The model

- Let $Y_1, \ldots, Y_n$ be independent variables having bernoulli distribution of parameter π_i
- Consider a set of covariates $X_1, X_2, \ldots, X_p$ forming the linear predictor

$$\eta_i = x_i^T \beta$$

- Choose as link function the probit link

$$\Phi^{-1}(\pi_i) = \eta_i$$

Bayesian analysis

- While resorting to a bayesian approach, a prior is assigned to the vector of parameters β, $\pi(\beta)$, and all inferences are based on the posterior distribution:

$$\pi(\beta|Y) \propto \pi(\beta) \prod_{i=1}^{n} (\Phi(\eta_i))^{y_i} (1 - \Phi(\eta_i))^{1-y_i}$$

which is analytically untractable

Solutions

- Several solutions have been suggested
- Approximate the posterior law resorting to a Normal density centered at the mode of $\pi(\beta|\mathbf{y})$ and having as covariance matrix the inverse of Fisher information evaluated at such mode
- Simulate $\pi(\beta|Y)$ with an importance sampling using as importance function a multivariate T distribution

Probit model: Metropolis algorithm

- The posterior distribution can be simulated resorting to a Metropolis algorithm using as proposal a multivariate normal distribution centered at the current update of β and with covariance matrix given by the inverse of Fisher information evaluated at the current update, that is

$$q(\beta^*|\beta_t) = N(\beta_t, \tau V),$$
$$V = (-l''(\beta_t))^{-1}$$

- So that the acceptance probability reduces to

$$\alpha(\beta_t, \beta^*) = min\left(1, \frac{\pi(\beta^*|Y)}{\pi(\beta_t|Y}\right)$$

Auxiliary variables Gibbs sampler

- We can simulate the posterior distribution $\pi(\beta|Y)$ resorting to an auxiliary variables Gibbs sampler, drawing on the representation of the probit model in terms of latent normal variables

Probit model: auxiliary variables

- Let $Z_1, Z_2, \ldots, Z_n$ be n auxiliary variables defined as follows
- $\forall i = 1, \cdots, n,\ Y_i = 1$ if and only if $Z_i > 0$
- $Z_1, Z_2, \ldots, Z_n$ are independent variables with normal density having expected value $x_i^T \beta$ and variance 1

Latent model

- Any probit model can be expressed in terms of a multivariate linear regression model of the latent variables on the covariates $Z_i'S$, as follows

$$Y_i = 1 \Leftrightarrow Z_i > 0$$

$$Z_i = x_i^{'}\beta + \epsilon_i$$

with $\epsilon_i \sim N(0, 1)$

- Values can be drawn from the posterior distribution of β by means of an auxiliary variables Gibbs sampler

Full conditionals

- The following full conditionals have to be worked out
- $\pi(\beta|Z, Y)$
- $\pi(Z|\beta, Y)$

$\pi(\beta|Z, Y)$

- $\pi(\beta|Z, Y)$ is the posterior distribution of the vector of parameters of a multivariate linear regression model

$$\pi(\beta|Z, Y) = \pi(\beta|Z) \propto \pi(\beta)\pi(Z|\beta)$$

- The posterior law is a Normal density, with parameters specified as follows according to the choice of the prior
- noninformative prior $E(\beta|Z, Y) = (X'X)^{-1}X'\mathrm{Z}$ and $cov(\beta|Z, Y) = (X'X)^{-1}$
- informative normal prior, with mean β_0 and covariance matrix V_0 $E(\beta|Z, Y) = (V_0^{-1} + X'X)^{-1}(V_0^{-1}\beta_0 + X'Z)$ $cov(\beta|Z, Y) = (V_0^{-1} + X'X)^{-1}$

$\pi(Z|\beta, \mathbf{y})$

- Conditionally on β and on Y, $Z_1, \ldots, Z_n$ are independent variables having Normal density, with expected value $x_i^T \beta$ and variance 1, truncated by 0
  - at the left if the corresponding observation y_i is equal to 1
  - at the right if the corresponding observation y_i is equal to 0

Generation from a truncated normal density

Gelfand et all (1990) Illustration of Bayesian Inference in Normal Data Models Using Gibbs Sampling, JASA, 85, 412, 972-985.

To draw a number from a Normal (c, d^2) restricted to (a, b):

- Draw U from a uniform on $(0, 1)$.
- Compute:

$$p = \Phi\left(\frac{a-c}{d}\right) + U\left(\Phi\left(\frac{b-c}{d}\right) - \Phi\left(\frac{a-c}{d}\right)\right)$$

- Set $Y = c + d\Phi^{-1}(p)$