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MODULE THEORY

UNIVERSITY OF DELHI

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Dedicated to my family and my best friend

Neeraj K. Gaud

Introduction

Warning :

This is my first document created using latex so it may be possible that there are several errors. if you notice any error then you can report it here.

<https://github.com/sirkapil/module-theory/issues/new>¹

¹ (may require a github account)

About :

This sample book discusses the course "Module Theory" being taught to Post-Graduate (M.Sc. Mathematics) students in Department of Mathematics under University of Delhi, Delhi.

All my \LaTeX documents are free and open-source. Each document is hosted in a github repository and can be found pinned here.

<https://github.com/sirkapil>

Contribution :

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Any suitable changes to document repository through pull requests are highly appreciated. You can create a new pull request here. Be sure to read *contribution file* in root/.github folder of repository before creating any pull-request.

<https://github.com/sirkapil/module-theory/compare>

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https://twitter.com/kapil_rc

Introduction to Modules

Definition of Module

Definition 1.1 (Left Module). Let R be a ring with identity and M be an abelian group with addition. We say M is a left R –module if there exists a mapping²

$$R \times M \rightarrow M$$

defined by

$$(a, x) \rightarrow ax$$

² often called as scalar multiplication.

$$\forall a \in R \text{ and } x \in M$$

satisfying following properties :

$$(a + b)x = ax + bx \quad (1.1)$$

$$a(x + y) = ax + ay \quad (1.2)$$

$$(ab)x = a(bx) \quad (1.3)$$

$$1x = x \quad (1.4)$$

$$\forall a, b \in R \\ x, y \in M$$

and denoted by ${}_R M$

Definition 1.2 (Right Module). Let R be a ring with identity and M be an abelian group with addition. We say M is a right R –module if there exists a mapping

$$M \times R \rightarrow M$$

defined by

$$(x, a) \rightarrow xa$$

$$\forall a \in R \text{ and } x \in M$$

satisfying following properties :

$$x(a + b) = xa + xb \quad (1.5)$$

$$(x + y)a = xa + ya \quad (1.6)$$

$$x(ab) = (xa)b \quad (1.7)$$

$$x1 = x \quad (1.8)$$

$$\forall a, b \in R \\ x, y \in M$$

and denoted by M_R .

Examples :

1. Let V be a vector space over a field F then V is a left as well as right F -Module.
2. Let G be any abelian group under addition , then G is a \mathbb{Z} -Module where \mathbb{Z} is set of integers.
3. Let R be ring and $M = R[x]$ where $R[x]$ is a group of all polynomials with coefficients in R then M is a left as well as a right R -Module with scalar multiplication being usual multiplication.
4. Let M be collection of all $m \times n$ matrices over ring R , then M is left R -Module where scalar multiplication being usual multiplication of a scalar to a matrix.

Suppose ring R is a field then R -Module $R[x]$ is a vector space over field R .

In particular, if M is a set of $1 \times n$ matrices over R or $M = R^n$ (set of n -tuples) then R^n is a left R -module.

Remark: 1. Let R be a commutative ring then every left R -module can be transformed to right R -module and vice-versa.

Proof. Let M be left R -module and R be a commutative ring.
so, \exists a mapping

$$R \times M \rightarrow M$$

defined by

$$(a, x) \rightarrow ax$$

for each $a \in R$ and $x \in M$ satisfying following properties :

$$\forall a, b \in R \text{ and } x, y \in M$$

$$(a + b)x = ax + bx$$

$$a(x + y) = ax + ay$$

$$(ab)x = a(bx)$$

$$1x = x$$

$\therefore R$ is a commutative ring.

