -e^x(1+y)^2$ $\therefore f_{xyy}(0,0) = -1$

$f_{yyy}(x,y) = 2e^x(1+y)^{-3}$ $\therefore f_{yyy}(0,0) = 2$

Now, Maclaurin's expansion of f (x,y) gives

$$f(x,y) = f(0,0) + x(f_x(0,0) + y f_y(0,0) + \frac{1}{2!}\{x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) +$$

$$y^2 f_{yy}(0,0)\} + \frac{1}{2!}\{x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\} + \text{----------}$$

$$\therefore e^x \log(1+y) = 0 + x.0 + y(1) + \frac{1}{2!}\{x2.0 + 2xy(1) + y2(-1)\}$$

$$+ \frac{1}{2!}\{x^3.0 + 3x^2 y(1) + 3xy^2(-1) + y^3(2)\} + \text{---------}$$

$$= y + xy - \frac{1}{2}y^2 + \frac{1}{2}(x^2 y = xy^2) + \frac{1}{2}y^3 + \text{----------}$$

23.   **Expand f (x,y) = e^x Cosy by Taylor's Theorem about the point $\left(1, \dfrac{\pi}{4}\right)$ up to the Second degree terms.**

Solution: f (x,y) = e^x Cosy and a = 1, b = $\dfrac{\pi}{4}$ $\therefore f = \left(1, \dfrac{\pi}{4}\right) = e/\sqrt{2}$

$f_x(x,y) = e^x \cos y$ $\therefore f\left(1, \dfrac{\pi}{4}\right) = \dfrac{e}{\sqrt{2}}$

$f_y(x,y) = -e^x \sin y$ $\therefore f_y\left(1, \dfrac{\pi}{4}\right) = -\dfrac{e}{\sqrt{2}}$

$f_{xx}(x,y) = e^x \cos y$ $\therefore f_{xx}\left(1, \dfrac{\pi}{4}\right) = \dfrac{e}{\sqrt{2}}$

$f_{xy}(x,y) = -e^x \sin y$ $\therefore f_{xy}\left(1, \dfrac{\pi}{4}\right) = -\dfrac{e}{\sqrt{2}}$

$f_{yy}(x,y) = -e^x \cos y$ $\therefore f_{yy}\left(1, \dfrac{\pi}{4}\right) = -\dfrac{e}{\sqrt{2}}$

Hence by Taylor's Theorem, we obtain

$$f(x,y) = f\left(1, \frac{\pi}{4}\right) + \left[(x-1)f_x + \left(y - \frac{\pi}{4}\right)fy\right] +$$

$$\frac{1}{2!}\left[(x-1)^2 f_x + 2(x-1)\left(y - \frac{\pi}{4}\right)^2 f_{yy}\right] + \text{------------}$$

i.e., $e^x \cos y = \dfrac{e}{\sqrt{2}} + \left[(x-1)\dfrac{e}{\sqrt{2}} + \left(y - \dfrac{\pi}{4}\right)\left(-\dfrac{e}{\sqrt{2}}\right)\right] + \dfrac{1}{2!}$

$$\left[(x-1)^2 \frac{e}{\sqrt{2}} + 2(x-1)\left(y - \frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right) + \left(y - \frac{\pi}{4}\right)^2\left(\frac{-e}{\sqrt{2}}\right)\right] + \text{------------}$$

$$e^x \cos y = \frac{e}{\sqrt{2}}\left[1 + (x-1) - \left(y - \frac{\pi}{4}\right)\right] + \frac{1}{2!}\left[(x-1)^2 - 2(x-1)\left(y - \frac{\pi}{4}\right) - \left(y - \frac{\pi}{4}\right)^2\right] + \text{-------}\}$$

Exercise:

1) Expand e^{xy} up to Second degree terms by using Maclaurin's theorem

2) Expand Log $(1 - x - y)$ up to Third degree terms by using Maclaurin's theorem

3) Expand $x^2 y$ about the point $(1,-2)$ by Taylor's expansion

4) Obtain the Taylor's expansion of e^x Siny about the point $\left(0, \frac{\pi}{2}\right)$ up to Second degree terms

5) Expand $e^{\sin x}$ up to the term containing x^4

Maxima and Minima:-

In mathematics, the maximum and minimum (plural: maxima and minima) of a function, known collectively as extrema (singular: extremum), are the largest and smallest value that the function takes at a point within a given neighborhood.

