KAPIL CHAUDHARY

MODULE THEORY

UNIVERSITY OF DELHI

Kapil Chaudhary

UNIVERSITY OF DELHI

https://contact.sirkapil.me/

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Contents

There are few persons without whom this was impossible.

They deserve credit for it so i would love to thank them.

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Dedicated to my family and my best friend

Neeraj K. Gaud

Introduction

Warning :

This is my first document created using latex so it may be possible that there are several errors. if you notice any error then you can report it here.

`https://github.com/sirkapil/module-theory/issues/new`[1]

[1] (may require a github account)

About :

This sample book discusses the course "Module Theory" being taught to Post-Graduate (M.Sc. Mathematics) students in Department of Mathematics under University of Delhi, Delhi.

All my LaTeX documents are free and open-source. Each document is hosted in a github repository and can be found pinned here.

`https://github.com/sirkapil`

Contribution :

If you find my work useful and want to contribute then you are welcome by heart.

Any suitable changes to document repository through pull requests are highly appreciated. You can create a new pull request here. Be sure to read *contribution file* in root/.github folder of repository before creating any pull-request.

`https://github.com/sirkapil/module-theory/compare`

If you don't have a github account or facing difficulty in creating a pull-request , then feel free to drop down a message here about that you are interested in contribution of this project.

`https://cont.netlify.com`

`https://twitter.com/kapil_rc`

Introduction to Modules

Defination of Module

Left Module:

Let R be a ring with identity and M be an abelian group with addition. We say M is a left $R-$module if there exists a mapping[2]

$$R \times M \to M$$

defined by

$$(a\ ,\ x) \to ax$$

$\forall\ a \in\ R$ and $x \in\ M$

satisfying following properties :

$$(a+b)x = ax + bx \tag{1.1}$$

$$a(x+y) = ax + ay \tag{1.2}$$

$$(ab)x = a(bx) \tag{1.3}$$

$$1x = x \tag{1.4}$$

$\forall\ a\ ,\ b \in R$ and $x\ , y\ \in M$

and denoted by ${}_RM$

[2] often called as scaler multiplication.

Right Module:

Let R be a ring with identity and M be an abelian group with addition. We say M is a right $R-$module if there exists a mapping

$$M \times R \to M$$

defined by

$$(x\ ,\ a) \to xa$$

$\forall\ a \in R$ and $x \in M$

satisfying following properties :

$$x(a+b) = xa + xb \tag{1.5}$$

$$(x+y)a = xa + ya \tag{1.6}$$

$$x(ab) = (xa)b \tag{1.7}$$

$$x1 = x \tag{1.8}$$

$\forall\ a\ ,\ b \in R$ and $x\ , y\ \in M$

and denoted by M_R.

Examples :

1. Let V be a vector space over a field F then V is a left as well as right $F-$Module.

2. Let G be any abelian group under addition , then G is a $\mathbb{Z}-$Module where $\mathbb{Z}$ is set of integers.

3. Let R be ring and $M = R[x]$ where $R[x]$ is a group of all polynomials with coefficents in R then M is a left as well as a right $R-$Module with scaler multiplication being usual multiplication.

Suppose ring R is a field then $R-$Module $R[x]$ is a vector space over field R.

4. Let M be collection of all $m \times n$ matrices over ring R , then M is left $R-$Module where scaler multiplication being usual multiplication of a scaler to a matrix.

   In particular, if M is a set of $1 \times n$ matrices over R or $M = R^n$(set of $n-$tuples) then R^n is a left $R-$module.

Remark: 2. *Let R be a commutative ring then every left $R-$module can be transformed to right $R-$module and vice-versa.*

Proof. Let M be left $R-$module and R be a commutative ring.
so, $\exists$ a mapping

$$R \times M \to M$$

defined by

$$(a\ ,\ x) \to ax$$

for each $a \in R$ and $x \in M$ satisfying following properties :

$\forall\ a\ ,\ b \in R$ and $x\ , y\ \in M$

$$\begin{aligned} (a+b)x &= ax+bx \\ a(x+y) &= ax+ay \\ (ab)x &= a(bx) \\ 1x &= x \end{aligned}$$

$\because$ R is a commutative ring.
Now, Define an another mapping

$$M \times R \to M$$

defined by

$$(x\ ,\ a) \to x * a\ =\ ax$$

To check M is a right $R-$Module , we need to verify properties number **??**-(1.8) *(i) Distribuitive Law*

$$\begin{aligned} x * (a+b) &= (a+b)x \\ &= ax + bx \\ &= (x*a) + (x*b) \end{aligned}$$

(ii) Distributive Law

$$\begin{aligned} (x+y) * a &= a(x+y) \\ &= ax + ay \\ &= (x*a) + (y*a) \end{aligned}$$

(iii)

$$\begin{aligned} x * (ab) &= (ab)x \\ &= (ba)x \\ &= b(ax) \\ &= (ax) * b \end{aligned}$$

(iv)

$$\begin{aligned} x * 1 &= 1x \\ &= x \end{aligned}$$

Thus, ${}_RM$ is transformed to M_R.