Now, Define an another mapping

$$M \times R \rightarrow M$$

defined by

$$(x, a) \rightarrow x * a = ax$$

To check M is a right R -Module , we need to verify properties number (1.5)-(1.8)

(i) *Distributive Law*

$$\begin{aligned} x * (a + b) &= (a + b)x \\ &= ax + bx \\ &= (x * a) + (x * b) \end{aligned}$$

(ii) *Distributive Law*

$$\begin{aligned} (x + y) * a &= a(x + y) \\ &= ax + ay \\ &= (x * a) + (y * a) \end{aligned}$$

(iii)

$$\begin{aligned} x * (ab) &= (ab)x \\ &= (ba)x \\ &= b(ax) \\ &= (ax) * b \end{aligned}$$

(iv)

$$\begin{aligned} x * 1 &= 1x \\ &= x \end{aligned}$$

Thus, ${}_R M$ is transformed to M_R .

Similarly, Converse statement can be verified.

■

Remark: 2. Let S be a subring of ring R then ${}_S M$ exists only if ${}_R M$ exists.

by existence means M is a valid left module over mentioned ring or subring. i.e. satisfying those four properties.

Remark: 3. Same Abelian group can have the structure of a Module for a number of different rings.

For Instance, The field \mathbb{R} is \mathbb{R} -module, \mathbb{Q} -module and \mathbb{Z} -module.

Remark: 4. Let I be left ideal of R then quotient ring R / I is a left R -module.

Here scalar multiplication is

verification: Left to reader

$$R \times R / I \rightarrow R / I$$

defined as

$$(a, x + I) \rightarrow ax + I$$

Hint: you need to verify those four properties: (1.1)-(1.4)

■

$$\forall a \in R \text{ and } \forall x + I \in R / I$$

Theorem 1.1. (Elementary Properties:)

Let M be a left R -module. Suppose 0_m and 0_r denotes additive identities of M and R respectively. Then, for each $x \in M$ and $r \in R$

(i)

$$0_m = 0_r x = r 0_m$$

(ii)

$$r(-x) = (-r)x = -rx$$

Proof. (i) As 0_m is the additive identity of M . so, $0_m = 0_m + 0_m$

$$\text{Consider } r(0_m + 0_m) = r 0_m = r 0_m + 0_m$$

$$\text{but, } r(0_m + 0_m) = (r 0_m) + (r 0_m)$$

so, we have

$$r 0_m + r 0_m = r 0_m + 0_m$$

as $(M, +)$ is an abelian group so left and right cancellation law holds.

$$\begin{aligned} \cancel{r 0_m} + r 0_m &= \cancel{r 0_m} + 0_m \\ r 0_m &= 0_m \end{aligned}$$

a similar argument can be used to prove $0_m = 0_r x$.

(ii) as M is a left R -module so $(r, x) \rightarrow rx \in M$

Now, Consider $(-r)x + rx$

using distributive law

$$\begin{aligned} (-r)x + rx &= (-r + r)x \\ &= 0_r x \\ &= 0_m \end{aligned}$$

i.e. $(-r)x$ is additive inverse of (rx) but additive inverse of (rx) is $-rx$ and it is unique for an abelian group (M here)

$$\therefore (-r)x = -rx$$

a similar argument can be used to prove that $r(-x) = -rx$. ■

Definition 1.3 (Ring Homomorphism). Let R and S be two rings with identities $1_r, 1_s$ respectively then a map (say f)

$$f : R \rightarrow S$$

is said to be a ring homomorphism or ring linear map if for every $a, b \in R$ following properties holds

$\therefore (r, 0_m) \rightarrow r 0_m \in M$
so, $r 0_m = r 0_m + 0_m$
 $\therefore M$ is a left R -module.
(using distributive property)

often called as ring homo

if $R = S$ then we call ring homo as ring endomorphism. For instance, let f be ring homo from R to R . we say f is endomorphism of R and denoted by $\text{End } R$

(i) *Preserves Addition*

$$(a + b)f = (a)f + (b)f$$

(ii) *Preservers Multiplication*

$$(ab)f = (a)f.(b)f$$

(iii) *Maps identity to identity*

$$(1_r)f = 1_s$$

Remark: 5. Such a mapping need not be bijective. if it is bijective then we say it is a ring isomorphism or rings are isomorphic.

