# Polynomial Functors in Lean4

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# Chapter 0

# Prelude

In this chapter we provide several different perspectives on polynomial functors.

As a first introduction, polynomial functors are categorfication of the usual polynomials from algebra. Categorification is the process of finding category-theoretic analogs of set-theoretic concepts by replacing sets with categories, elements by objects, functions with functors, equations between elements by isomorphisms between objects, and equations between functions by natural isomorphisms between functors.

| **Sets/Types** | **Categories** |
|---|---|
| $\mathbf{N}$ | The category $\mathbf{Set}_{\mathrm{fin}}$ of finite sets and functions |
| $2 \in \mathbf{N}$ | The set $\mathbf{2} = \{0, 1\}$ |
| Equality $1 + 1 = 2$ | Isomorphism $\mathbf{1} \coprod \mathbf{1} \cong \mathbf{2}$ in $\mathbf{Set}_{\mathrm{fin}}$ |
| Equality $2 \cdot 3 = 6$ | Isomorphism $\mathbf{2} \times \mathbf{3} \cong \mathbf{6}$ in $\mathbf{Set}_{\mathrm{fin}}$ |
| Addition operation<br>$+ : \mathtt{Nat} \times \mathtt{Nat} \to \mathtt{Nat}$ | The disjoint union functor<br>$+ : \mathbf{Set}_{\mathrm{fin}} \times \mathbf{Set}_{\mathrm{fin}} \to \mathbf{Set}_{\mathrm{fin}}$ |
| Multiplication operation<br>$\cdot : \mathtt{Nat} \times \mathtt{Nat} \to \mathtt{Nat}$ | The product functor<br>$\times : \mathbf{Set}_{\mathrm{fin}} \times \mathbf{Set}_{\mathrm{fin}} \to \mathbf{Set}_{\mathrm{fin}}$ |
| Commutativity of addition<br>$\forall (m, n : \mathbf{N}), m + n = n + m$ | Natural isomorphism<br>$\mathbf{m} + \mathbf{n} \cong \mathbf{n} + \mathbf{m}$ in $\mathbf{Set}_{\mathrm{fin}}$ |
| Commutativity of multiplication<br>$\forall (m, n : \mathbf{N}), m \cdot n = n \cdot m$ | Natural isomorphism<br>$\mathbf{m} \times \mathbf{n} \cong \mathbf{n} \times \mathbf{m}$ in $\mathbf{Set}_{\mathrm{fin}}$ |
| Associativity of addition<br>$\forall (m, n, k : \mathbf{N}), m + (n + k) = (m + n) + k$ | Natural isomorphism<br>$\mathbf{m} + (\mathbf{n} + \mathbf{k}) \cong (\mathbf{m} + \mathbf{n}) + \mathbf{k}$ in $\mathbf{Set}_{\mathrm{fin}}$ |
| Associativity of multiplication<br>$\forall (m, n, k : \mathbf{N}), m \cdot (n \cdot k) = (m \cdot n) \cdot k$ | Natural isomorphism<br>$\mathbf{m} \times (\mathbf{n} \times \mathbf{k}) \cong (\mathbf{m} \times \mathbf{n}) \times \mathbf{k}$ in $\mathbf{Set}_{\mathrm{fin}}$ |
| $\vdots$ | $\vdots$ |

In fact, $\mathbf{N}$ is a commutative semiring; it admits two monoid structure and can be seen as an additive commutative monoid $(\mathbf{N}, +, 0)$ and a multiplicative commutative monoid $(\mathbf{N}, \cdot, 1)$. The two operations are related by the distributive law $m \cdot (n+k) = m \cdot n + m \cdot k$. One category-level higher, $\mathbf{Set}_{\mathrm{fin}}$ can be seen as a commutative rig-category, that is a category with two monoidal structures, one for the product and one for the coproduct. The two monoidal structures are related by the distributive law $X \times (Y + Z) \cong (X \times Y) + (X \times Z)$. Iterating this process, we can climb the ladder of $n$-categories: The rig-categories are objects of a 2-category where 1-morphisms are monoidal functors and 2-morphisms are monoidal natural transformations.

Recall that a polynomial is a finite sum of monomials, where each monomial is a product of variables raised to some non-negative integer powers. A univariate polynomial is usually writted down as a formal sum

$$\sum_{i=0}^{n-1} a_i \, x^i \tag{0.0.1}$$

where $x$ is the variable, $a_i$ are coefficients in some ring $R$. To have a unique representation, we sort the monomials by their exponent, the terms of lower exponent come first, so although $x + x^2$ and $x^2 + x$ are the same polynomial, we choose the first one as the canonical representation.

Another way to think about this formal sum is to see the exponent $i$ as a function of the coefficients: In this view, a a polynomial is simply a function $R \to \mathbf{N}$ with finite support. So we can write 0.0.1 as

$$\sum_{c \in I} c \, x^{i(c)} \tag{0.0.2}$$

where $I$ is a finite support. This is how usual polynomials are defined in the Lean4's mathematical library `mathlib`.

```
structure Polynomial (R : Type*) [Semiring R] where ofFinsupp ::
toFinsupp : AddMonoidAlgebra R Nat
```

where `AddMonoidAlgebra R Nat` is defined as the type `R →₀ Nat` of functions from $R$ to $\mathbf{N}$ with finite support.

