Half Maximum Flux Diameter: The Discrete Case

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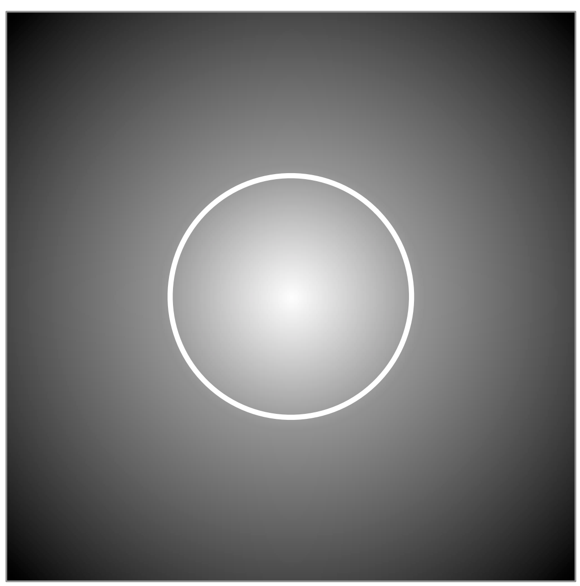
February 3, 2013

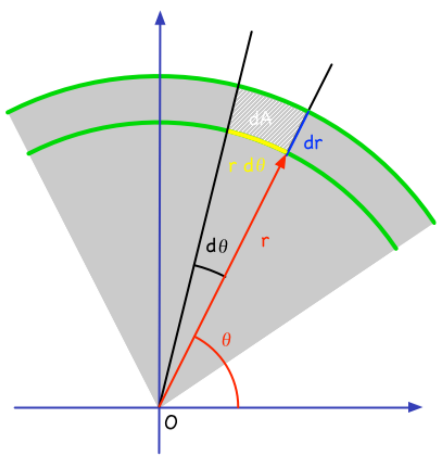
1 Definition

The *Half Maximum Flux Diameter* (HMFD or HFD) of an image is defined as the diameter of the circle centered on the image's brightness centroid such that half of the total brightness of the image lies within the circle, after the background has been filtered away. In the example illustrated below, the white circle includes half of the total image's brightness and, therefore, its diameter is the image's half maximum flux diameter. Note that, although the inside is much brighter than the outside, there are many more bright pixels outside than inside. The smaller the HFD, the more focused the image is.



2 Computing the HFD: Continuous case

How does one compute the HFD? The definition is quite clear so, in the case of a continuous distribution of brightness, defined such that $b(r,\theta)\, r dr\, d\theta$ is the brightness of the elemental area $dA = r dr\, d\theta$ between radii r and $r + dr$ and angles θ and $\theta + d\theta$ (see figure below), we'd have



$$\frac{\text{brightness up to radius } R}{\text{total brightness}}$$

$$= \frac{\displaystyle\int_0^{2\pi}\int_0^{R} b(r,\theta)\, r dr\, d\theta}{\displaystyle\int_0^{2\pi}\int_0^{+\infty} b(r,\theta)\, r dr\, d\theta} = \frac{1}{2}$$

when R equals the radius of the circle we're after. Suppose, for example, that the brightness per unit of area decreases according to a normal distribution, with circular symmetry, so that

$$b(r,\theta) = b_0\, \exp(-ar^2)$$

where $b_0 > 0$ and $a > 0$ are parameters describing the maximum brightness and the decay rate, respectively. Then,

$$\int_0^{2\pi}\int_0^{R} b(r,\theta)\, r dr\, d\theta = \pi\, \frac{b_0}{a}\, (1 - e^{-aR^2})$$

and

$$\int_0^{2\pi}\int_0^{+\infty} b(r,\theta)\, r dr\, d\theta = \pi\, \frac{b_0}{a}\, .$$

Therefore, imposing a ratio of 1/2 gives us

$$1 - e^{-aR^2} = \frac{1}{2} \qquad \Rightarrow \qquad R = \sqrt{\frac{\ln 2}{a}}\, .$$

The HFD is then twice that value. Different brightness distributions generally result in different values of R.

3 Computing the HFD: Discrete case

Unfortunately, real-world images are not continuous but, rather, are pixelated.[1] Each pixel occupies a small rectangular section of the image and the brightness of each pixel can be considered uniform. How do we compute the HFD then? The image used as an example above might actually look something like the image below.

We now have a grid of pixels, each pixel possibly not square in shape but rectangular, each of which associated with a brightness value. At least one reference on the internet finds the discrete HFD by computing a weighted average of the distance of each pixel to the centroid, using the pixel's brightness as its weight:

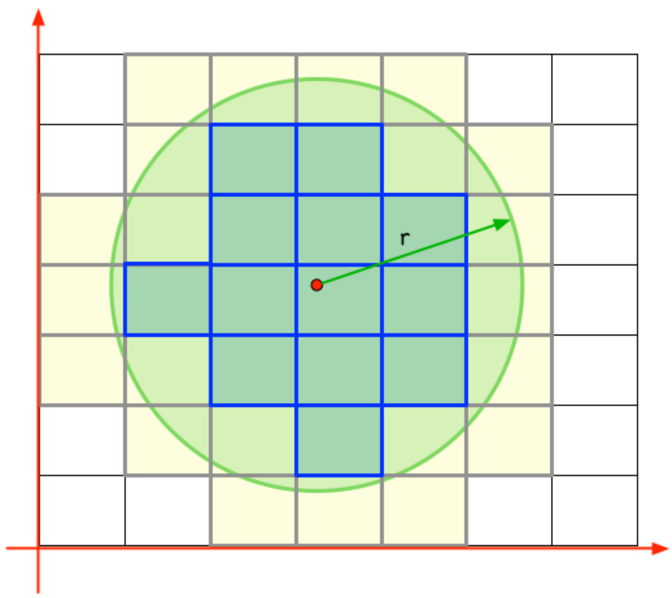
$$HFD = 2R = 2\,\frac{\sum b_i d_i}{\sum b_i}\,,$$

where both sums extend over all pixels of the image, b_i is the brightness of the i-th pixel and d_i is its distance to the brightness centroid. This method, as it turns out, is **incorrect**, as it assumes circular symmetry and is exact only for a single, very specific, and unrealistic brightness distribution. There is **no** single general prescription for numerically computing the discrete HFD that works for every image. The *only* way to accurately compute the HFD in the discrete case for any given actual image is to use the definition and numerically perform the two integrations at the top of the previous section.

[1] Moreover, even continuous distributions don't have to have circular symmetry.

3.1 Dealing with the grid

Suppose that we want to compute the total amount of brightness within a circle of a given radius, as illustrated below. Pixels are depicted as rectangular areas of the image, each with width w and height h. The red dot denotes the brightness centroid, which may or may not lie at the center of a pixel. We want to compute the total amount of brightness within the green circle (which, by definition, is centered on the brightness centroid).



