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# MODULE THEORY

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*Dedicated to my family and my best friend*

*Neeraj K. Gaud*

# Introduction

## Warning :

This is my first document created using latex so it may be possible that there are several errors. if you notice any error then you can report it here.

<https://github.com/sirkapil/module-theory/issues/new><sup>1</sup>

<sup>1</sup> (may require a github account)

## About :

This sample book discusses the course "Module Theory" being taught to Post-Graduate (M.Sc. Mathematics) students in Department of Mathematics under University of Delhi, Delhi.

All my  $\text{\LaTeX}$  documents are free and open-source. Each document is hosted in a github repository and can be found pinned here.

<https://github.com/sirkapil>

## Contribution :

If you find my work useful and want to contribute then you are welcome by heart.

Any suitable changes to document repository through pull requests are highly appreciated. You can create a new pull request here. Be sure to read *contribution file* in root/.github folder of repository before creating any pull-request.

<https://github.com/sirkapil/module-theory/compare>

If you don't have a github account or facing difficulty in creating a pull-request , then feel free to drop down a message here about that you are interested in contribution of this project.

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# Introduction to Modules

## Defination of Module

### Left Module:

Let  $R$  be a ring with identity and  $M$  be an abelian group with addition. We say  $M$  is a left  $R$ -module if there exists a mapping<sup>2</sup>

<sup>2</sup> often called as scalar multiplication.

$$R \times M \rightarrow M$$

defined by

$$(a, x) \rightarrow ax$$

$$\forall a \in R \text{ and } x \in M$$

satisfying following properties :

$$(a + b)x = ax + bx \quad (1.1)$$

$$a(x + y) = ax + ay \quad (1.2)$$

$$(ab)x = a(bx) \quad (1.3)$$

$$1x = x \quad (1.4)$$

$$\forall a, b \in R \text{ and } x, y \in M$$

and denoted by  ${}_R M$

### Right Module:

Let  $R$  be a ring with identity and  $M$  be an abelian group with addition. We say  $M$  is a right  $R$ -module if there exists a mapping

$$M \times R \rightarrow M$$

defined by

$$(x, a) \rightarrow xa$$

$$\forall a \in R \text{ and } x \in M$$

satisfying following properties :

$$x(a + b) = xa + xb \quad (1.5)$$

$$(x + y)a = xa + ya \quad (1.6)$$

$$x(ab) = (xa)b \quad (1.7)$$

$$x1 = x \quad (1.8)$$

$$\forall a, b \in R \text{ and } x, y \in M$$

and denoted by  $M_R$ .

*Examples :*

1. Let  $V$  be a vector space over a field  $F$  then  $V$  is a left as well as right  $F$ -Module.
2. Let  $G$  be any abelian group under addition , then  $G$  is a  $\mathbb{Z}$ -Module where  $\mathbb{Z}$  is set of integers.
3. Let  $R$  be ring and  $M = R[x]$  where  $R[x]$  is a group of all polynomials with coefficients in  $R$  then  $M$  is a left as well as a right  $R$ -Module with scalar multiplication being usual multiplication.
4. Let  $M$  be collection of all  $m \times n$  matrices over ring  $R$  , then  $M$  is left  $R$ -Module where scalar multiplication being usual multiplication of a scalar to a matrix.

Suppose ring  $R$  is a field then  $R$ -Module  $R[x]$  is a vector space over field  $R$ .

In particular, if  $M$  is a set of  $1 \times n$  matrices over  $R$  or  $M = R^n$  (set of  $n$ -tuples) then  $R^n$  is a left  $R$ -module.

**Remark: 2.** Let  $R$  be a commutative ring then every left  $R$ -module can be transformed to right  $R$ -module and vice-versa.

*Proof.* Let  $M$  be left  $R$ -module and  $R$  be a commutative ring.  
so,  $\exists$  a mapping

$$R \times M \rightarrow M$$

defined by

$$(a, x) \rightarrow ax$$

for each  $a \in R$  and  $x \in M$  satisfying following properties :

$$\forall a, b \in R \text{ and } x, y \in M$$

$$(a + b)x = ax + bx$$

$$a(x + y) = ax + ay$$

$$(ab)x = a(bx)$$

$$1x = x$$

$\therefore R$  is a commutative ring.

