Deviated pursuit based cooperative simultaneous interception against moving targets: Supplement

Abhinav Sinha, Dwaipayan Mukherjee, and Shashi Ranjan Kumar

Abstract

This work focuses on achieving cooperative simultaneous interception against moving targets using the concepts of deviated pursuit guidance strategy. Unlike most existing salvo guidance strategies which use estimates of time-to-go, based on proportional navigation guidance, the present strategy uses exact expression for time-to-go to ensure simultaneous interception. The guidance command is derived considering nonlinear engagement kinematics. Weighted consensus in time-to-go over an undirected graph is used to intercept a moving target cooperatively. It has been shown that through a judicious choice of these weights, the achievable set of interception times expands. Simulations are provided to vindicate the effectiveness of the proposed strategy.

Index Terms

Deviated pursuit, Cooperative control, Salvo, moving targets, consensus

This supplement provides some additional information related to the pseudo-undirected graphs proposed in the manuscript entitled *Deviated pursuit based cooperative simultaneous interception against moving targets*.

I. Pseudo-undirected complete graph

Figure 1 shows the pseudo-undirected complete graph. For this graph, the associated graph matrices are described below.

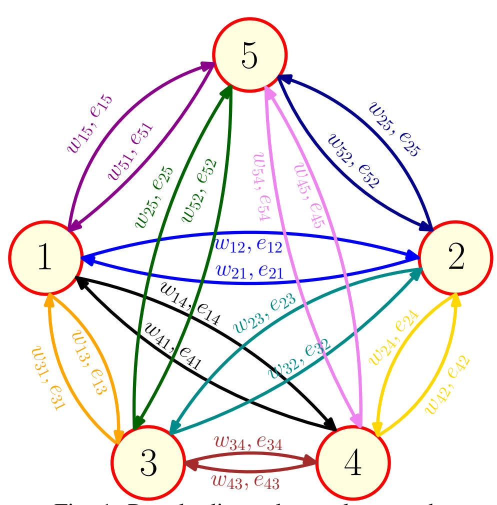

Fig. 1: Pseudo-directed complete graph.

A. Incidence matrix

$$E = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & -1 & 1 \end{bmatrix}. \quad (1)$$

A. Sinha and S.R. Kumar are with Intelligent Systems & Control Lab, Department of Aerospace Engineering, Indian Institute of Technology, Bombay (Mumbai), India. **emails:** abhinavsinha@aero.iitb.ac.in, srk@aero.iitb.ac.in.
D. Mukherjee is with Department of Electrical Engineering, Indian Institute of Technology, Bombay (Mumbai), India. **email:** dm@ee.iitb.ac.in.

B. Out-incidence matrix

$$E_\otimes = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}. \tag{2}$$

C. Edge weight (diagonal) matrix

$$\mathcal{W} = \mathrm{diag}\begin{bmatrix} 4 & 3 & 2 & 3 & 4 & 2 & 1 & 2 & 1 & 4 & 4 & 1 & 4 & 5 & 4 & 5 & 2 & 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & & & & & & & & & & & & & & & & & & & \\ & 3 & & & & & & & & & & & & & & & & & & \\ & & 2 & & & & & & & & & & & & & & & & & \\ & & & 3 & & & & & & & & & & & & & & & & \\ & & & & 4 & & & & & & & & & & & & & & & \\ & & & & & 2 & & & & & & & & & & & & & & \\ & & & & & & 1 & & & & & & & & & & & & & \\ & & & & & & & 2 & & & & & & & & & & & & \\ & & & & & & & & 1 & & & & & & & & & & & \\ & & & & & & & & & 4 & & & & & & & & & & \\ & & & & & & & & & & 4 & & & & & & & & & \\ & & & & & & & & & & & 1 & & & & & & & & \\ & & & & & & & & & & & & 4 & & & & & & & \\ & & & & & & & & & & & & & 5 & & & & & & \\ & & & & & & & & & & & & & & 4 & & & & & \\ & & & & & & & & & & & & & & & 5 & & & & \\ & & & & & & & & & & & & & & & & 2 & & & \\ & & & & & & & & & & & & & & & & & 2 & & \\ & & & & & & & & & & & & & & & & & & 3 & \\ & & & & & & & & & & & & & & & & & & & 4 \end{bmatrix}. \tag{3}$$

D. Laplacian matrix

$$\mathcal{L} = E_\otimes \mathcal{W} E^\top = \begin{bmatrix} 11 & -4 & -2 & -4 & -1 \\ -3 & 12 & -1 & -4 & -4 \\ -3 & -4 & 13 & -4 & -2 \\ -2 & -1 & -5 & 11 & -3 \\ -2 & -5 & -2 & -4 & 13 \end{bmatrix}. \tag{4}$$