A function $f(x, y)$ is said to have a Maximum value at (a,b) if their exists a neighborhood point of (a,b) (say $(a+h, b+k)$) such that $f(a, b) > f(a+h, b+k)$.
Similarly,

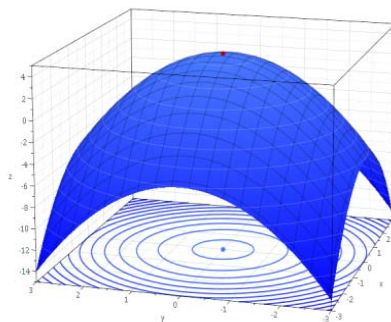Minimum value at (a,b) if there exists a neighborhood point of (a,b) (say $(a+h, b+k)$) such that $f(a, b) < f(a+h, b+k)$.
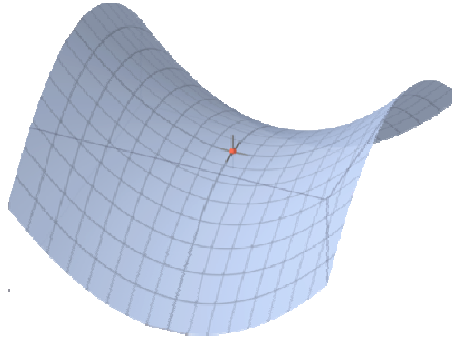
A Minimum point on the graph (in red) $f(x, y) = x^2 + y^2(1-x)^3$
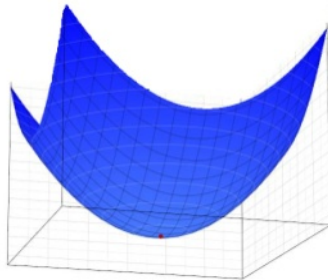
A Maximum point on the graph is at the top (in red)

A saddle point on the graph of z=x²−y² (in red)

Saddle point between two hills.
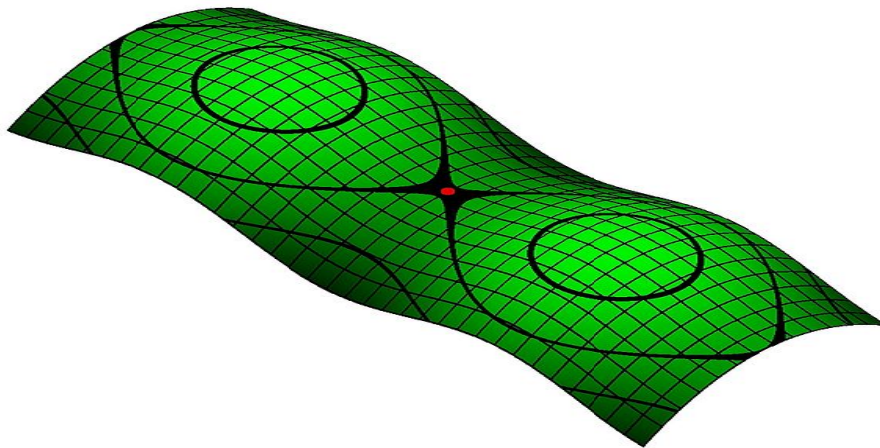
Necessary and Sufficient Condition:-