Theorem 1.2. Let R be a ring and M be any abelian group with addition. then M is a right R -module if and only if there exists a map which is ring homomorphism from R to $\text{End } M$

M is a right R -module

$$\begin{array}{c} \Downarrow \\ \exists f : R \xrightarrow[\text{Homo}]{\text{Ring}} \text{End } M \end{array}$$

Proof. (Forward Part) Let us suppose that M is a right R -module.

Claim: there exists a map which is ring homomorphism from R to $\text{End } M$

$\therefore M$ is a left R -module , so there exist a map

$$f : M \times R \rightarrow M$$

defined by

$$(x, a) \rightarrow ax$$

satisfying following properties:

$$(x + y)a = (x)a + (y)a$$

$$x(a + b) = xa + xb$$

$$x(ab) = (xa)b$$

$$x1 = x$$

$$\forall x, y \in M \text{ \& } a, b \in R$$

for each $a \in R$, define a map(say ϕ_a)

$$\phi_a : M \rightarrow M$$

such that for each $x \in M$

$$(x)\phi_a = xa \in M$$

Now, we'll show that $\phi_a \in \text{End } M$

Let $x, y \in M$

Consider $(x + y)\phi_a$

$$\begin{aligned} &= (x + y)a && \text{using definition of } \phi_a \\ &= xa + ya && \text{using (1.9)} \\ &= (x)\phi_a + (y)\phi_a \end{aligned}$$

so, ϕ_a preserves addition and is a group homo from M to M .

i.e. $\phi_a \in \text{End } M$

Now, we can define a map (say f)

$$f : R \rightarrow \text{End } M$$

defined as

$$(a)f \rightarrow \phi_a \quad \forall a \in R \text{ and } \phi_a \in \text{End } M$$

Now, We'll show that f is a ring homomorphism.

(A)

$$\begin{aligned} (a + b)f &= \phi_{a+b} \\ &= \phi_a + \phi_b \\ &= (a)f + (b)f \end{aligned} \quad \begin{aligned} &\text{for each } x \in M \text{ we have,} \\ (x)\phi_{a+b} &= x(a + b) = xa + xb \\ &= (x)\phi_a + (x)\phi_b \\ \therefore \phi_{a+b} &= \phi_a + \phi_b \end{aligned}$$

(B)

$$\begin{aligned} (ab)f &= \phi_{ab} \\ &= \phi_a \circ \phi_b \\ &= (a)f (b)f \end{aligned} \quad \begin{aligned} &\text{for each } x \in M \text{ we have,} \\ (x)\phi_{ab} &= x(ab) = (xa)b \\ &= (xa)\phi_b = (x)\phi_a \circ \phi_b \\ \therefore \phi_{ab} &= \phi_a \circ \phi_b \end{aligned}$$

(C)

$$(1)f = \phi_1 \quad \text{for each } x \in M \text{ we have,}$$

Thus, Forward Part is proved.

$$\begin{aligned} (x)\phi_1 &= x(1) \\ &= x \end{aligned}$$

$\therefore \phi_1$ is identity of $\text{End } M$

(Converse Part) Assume that \exists a ring homo. (say f)

$$f : R \xrightarrow[\text{Homo}]{\text{Ring}} \text{End } M$$

for any $a \in R$, we denote the $(a)f$ by $f_a \in \text{End } M$

Claim: M is a right R -module.

so let's define a map

$$R \times M \longrightarrow M$$

defined by

$$(a, x) \rightarrow x * a = (x)f_a$$

to prove M is a right R -module, we need to verify four properties (1.5)- (1.9) of right R -module.

