Polynomial Functors in Lean4

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Chapter 1

Locally Cartesian Closed Categories

Definition 1.0.1 (exponentiable morphism)**.** Suppose $\mathbb{C}$ is a category with pullbacks. A morphism $f\colon A \to B$ in $\mathbb{C}$ is **exponentiable** if the pullback functor $f^*\colon \mathbb{C}/B \to \mathbb{C}/A$ has a right adjoint f_*. Since f^* always has a left adjoint $f_!$, given by post-composition with f, an exponentiable morphism f gives rise to an adjoint triple

$$\begin{tikzcd}
\mathbb{C}/B \arrow[d, bend right=60, "f_!"'] \arrow[d, bend left=60, "f_*"] \\
\mathbb{C}/A \arrow[u, "{\dashv\ f^*\ \dashv}" description]
\end{tikzcd}$$

Definition 1.0.2 (pushforward functor)**.** Let $f\colon A \to B$ be an exponentiable morphism in a category $\mathbb{C}$ with pullbacks. We call the right adjoint f_* of the pullback functor f^* the **pushforward** functor along f.

Theorem 1.0.3 (exponentiable morphisms are exponentiable objects of the slices)**.** *A morphism $f\colon A \to B$ in a category $\mathbb{C}$ with pullbacks is exponentiable if and only if it is an exponentiable object, regarded as an object of the slice $\mathbb{C}/B$.*

Definition 1.0.4 (Locally cartesian closed categories)**.** A category with pullbacks is **locally cartesian closed** if is a category $\mathbb{C}$ with a terminal object 1 and with all slices $\mathbb{C}/A$ cartesian closed.

Chapter 2

Univaiate Polynomial Functors

In this section we develop some of the definitions and lemmas related to polynomial endofunctors that we will use in the rest of the notes.

Definition 2.0.1 (Polynomial endofunctor)**.** Let $\mathbb{C}$ be a locally Cartesian closed category (in our case, presheaves on the category of contexts). This means for each morphism $t : B \to A$ we have an adjoint triple

$$\begin{tikzcd}
\mathbb{C}/B \arrow[d, bend right=60, "t_!"'] \arrow[d, bend left=60, "t_*"] \\
\mathbb{C}/A \arrow[u, "\dashv\; t^*\; \dashv" description]
\end{tikzcd}$$

where t^* is pullback, and $t_!$ is composition with t.

Let $t : B \to A$ be a morphism in $\mathbb{C}$. Then define $P_t : \mathbb{C} \to \mathbb{C}$ be the composition

$$P_t := A_! \circ t_* \circ B^*$$

$$\mathbb{C} \xrightarrow{B^*} \mathbb{C}/B \xrightarrow{t_*} \mathbb{C}/A \xrightarrow{A_!} \mathbb{C}$$

Chapter 3

Multivariate Polynomial Functors

Let $\mathbb{C}$ be category with pullbacks and terminal object.

Definition 3.0.1 (multivariable polynomial functor)**.** A **polynomial** in $\mathbb{C}$ from I to O is a triple (i, p, o) where i, p and o are morphisms in $\mathbb{C}$ forming the diagram

$$I \xleftarrow{i} E \xrightarrow{p} B \xrightarrow{o} J.$$

The object I is the object of input variables and the object O is the object of output variables. The morphism p encodes the arities/exponents.

Definition 3.0.2 (extension of polynomial functors)**.** The **extension** of a polynomial $I \xleftarrow{i} B \xrightarrow{p} A \xrightarrow{o} J$ is the functor $\mathrm{P} = o_! \, f_* \, i^* \colon \mathbb{C}/I \to \mathbb{C}/O$. Internally, we can define P by

$$\mathrm{P}\,(\, X_i \mid i \in I \,) = \left(\sum_{b \in B_j} \prod_{e \in E_b} X_{s(b)} \;\middle|\; j \in J \right)$$

A **polynomial functor** is a functor that is naturally isomorphic to the extension of a polynomial.