Similarly, Converse statement can be verified.

■

Remark: 3. *Let S be a subring of ring R then ${}_SM$ exists only if ${}_RM$ exists.*

by existance means M is a valid left module over mentioned ring or subring. i.e. satisfying those four properties.

Remark: 4. *Same Abelian group can have the structure of a Module for a number of different rings.*

For Instance, The field $\mathbb{R}$ is $\mathbb{R}-$module,$\mathbb{Q}-$module and $\mathbb{Z}-$module.

Remark: 5. *Let I be left ideal of R then quotient ring R/I is a left R-module.*

verification: 'left to reader'

Hint: you need to verify those four properties: (1.1)-(1.4) ■

Here scaler multiplication is

$$R \times R/I \rightarrowtail R/I$$

defined as

$$(a\ ,\ x+I) \rightarrowtail ax + I$$

$\forall\ a \in R$ and $\forall\ x + I \in R/I$

Theorem 1.1. *(Elementry Properties:)*

Let M be a left R-module . Suppose 0_m and 0_r denotes additive identities of M and R respectively. Then, for each $x \in M$ and $r \in R$

(i)

$$0_m = 0_r\ x = r\ 0_m$$

(ii)

$$r(-x) = (-r)x = -rx$$

Proof. *(i)* As 0_m is the additive identity of M. so, $0_m = 0_m + 0_m$

Consider $r(0_m + 0_m) = r\ 0_m = r\ 0_m + 0_m$

$\because$ $(r\ ,\ 0_m) \rightarrowtail r\ 0_m \in M$
so, $r\ 0_m = r\ 0_m + 0_m$

but, $r(0_m + 0_m) = (r\ 0_m) + (r\ 0_m)$

$\because$ M is a left R-module.
(using distribuitive property)

so, we have

$$r\ 0_m + r\ 0_m = r\ 0_m + 0_m$$

as $(M, +)$ is an abelian group so left and right cancellation law holds.

$$\begin{aligned} \cancel{r\ 0_m} + r\ 0_m &= \cancel{r\ 0_m} + 0_m \\ r\ 0_m &= 0_m \end{aligned}$$

a similiar argument can be used to prove $0_m = 0_r\ x$.

(ii) as M is a left R-module so $(r\ , x) \rightarrowtail rx \in M$

Now, Consider $(-r)x + rx$

using distribuitive law

$$\begin{aligned} (-r)x + rx &= (-r + r)x \\ &= 0_r\ x \\ &= 0_m \end{aligned}$$

i.e. $(-r)x$ is additive inverse of (rx) but additive inverse of (rx) is $-rx$ and it is unique for an abelian group(M here)

$$\therefore\ (-r)x = -rx$$

a similar argument can be used to prove that $r(-x) = -rx$.

∎

Definition 1.1 (Ring Homomorphism)**.** *Let R and S be two rings with identities 1_r , 1_s respectively then a map(say f)*

often called as ring homo

$$f : R \to S$$

is said to be a ring homomorphism or ring linear map if for every a , $b \in R$ following properties holds

if R = S then we call ring homo as ring endomorphism. For instance , let f be ring homo from R to R . we say f is endomorphism of R and denoted by *End R* *(i) Preserves Addition*

$$(a + b)f = (a)f + (b)f$$

(ii) Preservers Multiplication

$$(ab)f = (a)f.(b)f$$

(iii) Maps identity to identity

$$(1_r)f = 1_s$$

Remark: 6. *Such a mapping need not to be bijective. if it is bijective then we say it is a ring isomorphism or rings are isomorphic.*

Theorem 1.2. *Let R be a ring and M be any abelian group with addition. then M is a right R-module if and only if there exists a map which is ring homomorphism from R to End M*

M is a right R-module
$\Updownarrow$
$\exists\, f : R \xrightarrow[\text{Homo}]{\text{Ring}} End\ M$

Proof. *(Forward Part)* Let us suppose that M is a right R-module.