E. Left eigenvector of the Laplacian

$$\mathbf{v}_L : \quad \mathbf{v}_L \mathcal{L}^\top = \mathbf{0} \tag{5}$$
$$\Rightarrow \mathbf{v}_L = \begin{bmatrix} 1.0980 & 1.2605 & 1.0341 & 1.5973 & 1 \end{bmatrix} \tag{6}$$

F. Right eigenvector of the Laplacian

$$\mathbf{v}_R : \quad \mathcal{L}\mathbf{v}_R = \mathbf{0} \tag{7}$$
$$\Rightarrow \mathbf{v}_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\top = \mathbf{1} \tag{8}$$

Note that $\mathbf{v}_L \neq \mathbf{v}_R$. Further, we have

$$p = \mathcal{W} E_\otimes^\top \mathbf{v}_R = \begin{bmatrix} 4 & 3 & 2 & 3 & 4 & 2 & 1 & 2 & 1 & 4 & 4 & 1 & 4 & 5 & 4 & 5 & 2 & 2 & 3 & 4 \end{bmatrix}^\top = w_{ii}, \tag{9}$$

where w_{ii} is corresponding the diagonal entry of the $\mathcal{W}$.

G. Other useful matrices

These computations have been done using MATLAB.

$$U^\top = E_\otimes \mathcal{W} = \begin{bmatrix} 4 & 0 & 2 & 0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 4 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 5 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 2 & 0 & 4 \end{bmatrix}. \tag{10}$$

$$\mathbf{V}: \quad E\mathbf{V} = 0 \tag{11}$$

$$\Rightarrow \mathbf{V} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

$$P_U = U(U^\top U)^{-1}U^\top \tag{13}$$

$$= \begin{bmatrix} 0.4324 & 0 & 0.2162 & 0 & 0.4324 & 0 & 0.1081 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2143 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0714 & 0 & 0.2857 & 0 & 0.2857 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2162 & 0 & 0.1081 & 0 & 0.2162 & 0 & 0.0541 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2000 & 0 & 0 & 0 & 0 & 0 & 0.2667 & 0 & 0 & 0 & 0 & 0.2667 & 0 & 0.1333 & 0 & 0 & 0 \\ 0.4324 & 0 & 0.2162 & 0 & 0.4324 & 0 & 0.1081 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1026 & 0 & 0 & 0 & 0 & 0 & 0.0513 & 0 & 0 & 0 & 0.2564 & 0 & 0 & 0.1538 & 0 \\ 0.1081 & 0 & 0.0541 & 0 & 0.1081 & 0 & 0.0270 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0816 & 0 & 0 & 0 & 0 & 0 & 0.2041 & 0 & 0 & 0 & 0.0816 & 0 & 0.1633 \\ 0 & 0.0714 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0238 & 0 & 0.0952 & 0 & 0.0952 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2667 & 0 & 0 & 0 & 0 & 0 & 0.3556 & 0 & 0 & 0 & 0 & 0.3556 & 0 & 0.1778 & 0 & 0 & 0 \\ 0 & 0.2857 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0952 & 0 & 0.3810 & 0 & 0.3810 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0513 & 0 & 0 & 0 & 0 & 0 & 0.0256 & 0 & 0 & 0 & 0.1282 & 0 & 0 & 0.0769 & 0 \\ 0 & 0.2857 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0952 & 0 & 0.3810 & 0 & 0.3810 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2041 & 0 & 0 & 0 & 0 & 0 & 0.5102 & 0 & 0 & 0 & 0.2041 & 0 & 0.4082 \\ 0 & 0 & 0 & 0.2667 & 0 & 0 & 0 & 0 & 0 & 0.3556 & 0 & 0 & 0 & 0 & 0.3556 & 0 & 0.1778 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2564 & 0 & 0 & 0 & 0 & 0 & 0.1282 & 0 & 0 & 0 & 0.6410 & 0 & 0 & 0.3846 & 0 \\ 0 & 0 & 0 & 0.1333 & 0 & 0 & 0 & 0 & 0 & 0.1778 & 0 & 0 & 0 & 0 & 0.1778 & 0 & 0.0889 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0816 & 0 & 0 & 0 & 0 & 0 & 0.2041 & 0 & 0 & 0 & 0.0816 & 0 & 0.1633 \\ 0 & 0 & 0 & 0 & 0 & 0.1538 & 0 & 0 & 0 & 0 & 0 & 0.0769 & 0 & 0 & 0 & 0.3846 & 0 & 0 & 0.2308 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1633 & 0 & 0 & 0 & 0 & 0 & 0.4082 & 0 & 0 & 0 & 0.1633 & 0 & 0.3265 \end{bmatrix}. \tag{14}$$

