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MODULE THEORY

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Contents

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Dedicated to my family and my best friend

Neeraj K. Gaur

Introduction

This sample book discusses the course `Module Theory` of `pure mathematics` being taught to `post-graduate` students in `University of Delhi`.

Introduction to Modules

Defination of Module

Left Module:

Let R be a ring with identity and M be an abelian group with addition. We say M is a left $R-$module if there exists a mapping[1]

[1] often called as scaler multiplication.

$$R \times M \to M$$

defined by

$$(a\ ,\ x) \to ax$$

for each $a \in R$ and $x \in M$ satisfying following properties :

$\forall\ a\ ,\ b \in R$ and $x\ ,y\ \in M$

$$(a+b)x = ax + bx \quad (1)$$
$$a(x+y) = ax + ay \quad (2)$$
$$(ab)x = a(bx) \quad (3)$$
$$1x = x \quad (4)$$

and denoted by ${}_RM$

Right Module:

Let R be a ring with identity and M be an abelian group with addition. We say M is a right $R-$module if there exists a mapping

$$M \times R \to M$$

defined by

$$(x\ ,\ a) \to xa$$

for each $a \in R$ and $x \in M$ satisfying following properties :

$\forall\ a\ ,\ b \in R$ and $x\ ,y\ \in M$

$$x(a+b) = xa + xb \quad (5)$$
$$(x+y)a = xa + ya \quad (6)$$
$$x(ab) = (xa)b \quad (7)$$
$$x1 = x \quad (8)$$

and denoted by M_R

Examples :

1. Let V be a vector space over a field F then V is a left as well as right $F-$Module.

2. Let G be any abelian group under addition , then G is a $\mathbb{Z}-$Module where $\mathbb{Z}$ is set of integers.

3. Let R be ring and $M = R[x]$ where $R[x]$ is a group of all polynomials with coefficents in R then M is a left as well as a right $R-$Module with scaler multiplication being usual multiplication.

Suppose ring R is a field then $R-$Module $R[x]$ is a vector space over field R.

4. Let M be collection of all $m \times n$ matrices over ring R , then M is left $R-$Module where scaler multiplication being usual multiplication of a scaler to a matrix.

   In particular, if M is a set of $1 \times n$ matrices over R or $M = R^n$(set of $n-$tuples) then R^n is a left $R-$module.

Remark. Let R be a commutative ring then every left $R-$module can be transformed to right $R-$module and vice-versa.

Proof. Let M be left $R-$module and R be a commutative ring.
so, $\exists$ a mapping

$$R \times M \to M$$

defined by

$$(a\ ,\ x) \to ax$$

for each $a \in R$ and $x \in M$ satisfying following properties :

$\forall\ a\ ,\ b \in R$ and $x\ , y\ \in M$

$$\begin{aligned} (a+b)x &= ax + bx \\ a(x+y) &= ax + ay \\ (ab)x &= a(bx) \\ 1x &= x \end{aligned}$$

$\because$ R is a commutative ring.
Now, Define an another mapping

$$M \times R \to M$$

defined by

$$(x\ ,\ a) \to x * a\ =\ ax$$

To check M is a right $R-$Module , we need to verify properties number (5)-(8)

1.

$$\begin{aligned} x * (a+b) &= (a+b)x \\ &= ax + bx \\ &= (x * a) + (x * b) \end{aligned}$$

2.

$$\begin{aligned} (x+y) * a &= a(x+y) \\ &= ax + ay \\ &= (x * a) + (y * a) \end{aligned}$$

3.

$$\begin{aligned} x * (ab) &= (ab)x \\ &= (ba)x \\ &= b(ax) \\ &= (ax) * b \end{aligned}$$

4.

$$\begin{aligned} x * 1 &= 1x \\ &= x \end{aligned}$$

Thus, ${}_RM$ is transformed to M_R.

Similarly, Converse statement can be verified.

■

Remark. Let S be a subring of ring R then ${}_SM$ exists only if ${}_RM$ exists.

by existance means M is a valid left module over mentioned ring or subring. i.e. satisfying those four properties.