KAPIL CHAUDHARY

MODULE THEORY

UNIVERSITY OF DELHI

Kapil Chaudhary

UNIVERSITY OF DELHI

https://contact.sirkapil.me/

First Print, January 2018

Contents

Introduction to Modules 9

There are few persons without whom this was impossible.

They deserve credit for it so i would love to thank them.

Special Thanks to :

- Dr. Anuj Bishnoi (Subject Teacher)

- Edward Tufte (LaTeX Tufte Templete)

Dedicated to my family and my best friend

Neeraj K. Gaur

Introduction

Preface

This sample book discusses the course `Module Theory` of `pure mathematics` being taught to `post-graduate` students in `University of Delhi`.

Reporting Problems

Introduction to Modules

`Defination of Module`

Left Module:

Let R be a ring with identity and M be an abelian group with addition. We say M is a left $R-$module if there exists a mapping[1]

[1] often called as scaler multiplication.

$$R \times M \to M$$

defined by

$$(a\ ,\ x) \to ax$$

$\forall\ a \in\ R$ and $x \in\ M$

satisfying following properties :

$$(a+b)x = ax + bx \tag{1.1}$$

$$a(x+y) = ax + ay \tag{1.2}$$

$$(ab)x = a(bx) \tag{1.3}$$

$$1x = x \tag{1.4}$$

$\forall\ a\ ,\ b \in R$ and $x\ , y\ \in M$

and denoted by ${}_RM$

Right Module:

Let R be a ring with identity and M be an abelian group with addition. We say M is a right $R-$module if there exists a mapping

$$M \times R \to M$$

defined by

$$(x\ ,\ a) \to xa$$

$\forall\ a \in R$ and $x \in M$

satisfying following properties :

$$x(a+b) = xa + xb \tag{1.5}$$

$$(x+y)a = xa + ya \tag{1.6}$$

$$x(ab) = (xa)b \tag{1.7}$$

$$x1 = x \tag{1.8}$$

$\forall\ a\ ,\ b \in R$ and $x\ , y\ \in M$

and denoted by M_R.

Examples :

1. Let V be a vector space over a field F then V is a left as well as right $F-$Module.

2. Let G be any abelian group under addition , then G is a $\mathbb{Z}-$Module where $\mathbb{Z}$ is set of integers.

3. Let R be ring and $M = R[x]$ where $R[x]$ is a group of all polynomials with coefficents in R then M is a left as well as a right $R-$Module with scaler multiplication being usual multiplication.

Suppose ring R is a field then $R-$Module $R[x]$ is a vector space over field R.

4. Let M be collection of all $m \times n$ matrices over ring R , then M is left $R-$Module where scaler multiplication being usual multiplication of a scaler to a matrix.

   In particular, if M is a set of $1 \times n$ matrices over R or $M = R^n$(set of $n-$tuples) then R^n is a left $R-$module.

Remark: 1.1. *Let R be a commutative ring then every left $R-$module can be transformed to right $R-$module and vice-versa.*

Proof. Let M be left $R-$module and R be a commutative ring.
so, $\exists$ a mapping

$$R \times M \to M$$

defined by

$$(a\ ,\ x) \to ax$$

for each $a \in R$ and $x \in M$ satisfying following properties :

$\forall\ a\ ,\ b \in R$ and $x\ ,y\ \in M$

$$\begin{aligned}(a+b)x &= ax+bx\\ a(x+y) &= ax+ay\\ (ab)x &= a(bx)\\ 1x &= x\end{aligned}$$

$\because$ R is a commutative ring.
Now, Define an another mapping

$$M \times R \to M$$

defined by

$$(x\ ,\ a) \to x * a\ =\ ax$$

To check M is a right $R-$Module , we need to verify properties number **??**-(1.8) *(i) Distribuitive Law*

$$\begin{aligned} x * (a+b) &= (a+b)x \\ &= ax + bx \\ &= (x*a) + (x*b) \end{aligned}$$

(ii) Distributive Law

$$\begin{aligned} (x+y) * a &= a(x+y) \\ &= ax + ay \\ &= (x*a) + (y*a) \end{aligned}$$

(iii)

$$\begin{aligned} x * (ab) &= (ab)x \\ &= (ba)x \\ &= b(ax) \\ &= (ax) * b \end{aligned}$$

(iv)

$$\begin{aligned} x * 1 &= 1x \\ &= x \end{aligned}$$

Thus, ${}_RM$ is transformed to M_R.

Similarly, Converse statement can be verified.

■

Remark: 1.2. *Let S be a subring of ring R then ${}_SM$ exists only if ${}_RM$ exists.*

by existance means M is a valid left module over mentioned ring or subring. i.e. satisfying those four properties.

Remark: 1.3. *Same Abelian group can have the structure of a Module for a number of different rings.*

For Instance, The field $\mathbb{R}$ is $\mathbb{R}-$module,$\mathbb{Q}-$module and $\mathbb{Z}-$module.