The green circle encompasses a number of 'full pixels', pixels that are entirely contained within it (the blue-framed pixels), but also receives contributions from pixels that are only partially contained within it (the grey-framed light-yellow pixels). We'll refer to these pixels as 'partial pixels.' The remaining pixels are entirely outside of the green circle and do not contribute to its brightness. These are 'ignorable pixels.'

3.2 Computing the brightness centroid

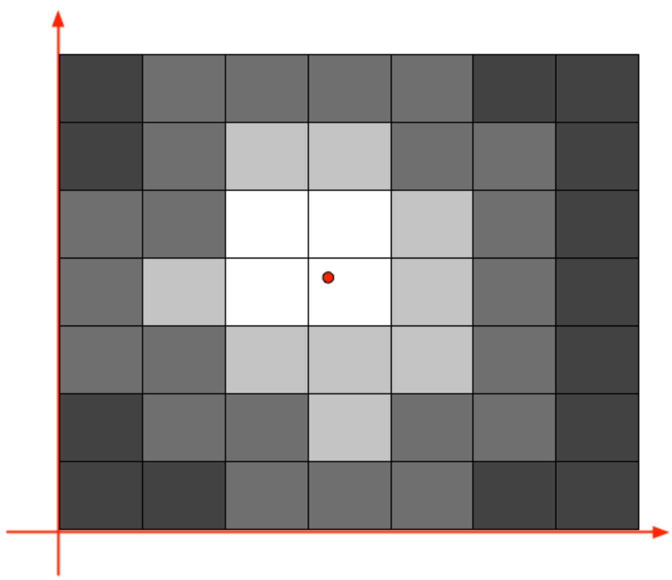
The first order of business is to compute the brightness centroid. Generally, the brightness might vary from point to point in which case the correct definition for the coordinates of the brightness centroid is

$$\bar{x} = \frac{\displaystyle\int b(x,y)\, x \;\, dx\, dy}{\displaystyle\int b(x,y) \;\, dx\, dy} \qquad \text{and} \qquad \bar{y} = \frac{\displaystyle\int b(x,y)\, y \;\, dx\, dy}{\displaystyle\int b(x,y) \;\, dx\, dy}\,,$$

where x and y are measured with respect to the coordinate system of interest (in our case, the image coordinate system). One might naively think that these reduce to

$$\bar{x} = \frac{\sum b_i\, x_i}{\sum b_i} \qquad \text{and} \qquad \bar{y} = \frac{\sum b_i\, y_i}{\sum b_i}$$

when, as in our case, the brightness is constant throughout rectangular sections (the pixels) of the image. In fact, they do, but how should we define x_i and y_i? Here, i is an index identifying each pixel but even that needs some clarification.



Each pixel can be identified by a pair of integer indices, (k_x, k_y), with $(0, 0)$ identifying the left-most bottom-most pixel of the image. Positive values of k_x identify pixels to the right of this corner pixel while positive values of k_y identify pixels above this corner pixel. Note that k_x and k_y are never negative. These k-indices identify pixels with respect to the image coordinate system and are the true meaning of i in the expressions above.

Consider now a particular pixel p_i, with index $i = (k_x, k_y)$. Its corners have coordinates

$$
\begin{array}{rrclcrcl}
\text{top-right:} & x & = & (k_x+1)\,w & \text{and} & y & = & (k_y+1)\,h \\
\text{top-left:} & x & = & k_x\,w & \text{and} & y & = & (k_y+1)\,h \\
\text{bottom-left:} & x & = & k_x\,w & \text{and} & y & = & k_y\,h \\
\text{bottom-right:} & x & = & (k_x+1)\,w & \text{and} & y & = & k_y\,h
\end{array}
$$

This pixel's contribution to the $\bar{x}$ integral above is easily computed, since the brightness has a constant value across the entire pixel, namely, b_i:

$$\int_{p_i} b(x,y)\,x\;dx\,dy = h\,b_i \int_{p_i} x\;dx = h\,b_i\,\frac{1}{2}\,x^2\Big|_{k_x\,w}^{(k_x+1)\,w} = h\,w^2\,b_i\,(k_x+1/2)\,.$$

Therefore,

$$\bar{x} = \frac{\int b(x,y)\,x\;dx\,dy}{\int b(x,y)\;dx\,dy} = \frac{\sum\int_{p_i} b(x,y)\,x\;dx\,dy}{\sum\int_{p_i} b(x,y)\;dx\,dy} = \frac{\sum h\,w^2\,b_i\,(k_x+1/2)}{\sum w\,h\,b_i} = \Big(\frac{\sum b_i\,k_x}{\sum b_i}+\frac{1}{2}\Big)\,w\,,$$

with an analogous result for $\bar{y}$. To summarize, the coordinates of the brightness centroid, in the image coordinate system, are given by

$$\bar{x} = \Big(\frac{\sum b_i\,k_x}{\sum b_i}+\frac{1}{2}\Big)\,w \qquad \text{and} \qquad \bar{y} = \Big(\frac{\sum b_i\,k_y}{\sum b_i}+\frac{1}{2}\Big)\,h\,,$$

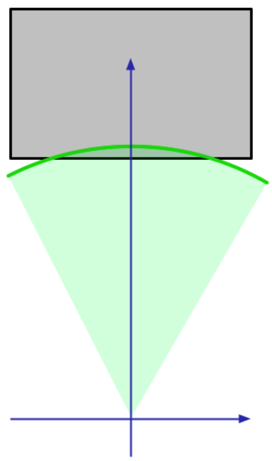
where b_i is the brightness of the pixel with index $i = (k_x, k_y)$.

3.3 Identifying full, partial, and ignorable pixels

With the coordinates of the corners of all pixels and having computed the brightness centroid, we can now classify each pixel as full, partial, or ignorable, for a given radius of interest. If all four corners fall within a circle of radius r centered on the brightness centroid then the entire pixel falls within that circle. Therefore, the condition for pixel (k_x, k_y) to be a full pixel is that

$$\Big[\,(k_x+\varepsilon_x)\,w-\bar{x}\,\Big]^2+\Big[\,(k_y+\varepsilon_y)\,h-\bar{y}\,\Big]^2 \le r^2$$

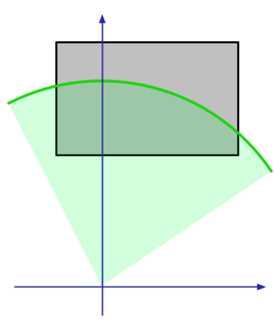
for all four combinations of $\varepsilon_x \in \{0,1\}$ and $\varepsilon_y \in \{0,1\}$. Note that all four corners being outside of the circle is *not* a sufficient condition for the pixel in question to be ignorable — see the figure above for an example. Therefore, every non-full pixel is a candidate partial pixel and must be tested for that status.



A partial pixel with all four corners outside of the circle of interest.

3.4 The brightness of each partial pixel

The brightness of a partial pixel depends on how much area it shares with the given circle of interest. Since we're assuming that the brightness of any pixel is uniform over its area, it follows that the brightness of a partial pixel is to the brightness of a full pixel (at the same location) as the shared area is to the area of the full pixel. The brightness of the partial pixel shown below is its full brightness times the ratio of the dark-green area to the area of the rectangle that makes up the pixel.