(i)

$$\begin{aligned} (x + y) * a &= (x + y)f_a \\ &= (x)f_a + (y)f_a \\ &= x * a + y * a \end{aligned} \quad \because f_a \in \text{End } M$$

(ii)

$$\begin{aligned} x * (a + b) &= (x)f_{a+b} \\ &= (x)(f_a + f_b) \\ &= (x)f_a + (x)f_b \\ &= (x * a) + (x * b) \end{aligned} \quad \because f_a, f_b \in \text{End } M$$

(iii)

$$\begin{aligned} x * (ab) &= (x)f_{ab} \\ &= (x)f_a \circ f_b \\ &= (xf_a)f_b \\ &= (x * a) * b \end{aligned}$$

(iv)

$$x * 1 = (x)f_1 = x$$

Thus, M is a right R -module. ■

$\because f_1$ is identity in $\text{End } M$

Definition 1.4 (Anti-Ring Homomorphism). Let R and S be two rings with identities 1_r and 1_s respectively. Define a map f

$$f : R \rightarrow S$$

satisfying following properties, for each $a, b \in R$

(i)

$$(a + b)f = (a)f + (b)f$$

(ii)

$$(ab)f = (b)f (a)f$$

(iii)

$$(1_r)f = 1_s$$

Then, f is called anti-ring homomorphism.

Theorem 1.3. Let R be a ring and M be any abelian group with addition. then M is a left R -module if and only if there exists a map which is anti-ring homomorphism from R to $\text{End } M$.

Proof. Left to reader. ■

$$\begin{array}{c} M \text{ is a left } R\text{-module} \\ \updownarrow \\ \exists f : R \xrightarrow[\text{Homo}]{\text{Anti-Ring}} \text{End } M \end{array}$$

Definition 1.5 (SubModule). Let M be a left (right) R -module then a subset N of M is called a submodule of M if N is a left (right) R -module under the operation induced from M .

In other words, A subset N of M is called submodule of M if

- (i) N is subgroup of M .
- (ii) N is closed under induced scalar multiplication from M .

Theorem 1.4 (Criterion for Checking Modules). Let M be a left (right) R -module and N be a subset of M then N is a submodule of M if and only if

(i)

$$x - y \in N$$

$$\forall x, y \in N$$

(ii)

$$ax \in N$$

$$\forall a \in R \text{ \& } x \in N$$

Proof. Left to reader. ■

Examples:

1. As every Vector Space V over a Field F is a F -module. So, submodules of V are subspaces of V .
2. As every abelian group G is a \mathbb{Z} -module. So, all subgroups of G are submodules.
3. Let R be a ring then R is a left as well as right R -module then left (right) ideals of R are left (right) submodules of R .
4. $\{0\}$ and M are trivial submodules of any left (right) R -module M .

Remark: 6.

1. Union of two submodules need not to be a submodule.

Think an example !

2. Intersection of any number of submodules is again a submodule.

Hint: Verify using criterion for checking modules.

Remark: 7. (Smallest Submodule containing a set)

Let M be any left (right) R -module and S be any subset of M . Suppose \mathcal{F} be the family of all submodules of M containing S .

$$\text{Let } P = \bigcap_{N \in \mathcal{F}} N$$

then P is a submodule of M containing S as being intersection of an indexed family of submodules containing S .

Moreover, P is the smallest submodule of M containing S . i.e. for any arbitrary submodule $K \in \mathcal{F}$, we have $P \subseteq K$. Such submodule P of M is said to be generated by set S and is denoted by

$$P = \langle S \rangle = (S)$$

Remark: 8.

Let S be any subset of left R -module M and $\langle S \rangle$ is the smallest submodule of M containing S .

1. if S is non-empty and finite, $S = \{x_1, x_2, x_3, \dots, x_n\}$

$$\langle S \rangle = \langle \{x_1, x_2, x_3, \dots, x_n\} \rangle = \langle x_1, x_2, x_3, \dots, x_n \rangle$$

is said to be a finitely generated by S and is smallest submodule of M containing S .