Remark: 1.4. *Let I be left ideal of R then quotient ring R/I is a left R-module.*

verification: 'left to reader'

Hint: you need to verify those four properties: (1.1)-(1.4) ■

Here scaler multiplication is

$$R \times R/I \rightarrowtail R/I$$

defined as

$$(a\ ,\ x+I) \rightarrowtail ax + I$$

$\forall\, a \in R$ and $\forall\, x + I \in R/I$

Theorem 1.5. *(Elementry Properties:)*

Let M be a left R-module . Suppose 0_m and 0_r denotes additive identities of M and R respectively. Then, for each $x \in M$ and $r \in R$

(i)
$$0_m = 0_r\ x = r\ 0_m$$

(ii)
$$r(-x) = (-r)x = -rx$$

Proof. *(i)* As 0_m is the additive identity of M. so, $0_m = 0_m + 0_m$

Consider $r(0_m + 0_m) = r\ 0_m = r\ 0_m + 0_m$

but, $r(0_m + 0_m) = (r\ 0_m) + (r\ 0_m)$

so, we have

$$r\ 0_m + r\ 0_m = r\ 0_m + 0_m$$

$\because\ (r\ ,\ 0_m) \rightarrowtail r\ 0_m \in M$
so, $r\ 0_m = r\ 0_m + 0_m$
$\because\ M$ is a left R-module.
(using distribuitive property)

as $(M, +)$ is an abelian group so left and right cancellation law holds.

$$\begin{aligned} \cancel{r\ 0_m} + r\ 0_m &= \cancel{r\ 0_m} + 0_m \\ r\ 0_m &= 0_m \end{aligned}$$

a similiar argument can be used to prove $0_m = 0_r\ x$.

(ii) as M is a left R-module so $(r\ , x) \rightarrowtail rx \in M$

Now, Consider $(-r)x + rx$

using distribuitive law

$$\begin{aligned} (-r)x + rx &= (-r + r)x \\ &= 0_r\ x \\ &= 0_m \end{aligned}$$

i.e. $(-r)x$ is additive inverse of (rx) but additive inverse of (rx) is $-rx$ and it is unique for an abelian group(M here)

$$\therefore\ (-r)x = -rx$$

a similar argument can be used to prove that $r(-x) = -rx$.

∎

Definition 1.6 (Ring Homomorphism)**.** *Let R and S be two rings with identities 1_r , 1_s respectively then a map(say f)*

often called as ring homo

$$f : R \to S$$

is said to be a ring homomorphism or ring linear map if for every a , $b \in R$ following properties holds

if R = S then we call ring homo as ring endomorphism. For instance , let f be ring homo from R to R . we say f is endomorphism of R and denoted by *End R* *(i) Preserves Addition*

$$(a+b)f = (a)f + (b)f$$

(ii) Preservers Multiplication

$$(ab)f = (a)f.(b)f$$

(iii) Maps identity to identity

$$(1_r)f = 1_s$$

Remark: 1.7. *Such a mapping need not to be bijective. if it is bijective then we say it is a ring isomorphism or rings are isomorphic.*

Theorem 1.8. *Let R be a ring and M be any abelian group with addition. then M is a right R-module if and only if there exists a map which is ring homomorphism from R to End M*

M is a right R-module

$$\Updownarrow$$

$$\exists\, f : R \xrightarrow[\text{Homo}]{\text{Ring}} End\ M$$

Proof. *(Forward Part)* Let us suppose that M is a right R-module.

Claim: there exists a map which is ring homomorphism from R to $End\ M$

$\because$ M is a left R-module , so there exist a map

$$f : M \times R \rightarrowtail M$$

defined by

$$(x\ , a) \rightarrowtail ax$$

satisfying following properties:

$$\begin{aligned}(x+y)a &= (x)a + (y)a\\ x(a+b) &= xa + xb\\ x(ab) &= (xa)b\\ x1 &= x\end{aligned}$$

$\forall\ x, y\ \in M\ \&\ a, b \in R$

for each $a \in R$, define a map(say ϕ_a)

$$\phi_a : M \rightarrowtail\ M$$

such that for each $x \in M$

$$(x)\phi_a = xa \in\ M$$

Now, we'll show that $\phi_a \in\ End\ M$

Let $x, y \in M$

Consider $(x+y)\phi_a$

$$
\begin{aligned}
&= (x+y)a && \text{using defination of } \phi_a \\
&= xa + ya && \text{using (1.9)} \\
&= (x)\phi_a + (y)\phi_a
\end{aligned}
$$

so, ϕ_a preserves addition and is a group homo from M to M.

i.e. $\phi_a \in End\ M$

Now, we can define a map (say f)

$$f : R \rightarrowtail End\ M$$

defined as

$$a \rightarrowtail \phi_a \qquad \forall a \in R \text{ and } \phi_a \in End\ M$$

■