$$P_V = V(V^\top V)^{-1}V^\top \tag{15}$$

$$= \begin{bmatrix} 0.8 & 0.2 & -0.1 & 0.1 & -0.1 & 0.1 & -0.1 & 0.1 & 0.1 & -0.1 & 0.1 & -0.1 & 0.1 & -0.1 & -0 & -0 & 0 & 0 & 0 & -0 \\ 0.2 & 0.8 & 0.1 & -0.1 & 0.1 & -0.1 & 0.1 & -0.1 & -0.1 & 0.1 & -0.1 & 0.1 & -0.1 & 0.1 & -0 & 0 & -0 & 0 & -0 & 0 \\ -0.1 & 0.1 & 0.8 & 0.2 & -0.1 & 0.1 & -0.1 & 0.1 & -0.1 & 0.1 & 0 & 0 & -0 & -0 & 0.1 & -0.1 & 0.1 & -0.1 & 0 & -0 \\ 0.1 & -0.1 & 0.2 & 0.8 & 0.1 & -0.1 & 0.1 & -0.1 & 0.1 & -0.1 & -0 & 0 & -0 & 0 & -0.1 & 0.1 & -0.1 & 0.1 & -0 & 0 \\ -0.1 & 0.1 & -0.1 & 0.1 & 0.8 & 0.2 & -0.1 & 0.1 & -0 & -0 & -0.1 & 0.1 & -0 & -0 & -0.1 & 0.1 & 0 & -0 & 0.1 & -0.1 \\ 0.1 & -0.1 & 0.1 & -0.1 & 0.2 & 0.8 & 0.1 & -0.1 & 0 & 0 & 0.1 & -0.1 & 0 & 0 & 0.1 & -0.1 & -0 & 0 & -0.1 & 0.1 \\ -0.1 & 0.1 & -0.1 & 0.1 & -0.1 & 0.1 & 0.8 & 0.2 & 0 & -0 & -0 & 0 & -0.1 & 0.1 & 0 & 0 & -0.1 & 0.1 & -0.1 & 0.1 \\ 0.1 & -0.1 & 0.1 & -0.1 & 0.1 & -0.1 & 0.2 & 0.8 & -0 & 0 & 0 & -0 & 0.1 & -0.1 & -0 & 0 & 0.1 & -0.1 & 0.1 & -0.1 \\ 0.1 & -0.1 & -0.1 & 0.1 & -0 & 0 & 0 & -0 & 0.8 & 0.2 & -0.1 & 0.1 & -0.1 & 0.1 & 0.1 & -0.1 & 0.1 & -0.1 & -0 & 0 \\ -0.1 & 0.1 & 0.1 & -0.1 & -0 & 0 & -0 & 0 & 0.2 & 0.8 & 0.1 & -0.1 & 0.1 & -0.1 & -0.1 & 0.1 & -0.1 & 0.1 & 0 & -0 \\ 0.1 & -0.1 & 0 & -0 & -0.1 & 0.1 & -0 & 0 & -0.1 & 0.1 & 0.8 & 0.2 & -0.1 & 0.1 & -0.1 & 0.1 & -0 & 0 & 0.1 & -0.1 \\ -0.1 & 0.1 & 0 & 0 & 0.1 & -0.1 & 0 & -0 & 0.1 & -0.1 & 0.2 & 0.8 & 0.1 & -0.1 & 0.1 & -0.1 & 0 & 0 & -0.1 & 0.1 \\ 0.1 & -0.1 & -0 & -0 & -0 & 0 & -0.1 & 0.1 & -0.1 & 0.1 & -0.1 & 0.1 & 0.8 & 0.2 & 0 & 0 & -0.1 & 0.1 & -0.1 & 0.1 \\ -0.1 & 0.1 & -0 & 0 & -0 & 0 & 0.1 & -0.1 & 0.1 & -0.1 & 0.1 & -0.1 & 0.2 & 0.8 & -0 & -0 & 0.1 & -0.1 & 0.1 & -0.1 \\ -0 & -0 & 0.1 & -0.1 & -0.1 & 0.1 & 0 & -0 & 0.1 & -0.1 & -0.1 & 0.1 & 0 & -0 & 0.8 & 0.2 & -0.1 & 0.1 & 0.1 & -0.1 \\ -0 & 0 & -0.1 & 0.1 & 0.1 & -0.1 & 0 & 0 & -0.1 & 0.1 & 0.1 & -0.1 & 0 & -0 & 0.2 & 0.8 & 0.1 & -0.1 & -0.1 & 0.1 \\ 0 & -0 & 0.1 & -0.1 & 0 & -0 & -0.1 & 0.1 & 0.1 & -0.1 & -0 & 0 & -0.1 & 0.1 & -0.1 & 0.1 & 0.8 & 0.2 & -0.1 & 0.1 \\ -0 & 0 & -0.1 & 0.1 & -0 & 0 & 0.1 & -0.1 & -0.1 & 0.1 & 0 & 0 & 0.1 & -0.1 & 0.1 & -0.1 & 0.2 & 0.8 & 0.1 & -0.1 \\ 0 & -0 & 0 & -0 & 0.1 & -0.1 & -0.1 & 0.1 & -0 & 0 & 0.1 & -0.1 & -0.1 & 0.1 & 0.1 & -0.1 & -0.1 & 0.1 & 0.8 & 0.2 \\ -0 & 0 & 0 & 0 & -0.1 & 0.1 & 0.1 & -0.1 & 0 & -0 & -0.1 & 0.1 & 0.1 & -0.1 & -0.1 & 0.1 & 0.1 & -0.1 & 0.2 & 0.8 \end{bmatrix}. \tag{16}$$