- If $f_x = 0$ and $f_y = 0$ (Necessary Condition)
- Function will be minimum if $AC\text{-}B^2 > 0 \ and \ A > 0$
- Function will be maximum if $AC\text{-}B^2 > 0 \ and \ A < 0$
- Function will be neither maxima nor minima if $AC\text{-}B^2 < 0$
- If $AC\text{-}B^2 = 0$ we cannot make any conclusion without any further analysis

$$\text{where } A = f_{xx}, \ B = f_{xy}, \ C = f_{yy}$$

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Working Procedure:-

- First we find Stationary points by considering
  $f_x = 0$ and $f_y = 0$.
- Function will be minimum if $AC\text{-}B^2 > 0 \; and \; A > 0$ at that stationary point
- Function will be maximum if $AC\text{-}B^2 > 0 \; and \; A < 0$ at that stationary point
- Function will be neither maximum nor minimum if $AC\text{-}B^2 < 0$ at that stationary point and it is called as *SADDLE POINT*.

25. **Explain Maxima & Minima for Functions of Two Variables& hence obtain the Necessary Conditions for Maxima, Minima.**
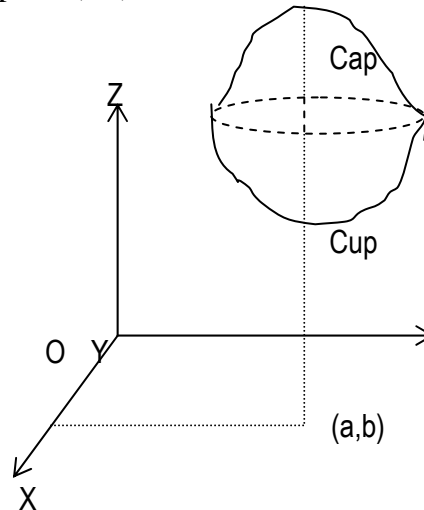
Solution: Let Z = f (x,y) be a given function of two independent variables x & y. The above equation represents a surface in 3D.
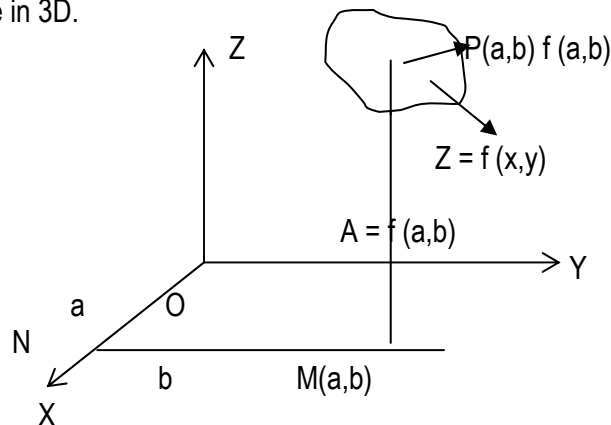
A given points (a,b) on the surface has Co-ordinates [a,b, f (a,b)]

Definition:

The function Z = f (x,y) is said to be a maximum at the point (a,b) if f (x,y) < f (a,b) in the neighborhood of the point (a,b)

Definition:

The function Z = f (x,y) is said to posses a minimum at the point (a,b) if f (x,y) > f (a,b) in the neighborhood of the point (a,b)

Necessary Condition for Maxima, Minima:

If Z = f (x,y) has a max or min at (a,b) then f_x (a,b) = 0, f_y (a,b) = 0

Sufficient Conditions for Maxima, Minima:

Put R = f_{xx} (a,b), S = f_{xy} (a,b), T = f_{yy} (a,b)

(1) Suppose $S^2 - RT > 0$

There is no maxima or minima at (a,b)

(2) Suppose $S^2 - RT < 0$

Thus there is maxima or minima at (a,b) according as R < 0 Or R > 0

(3) Suppose $S^2 - RT = 0$, Then there is a saddle point at (a,b)

26. Find the maxima and minima of the functions $f(x,y) = x^3 + y^3 - 3axy$, a > 0 is constant.

Solution: Given $f(x,y) = x^3 + y^3 - 3axy$

$f_x = 3x^2 - 3ay$, $f_y = 3y^2 - 3ax$

$f_{xx} = 6x$　$f_{yy} = 6y$.

