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MODULE THEORY

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Dedicated to my family and my best friend

Neeraj K. Gaud

Introduction

Warning :

This is my first document created using latex so it may be possible that there are several errors. if you notice any error then you can report it here.

<https://github.com/sirkapil/module-theory/issues/new>¹

¹ (may require a github account)

About :

This sample book discusses the course "Module Theory" being taught to Post-Graduate (M.Sc. Mathematics) students in Department of Mathematics under University of Delhi, Delhi.

All my \LaTeX documents are free and open-source. Each document is hosted in a github repository and can be found pinned here.

<https://github.com/sirkapil>

Contribution :

If you find my work useful and want to contribute then you are welcome by heart.

Any suitable changes to document repository through pull requests are highly appreciated. You can create a new pull request here. Be sure to read *contribution file* in root/.github folder of repository before creating any pull-request.

<https://github.com/sirkapil/module-theory/compare>

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https://twitter.com/kapil_rc

Introduction to Modules

Defination of Module

Left Module:

Let R be a ring with identity and M be an abelian group with addition. We say M is a left R -module if there exists a mapping²

² often called as scalar multiplication.

$$R \times M \rightarrow M$$

defined by

$$(a, x) \rightarrow ax$$

$$\forall a \in R \text{ and } x \in M$$

satisfying following properties :

$$(a + b)x = ax + bx \quad (1.1)$$

$$a(x + y) = ax + ay \quad (1.2)$$

$$(ab)x = a(bx) \quad (1.3)$$

$$1x = x \quad (1.4)$$

$$\forall a, b \in R \text{ and } x, y \in M$$

and denoted by ${}_R M$

Right Module:

Let R be a ring with identity and M be an abelian group with addition. We say M is a right R -module if there exists a mapping

$$M \times R \rightarrow M$$

defined by

$$(x, a) \rightarrow xa$$

$$\forall a \in R \text{ and } x \in M$$

satisfying following properties :

$$x(a + b) = xa + xb \quad (1.5)$$

$$(x + y)a = xa + ya \quad (1.6)$$

$$x(ab) = (xa)b \quad (1.7)$$

$$x1 = x \quad (1.8)$$

$$\forall a, b \in R \text{ and } x, y \in M$$

and denoted by M_R .

Examples :

1. Let V be a vector space over a field F then V is a left as well as right F -Module.
2. Let G be any abelian group under addition , then G is a \mathbb{Z} -Module where \mathbb{Z} is set of integers.
3. Let R be ring and $M = R[x]$ where $R[x]$ is a group of all polynomials with coefficients in R then M is a left as well as a right R -Module with scalar multiplication being usual multiplication.
4. Let M be collection of all $m \times n$ matrices over ring R , then M is left R -Module where scalar multiplication being usual multiplication of a scalar to a matrix.

Suppose ring R is a field then R -Module $R[x]$ is a vector space over field R .

In particular, if M is a set of $1 \times n$ matrices over R or $M = R^n$ (set of n -tuples) then R^n is a left R -module.

Remark: 2. Let R be a commutative ring then every left R -module can be transformed to right R -module and vice-versa.

Proof. Let M be left R -module and R be a commutative ring.
so, \exists a mapping

$$R \times M \rightarrow M$$

defined by

$$(a, x) \rightarrow ax$$

for each $a \in R$ and $x \in M$ satisfying following properties :

$$\forall a, b \in R \text{ and } x, y \in M$$

$$(a + b)x = ax + bx$$

$$a(x + y) = ax + ay$$

$$(ab)x = a(bx)$$

$$1x = x$$

$\therefore R$ is a commutative ring.

Now, Define an another mapping

$$M \times R \rightarrow M$$

defined by

$$(x, a) \rightarrow x * a = ax$$

To check M is a right R -Module , we need to verify properties number ??-(1.8)

(i) Distributive Law

$$\begin{aligned} x * (a + b) &= (a + b)x \\ &= ax + bx \\ &= (x * a) + (x * b) \end{aligned}$$

(ii) Distributive Law

$$\begin{aligned} (x + y) * a &= a(x + y) \\ &= ax + ay \\ &= (x * a) + (y * a) \end{aligned}$$

(iii)

$$\begin{aligned} x * (ab) &= (ab)x \\ &= (ba)x \\ &= b(ax) \\ &= (ax) * b \end{aligned}$$

(iv)

$$\begin{aligned} x * 1 &= 1x \\ &= x \end{aligned}$$

Thus, ${}_R M$ is transformed to M_R .

