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MODULE THEORY

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Introduction to Modules 9

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Dedicated to my family and my best friend

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Introduction

This sample book discusses the course `Module Theory` of `pure mathematics` being taught to `post-graduate` students in `University of Delhi`.

Introduction to Modules

Defination of Module

Left Module:

Let R be a ring with identity and M be an abelian group with addition. We say M is a left $R-$module if there exists a mapping[1]

[1] often called as scaler multiplication.

$$R \times M \to M$$

defined by

$$(a\ ,\ x) \to ax$$

for each $a \in R$ and $x \in M$ satisfying following properties :

$\forall\ a\ ,\ b \in R$ and $x\ ,y\ \in M$

$$\begin{align} (a+b)x &= ax+bx \tag{1}\\ a(x+y) &= ax+ay \tag{2}\\ (ab)x &= a(bx) \tag{3}\\ 1x &= x \tag{4} \end{align}$$

and denoted by $_RM$

Right Module:

Let R be a ring with identity and M be an abelian group with addition. We say M is a right $R-$module if there exists a mapping

$$M \times R \to M$$

defined by

$$(x\ ,\ a) \to xa$$

for each $a \in R$ and $x \in M$ satisfying following properties :

$\forall\ a\ ,\ b \in R$ and $x\ ,y\ \in M$

$$\begin{align} x(a+b) &= xa+xb \tag{5}\\ (x+y)a &= xa+ya \tag{6}\\ x(ab) &= (xa)b \tag{7}\\ x1 &= x \tag{8} \end{align}$$

and denoted by M_R

Examples :

1. Let V be a vector space over a field F then V is a left as well as right $F-$Module.

2. Let G be any abelian group under addition , then G is a $\mathbb{Z}-$Module where $\mathbb{Z}$ is set of integers.

3. Let R be ring and $M = R[x]$ where $R[x]$ is a group of all polynomials with coefficents in R then M is a left as well as a right $R-$Module with scaler multiplication being usual multiplication.

Suppose ring R is a field then $R-$Module $R[x]$ is a vector space over field R.

4. Let M be collection of all $m \times n$ matrices over ring R , then M is left $R-$Module where scaler multiplication being usual multiplication of a scaler to a matrix.

   In particular, if M is a set of $1 \times n$ matrices over R or $M = R^n$(set of $n-$tuples) then R^n is a left $R-$module.

Remark: 1. *Let R be a commutative ring then every left $R-$module can be transformed to right $R-$module and vice-versa.*

Proof. Let M be left $R-$module and R be a commutative ring.
so, $\exists$ a mapping

$$R \times M \to M$$

defined by

$$(a\ ,\ x) \to ax$$

for each $a \in R$ and $x \in M$ satisfying following properties :

$\forall\ a\ ,\ b \in R$ and $x\ ,y\ \in M$

$$\begin{aligned}(a+b)x &= ax+bx\\ a(x+y) &= ax+ay\\ (ab)x &= a(bx)\\ 1x &= x\end{aligned}$$

$\because$ R is a commutative ring.
Now, Define an another mapping

$$M \times R \to M$$

defined by

$$(x\ ,\ a) \to x * a\ =\ ax$$

To check M is a right $R-$Module , we need to verify properties number (5)-(8)

1.

$$\begin{aligned} x * (a+b) &= (a+b)x \\ &= ax + bx \\ &= (x*a) + (x*b) \end{aligned}$$

2.

$$\begin{aligned} (x+y)*a &= a(x+y) \\ &= ax + ay \\ &= (x*a) + (y*a) \end{aligned}$$

3.

$$\begin{aligned} x*(ab) &= (ab)x \\ &= (ba)x \\ &= b(ax) \\ &= (ax)*b \end{aligned}$$

4.

$$\begin{aligned} x*1 &= 1x \\ &= x \end{aligned}$$

Thus, ${}_RM$ is transformed to M_R.

Similarly, Converse statement can be verified.

■

Remark: 2. *Let S be a subring of ring R then ${}_SM$ exists only if ${}_RM$ exists.*

by existance means M is a valid left module over mentioned ring or subring. i.e. satisfying those four properties.

Remark: 3. *Same Abelian group can have the structure of a Module for a number of different rings.*

For Instance, The field $\mathbb{R}$ is $\mathbb{R}-$module,$\mathbb{Q}-$module and $\mathbb{Z}-$module.

Remark: 4. *Let I be left ideal of R then quotient ring ${}^{R}/_{I}$ is a left R-module.*

verification: 'left to reader'

Hint: you need to verify those four properties: (1)-(4) ■

Here scaler multiplication is

$$R \times {}^{R}/_{I} \rightarrowtail {}^{R}/_{I}$$

defined as

$$(a\ ,\ x+I) \rightarrowtail ax+I$$

$\forall\ a \in R$ and $\forall\ x+I \in {}^{R}/_{I}$

Theorem 0.0.1. *(Elementry Properties:)*

Let M be a left R-module . Suppose 0_m and 0_r denotes additive identities of M and R respectively. Then, for each $x \in M$ and $r \in R$

(i)
$$0_m = 0_r\ x = r\ 0_m$$

(ii)
$$r(-x) = (-r)x = -rx$$

Proof. *(i)* As 0_m is the additive identity of M. so, $0_m = 0_m + 0_m$

Consider $r(0_m + 0_m) = r\ 0_m = r\ 0_m + 0_m$

but, $r(0_m + 0_m) = (r\ 0_m) + (r\ 0_m)$

so, we have

$$r\ 0_m + r\ 0_m = r\ 0_m + 0_m$$

$\because r\ 0_m \in M$
so, $r\ 0_m = r\ 0_m + 0_m$
$\because$ M is a left R-module.
(using distribuitive property)

as $(M, +)$ is an abelian group so left and right cancellation law holds.

$$\begin{aligned} \cancel{r\ 0_m} + r\ 0_m &= \cancel{r\ 0_m} + 0_m \\ r\ 0_m &= 0_m \end{aligned}$$

a similiar argument can be used to prove $0_m = 0_r\ x$.

(ii)

■