Now, Define an another mapping

$$M \times R \rightarrow M$$

defined by

$$(x, a) \rightarrow x * a = ax$$

To check  $M$  is a right  $R$ -Module , we need to verify properties number ??-(1.8)

(i) Distributive Law

$$\begin{aligned} x * (a + b) &= (a + b)x \\ &= ax + bx \\ &= (x * a) + (x * b) \end{aligned}$$

(ii) Distributive Law

$$\begin{aligned} (x + y) * a &= a(x + y) \\ &= ax + ay \\ &= (x * a) + (y * a) \end{aligned}$$

(iii)

$$\begin{aligned} x * (ab) &= (ab)x \\ &= (ba)x \\ &= b(ax) \\ &= (ax) * b \end{aligned}$$

(iv)

$$\begin{aligned} x * 1 &= 1x \\ &= x \end{aligned}$$

Thus,  ${}_R M$  is transformed to  $M_R$ .

Similarly, Converse statement can be verified.

■

**Remark: 3.** Let  $S$  be a subring of ring  $R$  then  ${}_S M$  exists only if  ${}_R M$  exists.

by existence means  $M$  is a valid left module over mentioned ring or subring. i.e. satisfying those four properties.

**Remark: 4.** Same Abelian group can have the structure of a Module for a number of different rings.

For Instance, The field  $\mathbb{R}$  is  $\mathbb{R}$ -module,  $\mathbb{Q}$ -module and  $\mathbb{Z}$ -module.

**Remark: 5.** Let  $I$  be left ideal of  $R$  then quotient ring  $R/I$  is a left  $R$ -module.

Here scalar multiplication is

verification: 'left to reader'

$$R \times R/I \rightarrow R/I$$

defined as

$$(a, x + I) \mapsto ax + I$$

$$\forall a \in R \text{ and } \forall x + I \in R/I$$

**Hint:** you need to verify those four properties: (1.1)-(1.4)

■

**Theorem 1.1.** (Elementary Properties:)

Let  $M$  be a left  $R$ -module. Suppose  $0_m$  and  $0_r$  denotes additive identities of  $M$  and  $R$  respectively. Then, for each  $x \in M$  and  $r \in R$

(i)

$$0_m = 0_r x = r 0_m$$

(ii)

$$r(-x) = (-r)x = -rx$$

*Proof.* (i) As  $0_m$  is the additive identity of  $M$ . so,  $0_m = 0_m + 0_m$

$$\text{Consider } r(0_m + 0_m) = r 0_m = r 0_m + r 0_m$$

$$\text{but, } r(0_m + 0_m) = (r 0_m) + (r 0_m)$$

so, we have

$$r 0_m + r 0_m = r 0_m + 0_m$$

as  $(M, +)$  is an abelian group so left and right cancellation law holds.

$$\begin{aligned} \cancel{r 0_m} + r 0_m &= \cancel{r 0_m} + 0_m \\ r 0_m &= 0_m \end{aligned}$$

a similar argument can be used to prove  $0_m = 0_r x$ .

(ii) as  $M$  is a left  $R$ -module so  $(r, x) \mapsto rx \in M$

Now, Consider  $(-r)x + rx$

using distributive law

$$\begin{aligned} (-r)x + rx &= (-r + r)x \\ &= 0_r x \\ &= 0_m \end{aligned}$$

i.e.  $(-r)x$  is additive inverse of  $(rx)$  but additive inverse of  $(rx)$  is  $-rx$  and it is unique for an abelian group ( $M$  here)

$$\therefore (-r)x = -rx$$

a similar argument can be used to prove that  $r(-x) = -rx$ . ■

**Definition 1.1** (Ring Homomorphism). Let  $R$  and  $S$  be two rings with identities  $1_r, 1_s$  respectively then a map (say  $f$ )

$$f : R \rightarrow S$$

is said to be a ring homomorphism or ring linear map if for every  $a, b \in R$  following properties holds