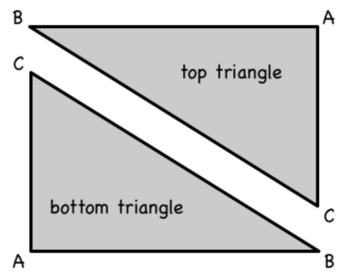
$$\frac{\text{partial pixel brightness}}{\text{full pixel brightness}}$$

$$= \frac{\text{area of the intersection}}{\text{area of the rectangle}}$$

That means we need a way to compute the area of the intersection of a rectangle with a circle. Since a rectangle can always be split into two triangles and since it's easier to compute the intersection of triangles and circles than the intersection of rectangles and circles, we'll proceed to look at the former. There are four possible situations to consider: the triangle lies entirely outside the circle (so it contributes nothing), or entirely inside (so it contributes its full brightness), or only one of its vertices lies inside, or two of its vertices lie inside. In every case, we'll need first a means of determining whether a point lies inside, on the boundary, or outside of a circle of a given center and radius.

It's important to note that these triangles are not general ones but, rather, *axis-aligned rectangular* triangles. This makes a great deal of difference, both in terms of the complexity of the various relative positions between the triangles and the circle, and in terms of the computational cost of determining the area they share with the circle.

We'll establish the convention that the rectangle for any given partial pixel shall be split into two triangles along the top-left/bottom-right diagonal, as depicted below. Moreover, the vertex with the 90-degree angle shall always be named A, the vertex along the horizontal direction from A shall be named B, and the remainging vertex, along the vertical direction from A, shall be called C. Note that the top triangle and the bottom triangle have different names for their vertices but the letters ABC always run in a counter-clockwise direction.



With the conventions above, we can write the following expressions for the coordinates of the various vertices, for the pixel with indices (k_x, k_y):

$$\text{top } \Delta: \quad \begin{cases} A\Big[\,(k_x+1)\,w, (k_y+1)\,h\,\Big] \\ B\Big[\,k_x\,w, (k_y+1)\,h\,\Big] \\ C\Big[\,(k_x+1)\,w, k_y\,h\,\Big] \end{cases} \qquad \text{bottom } \Delta: \quad \begin{cases} A\Big[\,k_x\,w, k_y\,h\,\Big] \\ B\Big[\,(k_x+1)\,w, k_y\,h\,\Big] \\ C\Big[\,k_x\,w, (k_y+1)\,h\,\Big] \end{cases}$$

Moreover, $|AB| = w$, $|AC| = h$, $|BC| = \sqrt{w^2 + h^2}$, and the area of each triangle is simply $wh/2$.

3.5 The relative position of a point and a circle

We've already seen how to determine whether or not a corner of the pixel rectangle lies inside of the circle of interest. In general, a point $P(x, y)$ lies inside, on the boundary, or outside of a circle of center $C(x_0, y_0)$ and radius r when one of the following conditions is true:

$$\begin{aligned} (x-x_0)^2+(y-y_0)^2 &< r^2 \quad \Leftrightarrow \quad P \text{ inside the circle} \\ (x-x_0)^2+(y-y_0)^2 &= r^2 \quad \Leftrightarrow \quad P \text{ on the boundary} \\ (x-x_0)^2+(y-y_0)^2 &> r^2 \quad \Leftrightarrow \quad P \text{ outside the circle.} \end{aligned}$$

3.6 The intersection of a segment and a circle

In general, a point P on the line connecting points A and B can be parameterized by a real value λ such that

$$P = A + \lambda\,(B - A)\,.$$

Note the continuity of the change in the value of λ and the change in the position of P relative to A and B:

$$\begin{array}{rcccccl} & & \lambda & < & 0 & \Leftrightarrow & P \text{ falls strictly outside } \overline{AB} \\ & & \lambda & = & 0 & \Leftrightarrow & P \text{ coincides with } A \\ 0 & < & \lambda & < & 1 & \Leftrightarrow & P \text{ falls strictly inside } \overline{AB} \\ & & \lambda & = & 1 & \Leftrightarrow & P \text{ coincides with } B \\ & & \lambda & > & 1 & \Leftrightarrow & P \text{ falls strictly outside } \overline{AB}. \end{array}$$

If the line spanned by the segment $\overline{AB}$ intersects the circle of interest at point P, then P must lie both on that line and on the boundary of the circle. We could try to massage the equations to extract the coordinates of P given the points A and B, the center C, and the radius r but there is a much more efficient and elegant way of achieving the same goal. Since $P = A + \lambda\,(B - A)$, it follows that

$$(P - C) = (A - C) + \lambda\,(B - A)\,.$$

In vector notation, this reads $\overrightarrow{CP} = \overrightarrow{CA} + \lambda\,\overrightarrow{AB}$. Since P lies on the boundary of the circle centered at C with radius r, then it must be true that the magnitude squared of the vector $\overrightarrow{CP}$ is r^2. Thus,

$$|\overrightarrow{CP}|^2 = |\overrightarrow{CA} + \lambda\,\overrightarrow{AB}|^2 = |\overrightarrow{CA}|^2 + \lambda^2\,|\overrightarrow{AB}|^2 + 2\,\lambda\,\overrightarrow{CA}\cdot\overrightarrow{AB} = r^2\,,$$

which gives us a quadratic equation for λ:

$$|\overrightarrow{AB}|^2\,\lambda^2 + 2\,(\overrightarrow{CA}\cdot\overrightarrow{AB})\,\lambda + (|\overrightarrow{CA}|^2 - r^2) = 0\,.$$

In the above, $\cdot$ denotes the *dot product* between vectors. If this equation has no real solutions, then we know that the segment $\overline{AB}$ does not intersect the circle. Otherwise, there is one real solution if the line spanned by the segment intersects the circle at only one point or there are two real solutions if the line spanned by the segment intersects the circle at two distinct points. Whether or not those solutions correspond to points inside the segment depends on whether or not their λ values are in the closed range $[0, 1]$.

In order to have real solutions, the discriminant Δ of the equation must be non-negative:

$$\frac{\Delta}{4} \equiv (\overrightarrow{CA} \cdot \overrightarrow{AB})^2 - |\overrightarrow{AB}|^2 \, (|\overrightarrow{CA}|^2 - r^2) \geq 0 \, .$$

If it equals zero, there is a single real solution; if it's positive, there are two distinct real solutions. In any case, the solutions are

$$\lambda_{1,2} = \frac{-(\overrightarrow{CA} \cdot \overrightarrow{AB}) \pm \sqrt{\Delta/4}}{|\overrightarrow{AB}|^2} \, .$$

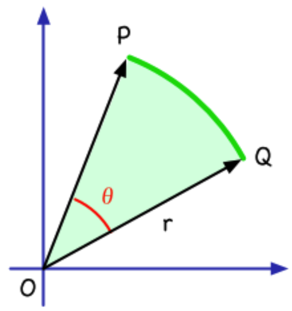
Having found one or both real solutions, if they exist, and having ascertained that they lie in the range $[0, 1]$, then the point or points where the segment $\overline{AB}$ intersects the circle of interest can be obtained by using $P = A + \lambda \, (B - A)$.