Similarly, Converse statement can be verified.

■

Remark: 3. Let S be a subring of ring R then ${}_S M$ exists only if ${}_R M$ exists.

by existence means M is a valid left module over mentioned ring or subring. i.e. satisfying those four properties.

Remark: 4. Same Abelian group can have the structure of a Module for a number of different rings.

For Instance, The field \mathbb{R} is \mathbb{R} -module, \mathbb{Q} -module and \mathbb{Z} -module.

Remark: 5. Let I be left ideal of R then quotient ring R / I is a left R -module.

Here scalar multiplication is

verification: 'left to reader'

Hint: you need to verify those four properties: (1.1)-(1.4)

■

$R \times R / I \rightarrow R / I$
defined as

$$\begin{aligned} (a, x + I) &\mapsto ax + I \\ \forall a \in R \text{ and } \forall x + I \in R / I \end{aligned}$$

Theorem 1.1. (Elementary Properties:)

Let M be a left R -module. Suppose 0_m and 0_r denotes additive identities of M and R respectively. Then, for each $x \in M$ and $r \in R$

(i)

$$0_m = 0_r x = r 0_m$$

(ii)

$$r(-x) = (-r)x = -rx$$

Proof. (i) As 0_m is the additive identity of M . so, $0_m = 0_m + 0_m$

$$\text{Consider } r(0_m + 0_m) = r 0_m = r 0_m + r 0_m$$

$$\text{but, } r(0_m + 0_m) = (r 0_m) + (r 0_m)$$

so, we have

$$r 0_m + r 0_m = r 0_m + 0_m$$

as $(M, +)$ is an abelian group so left and right cancellation law holds.

$$\begin{aligned} \cancel{r 0_m} + r 0_m &= \cancel{r 0_m} + 0_m \\ r 0_m &= 0_m \end{aligned}$$

a similar argument can be used to prove $0_m = 0_r x$.

(ii) as M is a left R -module so $(r, x) \mapsto rx \in M$

Now, Consider $(-r)x + rx$

using distributive law

$$\begin{aligned} (-r)x + rx &= (-r + r)x \\ &= 0_r x \\ &= 0_m \end{aligned}$$

i.e. $(-r)x$ is additive inverse of (rx) but additive inverse of (rx) is $-rx$ and it is unique for an abelian group (M here)

$$\therefore (-r)x = -rx$$

a similar argument can be used to prove that $r(-x) = -rx$. ■

Definition 1.1 (Ring Homomorphism). Let R and S be two rings with identities $1_r, 1_s$ respectively then a map (say f)

$$f : R \rightarrow S$$

is said to be a ring homomorphism or ring linear map if for every $a, b \in R$ following properties holds

$\therefore (r, 0_m) \mapsto r 0_m \in M$
so, $r 0_m = r 0_m + 0_m$
 $\therefore M$ is a left R -module.
(using distributive property)

often called as ring homo

if $R = S$ then we call ring homo as ring endomorphism. For instance, let f be ring homo from R to R . we say f is endomorphism of R and denoted by $\text{End } R$

(i) *Preserves Addition*

$$(a + b)f = (a)f + (b)f$$

(ii) *Preservers Multiplication*

$$(ab)f = (a)f.(b)f$$

(iii) *Maps identity to identity*

$$(1_r)f = 1_s$$

Remark: 6. Such a mapping need not to be bijective. if it is bijective then we say it is a ring isomorphism or rings are isomorphic.

Theorem 1.2. Let R be a ring and M be any abelian group with addition. then M is a right R -module if and only if there exists a map which is ring homomorphism from R to $\text{End } M$

M is a right R -module

$$\begin{array}{c} \Downarrow \\ \exists f : R \xrightarrow[\text{Homo}]{\text{Ring}} \text{End } M \end{array}$$

Proof. (Forward Part) Let us suppose that M is a right R -module.