$\therefore (r, 0_m) \mapsto r 0_m \in M$   
so,  $r 0_m = r 0_m + 0_m$   
 $\therefore M$  is a left  $R$ -module.  
(using distributive property)

often called as ring homo

if  $R = S$  then we call ring homo as ring endomorphism. For instance, let  $f$  be ring homo from  $R$  to  $R$ . we say  $f$  is endomorphism of  $R$  and denoted by  $\text{End } R$

(i) Preserves Addition

$$(a + b)f = (a)f + (b)f$$

(ii) Preservers Multiplication

$$(ab)f = (a)f.(b)f$$

(iii) Maps identity to identity

$$(1_r)f = 1_s$$

**Remark: 6.** Such a mapping need not to be bijective. if it is bijective then we say it is a ring isomorphism or rings are isomorphic.

**Theorem 1.2.** Let  $R$  be a ring and  $M$  be any abelian group with addition. then  $M$  is a right  $R$ -module if and only if there exists a map which is ring homomorphism from  $R$  to  $\text{End } M$

$M$  is a right  $R$ -module

$$\begin{array}{c} \Downarrow \\ \exists f : R \xrightarrow[\text{Homo}]{\text{Ring}} \text{End } M \end{array}$$

*Proof. (Forward Part)* Let us suppose that  $M$  is a right  $R$ -module.

**Claim:** there exists a map which is ring homomorphism from  $R$  to  $\text{End } M$

$\therefore M$  is a left  $R$ -module, so there exist a map

$$f : M \times R \rightarrow M$$

defined by

$$(x, a) \mapsto ax$$

satisfying following properties:

$$\begin{aligned} (x + y)a &= (x)a + (y)a &= xa + ya \\ x(ab) &= (xa)b \\ x1 &= x \end{aligned}$$

$$\forall x, y \in M \text{ \& } a, b \in R$$

for each  $a \in R$ , define a map (say  $\phi_a$ )

$$\phi_a : M \rightarrow M$$

such that for each  $x \in M$

$$(x)\phi_a = xa \in M$$

Now, we'll show that  $\phi_a \in \text{End } M$

Let  $x, y \in M$

Consider  $(x + y)\phi_a$

$$\begin{aligned} &= (x + y)a && \text{using definition of } \phi_a \\ &= xa + ya && \text{using (1.9)} \\ &= (x)\phi_a + (y)\phi_a \end{aligned}$$

so,  $\phi_a$  preserves addition and is a group homo from  $M$  to  $M$ .

i.e.  $\phi_a \in \text{End } M$

Now, we can define a map (say  $f$ )

$$f : R \rightarrow \text{End } M$$

defined as

$$(a)f \mapsto \phi_a \quad \forall a \in R \text{ and } \phi_a \in \text{End } M$$

Now, We'll show that  $f$  is a ring homomorphism.

(A)

$$\begin{aligned} (a + b)f &= \phi_{a+b} \\ &= \phi_a + \phi_b \\ &= (a)f + (b)f \end{aligned}$$

for each  $x \in M$  we have,

$$\begin{aligned} (x)\phi_{a+b} &= x(a + b) = xa + xb \\ &= (x)\phi_a + (x)\phi_b \\ \therefore \phi_{a+b} &= \phi_a + \phi_b \end{aligned}$$

(B)

$$\begin{aligned} (ab)f &= \phi_{ab} \\ &= \phi_a \cdot \phi_b \\ &= (a)f (b)f \end{aligned}$$

for each  $x \in M$  we have,

$$\begin{aligned} (x)\phi_{ab} &= x(ab) = (xa)b \\ &= (xa)\phi_b = (x)\phi_a \cdot \phi_b \\ \therefore \phi_{ab} &= \phi_a \cdot \phi_b \end{aligned}$$

(C)

$$(1)f = \phi_1$$

for each  $x \in M$  we have,

Thus, Forward Part is proved.

$$\begin{aligned} (x)\phi_1 &= x(1) \\ &= x \end{aligned}$$

$\therefore \phi_1$  is identity of  $\text{End } M$

(Converse Part) Assume that  $\exists$  a ring homo. (say  $f$ )

$$f : R \xrightarrow[\text{Homo}]{\text{Ring}} \text{End } M$$

for any  $a \in R$ , we denote the  $(a)f$  by  $f_a \in \text{End } M$

**Claim:**  $M$  is a right  $R$ -module.

so let's define a map

$$R \times M \longrightarrow M$$

defined by

$$(a, x) \mapsto x * a = (x)f_a$$

to prove  $M$  is a right  $R$ -module, we need to verify four properties (1.5)- (1.9) of right  $R$ -module.