A final word on computing the solutions to the quadratic equation. The expression above for λ turns out to be numerically unsuitable when the two real roots are close to one another in magnitude, because the difference between the magnitude of $\overrightarrow{CA} \cdot \overrightarrow{AB}$ and the magnitude of $\sqrt{\Delta/4}$ may be smaller than the floating-point accuracy available for the computation. The correct numerical way to compute the two roots is to compute the largest-magnitude root by means of the expression above and then compute the smaller-magnitude root by exploiting a property of the quadratic equation, namely, that the product of the two roots equals a particular combination of the coefficients, the λ-independent coefficient divided by the λ^2-coefficient:

$$\lambda_1 \, \lambda_2 = \frac{|\overrightarrow{CA}|^2 - r^2}{|\overrightarrow{AB}|^2} \, .$$

3.7 The area spanned by a circular sector

Consider the situation depicted in the figure below. The area contained within the circular sector $\widehat{POQ}$ of radius r centered at O is given by $a(\widehat{POQ}) = \frac{\theta}{2} \, r^2$, where θ must be in the range $[0, 2\pi]$. Since the dot product of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ equals $|\overline{OP}| \, |\overline{OQ}| \cos\theta$ and the magnitudes of those two vectors are simply r, it follows that



$$a(\widehat{POQ}) = \frac{\theta}{2}\, r^2 = \arccos(\frac{\overrightarrow{OP} \cdot \overrightarrow{OQ}}{r^2})\, \frac{r^2}{2}\,.$$

3.8 The area of an isosceles triangle

As we shall see in the sections to follow, we will need to compute the areas of some isosceles triangles. Computing the area of a general triangle is actually quite simple: it's half the magnitude of the *cross product* of the vectors defining any two sides of the triangle:

$$a(\Delta POQ) = \frac{1}{2}\, r^2 \sin\theta = \frac{1}{2}\, |\overrightarrow{OP} \times \overrightarrow{OQ}|\,,$$

where θ is the angle at vertex O (see figure above). This isn't always the most accurate way of computing the area of a triangle because the magnitude of the cross product involves the difference of two terms which might be very close in magnitude, corresponding to θ being very small. The most accurate way of computing the area of a triangle involves using a variant of *Heron's formula*, as follows. If a, b, and c are the three sides of the triangle, arranged such that $a \geq b \geq c \geq 0$, then

$$\text{area}(\Delta abc) = \frac{1}{4}\sqrt{[a + (b + c)]\,[c - (a - b)]\,[c + (a - b)]\,[a + (b - c)]}\,,$$

where the various terms are computed in the order mandated by the parentheses, with the four outer bracketed terms being multiplied in the order shown, from left to right.

For our cases of interest, two of the sides are always going to be equal in length, $|\overline{OP}| = |\overline{OQ}| = r$ (that is, our triangles will be isosceles) so there are only two cases to consider: $r > |\overline{PQ}|$ and $r < |\overline{PQ}|$. In general, then, we may compute the area of the triangle $\Delta(POQ)$ by first computing the terms involved in the cross product. If their magnitudes are not comparable, then we go ahead and use the cross product to compute the area. If they are comparable in magnitude, then we use one of the following two formulas, depending on whether $r > |\overline{PQ}|$ or $r < |\overline{PQ}|$:

$$\text{area}(\Delta POQ) = \begin{cases} \dfrac{|\overline{PQ}|}{4}\sqrt{[r + (r + |\overline{PQ}|)]\,[r + (r - |\overline{PQ}|)]}\,, & \text{if } r > |\overline{PQ}| \\ \dfrac{|\overline{PQ}|}{4}\sqrt{[2r + |\overline{PQ}|]\,[2r - |\overline{PQ}|]}\,, & \text{if } r < |\overline{PQ}|\,. \end{cases}$$

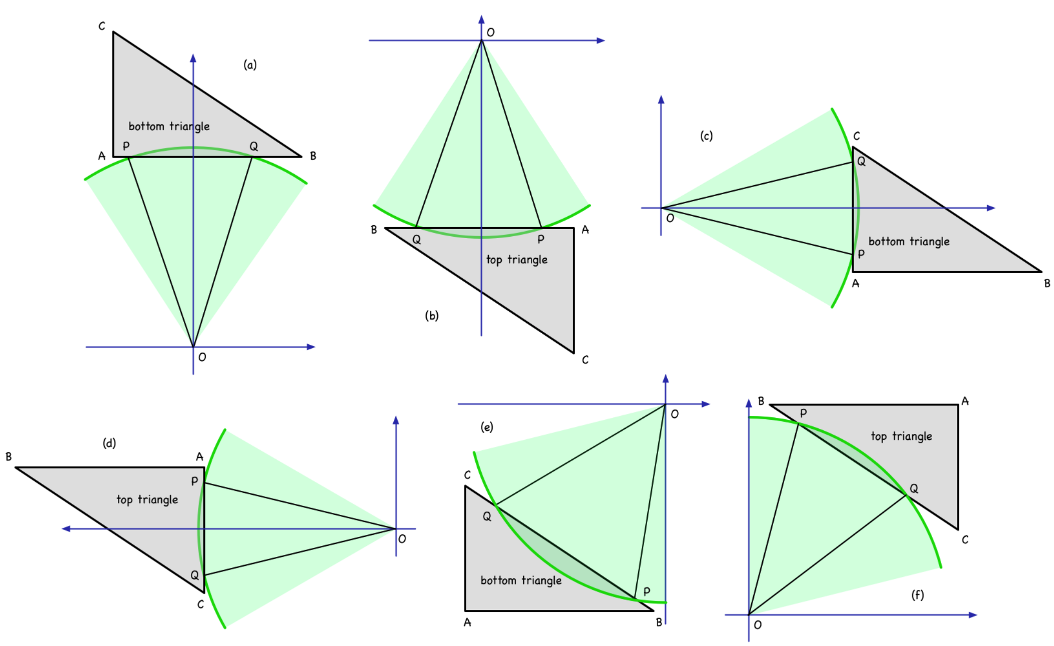
Note that the second formula can only be used if $|\overline{PQ}| \leq 2r$. This will always be the case for our triangles since we will have ascertained that there are indeed two distinct points of intersection before we even consider using Heron's formulas above. Note also that Heron's formulas are considerably more computationally expensive than the cross product since they involve several more additions, subtractions, divisions, and multiplications, and two square roots. That's the cost we have to pay for a more accurate result.

3.9 The area of the intersection of an axis-aligned rectangular triangle and a circle

As mentioned previously, we're only interested in considering axis-aligned rectangular triangles. Those triangles for which all three vertices fall inside the circle of interest, *full triangles*, contribute their full areas to the computation of their brightness values. Thus, the difficult cases are those for which the triangles intercept the circle of interest only partially.