Put $f_x = 0$, $f_y = 0$ and solve

i.e., $3x^2 - 3ay = 0$ & $3y^2 - 3ax = 0$

i.e., $x^2 = ay$ & $y^2 = ax$

$\Rightarrow y = \dfrac{x^2}{a}$　　　　　$\therefore \Rightarrow \left(\dfrac{x^2}{a}\right)^2 = ax$　　　　$(\because x^2 = ay)$

$\therefore \dfrac{x^4}{a^2} = ax$

$\therefore x^4 = a^3 x$

i.e., $x(x^3 - a^3) = 0$

$\therefore x = 0$, $x = a$

$\Rightarrow y = 0$, $y = \pm a$

$\therefore$ The critical Or stationary points are (0,0), (a,a) and (a,-a)

(1) At (0,0)

$R = f_{xx}(0,0) = 0$

$S = f_{xy}(0,0) = -3a$

$T = f_{yy}(0,0) = 0$

$\therefore S^2 - RT = 9a^2 - 0 = 9a^2 > 0$

$\therefore$ There is neither a maximum or a minimum at (0,0)

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27. Examine the following functions for extreme values $f = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Solution:

$f_x = 4x^3 - 4x + 4y$

$f_y = 4y^3 - 4x - 4y$

$f_{xy} = 4, \quad f_{xx} = 12x^2 - 4, \quad f_{yy} = 12y^2 - 4$

Put $f_x = 0$, $f_y = 0$ and solve

i.e., $4x^3 - 4x + 4y = 0$ $\qquad \rightarrow$ (1)

$\quad 4y^3 + 4x - 4y = 0$ $\qquad \rightarrow$ (2)

Adding (1) & (2), we get

$4(x^3 + y^3) = 0$

i.e., $x^3 + y^3 = 0$
i.e., $y = -x$

Substitute $y = -x$ in (1), we get

$4x^3 - 4x - 4x = 0$

i.e., $4x^3 - 8x = 0$

i.e., $x^3 - 2x = 0 \implies x(x^2 - 2) = 0$

i.e., $x = 0$ & $x^2 - 2 = 0$

i.e., $x = 0$ & $x = \pm\sqrt{2}$

$x = \sqrt{2}, -\sqrt{2}$

$\therefore x = 0, \sqrt{2}, -\sqrt{2}$ and corresponding values of y are $y = 0, -\sqrt{2}, \sqrt{2}$

$\therefore$ The critical points are $(0,0)$, $\left(\sqrt{2}, -\sqrt{2}\right)$, $\left(-\sqrt{2}, \sqrt{2}\right)$

(1) at $(0,0)$

$R = f_{xx}(0,0) = -4$

$S = f_{xy}(0,0) = 4$

$T = f_{yy}(0,0) = -4$

$\therefore S^2 - RT = 16 - (-4)(-4) = 16 - 16 = 0$

i.e., $S^2 - RT = 0$, These is a saddle point at $(0,0)$

(2) at $\left(\sqrt{2}, -\sqrt{2}\right)$

$R = f_{xx}\left(\sqrt{2}, -\sqrt{2}\right) = 24 - 4 = 20$

$S = f_{xy}\left(\sqrt{2}, -\sqrt{2}\right) = 4$

$T = f_{yy}\left(\sqrt{2}, -\sqrt{2}\right) = 20$

$\therefore S^2 - RT = 16 - (20)(20) = 16 - 400 = -384 < 0$

Thus these is neither maximum nor minimum according to R < 0 or R > 0 at $\left(\sqrt{2}, -\sqrt{2}\right)$

Hence R = 20 > 0

$\therefore$ There is a minimum at $\left(\sqrt{2}, -\sqrt{2}\right)$

$\therefore f_{min} = \left(\sqrt{2}\right)^4 + \left(-\sqrt{2}\right)^4 - 2\left(\sqrt{2}\right)^2 + 4\sqrt{2}\left(-\sqrt{2}\right) - 2\left(-\sqrt{2}\right)^2$