These two views are equivalent. For simplicity, let's restrict ourselves to the case of polynomials with coefficients in $\mathbf{N}$ so that when expanded all the coefficients become 1 and instead we will have multiplicities of the monomials. So we can write the polynomial $x + 3 * x^2$ as $x + x^2 + x^2 + x^2$. In this presentation we encode a polynomial by its degree (a natural number $n$) and a finite list of multiplicities $m_0, m_1, \ldots, m_{n-1}$. This is just a function $m : \mathbf{N} \to \mathbf{N}$ with finite support.

A multivariate polynomial is a polynomial in several variables, say $x_1, \ldots, x_n$.

They are usually denoted by a sum of the form

$$\sum_{i \in I} \prod_{j \in J} a_{ij} x_j^{e_{ij}}$$

where $I$ and $J$ are finite sets, $a_{ij}$ are coefficients, and $e_{ij}$ are non-negative integers.

# Chapter 1

# Locally Cartesian Closed Categories

**Definition 1.0.1** (exponentiable morphism)**.** Suppose $\mathbb{C}$ is a category with pullbacks. A morphism $f\colon A \to B$ in $\mathbb{C}$ is **exponentiable** if the pullback functor $f^*\colon \mathbb{C}/B \to \mathbb{C}/A$ has a right adjoint $f_*$. Since $f^*$ always has a left adjoint $f_!$, given by post-composition with $f$, an exponentiable morphism $f$ gives rise to an adjoint triple

$$\begin{tikzcd}
\mathbb{C}/B \arrow[d, bend right=60, "f_!"'] \arrow[d, bend left=60, "f_*"] \arrow[d, phantom, "\dashv \quad\quad \dashv"] \\
\mathbb{C}/A \arrow[u, "f^*"]
\end{tikzcd}$$

**Definition 1.0.2** (pushforward functor)**.** Let $f\colon A \to B$ be an exponentiable morphism in a category $\mathbb{C}$ with pullbacks. We call the right adjoint $f_*$ of the pullback functor $f^*$ the **pushforward** functor along $f$.

**Theorem 1.0.3** (exponentiable morphisms are exponentiable objects of the slices)**.** *A morphism $f\colon A \to B$ in a category $\mathbb{C}$ with pullbacks is exponentiable if and only if it is an exponentiable object, regarded as an object of the slice $\mathbb{C}/B$.*

**Definition 1.0.4** (Locally cartesian closed categories)**.** A category with pullbacks is **locally cartesian closed** if is a category $\mathbb{C}$ with a terminal object 1 and with all slices $\mathbb{C}/A$ cartesian closed.

# Chapter 2

# Univaiate Polynomial Functors

In this section we develop some of the definitions and lemmas related to polynomial endofunctors that we will use in the rest of the notes.

**Definition 2.0.1** (Polynomial endofunctor)**.** Let $\mathbb{C}$ be a locally Cartesian closed category (in our case, presheaves on the category of contexts). This means for each morphism $t : B \to A$ we have an adjoint triple

$$\begin{tikzcd}
\mathbb{C}/B \arrow[d, bend right=60, "t_!"'] \arrow[d, bend left=60, "t_*"] \\
\mathbb{C}/A \arrow[u, "\dashv \; t^* \; \dashv" description]
\end{tikzcd}$$

where $t^*$ is pullback, and $t_!$ is composition with $t$.

Let $t : B \to A$ be a morphism in $\mathbb{C}$. Then define $P_t : \mathbb{C} \to \mathbb{C}$ be the composition

$$P_t := A_! \circ t_* \circ B^*$$

$$\mathbb{C} \xrightarrow{B^*} \mathbb{C}/B \xrightarrow{t_*} \mathbb{C}/A \xrightarrow{A_!} \mathbb{C}$$

# Chapter 3

# Multivariate Polynomial Functors

Let $\mathbb{C}$ be category with pullbacks and terminal object.

**Definition 3.0.1** (multivariable polynomial functor)**.** A **polynomial** in $\mathbb{C}$ from $I$ to $O$ is a triple $(i, p, o)$ where $i$, $p$ and $o$ are morphisms in $\mathbb{C}$ forming the diagram

$$I \xleftarrow{i} E \xrightarrow{p} B \xrightarrow{o} J.$$

The object $I$ is the object of input variables and the object $O$ is the object of output variables. The morphism $p$ encodes the arities/exponents.

**Definition 3.0.2** (extension of polynomial functors)**.** The **extension** of a polynomial $I \xleftarrow{i} B \xrightarrow{p} A \xrightarrow{o} J$ is the functor $\mathrm{P} = o_! \, f_* \, i^* \colon \mathbb{C}/I \to \mathbb{C}/O$. Internally, we can define P by

$$\mathrm{P}\,(\, X_i \mid i \in I \,) = \left( \sum_{b \in B_j} \prod_{e \in E_b} X_{s(b)} \,\middle|\, j \in J \right)$$

A **polynomial functor** is a functor that is naturally isomorphic to the extension of a polynomial.