2. if $S = \phi$ i.e. S is an empty set

$$\langle S \rangle = \langle \phi \rangle = \{0\}$$

3. if $S = \{a\}$ i.e. S is singleton then $\langle S \rangle = \langle a \rangle$ is said to be a cyclic submodule.

Definition 1.6 (Cyclic module). ³ A module M is said to be a cyclic module if it can be generated by a single element.

³ P. M. Cohn. *Basic Algebra*. Springer, 2 edition, 2005. ISBN 978-1-4471-1060-6

For Example: A ring R over itself is a module and can be generated by identity element $\{1\}$ so is a cyclic module.

Theorem 1.5. Let M be left R module and S being any subset of M .

$$\langle S \rangle = \begin{cases} \{0\} & \text{if } S = \phi \\ \left\{ \sum_{i \in I_n} a_i x_i \mid a_i \in R, x_i \in S \right\} & \text{otherwise} \end{cases}$$

Proof. Case-I Let us suppose that $S = \phi$, as $\langle S \rangle$ is the intersection of all the submodules of M containing S .

i.e. Every submodule of M will contain S

In particular, $\{0\}$ also contains S i.e.

$$\{0\} \in \mathcal{F}$$

so,

$$\begin{aligned}\langle S \rangle &= \bigcap_{N \in \mathcal{F}} N \\ &= \{0\}\end{aligned}$$

$\therefore \mathcal{F}$ is a collection of all submodules of M containing S

Case-II Suppose S is non-empty and let

$$P = \left\{ \sum_{i \in J_n} a_i x_i \mid a_i \in R, x_i \in S \right\}$$

First, we'll show that $S \subseteq P$

Let $x \in S$ then it can be expressed in following form:

$$x = 1.x = \sum_{i \in J_1} a_i x_i$$

with $a_1 = 1$ and $x_1 = x$

$$\therefore x \in P \Rightarrow S \subseteq P$$

$\therefore x$ was chosen arbitrary.

Now, we'll show that P is a submodule of M using submodule criterion.

Let $u, v \in P$. so, we need to show $u + \alpha v \in P$

for any $\alpha \in R$

$$u = \sum_{i \in J_n} a_i x_i$$

$\forall x_i \in S \text{ \& } a_i \in R$

$$v = \sum_{j \in J_m} b_j y_j$$

$\forall y_j \in S \text{ \& } b_j \in R$

define, for any $\alpha \in R$

$$z_k = x_k, \quad c_k = a_k$$

$k \in J_n$

$$z_{k+j} = y_j, \quad c_{k+j} = \alpha b_j$$

$j \in J_m$

Thus, we have

$$\begin{aligned}u + \alpha v &= \sum_{i \in J_n} a_i x_i + \alpha \sum_{j \in J_m} b_j y_j \\ &= \sum_{i \in J_n} a_i x_i + \sum_{j \in J_m} \alpha b_j y_j \\ &= \sum_{k \in J_n} c_k z_k + \sum_{k=n+1}^{n+m} c_k z_k \\ &= \sum_{k \in J_{n+m}} c_k z_k\end{aligned}$$

so, P is a submodule of M containing S .

Now, we'll show that P is smallest submodule of M containing S .

Let K be any arbitrary submodule of M containing S

$$\text{i.e. } K \in \mathcal{F}$$

$\therefore K$ is a submodule and $S \subseteq K$

$\therefore K$ is closed under scalar multiplication and addition.

$$\text{i.e. } \sum_{i \in J_n} a_i x_i \in K \quad \forall a_i \in R \text{ \& } x_i \in S$$

so ,

$$P = \langle S \rangle \subseteq K$$

Hence, P is smallest submodule of M containing S . ■

Definition 1.7 (Generating Set / Set of Generators). *A set of generators for a left (right) R -module M is a subset S of M such that*

$$M = \langle S \rangle$$

if no proper submodule of M contains S then S generates M (verify ?)