$= 4 + 4 - 4 - 8 - 4$

$= -8$

(3) at $\left(-\sqrt{2}, \sqrt{2}\right)$

$R = f_{xx}\left(-\sqrt{2}, \sqrt{2}\right) = 20 > 0$

$S = f_{xy}\left(-\sqrt{2}, \sqrt{2}\right) = 4$

$T = f_{yy}\left(-\sqrt{2}, \sqrt{2}\right) = 20$

$\therefore S^2 - RT = 16 - 400 = -384 < 0$

Since R > 0, ∴ There is minima at $\left(-\sqrt{2}, \sqrt{2}\right)$

∴ f_{min} = - 8 at $\left(-\sqrt{2}, \sqrt{2}\right)$

∴ Extreme Value = - 8 at $\left(-\sqrt{2}, \sqrt{2}\right)$ & $\left(-\sqrt{2}, \sqrt{2}\right)$

Exercise:

1) **Find the extreme values of f = x³ y² (1 – x – y)**
2) **Determine the maxima or minima of the function Sin x + Sin y + Sin (x + y)**
3) **Examine the function f(x,y) = 1+ sin(x² + y²) for extremum.**

28.   **If PV² = K and if the relative errors in P is 0.05 and in V is 0.025 show that the error in K is 10%.**

Solution : PV^2 $=$ K by data. Also $\dfrac{\delta P}{P} = 0.05$ and $\dfrac{\delta V}{V} = 0.025$

$\Rightarrow$ $\log\ P + 2\ \log\ V = \log\ K$

$\Rightarrow \delta(\log P) + 2\delta\ (\log V) = \delta(\log\ K)$

$i.e.,\ \dfrac{1}{p}\delta P + 2\cdot\dfrac{1}{V}\delta V = \dfrac{1}{K}\delta K$

$i.e.,\ \ 0.05 + 2(0.025) = \dfrac{\delta K}{K}\ or\ \dfrac{\delta K}{K} = 0.1$

$\therefore\ \dfrac{\delta K}{K} \times 100 = (0.1) \times 100 = 10$

Thus the error in **K** is 10%.

29. The time T of a complete oscillation of a simple pendulum is given by the formula $T = 2\pi\sqrt{l/g}$.

(i) If g is a constant find the error in the calculated value of T due to an error of 3% in the value of l.

(ii) Find the maximum error in T due to possible errors upto 1% in l and 3% in g.

Solution :

(i) $T = 2\pi\sqrt{l/g}, \qquad g = \text{Constant}, \qquad \dfrac{\delta l}{l} \times 100 = 3$

$\Rightarrow \log T = \log 2\pi + \dfrac{1}{2}(\log l - \log g)$

$\Rightarrow \delta(\log T) = \delta(\log 2\pi) + \dfrac{1}{2}\delta(\log g)$

$i.e., \dfrac{\delta T}{T} = 0 + \dfrac{1}{2}\dfrac{\delta l}{l} - 0$

$or \dfrac{\delta T}{T} \times 100 = \dfrac{1}{2}\left(\dfrac{\delta l}{l} \times 100\right) = \dfrac{1}{2}(3) = 1.5$

∴ **the error in T = 1.5%.**

(ii) If g is not a constant we have,

$$\dfrac{\delta T}{T} \times 100 = \dfrac{1}{2}\left(\dfrac{\delta l}{l} \times 100\right) - \dfrac{1}{2}\left(\dfrac{\delta g}{g} \times 100\right)$$

The error in T will be maximum if the error in l is positive and the error in g is negative (or vice-versa) as the difference in errors converts in to a sum.

$$\therefore \max\left(\dfrac{\delta T}{T} \times 100\right) = \dfrac{1}{2}(+1) - \dfrac{1}{2}(-3) = 2$$

∴ **the maximum error in T is 2%.**

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30.   The current measured by a tangent galvanometer is given by the relation

c = k tan θ where θ is the angle of deflection. Show that the relative error in c due to a

given error in θ is minimum when θ = 45^0.