(i)

$$\begin{aligned} (x + y) * a &= (x + y)f_a \\ &= (x)f_a + (y)f_a \\ &= x * a + y * a \end{aligned} \quad \because f_a \in \text{End } M$$

(ii)

$$\begin{aligned} x * (a + b) &= (x)f_{a+b} \\ &= (x)(f_a + f_b) \\ &= (x)f_a + (x)f_b \\ &= (x * a) + (x * b) \end{aligned} \quad \because f_a, f_b \in \text{End } M$$

(iii)

$$\begin{aligned} x * (ab) &= (x)f_{ab} \\ &= (x)f_a \circ f_b \\ &= (xf_a)f_b \\ &= (x * a) * b \end{aligned}$$

(iv)

$$x * 1 = (x)f_1 = x$$

Thus,  $M$  is a right  $R$ -module. ■

$\because f_1$  is identity in  $\text{End } M$

**Definition 1.2** (Anti-Ring Homomorphism). Let  $R$  and  $S$  be two rings with identities  $1_r$  and  $1_s$  respectively. Define a map  $f$

$$f : R \rightarrow S$$

satisfying following properties, for each  $a, b \in R$

(i)

$$(a + b)f = (a)f + (b)f$$

(ii)

$$(ab)f = (b)f (a)f$$

(iii)

$$(1_r)f = 1_s$$

Then,  $f$  is called anti-ring homomorphism.

**Theorem 1.3.** Let  $R$  be a ring and  $M$  be any abelian group with addition. then  $M$  is a left  $R$ -module if and only if there exists a map which is anti-ring homomorphism from  $R$  to  $\text{End } M$ .

*Proof.* Left to reader. ■

$$\begin{array}{c} M \text{ is a left } R\text{-module} \\ \updownarrow \\ \exists f : R \xrightarrow[\text{Homo}]{\text{Anti-Ring}} \text{End } M \end{array}$$

## Submodules

**Definition 1.3** (SubModule). Let  $M$  be a left (right)  $R$ -module then a subset  $N$  of  $M$  is called a submodule of  $M$  if  $N$  is a left (right)  $R$ -module under the operation induced from  $M$ .

In other words, A subset  $N$  of  $M$  is called submodule of  $M$  if

- (i)  $N$  is subgroup of  $M$ .
- (ii)  $N$  is closed under induced scalar multiplication from  $M$ .

**Theorem 1.4** (Criterion for Checking Modules). Let  $M$  be a left (right)  $R$ -module and  $N$  be a subset of  $M$  then  $N$  is a submodule of  $M$  if and only if

(i)

$$x - y \in N$$

$$\forall x, y \in N$$

(ii)

$$ax \in N$$

$$\forall a \in R \ \& \ x \in N$$

*Proof.* Left to reader. ■

*Examples:*

1. As every Vector Space  $V$  over a Field  $F$  is a  $F$ -module. So, submodules of  $V$  are subspaces of  $V$ .
2. As every abelian group  $G$  is a  $\mathbb{Z}$ -module. So, all subgroups of  $G$  are submodules.
3. Let  $R$  be a ring then  $R$  is a left as well as right  $R$ -module then left (right) ideals of  $R$  are left (right) submodules of  $R$ .
4.  $\{0\}$  and  $M$  are trivial submodules of any left (right)  $R$ -module  $M$ .



**Remark: 7.**

1. Union of two submodules need not to be a submodule.
2. Intersection of any number of submodules is again a submodule.

Think an example !

