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MODULE THEORY

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Contents

There are few persons without whom this was impossible.

They deserve credit for it so i would love to thank them.

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Dedicated to my family and my best friend

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Introduction

Warning :

This is my first document created using latex so it may be possible that there are typo errors and other errors. if you notice any such error then you can report it here.

https://github.com/sirkapil/module-theory/issues/new[1]

[1] (may require a github account)

About :

This sample book discusses the course "Module Theory" being taught to Post-Graduate (M.Sc. Mathematics) students in Department of Mathematics under University of Delhi, Delhi.

All my LaTeX documents are free, open-source and can be found pinned here :

https://github.com/sirkapil

Contribution :

any contribution will be welcomed.

Introduction to Modules

Defination of Module

Left Module:

Let R be a ring with identity and M be an abelian group with addition. We say M is a left $R-$module if there exists a mapping[2]

[2] often called as scaler multiplication.

$$R \times M \rightarrow M$$

defined by

$$(a\ ,\ x) \rightarrow ax$$

$\forall\, a \in\ R$ and $x \in\ M$

satisfying following properties :

$$(a+b)x = ax + bx \tag{1.1}$$
$$a(x+y) = ax + ay \tag{1.2}$$
$$(ab)x = a(bx) \tag{1.3}$$
$$1x = x \tag{1.4}$$

$\forall\, a\ ,\ b \in R$ and $x\ ,y\ \in M$

and denoted by ${}_RM$

Right Module:

Let R be a ring with identity and M be an abelian group with addition. We say M is a right $R-$module if there exists a mapping

$$M \times R \rightarrow M$$

defined by

$$(x\ ,\ a) \rightarrow xa$$

$\forall\, a \in R$ and $x \in M$

satisfying following properties :

$$x(a+b) = xa + xb \tag{1.5}$$
$$(x+y)a = xa + ya \tag{1.6}$$
$$x(ab) = (xa)b \tag{1.7}$$
$$x1 = x \tag{1.8}$$

$\forall\, a\ ,\ b \in R$ and $x\ ,y\ \in M$

and denoted by M_R.

Examples :

1. Let V be a vector space over a field F then V is a left as well as right $F-$Module.

2. Let G be any abelian group under addition , then G is a $\mathbb{Z}-$Module where $\mathbb{Z}$ is set of integers.

3. Let R be ring and $M = R[x]$ where $R[x]$ is a group of all polynomials with coefficents in R then M is a left as well as a right $R-$Module with scaler multiplication being usual multiplication.

Suppose ring R is a field then $R-$Module $R[x]$ is a vector space over field R.

4. Let M be collection of all $m \times n$ matrices over ring R , then M is left $R-$Module where scaler multiplication being usual multiplication of a scaler to a matrix.

   In particular, if M is a set of $1 \times n$ matrices over R or $M = R^n$(set of $n-$tuples) then R^n is a left $R-$module.

Remark: 1.1. *Let R be a commutative ring then every left $R-$module can be transformed to right $R-$module and vice-versa.*

Proof. Let M be left $R-$module and R be a commutative ring.
so, $\exists$ a mapping

$$R \times M \to M$$

defined by

$$(a\ ,\ x) \to ax$$

for each $a \in R$ and $x \in M$ satisfying following properties :

$\forall\ a\ ,\ b \in R$ and $x\ ,y\ \in M$

$$\begin{aligned} (a+b)x &= ax+bx \\ a(x+y) &= ax+ay \\ (ab)x &= a(bx) \\ 1x &= x \end{aligned}$$

$\because$ R is a commutative ring.
Now, Define an another mapping

$$M \times R \to M$$

defined by

$$(x\ ,\ a) \to x * a\ =\ ax$$

To check M is a right $R-$Module , we need to verify properties number **??**-(1.8) *(i) Distribuitive Law*

$$\begin{aligned} x * (a + b) &= (a + b)x \\ &= ax + bx \\ &= (x * a) + (x * b) \end{aligned}$$

(ii) Distributive Law

$$\begin{aligned} (x + y) * a &= a(x + y) \\ &= ax + ay \\ &= (x * a) + (y * a) \end{aligned}$$

(iii)

$$\begin{aligned} x * (ab) &= (ab)x \\ &= (ba)x \\ &= b(ax) \\ &= (ax) * b \end{aligned}$$

(iv)

$$\begin{aligned} x * 1 &= 1x \\ &= x \end{aligned}$$

Thus, ${}_R M$ is transformed to M_R.

Similarly, Converse statement can be verified.

■

Remark: 1.2. *Let S be a subring of ring R then ${}_S M$ exists only if ${}_R M$ exists.*

by existance means M is a valid left module over mentioned ring or subring. i.e. satisfying those four properties.

Remark: 1.3. *Same Abelian group can have the structure of a Module for a number of different rings.*

For Instance, The field $\mathbb{R}$ is $\mathbb{R}-$module,$\mathbb{Q}-$module and $\mathbb{Z}-$module.