Solution : Consider c = k tan θ. K is taken as a constant.

$$\Rightarrow \log c = \log k + \log (\tan \theta)$$

$$\Rightarrow \delta(\log c) = \delta(\log k) + \delta \log (\tan \theta)$$

$$i.e., \frac{1}{c} \delta c = 0 + \frac{\sec^2 \theta}{\tan \theta} \delta\theta$$

$$i.e., \frac{\delta c}{c} = \frac{\cos \theta}{\sin \theta} \cdot \frac{1}{\cos^2 \theta} \delta\theta \quad or \quad \frac{\delta\theta}{c} = \frac{\delta\theta}{\sin \theta \cos \theta}$$

$$i.e., \frac{\delta c}{c} = \frac{2}{\sin 2\theta} \delta\theta$$

The relative error in c being $\delta c / c$ minimum when the denominator of the R.H.S. is maximum

and the maximum value of a sine function is 1.

$$\therefore \sin 2\theta = 1 \Rightarrow 2\theta = 90^0 \ or \ \theta = 45^0 \cdot$$

Thus the relative error in c is minimum when $\theta = 45^0$

31. $If \ T = \frac{1}{2} mv^2$ is the kinetic energy, find approximately the change in T as m changes

from 49 to 49.5 and v changes from 1600 to 1590.   6 Marks

Solution : We have by data $T = \frac{1}{2} mv^2$ and

$$m = 49, m + \delta m = 49.5 \quad \therefore \quad \delta m = 0.5$$

$$v = 1600, v + \delta v = 1590 \therefore \delta v = -10$$

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We have to find δT.   (logarithm is not required)

$$\therefore \ \delta T = \frac{1}{2}\delta(mv^2)$$

$$\frac{1}{2}\left\{m(2v\delta v) + \delta m.v^2\right\}$$

$$i.e, = \frac{1}{2}\left\{(49)\ (2)\ (1600)\ (-10) + (0.5)\ (1600)^2\right\} = 1,44,000$$

Thus the change in $T = \delta T = -1,44,000$

32.   **The pressure p and the volume v of a gas are concentrated by the relation** $pv^{1.4} = cons\tan t$. **Find the percentage increase in pressure corresponding to a diminution of ½% in volume.**

Solution :

$pv^{1.4} = \text{Constant} = c(say)$, by data.

$\Rightarrow \ \log p + 1.4 \ \log v = \log c$

$\Rightarrow \delta(\log p) + 1.4\delta(\log v) = \delta(\log c)$

$i.e, \ \dfrac{\delta p}{p} + 1.4\left(\dfrac{\delta v}{v}\right) = 0; \ \text{But} \ \dfrac{\delta v}{v} \times 100 = -\dfrac{1}{2}, \ \text{by data.}$

$\therefore \dfrac{\delta p}{p} \times 100 + 1.4\left(\dfrac{\delta v}{v} \times 100\right) = 0 \ \text{or} \ \dfrac{\delta p}{p} \times 100 = +0.7.$

Thus the percentage increase in pressure = 0.7

33.   **Find the percentage error in the area of an ellipse when an error of +1% is made in measuring the major and minor axis.**

Solution : For the ellipse $x^2/a^2 + y^2/b^2 = 1$ the area (A) is given by π ab where 2a and 2b are the lengths of the major and minor axis.

Let 2a = x and 2b = y.