3.9.1 No vertices inside the circle

There are only six possible ways the top and bottom rectangular triangles defined previously can have all three of their vertices outside the circle of interest and still intersect it. They are shown in the figure below.



Of course, their exact positions can vary and there may even be a single intersection, when an edge is tangent to the circle. Using the methods described in earlier sections, it's easy to determine which segment intersects the circle as well as the points of intersection. If there is a single point of intersection, the area of the intersection is zero and we can move on to the next triangle. If, however, there are two points of intersection, the result is always that the area of the intersection equals the area of the circular sector minus the area of an isosceles triangle entirely contained in the circle:

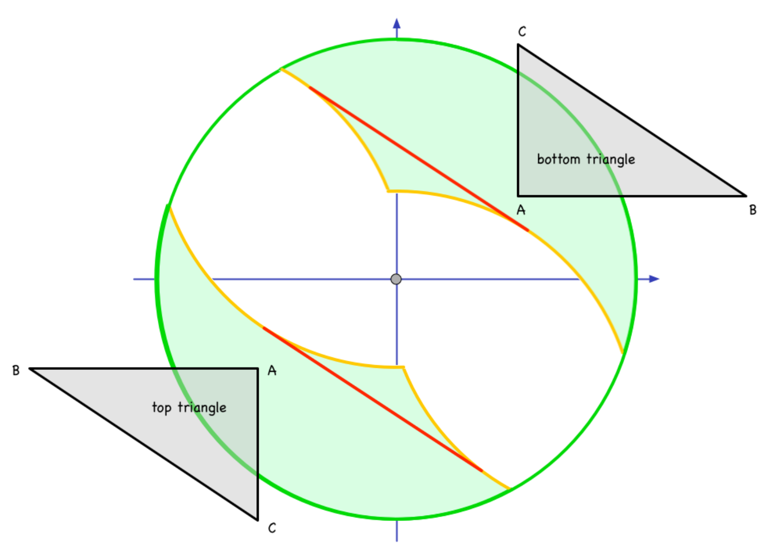
$$a(\text{intersection}) = \left[\, a(\widehat{POQ}) - a(\Delta POQ) \,\right].$$

Thanks to results shown earlier, we already know how to compute both terms.

3.9.2 One vertex inside the circle

When there is a single vertex inside the circle, there may be two or four intersections between the triangle and the circle. There are only two intersections when the triangle doesn't penetrate so far into the circle as to cause the edge opposite to the vertex that's inside to intersect the circle.

3.9.2.1 Vertex A inside the circle Recall that our conventions require the vertex with the right angle to be named A. When this vertex is the only one inside the circle, there are two regions of interest for each of the two kinds of triangles (top and bottom) under consideration. See the figure below.



The gold-colored lines indicate the boundaries separating the regions where A is the only vertex inside the circle from the regions where either of the other vertices also enters the circle. For the triangle we call 'bottom triangle' ('top triangle'), A must be above (below) the gold-colored curves, but within the dark-green circle, in order to be the only vertex inside the circle. Moreover, if A enters the region between the red line and the gold curves, there will be four intersections rather than two. Where do these curves come from?

It's easy to see that if we move the triangle marked 'bottom triangle' while keeping its vertex C on the boundary of the circle, vertex A will also travel along a circle, of the same radius but centered a distance equal to $|\overline{AC}| = h$ below the green circle's center. Similarly, moving the same triangle such that vertex B stays on the green circumference, we find that A now travels along a circle centered a distance $|\overline{AB}| = w$ to the left of the green circle's center. The curves bounding the intersection of these two circles with the green circle bound the region where A is the only vertex inside. As for the red line, it has the same slope as the line spanned by the segment $\overline{BC}$ and is constructed by tracking the motion of vertex A when the triangle is moved in such a way as to keep $\overline{BC}$ always tangent to the green circle. If A falls below that line then $\overline{BC}$ will intersect the green circle on two distinct points.

We can find whether A falls below the red line by attempting to find intersections between $\overline{BC}$ and the green circle but it may be simpler to test A's coordinates directly against the equation for the red line:

$$y(x) = +\Big[\,\frac{r}{w}\,|\overline{BC}| - h\,\Big] - \frac{h}{w}\,x\,.$$

If $y(x_A) < y_A$ then A falls below the red line and, therefore, $\overline{BC}$ intersects the green circle in two distinct points, for a total of four intersections.

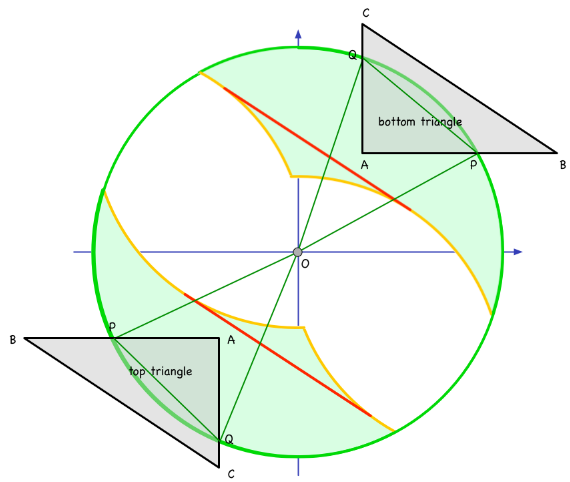
All the arguments above have counterparts for the triangle marked 'top triangle'. In particular, its $\overline{BC}$ segment will intersect the green circle in two distinct points when its A vertex falls *above* the lower red line ($y(x_A) > y_A$), whose equation is

$$y(x) = -\Big[\,\frac{r}{w}\,|\overline{BC}| - h\,\Big] - \frac{h}{w}\,x\,.$$

3.9.2.2 Vertex A inside the circle: 2 intersections Assuming that we have already determined that the current triangle of interest has only its A vertex inside the circle of interest and assuming that we have also already found that A does not fall in the region that causes four intersections to happen, what is the area of the intersection of the triangle and the circle? The figure below shows typical triangles whose A vertices are such that there are only two intersections, rather than four.

Note that the area of the triangle ΔPAQ can be computed quite easily:

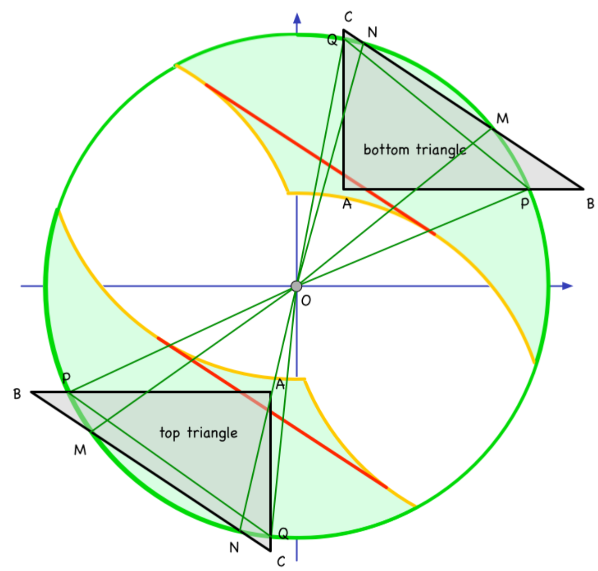
$$a(\Delta PAQ) = \frac{1}{2}\,|\overline{PA}|\,|\overline{AQ}| = \frac{1}{2}\,|x_P - x_A|\,|y_Q - y_A|\,.$$



$$a(\text{intersection}) =$$
$$a(\Delta PAQ) + \left[a(\widehat{POQ}) - a(\Delta POQ) \right].$$

3.9.2.3 Vertex A inside the circle: 4 intersections

If A is the only vertex inside the circle of interest and is located such that segment $\overline{BC}$ intersects the circle, then there are four intersections, as shown below.