Claim: there exists a map which is ring homomorphism from R to $\text{End } M$

$\therefore M$ is a left R -module, so there exist a map

$$f : M \times R \rightarrow M$$

defined by

$$(x, a) \mapsto ax$$

satisfying following properties:

$$\begin{aligned} (x + y)a &= (x)a + (y)a &= xa + ya \\ x(ab) &= (xa)b \\ x1 &= x \end{aligned}$$

$$\forall x, y \in M \text{ \& } a, b \in R$$

for each $a \in R$, define a map (say ϕ_a)

$$\phi_a : M \rightarrow M$$

such that for each $x \in M$

$$(x)\phi_a = xa \in M$$

Now, we'll show that $\phi_a \in \text{End } M$

Let $x, y \in M$

Consider $(x + y)\phi_a$

$$\begin{aligned} &= (x + y)a && \text{using definition of } \phi_a \\ &= xa + ya && \text{using (1.9)} \\ &= (x)\phi_a + (y)\phi_a \end{aligned}$$

so, ϕ_a preserves addition and is a group homo from M to M .

i.e. $\phi_a \in \text{End } M$

Now, we can define a map (say f)

$$f : R \rightarrow \text{End } M$$

defined as

$$(a)f \mapsto \phi_a \quad \forall a \in R \text{ and } \phi_a \in \text{End } M$$

Now, We'll show that f is a ring homomorphism.

(A)

$$\begin{aligned} (a + b)f &= \phi_{a+b} \\ &= \phi_a + \phi_b \\ &= (a)f + (b)f \end{aligned}$$

for each $x \in M$ we have,

$$\begin{aligned} (x)\phi_{a+b} &= x(a + b) = xa + xb \\ &= (x)\phi_a + (x)\phi_b \\ \therefore \phi_{a+b} &= \phi_a + \phi_b \end{aligned}$$

(B)

$$\begin{aligned} (ab)f &= \phi_{ab} \\ &= \phi_a \cdot \phi_b \\ &= (a)f (b)f \end{aligned}$$

for each $x \in M$ we have,

$$\begin{aligned} (x)\phi_{ab} &= x(ab) = (xa)b \\ &= (xa)\phi_b = (x)\phi_a \cdot \phi_b \\ \therefore \phi_{ab} &= \phi_a \cdot \phi_b \end{aligned}$$

(C)

$$(1)f = \phi_1$$

for each $x \in M$ we have,

$$\begin{aligned} (x)\phi_1 &= x(1) \\ &= x \end{aligned}$$

$\therefore \phi_1$ is identity of $\text{End } M$

Thus, Forward Part is proved.

(Converse Part) Assume that \exists a ring homo. (say f)

$$f : R \xrightarrow[\text{Homo}]{\text{Ring}} \text{End } M$$

for any $a \in R$, we denote the $(a)f$ by $f_a \in \text{End } M$

Claim: M is a right R -module.

so let's define a map

$$R \times M \longrightarrow M$$

defined by

$$(a, x) \mapsto x * a = (x)f_a$$

to prove M is a right R -module, we need to verify four properties (1.5)- (1.9) of right R -module.

(i)

$$\begin{aligned} (x + y) * a &= (x + y)f_a \\ &= (x)f_a + (y)f_a \\ &= x * a + y * a \end{aligned} \quad \because f_a \in \text{End } M$$

(ii)

$$\begin{aligned} x * (a + b) &= (x)f_{a+b} \\ &= (x)(f_a + f_b) \\ &= (x)f_a + (x)f_b \\ &= (x * a) + (x * b) \end{aligned} \quad \because f_a, f_b \in \text{End } M$$

(iii)

$$\begin{aligned} x * (ab) &= (x)f_{ab} \\ &= (x)f_a \circ f_b \\ &= (xf_a)f_b \\ &= (x * a) * b \end{aligned}$$

(iv)

$$x * 1 = (x)f_1 = x$$

Thus, M is a right R -module. ■

$\because f_1$ is identity in $\text{End } M$

Definition 1.2 (Anti-Ring Homomorphism). Let R and S be two rings with identities 1_r and 1_s respectively. Define a map f

$$f : R \rightarrow S$$

satisfying following properties, for each $a, b \in R$

(i)

$$(a + b)f = (a)f + (b)f$$

(ii)

$$(ab)f = (b)f (a)f$$

(iii)

$$(1_r)f = 1_s$$

Then, f is called anti-ring homomorphism.