Claim: there exists a map which is ring homomorphism from R to *End M*

$\because$ M is a left R-module , so there exist a map

$$f : M \times R \rightarrowtail M$$

defined by

$$(x\ , a) \rightarrowtail ax$$

satisfying following properties:

$$\begin{aligned} (x + y)a &= (x)a + (y)a x(a + b) && = xa + xb \\ x(ab) &= (xa)b \\ x1 &= x \end{aligned}$$

$\forall\, x, y\ \in M\ \&\ a, b \in R$

for each $a \in R$, define a map(say ϕ_a)

$$\phi_a : M \rightarrowtail\ M$$

such that for each $x \in M$

$$(x)\phi_a = xa \in\ M$$

Now, we'll show that $\phi_a \in$ *End M*

Let $x, y \in M$

Consider $(x+y)\phi_a$

$$= (x+y)a \qquad \text{using defination of } \phi_a$$
$$= xa + ya \qquad \text{using (1.9)}$$
$$= (x)\phi_a + (y)\phi_a$$

so, ϕ_a preserves addition and is a group homo from M to M.

i.e. $\phi_a \in End\ M$

Now, we can define a map (say f)

$$f : R \rightarrowtail End\ M$$

defined as

$$(a)f \rightarrowtail \phi_a \qquad \forall a \in R \text{ and } \phi_a \in End\ M$$

Now, We'll show that f is a ring homomorphism.

(A)

$$(a+b)f = \phi_{a+b}$$
$$= \phi_a + \phi_b$$
$$= (a)f + (b)f$$

for each $x \in M$ we have,

$$(x)\phi_{a+b} = x(a+b) = xa + xb$$
$$= (x)\phi_a + (y)\phi_b$$
$$\therefore\ \phi_{a+b} = \phi_a + \phi_b$$

(B)

$$(ab)f = \phi_{ab}$$
$$= \phi_a \cdot \phi_b$$
$$= (a)f\ (b)f$$

for each $x \in M$ we have,

$$(x)\phi_{ab} = x(ab) = (xa)b$$
$$= (xa)\phi_b = (x)\phi_a \cdot \phi_b$$
$$\therefore\ \phi_{ab} = \phi_a \cdot \phi_b$$

(C)

$$(1)f = \phi_1$$

for each $x \in M$ we have,

$$(x)\phi_1 = x(1)$$
$$= x$$

$\therefore\ \phi_1$ is identity of$End\ M$

Thus, Forward Part is proved.

(Converse Part) Assume that $\exists$ a ring homo.(say f)

$$f : R \xrightarrow[\text{Homo}]{\text{Ring}} End\ M$$

for any $a \in R$,we denote the $(a)f$ by $f_a \in End\ M$

Claim: M is a right R-module.

so let's define a map

$$R \times M \longrightarrow M$$

defined by

$$(a, x) \rightarrowtail x * a = (x) f_a$$

to prove M is a right R-module , we need to verify four properties (1.5)- (1.9) of right R-module.

(i)

$$\begin{aligned} (x+y) * a &= (x+y) f_a \\ &= (x) f_a + (y) f_a \\ &= x * a + y * a \end{aligned}$$

$\because\ f_a\ \in End\ M$

(ii)

$$\begin{aligned} x * (a+b) &= (x) f_{a+b} \\ &= (x)(f_a + f_b) \\ &= (x) f_a + (x) f_b \\ &= (x * a) + (x * b) \end{aligned}$$

$\because\ f_a\ ,\ f_b \in End\ M$

(iii)

$$\begin{aligned} x * (ab) &= (x) f_{ab} \\ &= (x) f_a \cdot f_b \\ &= (x f_a) f_b \\ &= (x * a) * b \end{aligned}$$

(iv)

$$x * 1 = (x) f_1 = x$$

Thus, M is a right R-module.

$\because\ f_1$is identity in$End\ M$

■

Definition 1.2 (Anti-Ring Homomorphism). *Let R and S be two rings with identities 1_r and 1_s respectively. Define a map f*

$$f : R \rightarrowtail S$$

satisfying following properties, for each $a, b \in R$

(i)

$$(a+b) f = (a) f + (b) f$$

(ii)

$$(ab) f = (b) f\ (a) f$$

(iii)

$$(1_r) f = 1_s$$

Then, f is called anti-ring homomorphism.

Theorem 1.3. *Let R be a ring and M be any abelian group with addition. then M is a left R-module if and only if there exists a map which is anti-ring homomorphism from R to $End\ M$.*

M is a left R-module

$\Updownarrow$

$\exists\ f : R \xrightarrow[\text{Homo}]{\text{Anti-Ring}} End\ M$

Proof. Left to reader. ∎

Submodules

Definition 1.3 (SubModule). *Let M be a left (right) R-module then a subset N of M is called a submodule of M if N is a left (right) R-module under the operation induced from M.*

In other words, A subset N of M is called submodule of M if

(i) N is subgroup of M.