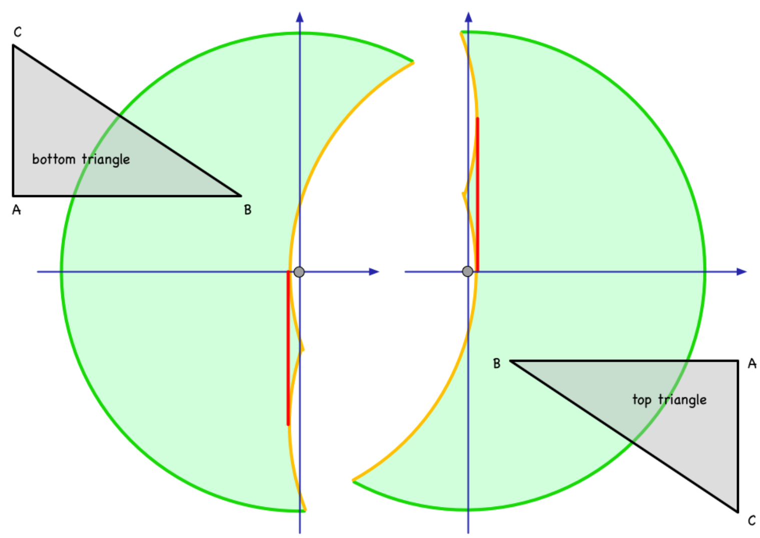


The intersection area is now

$$a(\text{intersection}) = a(\Delta PAQ) + \left[a(\widehat{POQ}) - a(\Delta POQ) \right] - \left[a(\widehat{MON}) - a(\Delta MON) \right].$$

Note that triangle ΔMON is isosceles, just as ΔPOQ. Note also that the expression above reduces to that for two intersections if points M and N coincide, that is, if segment $\overline{BC}$ is tangent to the circle.

3.9.2.4 Vertex B inside the circle When vertex B (the vertex on the horizontal edge that is not the right-angle vertex) is the only vertex inside the circle, there are again two regions of interest for each kind of triangle. See figure below. The meanings of the various curves are the same as when A was the lone vertex inside, and are constructed the same way. Thus, if the bottom (top) triangle's B vertex is to the right (left) of the red line, $x_B > w - r$ ($x_B < r - w$), the segment $\overline{AC}$ will intersect the circle of interest and there will be four intersections rather than two.

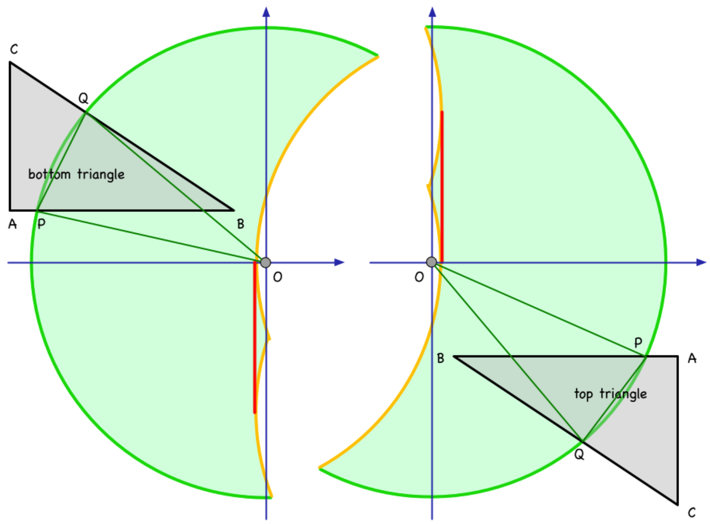


3.9.2.5 Vertex B inside the circle: 2 intersections Inspection of the figure below shows that the area of the intersection between the triangle and the circle, in either case, is given by

$$a(\text{intersection}) = a(\Delta PBQ) + \left[a(\widehat{POQ}) - a(\Delta POQ) \right].$$

Here, too, the area of the triangle ΔPBQ can be computed rather simply:

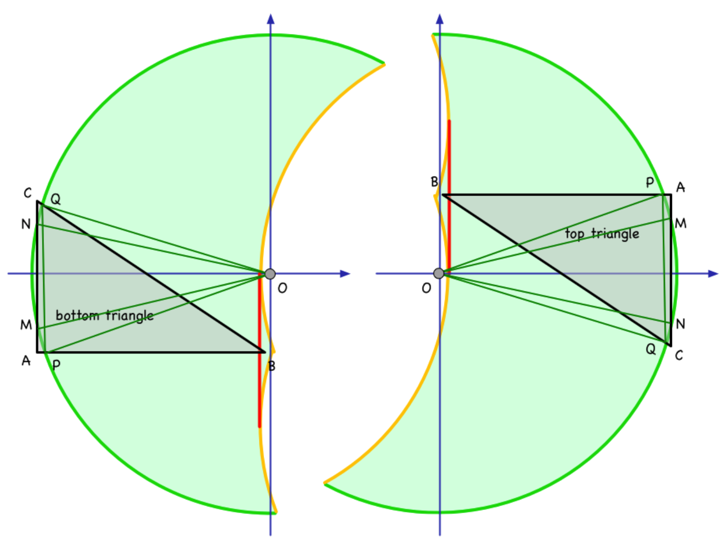
$$a(\Delta PBQ) = \frac{1}{2} |x_P - x_B| \, |y_Q - y_B| \, .$$



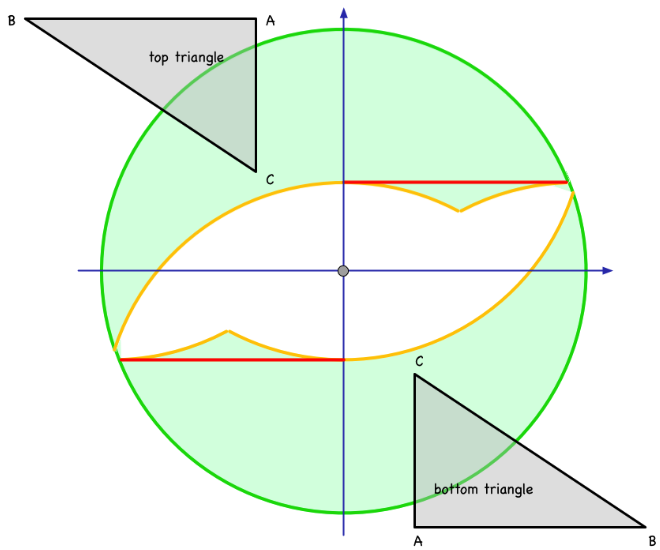
3.9.2.6 **Vertex B inside the circle: 4 intersections**