Theorem 1.3. Let R be a ring and M be any abelian group with addition. then M is a left R -module if and only if there exists a map which is anti-ring homomorphism from R to $\text{End } M$.

Proof. Left to reader. ■

$$\begin{array}{c} M \text{ is a left } R\text{-module} \\ \updownarrow \\ \exists f : R \xrightarrow[\text{Homo}]{\text{Anti-Ring}} \text{End } M \end{array}$$

Submodules

Definition 1.3 (SubModule). Let M be a left (right) R -module then a subset N of M is called a submodule of M if N is a left (right) R -module under the operation induced from M .

In other words, A subset N of M is called submodule of M if

- (i) N is subgroup of M .
- (ii) N is closed under induced scalar multiplication from M .

Theorem 1.4 (Criterion for Checking Modules). Let M be a left (right) R -module and N be a subset of M then N is a submodule of M if and only if

(i)

$$x - y \in N$$

$$\forall x, y \in N$$

(ii)

$$ax \in N$$

$$\forall a \in R \ \& \ x \in N$$

Proof. Left to reader. ■

Examples:

1. As every Vector Space V over a Field F is a F -module. So, submodules of V are subspaces of V .
2. As every abelian group G is a \mathbb{Z} -module. So, all subgroups of G are submodules.
3. Let R be a ring then R is a left as well as right R -module then left (right) ideals of R are left (right) submodules of R .
4. $\{0\}$ and M are trivial submodules of any left (right) R -module M .

Remark: 7.

1. Union of two submodules need not to be a submodule.
2. Intersection of any number of submodules is again a submodule.

Think an example !

Hint: Verify using criterion for checking modules.

Remark: 8. (Smallest Submodule containing a set)

Let M be any left (right) R -module and S be any subset of M . Suppose \mathcal{F} be the family of all submodules of M containing S .

$$\text{Let } P = \bigcap_{N \in \mathcal{F}} N$$

then P is a submodule of M containing S as being intersection of an indexed family of submodules containing S .

Moreover, P is the smallest submodule of M containing S . i.e. for any arbitrary submodule $K \in \mathcal{F}$, we have $P \subseteq K$. Such submodule P of M is said to be generated by set S and is denoted by

$$P = \langle S \rangle = (S)$$

Remark: 9.

Let S be any subset of left R -module M and $\langle S \rangle$ is the smallest submodule of M containing S .

1. if S is non-empty and finite, $S = \{x_1, x_2, x_3, \dots, x_n\}$

$$\langle S \rangle = \langle \{x_1, x_2, x_3, \dots, x_n\} \rangle = \langle x_1, x_2, x_3, \dots, x_n \rangle$$

is said to be a finitely generated by S and is smallest submodule of M containing S .

2. if $S = \phi$ i.e. S is an empty set

$$\langle S \rangle = \langle \phi \rangle = \{0\}$$

3. if $S = \{a\}$ i.e. S is singleton then $\langle S \rangle = \langle a \rangle$ is said to be a cyclic submodule.

Definition 1.4 (Cyclic module). A module M is said to be a cyclic module³ if it can be generated by a single element.

For Example: A ring R over itself is a module and can be generated by identity element $\{1\}$ so is a cyclic module.

³ P. M. Cohn. *Basic Algebra*. Springer, 2 edition, 2005. ISBN 978-1-4471-1060-6

Theorem 1.5. *Let M be left R module and S being any subset of M .*

$$\langle S \rangle = \begin{cases} \{0\} & \text{if } S = \phi \\ \left\{ \sum_{i \in J_n} a_i x_i \mid a_i \in R, x_i \in S \right\} & \text{otherwise} \end{cases}$$

Proof. Case-I Let us suppose that $S = \phi$, as $\langle S \rangle$ is the intersection of all the submodules of M containing S .

In particular, $\{0\}$ also contains S i.e.

$$\{0\} \in \mathcal{F}$$

so,

$$\begin{aligned} \langle S \rangle &= \bigcap_{N \in \mathcal{F}} N \\ &= \{0\} \end{aligned}$$

■

Stay Tuned for next chapters!

Bibliography

P. M. Cohn. *Basic Algebra*. Springer, 2 edition, 2005. ISBN 978-1-4471-1060-6.