**Hint:** Verify using criterion for checking modules.

**Remark: 8. (Smallest Submodule containing a set)**

Let  $M$  be any left (right)  $R$ -module and  $S$  be any subset of  $M$ . Suppose  $\mathcal{F}$  be the family of all submodules of  $M$  containing  $S$ .

$$\text{Let } P = \bigcap_{N \in \mathcal{F}} N$$

then  $P$  is a submodule of  $M$  containing  $S$  as being intersection of an indexed family of submodules containing  $S$ .

Moreover,  $P$  is the smallest submodule of  $M$  containing  $S$ . i.e. for any arbitrary submodule  $K \in \mathcal{F}$ , we have  $P \subseteq K$ . Such submodule  $P$  of  $M$  is said to be generated by set  $S$  and is denoted by

$$P = \langle S \rangle = (S)$$

**Remark: 9.**

Let  $S$  be any subset of left  $R$ -module  $M$  and  $\langle S \rangle$  is the smallest submodule of  $M$  containing  $S$ .

1. if  $S$  is non-empty and finite,  $S = \{x_1, x_2, x_3, \dots, x_n\}$

$$\langle S \rangle = \langle \{x_1, x_2, x_3, \dots, x_n\} \rangle = \langle x_1, x_2, x_3, \dots, x_n \rangle$$

is said to be a finitely generated by  $S$  and is smallest submodule of  $M$  containing  $S$ .

2. if  $S = \phi$  i.e.  $S$  is an empty set

$$\langle S \rangle = \langle \phi \rangle = \{0\}$$

3. if  $S = \{a\}$  i.e.  $S$  is singleton then  $\langle S \rangle = \langle a \rangle$  is said to be a cyclic submodule.

**Definition 1.4** (Cyclic module). A module  $M$  is said to be a cyclic module<sup>3</sup> if it can be generated by a single element.

For Example: A ring  $R$  over itself is a module and can be generated by identity element  $\{1\}$  so is a cyclic module.

<sup>3</sup> P. M. Cohn. *Basic Algebra*. Springer, 2 edition, 2005. ISBN 978-1-4471-1060-6

**Theorem 1.5.** Let  $M$  be left  $R$  module and  $S$  being any subset of  $M$ .

$$\langle S \rangle = \begin{cases} \{0\} & \text{if } S = \phi \\ \left\{ \sum_{i \in J_n} a_i x_i \mid a_i \in R, x_i \in S \right\} & \text{otherwise} \end{cases}$$

*Proof. Case-I* Let us suppose that  $S = \phi$ , as  $\langle S \rangle$  is the intersection of all the submodules of  $M$  containing  $S$ .

i.e. Every submodule of  $M$  will contain  $S$

In particular,  $\{0\}$  also contains  $S$  i.e.

$$\{0\} \in \mathcal{F}$$

so,

$$\begin{aligned} \langle S \rangle &= \bigcap_{N \in \mathcal{F}} N \\ &= \{0\} \end{aligned}$$

$\therefore \mathcal{F}$  is a collection of all submodules of  $M$  containing  $S$

*Case-II* Suppose  $S$  is non-empty and let

$$P = \left\{ \sum_{i \in J_n} a_i x_i \mid a_i \in R, x_i \in S \right\}$$

First, we'll show that  $S \subseteq P$

Let  $x \in S$  then  $x$  can be expressed in following form:

$$x = 1.x = \sum_{i \in J_1} a_i x_i$$

with  $a_1 = 1$  and  $x_1 = x$

$$\therefore x \in P \Rightarrow S \subseteq P$$

$\therefore x$  was chosen arbitrary.

Now, we'll show that  $P$  is a submodule of  $M$  using submodule criterion.

Let  $u, v \in P$ . so, we need to show  $u + \alpha v \in P$

for any  $\alpha \in R$

$$u = \sum_{i \in J_n} a_i x_i$$

$\forall x_i \in S \text{ \& } a_i \in R$

$$v = \sum_{j \in J_m} b_j y_j$$

$\forall y_j \in S \text{ \& } b_j \in R$

■

*Stay Tuned for next chapters!*



## *Bibliography*

P. M. Cohn. *Basic Algebra*. Springer, 2 edition, 2005. ISBN 978-1-4471-1060-6.