Remark: 1.4. *Let I be left ideal of R then quotient ring R/I is a left R-module.*

verification: 'left to reader'

Hint: you need to verify those four properties: (1.1)-(1.4) ■

Here scaler multiplication is

$$R \times R/I \rightarrowtail R/I$$

defined as

$$(a\ ,\ x + I) \rightarrowtail ax + I$$

$\forall\, a \in R$ and $\forall\, x + I \in R/I$

Theorem 1.5. *(Elementry Properties:)*

Let M be a left R-module . Suppose 0_m and 0_r denotes additive identities of M and R respectively. Then, for each $x \in M$ and $r \in R$

(i)

$$0_m = 0_r\, x = r\, 0_m$$

(ii)

$$r(-x) = (-r)x = -rx$$

Proof. *(i)* As 0_m is the additive identity of M. so, $0_m = 0_m + 0_m$

Consider $r(0_m + 0_m) = r\, 0_m = r\, 0_m + 0_m$

but, $r(0_m + 0_m) = (r\, 0_m) + (r\, 0_m)$

so, we have

$$r\, 0_m + r\, 0_m = r\, 0_m + 0_m$$

$\because$ $(r\ ,\ 0_m) \rightarrowtail r\, 0_m \in M$
so, $r\, 0_m = r\, 0_m + 0_m$
$\because$ M is a left R-module.
(using distribuitive property)

as $(M, +)$ is an abelian group so left and right cancellation law holds.

$$\begin{aligned} \cancel{r\, 0_m} + r\, 0_m &= \cancel{r\, 0_m} + 0_m \\ r\, 0_m &= 0_m \end{aligned}$$

a similiar argument can be used to prove $0_m = 0_r\, x$.

(ii) as M is a left R-module so $(r\ , x) \ \rightarrowtail rx \in M$

Now, Consider $(-r)x + rx$

using distribuitive law

$$\begin{aligned} (-r)x + rx &= (-r + r)x \\ &= 0_r\, x \\ &= 0_m \end{aligned}$$

i.e. $(-r)x$ is additive inverse of (rx) but additive inverse of (rx) is $-rx$ and it is unique for an abelian group(M here)

$$\therefore\ (-r)x = -rx$$

a similar argument can be used to prove that $r(-x) = -rx$.

■

Definition 1.6 (Ring Homomorphism)**.** *Let R and S be two rings with identities 1_r , 1_s respectively then a map(say f)*

often called as ring homo

$$f : R \to S$$

is said to be a ring homomorphism or ring linear map if for every a , $b \in R$ following properties holds

if R = S then we call ring homo as ring endomorphism. For instance , let f be ring homo from R to R . we say f is endomorphism of R and denoted by *End R* *(i) Preserves Addition*

$$(a+b)f = (a)f + (b)f$$

(ii) Preservers Multiplication

$$(ab)f = (a)f.(b)f$$

(iii) Maps identity to identity

$$(1_r)f = 1_s$$

Remark: 1.7. *Such a mapping need not to be bijective. if it is bijective then we say it is a ring isomorphism or rings are isomorphic.*

Theorem 1.8. *Let R be a ring and M be any abelian group with addition. then M is a right R-module if and only if there exists a map which is ring homomorphism from R to End M*

M is a right R-module

$\Updownarrow$

$\exists\, f : R \xrightarrow[\text{Homo}]{\text{Ring}} End\ M$

Proof. *(Forward Part)* Let us suppose that M is a right R-module.

Claim: there exists a map which is ring homomorphism from R to $End\ M$

$\because$ M is a left R-module , so there exist a map

$$f : M \times R \rightarrowtail M$$

defined by

$$(x\ , a) \rightarrowtail ax$$

satisfying following properties:

$$\begin{aligned} (x+y)a &= (x)a + (y)a \\ x(a+b) &= xa + xb \\ x(ab) &= (xa)b \\ x1 &= x \end{aligned}$$

$\forall\ x, y\ \in M\ \&\ a, b \in R$

for each $a \in R$, define a map(say ϕ_a)

$$\phi_a : M \rightarrowtail\ M$$

such that for each $x \in M$

$$(x)\phi_a = xa \in\ M$$

Now, we'll show that $\phi_a \in\ End\ M$

Let $x, y \in M$

Consider $(x+y)\phi_a$

$$\begin{aligned} &= (x+y)a && \text{using defination of } \phi_a \\ &= xa + ya && \text{using (1.9)} \\ &= (x)\phi_a + (y)\phi_a \end{aligned}$$

so, ϕ_a preserves addition and is a group homo from M to M.

i.e. $\phi_a \in End\ M$

Now, we can define a map (say f)

$$f : R \rightarrowtail End\ M$$

defined as

$$a \rightarrowtail \phi_a \qquad \forall a \in R \text{ and } \phi_a \in End\ M$$

$\blacksquare$