By data $\dfrac{\delta x}{x} \times 100 = 1, \dfrac{\delta y}{y} \times 100 = 1.$

$$A = \pi ab = \pi \cdot \frac{x}{2} \cdot \frac{y}{2} = \frac{\pi}{4} xy$$

$$\therefore \log A = \log (\pi / 4) + \log x + \log y$$

$$\Rightarrow \delta(\log A) = \delta \log (\pi / 4) + \delta(\log x) + \delta(\log y)$$

$$i.e., \frac{\delta A}{A} = 0 + \frac{\delta x}{x} + \frac{\delta y}{y} \ or \ \frac{\delta A}{A} \times 100 = \frac{\delta x}{x} \times 100 + \frac{\delta y}{y} \times 100$$

$$\therefore \frac{\delta A}{A} \times 100 = 1 + 1 = 2$$

Thus error in the area = 2%

27.   If the sides and angles of a triangle ABC vary in such way that the circum radius remains constant, prove that

$$\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos c} = 0$$

Solution : If the triangle ABC is inscribed in a circle of radius r and if a,b,c respectively denotes the sides opposite to the angles A,B,C we have the sine rule (formula) given by

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$$

or $a = 2r \sin A, b = 2r \sin B, c = 2r \sin C$

$$\Rightarrow \delta a = 2r \delta \ (\sin A), \ \delta b = 2r \delta(\sin B), \ \delta c = 2r \delta(\sin C)$$

$$i.e., \ \delta a = 2r \cos A \delta A, \ \delta b = 2r \cos B \delta B, \ \delta c = 2r \cos C \delta C$$

or $\dfrac{\delta a}{\cos A} = 2r \delta A, \dfrac{\delta b}{\cos B} = 2r \delta B, \dfrac{\delta c}{\cos C} = 2r \delta C$

Adding all these results we get,

$$\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 2r(\delta A + \delta B + \delta C) = 2r\delta(A + B + C)$$

But A+B+C = 180 = π radians = constant.

$$\Rightarrow \delta(A + B + C) = \delta \ (constant) = 0$$

Thus $\dfrac{\delta a}{\cos A} + \dfrac{\delta b}{\cos B} + \dfrac{\delta c}{\cos C} = 0$

Multiple choice Questions:

DIFFERENTIAL CALCULUS

1) The radius of curvature at any point of catenary S=C tan Φ is
a) c $\sec^2$ Φ   b) c $\cos^2$ Φ
c) c $\tan^{-2}$ Φ   d)none

2) Stationary points of f(x,y) = sin x+ sin y+ sin (x + y) is
a) (п/3, п/3)   b) (п/6, п/6)
c) (п/4, п/4)   d) none

3) If the curvature is zero, that point is known as--------------------
a) Point of inflection   b)   Stationary point
c) (a) or (b)   d)   none

4) The radius of curvature of the curve y = 4 sin x- sin 2x at x=п/2 is
a) 5√5/4   b) -5√5/4
c)5√5/2   d) none

5) The function x^2+2xy+ $2y^2$+2x+2y has a minimum value at

a) (-3/2,1/2)   b)(3/2, 1/2)   c) (3/2,-1/2)   d)none

6) The stationary point of f(x,y,z)=x^2+y^2+z^2 where x,y,z are connected by x+y+z=a is
a) (a,a)   b) (-a,-a)   c) (2a, 0)   d) none

7) The radius of curvature of the curve √x+√y =1 at the point (1/4,1/4) is ------------
a) ρ =1/√2   b)   ρ=√2   c) ρ= -1/√2   d) none)

8) The expression for the derivative of arc length in Cartesian form is given by
 a) ds/d = $\sqrt{r^2 + (dr/d\theta)^2}$ b) ds/dx= $\sqrt{1+(dy/dx)^2}$

 c) ds/dr= $\sqrt{1+(d?/dr)^2}$ d) ds/dt= $1+(dy/dx)^2$

9)The formulae for radius of curvature in Cartesian form is

 a) ρ = (x^1)^2 + (y^1)^2)$^{3/2}$ /x^1y^{11}+$x^{11}y^1$

 b) ρ = $\dfrac{\left(1+y_1^2\right)^{3/2}}{y_2}$

 c) ρ = [(x^1)^2 + (y^1)^2)$^{3/2}$]

d) ρ = (1+(y¹)²)³/²

10) The function for which Rolle's theorem is true is:
a) f(x)=log x in the interval [1/2,2]
b) f(x)=|x+1| in the interval [-2,2]
c) f(x)=| x |in the interval [-1,1]
d) Non of the above