Examples

1. A ring R considered as left (right) R -module is generated by identity element $\{1\}$
2. $\mathbb{Z} \times \mathbb{Z}$ over \mathbb{Z} can be generated by

$$S = \{(0, 1), (1, 0)\}$$

3. All finite dimensional vector space can be generated by it's basis (finite), so is finitely generated submodule.
4. Let R be a ring, I be left(right) ideal of R then it is a left(right) R -module. So, every finitely generated left(right) ideals of R are finitely generated submodule.
5. A submodule of left R -module is cyclic iff it is principal ideal of ${}_R R$.

Definition 1.8. Let M be a left R -module and $\{N_\alpha\}_{\alpha \in \Omega}$ be family of submodules of M then sum $\sum_{\alpha \in \Omega} N_\alpha$ is defined to be a submodule of M generated by $\bigcup_{\alpha \in \Omega} N_\alpha$

$$\left\langle \bigcup_{\alpha \in \Omega} N_\alpha \right\rangle = \sum_{\alpha \in \Omega} N_\alpha$$

Moreover, $\sum_{\alpha \in \Omega} N_\alpha$ is smallest submodule of M containing N_α

where Ω is indexing set.

for each $\alpha \in \Omega$

Proposition 1.6. Let M be a left R -module and $\{N_\alpha\}_{\alpha \in \Omega}$ be family of submodules of M then sum

$$\sum_{\alpha \in \Omega} N_\alpha = \left\{ \sum_{\alpha \in \Omega} x_\alpha \mid x_\alpha \in N_\alpha, x_\alpha = 0 \text{ for almost all } \alpha \right\}$$

where Ω is indexing set.

for each $\alpha \in \Omega$

Proof. Let

$$P = \left\{ \sum_{\alpha \in \Omega} x_\alpha \mid x_\alpha \in N_\alpha, x_\alpha = 0 \text{ for almost all } \alpha \right\}$$

We need to show that P is smallest submodule of M containing each N_α

Claim 1 : P is submodule of M

Clearly, P is non-empty. Taking $x_\alpha = 0$ for each $\alpha \in \Omega$, we have

$$\Rightarrow 0 \in P$$

Also, for a fixed but arbitrary $\alpha \in \Omega$

Let $x \in N_\alpha$ and choose

$$x_i = \begin{cases} x, & \text{if } i = \alpha \\ 0, & \text{otherwise.} \end{cases}$$

so $x = \sum x_i \in P$ we have $N_\alpha \subseteq P \quad \forall \alpha \in \Omega$

$\therefore \alpha$ was arbitrary chosen

Now, we'll show that P is submodule of M using submodule criterion.

P is closed under addition and scalar multiplication Let u, v be two elements of P

$$u = \sum_{\alpha} x_\alpha$$

where $x_\alpha \in N_\alpha$ and $x_\alpha = 0$ for almost all α

$$v = \sum_{\beta} y_\beta$$

where $y_\beta \in N_\beta$ and $y_\beta = 0$ for almost all β

Let Ω_1, Ω_2 be finite subsets of Ω for which x_α and y_β are non-zero respectively.

$$x_\alpha = \begin{cases} \text{non-zero,} & \text{if } \alpha \in \Omega_1 \\ 0, & \text{otherwise.} \end{cases}, \quad y_\beta = \begin{cases} \text{non-zero,} & \text{if } \beta \in \Omega_2 \\ 0, & \text{otherwise.} \end{cases}$$

Also for any arbitrary scalar $c \in R$, define

$$z_r = \begin{cases} x_r, & \text{if } r \in \Omega_1 \\ cy_r, & \text{if } r \in \Omega_2. \end{cases} \Rightarrow z_r = x_r + cy_r \quad \forall r \in \Omega_1 \cup \Omega_2$$