Inspection of the figure below shows that the area of the intersection between the triangle and the circle, in either case, is given by

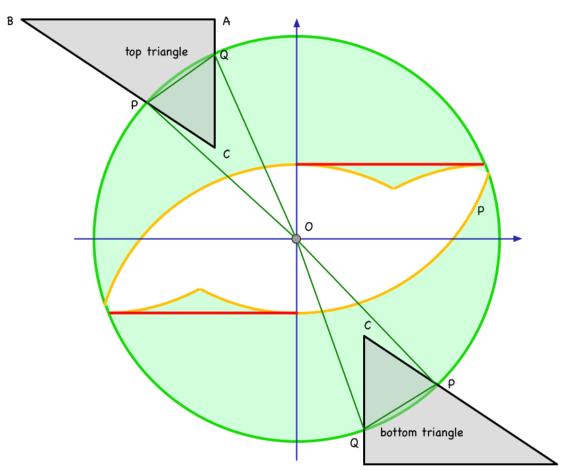
$$a(\text{intersection}) = a(\Delta PBQ) + \Big[a(\widehat{POQ}) - a(\Delta POQ)\Big] - \Big[a(\widehat{MON}) - a(\Delta MON)\Big].$$



3.9.2.7 Vertex C inside the circle Unsurprisingly, this is not much different from the previous cases. If the bottom (top) triangle's C vertex is above (below) the red line, $y_C > h - r$ ($y_C < r - h$), the segment $\overline{AB}$ will intersect the circle of interest and there will be four intersections rather than two.



3.9.2.8 Vertex C inside the circle: 2 intersections The area of the intersection between the triangle and the circle, for both kinds of triangles, is now given by

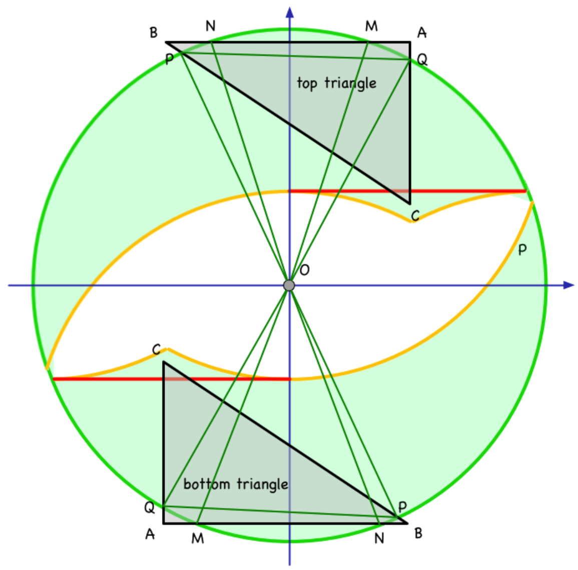


$$a(\text{intersection}) =$$

$$a(\Delta PCQ) + \left[a(\widehat{POQ}) - a(\Delta POQ) \right]$$

$$a(\Delta PCQ) = \frac{1}{2} |x_P - x_C| \, |y_Q - y_C|$$

3.9.2.9 Vertex C inside the circle: 4 intersections



$$a(\text{intersection}) = a(\Delta PCQ) + \left[a(\widehat{POQ}) - a(\Delta POQ) \right] - \left[a(\widehat{MON}) - a(\Delta MON) \right].$$

3.9.3 Two vertices inside the circle

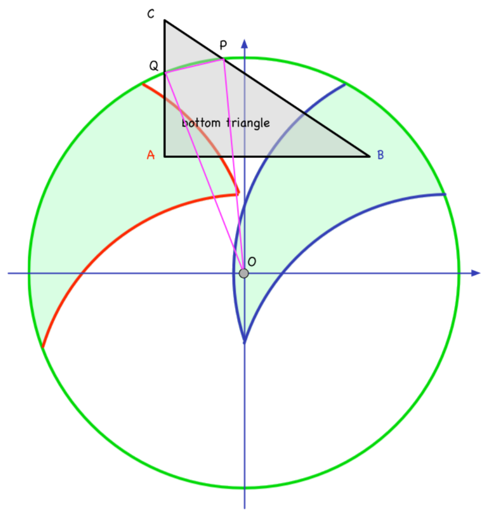
When two of the vertices must be inside and the third outside, the available regions for each vertex are more restricted. Moreover, there can only be two intersections now.

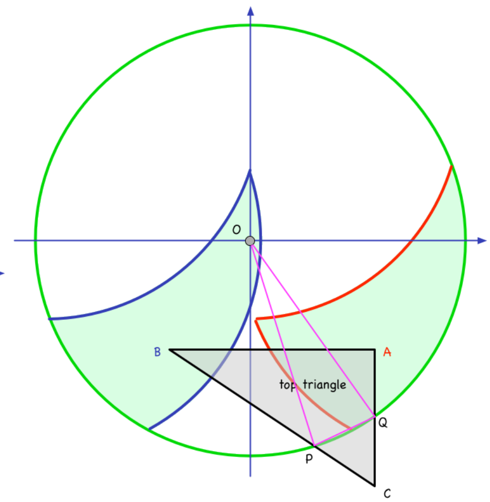
3.9.3.1 Vertices A and B inside the circle When A and B must be inside, but not C, A has to be within the region bounded by the red curves in the figure below while B has to be within the region bounded by the blue curves. These curves are constructed much the same way as in previous sections and represent the positions of the points A and B, respectively, which cause the other two points (B and C, or A and C, respectively) to be on the verge of being in the wrong side of the circle.

It is clear from the figure that the area of the intersection between either triangle and the circle is given by

$$a(\text{intersection}) = \left[a(\Delta ABC) - a(\Delta PCQ) \right] + \left[a(\widehat{POQ}) - a(\Delta POQ) \right].$$

As we've seen before, the area of the triangle ΔABC is simply $hw/2$ while the area of