11) The value of 'c'in Rolle's theorem ,where $-\pi/2 < c < \pi/2$ and f(x)= cosx is equal to:
a $\pi/4$ b) $\pi/3$ c) π d)0

12).The expansion of tanx in powers of x by Maclaurin's theorem is valid in the interval :
a) $(-\infty,\infty)$ b) $(-3\pi/2, 3\pi/2)$ c) $(-\pi,\pi)$ d) $(-\pi/2, \pi/2)$

13) The value of 'c'in Lagrange,s mean value theorem ,where [1,2]
and f(x)= x(x-1) is :
a)5/4   b)3/2   c)7/4   d) 11/6

14) The value of 'c'in Rolle's theorem, where [0, π]
and f(x)= sinx is equal to:
a) $\pi/6$ b) $\pi/3$ c) $\pi/2$ d) Non of these

15) The maximum value of logx/x is:
a) 1   b)e   c)2/e   d) 1/e

16) The maximum value of (l/x)ˣ is equal to:
a)e   b)1   c) e^(1/e)   d) (1/e)^e

17) The difference bwtween the maximum and minimum values of the function
a sinx+bcosx is:
a) $2\sqrt{a^2+b^2}$ b) $2(a^2+b^2)$ c) a^2+b^2 d) $-\sqrt{a^2+b^2}$

18) Which one of the following statement is correct for the function f(x)=x³
a) f(x) has a maximum value at x=0
b) f(x) has a manimum value at x=0
c) f(x) has a neither a maximum nor a manimum value at x=0
d) f(x) has no point of inflexion

19) which one of the following is not an indeterminate form
a) $\infty+\infty$ b) $\infty-\infty$ c) ∞/∞ d) $0X\infty$

20) In Lagrange,s mean value theorem , f¹(c)=
a) f(a) – f(b)/ b -a   b) f(b) – f(a) / a -b
c) f(b) – f(a) / b -a   d) none

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21) In Cauchy's mean value theorem , $f^1(c) / g^1(c) =$
a) $f(a) - f(b)/ g(b) - g(a)$ b) $f(b) - f(a) / g(b) - g(a)$
.c) $f(b) + f(a) / g(b) + g(a)$ d) none

22) $\lim\limits_{x \to 0} \dfrac{a^x - b^x}{x}$ **is equal to :**

a) 0   b) ∞   c) log (a/b)   d) log (a-b)

23) The value of $\lim\limits_{x \to 0} \dfrac{\log x}{x-1}$ **is equal to:**

a) -1   b) ∞   c) 1   d) 0

24) The value of $\lim\limits_{x \to 0}$ **$(1 - \cos x / 3 x^2)$is equal to:**

a) 3   b) 1/3   c) 1/6   d) 1/9

25) $\lim\limits_{x \to 0}$ **$(e^x + e^{-x} - 20) / x^2$ is equal to :**

a) 1   b) -1   c) 1/2   d) -1/2

26) The value of $\lim\limits_{x \to 0}$ **$(1 + x)^{1/x}$ is :**

a) 1   b) -1   c) 1/e   d) e

27) The value of $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = ...$

a) 1   b) 2   c) 3   d)0
28) The formulae for radius of curvature in polar form is

a) $\rho = r dr/dp$ b) $\rho = [(r^2 + (r_1)^2)^{3/2}] / [r^2 + 2 (r_1)^2 - r r_2]$

c) $\rho = (r^2 + (r^1)^2)^{3/2}$ d) None

29) The value of $\lim\limits_{x \to 0}$ **$1^{1/x} = ...$**

a) 1   b) ∞   c) -∞   d) 0

KEY ANSWERS:

1-a	2-a	3-a	4-a	5-a	6-a	7-a	8-b	9-b	10-a
11-d	12-d	13-b	14-c	15-d	16-c	17-a	18-c	19-a	20-c
21-b	22-c	23-c	24-c	25-a	26-c	27-a	28-b	29-b	--