Now ,

$$\begin{aligned} u + cv &= \sum_{\alpha} x_{\alpha} + c \sum_{\beta} y_{\beta} \\ &= \sum_{\alpha \in \Omega_1} x_{\alpha} + c \sum_{\beta \in \Omega_2} y_{\beta} \\ &= \sum_{\alpha \in \Omega_1} x_{\alpha} + \sum_{\beta \in \Omega_2} cy_{\beta} \\ &= \sum_{r \in \Omega_1} x_r + \sum_{r \in \Omega_2} cy_r + \sum_{r \in \Omega_1 \cap \Omega_2} (x_r + cy_r) \end{aligned}$$

Thus , We have

$$u + cv = \sum_{r \in \Omega_1 \cup \Omega_2} z_r \in P$$

$z_r = 0$ for almost all r
 $\therefore \Omega_1 \cup \Omega_2$ is also finite.

Claim 2 : P is smallest submodule of M containing each N_{α} .

Let N be any submodule of M containing each N_{α}

$\therefore N$ is closed under addition so N contains all finite sum of the form $\sum_{\alpha} x_{\alpha}$ where $x_{\alpha} \in N_{\alpha}$ & $x_{\alpha} = 0$, for almost all α

It follows that $P \subseteq N$

$\therefore N$ was chosen arbitrary

Thus, P is smallest submodule of M containing each N_{α} ■

Definition 1.9 (Maximal Submodule). Let N be a submodule of left R -module M then N is said to be a maximal submodule of M if there does not exist any proper submodule of M .

In other words , for any submodule K of M satisfying $N \subseteq K \subseteq M$ we must have

$$\text{either } N = K \text{ or } K = M$$

for N to be a maximal submodule of M .

Theorem 1.7. *Let M be a finitely generated left R -module then every proper submodule of M is contained in maximal submodule of M .*

In particular , if M is non-trivial then M contains a maximal submodule.

Proof. Left to Reader. ■

Remark: 9. \mathbb{Q} is not finitely generated \mathbb{Z} -module.

Verification: Let \mathbb{Q} is finitely generated over \mathbb{Z} by

$$\left\{ \frac{p_i}{q_i} \mid p_i, q_i \in \mathbb{Z}, q_i \neq 0 \quad \forall i \in J_n \right\}$$

Without loss of generality , Assume that

$$q_1, q_2, q_3, \dots, q_n > 0$$

Stay Tuned for next chapters!

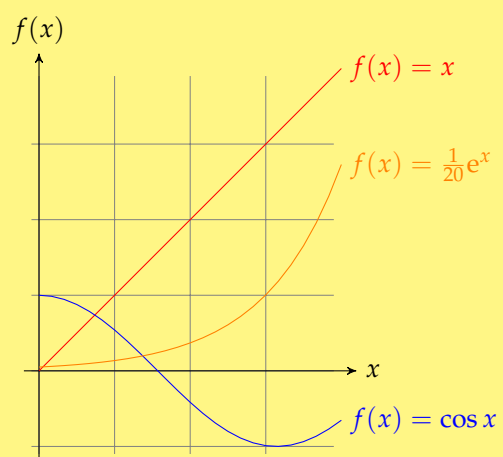
$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C \longrightarrow 0 \\
 & & \downarrow \eta_1 & & \downarrow \eta_2 & & \downarrow \eta_3 \\
 0 & \longrightarrow & A' & \xrightarrow{\varphi'} & B' & \xrightarrow{\psi'} & C' \longrightarrow 0
 \end{array}$$

$$\begin{array}{ccc}
 A & \xrightarrow{\phi} & B \\
 & \searrow & \downarrow \\
 & & C
 \end{array}$$

$$\begin{array}{ccc}
 & B & \\
 A & \nearrow & \downarrow \\
 & C &
 \end{array}$$

$$\begin{array}{ccc}
 & A & \\
 \swarrow & & \searrow \\
 B & \longrightarrow & C
 \end{array}$$

$$\begin{array}{ccc}
 A & \xrightarrow{\text{red}} & B \\
 \downarrow \text{red} & \searrow \text{blue} & \\
 C & & D
 \end{array}$$



Bibliography

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