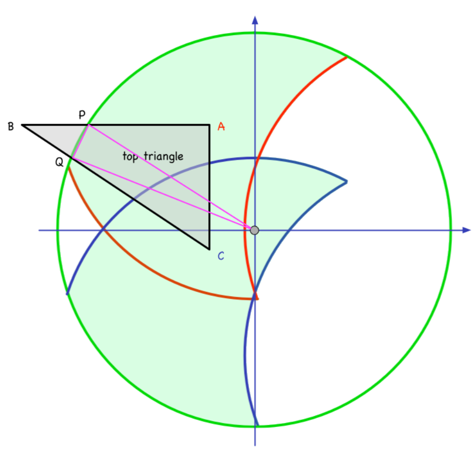


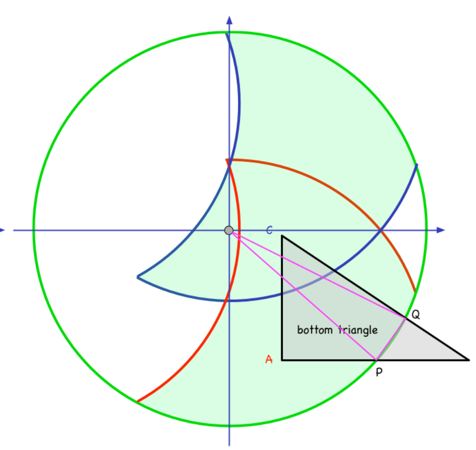


triangle ΔPCQ is still

$$a(\Delta PCQ) = \frac{1}{2}\left|x_P - x_C\right|\left|y_Q - y_C\right|.$$

3.9.3.2 Vertices A and C inside the circle



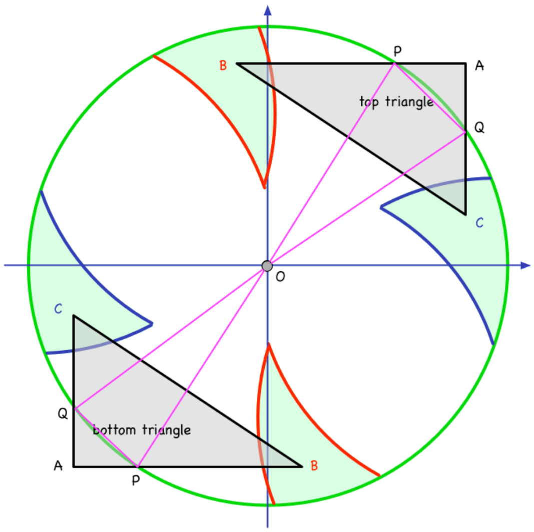


$$a(\text{intersection}) = \Big[\, a(\Delta ABC) - a(\Delta PBQ) \Big] + \Big[\, a(\widehat{POQ}) - a(\Delta POQ) \Big],$$

with

$$a(\Delta PBQ) = \frac{1}{2}\left|x_P - x_B\right|\left|y_Q - y_B\right|.$$

3.9.3.3 Vertices B and C inside the circle



$$a(\text{intersection}) =$$

$$\Big[a(\Delta ABC) - a(\Delta PAQ) \Big] +$$

$$\Big[a(\widehat{POQ}) - a(\Delta POQ) \Big]$$

$$a(\Delta PAQ) = \tfrac{1}{2} \left| x_P - x_A \right| \left| y_Q - y_A \right|$$

3.10 A summary of the results on how to compute the area of the intersection of a given pixel triangle and a given circle of interest

As it happens, given the conventions we've established, it is possible to group the results above, as follows:

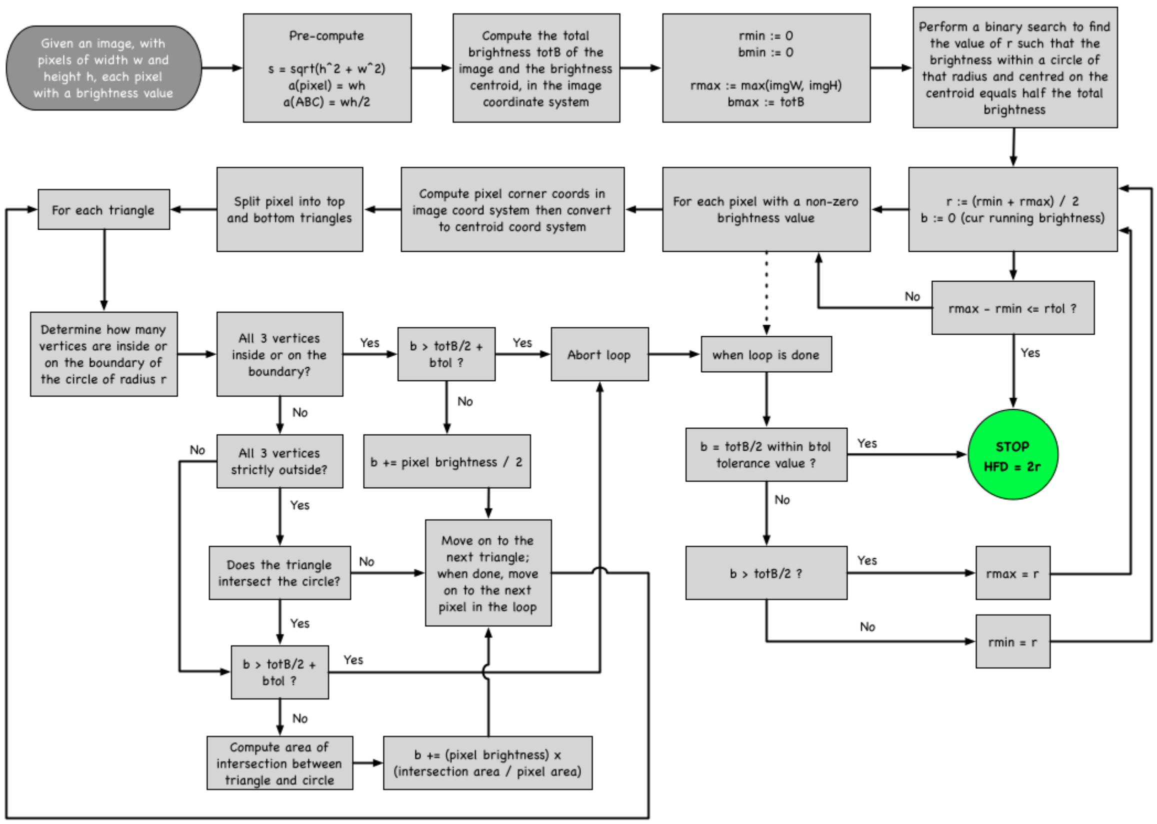
- **Zero or one vertex inside**. Let Z be the vertex that is **inside**, if any. Then,
$$a(\text{intersection}) = a(\Delta PZQ) + \Big[a(\widehat{POQ}) - a(\Delta POQ) \Big] - \Big[a(\widehat{MON}) - a(\Delta MON) \Big] .$$
- **Two or three vertices inside**. Let Z be the vertex that is **outside**, if any. Then,
$$a(\text{intersection}) = a(\Delta ABC) - a(\Delta PZQ) + \Big[a(\widehat{POQ}) - a(\Delta POQ) \Big] .$$

In all cases, P and Q are the intersections (with the circle) of the two edges sharing the vertex Z, and M and N are the intersections (with the circle) of the remaining edge (the edge opposite to Z). The triangles ΔPOQ and ΔMON, if they exist, are isosceles. Also in all cases,

$$a(\Delta PZQ) = \frac{1}{2} \left| x_P - x_Z \right| \left| y_Q - y_Z \right| .$$

Note that the terms in brackets are all of the same form, namely, the area of a circular sector minus the area of the associated isosceles triangle.

3.11 Putting everything together: A numerical algorithm for accurately computing the HFD for any given image



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