II. Pseudo-undirected star graph

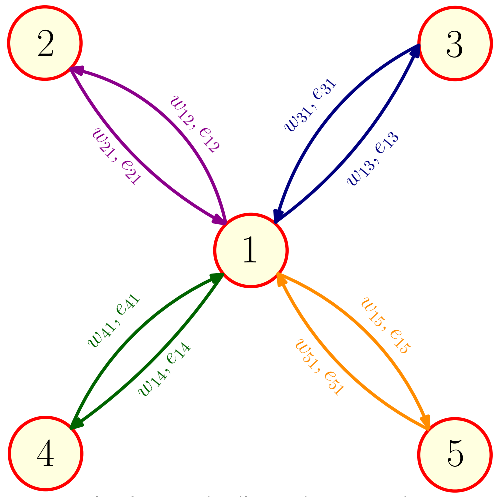

Fig. 2: Pseudo-directed star graph.

A. Incidence matrix

$$E = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}. \tag{17}$$

B. Out-incidence matrix

$$E_{\otimes} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{18}$$

C. Edge weight (diagonal) matrix

$$\mathcal{W} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}. \tag{19}$$

D. Laplacian matrix

$$\mathcal{L} = E_{\otimes} \mathcal{W} E^{\top} = \begin{bmatrix} 12 & -4 & -3 & -3 & -2 \\ -2 & 2 & 0 & 0 & 0 \\ -4 & 0 & 4 & 0 & 0 \\ -4 & 0 & 0 & 4 & 0 \\ -3 & 0 & 0 & 0 & 3 \end{bmatrix}. \tag{20}$$

E. Left eigenvector of the Laplacian

$$
\begin{aligned}
\mathbf{v}_L: \quad \mathbf{v}_L \mathcal{L}^\top = \mathbf{0} \qquad (21)\\
\Rightarrow \mathbf{v}_L = \begin{bmatrix} 1.5 & 3 & 1.1250 & 1.1250 & 1 \end{bmatrix} \qquad (22)
\end{aligned}
$$

F. Right eigenvector of the Laplacian

$$
\begin{aligned}
\mathbf{v}_R: \quad \mathcal{L}\mathbf{v}_R = \mathbf{0} \qquad (23)\\
\Rightarrow \mathbf{v}_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\top = \mathbf{1} \qquad (24)
\end{aligned}
$$

Note that $\mathbf{v}_L \neq \mathbf{v}_R$. Further, we have

$$
p = \mathcal{W}E_\otimes^\top \mathbf{v}_R = \begin{bmatrix} 4 & 2 & 3 & 4 & 3 & 4 & 2 & 3 & \end{bmatrix}^\top = w_{ii}, \qquad (25)
$$

where w_{ii} is corresponding the diagonal entry of the $\mathcal{W}$.

G. Other useful matrices

These computations have been done using MATLAB.

$$
U = \mathcal{W}E_\otimes^\top = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}. \qquad (26)
$$

$$
\begin{aligned}
\mathbf{V}: \quad E\mathbf{V} = 0 \qquad (27)\\
\Rightarrow \mathbf{V} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
P_U &= U(U^\top U)^{-1}U^\top \qquad (29)\\
&= \begin{bmatrix} 0.4211 & 0 & 0.3158 & 0 & 0.3158 & 0 & 0.2105 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3158 & 0 & 0.2368 & 0 & 0.2368 & 0 & 0.1579 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0.3158 & 0 & 0.2368 & 0 & 0.2368 & 0 & 0.1579 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.2105 & 0 & 0.1579 & 0 & 0.1579 & 0 & 0.1053 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
P_V &= V(V^\top V)^{-1}V^\top \qquad (31)\\
&= \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}. \qquad (32)
\end{aligned}
$$