(ii) N is closed under induced scaler multiplication from M.

Theorem 1.4 (Criterion for Checking Modules). *Let M be a left (right) R-module and N be a subset of M then N is a submodule of M if and only if*

(i)

$$x - y \in N$$

$\forall\ x, y \in N$

(ii)

$$ax \in N$$

$\forall\ a \in R\ \&\ x \in N$

Proof. Left to reader. ∎

Examples:

1. As every Vector Space V over a Field F is a F-module. So, submodules of V are subspaces of V.

2. As every abelian group G is a $\mathbb{Z}$-module. So, all subgroups of G are submodules.

3. Let R be a ring then R is a left as well as right R-module then left (right) ideals of R are left (right) submodules of R.

4. $\{0\}$ and M are trivial submodules of any left (right) R-module M.

Remark: 7.

1. *Union of two submodules need not to be a submodule.*

Think an example !

2. *Intersection of any number of submodules is again a submodule.*

Hint: Verify using criterion for checking modules.

Remark: 8. ***(Smallest Submodule containing a set)***

Let M be any left (right) R-module and S be any subset of M. Suppose $\mathcal{F}$ be the family of all submodules of M containing S.

$$\text{Let } P = \bigcap_{N \in \mathcal{F}} N$$

then P is a submodule of M containing S as being intersection of an indexed family of submodules containing S.

Moreover, P is the smallest submodule of M containing S. i.e. for any arbitrary submodule $K \in \mathcal{F}$, we have $P \subseteq K$. Such submodule P of M is said to be generated by set S and is denoted by

$$P = \langle S \rangle = (S)$$

Remark: 9.

Let S be any subset of left R-module M and $\langle S \rangle$ is the smallest submodule of M containing S.

1. *if S is non-empty and finite , $S = \{x_1, x_2, x_3, \cdots, x_n\}$*

$$\langle S \rangle = \langle \{x_1, x_2, x_3, \cdots, x_n\} \rangle = \langle x_1, x_2, x_3, \cdots, x_n \rangle$$

   is said to be a finitely generated by S and is smallest submodule of M containing S.

2. *if $S = \phi$ i.e. S is an empty set*

$$\langle S \rangle = \langle \phi \rangle = \{0\}$$

3. *if $S = \{a\}$ i.e. S is singleton then $\langle S \rangle = \langle a \rangle$ is said to be a cyclic submodule.*

Definition 1.4 (Cyclic module). *A module M is said to be a cyclic module[3] if it can be generated by a single element.*

[3] P. M. Cohn. *Basic Algebra*. Springer, 2 edition, 2005. ISBN 978-1-4471-1060-6

For Example: A ring R over itself is a module and can be generated by identity element $\{1\}$ so is a cyclic module.

Theorem 1.5. *Let M be left R module and S being any subset of M.*

$$\langle S\rangle = \begin{cases} \{0\} & \text{if } S = \phi \\ \left\{ \sum_{i\in J_n} a_i x_i \mid a_i \in R \, , x_i \in S \right\} & \text{otherwise} \end{cases}$$

Proof. *Case-I* Let us suppose that $S = \phi$,as $\langle S\rangle$ is the intersection of all the submodules of M containing S.

i.e. Every submodule of M will contain S

In particular, $\{0\}$ also contains S i.e.

$$\{0\} \in \mathcal{F}$$

$\because$ $\mathcal{F}$ is a collection of all submodules of M containing S

so,

$$\begin{aligned}\langle S\rangle &= \bigcap_{N\in\mathcal{F}} N \\ &= \{0\}\end{aligned}$$

Case-II Suppose S is non-empty and let

$$P = \left\{ \sum_{i\in J_n} a_i x_i \mid a_i \in R \, , x_i \in S \right\}$$

First , we'll show that $S \subseteq P$

Let $x \in S$ then it can be expressed in following form:

$$x = 1.x = \sum_{i\in J_1} a_i x_i$$

with $a_1 = 1$ and $x_1 = x$

$$\therefore x \in P \Rightarrow S \subseteq P$$

$\because$ x was chosen arbitrary.

Now ,we'll show that P is a submodule of M using submodule criterion.

Let u , $v \in P$. so, we need to show $u + \alpha v \in P$

for any $\alpha \in R$

$$u = \sum_{i\in J_n} a_i x_i$$

$\forall x_i \in S$ & $a_i \in R$

$$v = \sum_{j\in J_m} b_j y_j$$

$\forall y_j \in S$ & $b_j \in R$

∎

Stay Tuned for next chapters!

Bibliography

P. M. Cohn. *Basic Algebra*. Springer, 2 edition, 2005. ISBN 978-1-4471-1060-6.