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# MODULE THEORY

UNIVERSITY OF DELHI

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*Dedicated to my family and my best friend*

*Neeraj K. Gaur*

# *Introduction*

This sample book discusses the course Module Theory of pure mathematics being taught to post-graduate students in University of Delhi.





# Introduction to Modules

## Defination of Module

### Left Module:

Let  $R$  be a ring with identity and  $M$  be an abelian group with addition. We say  $M$  is a left  $R$ -module if there exists a mapping<sup>1</sup>

<sup>1</sup> often called as scaler multiplication.

$$R \times M \rightarrow M$$

defined by

$$(a, x) \rightarrow ax$$

for each  $a \in R$  and  $x \in M$  satisfying following properties :

$$\forall a, b \in R \text{ and } x, y \in M$$

$$(a + b)x = ax + bx \quad (1)$$

$$a(x + y) = ax + ay \quad (2)$$

$$(ab)x = a(bx) \quad (3)$$

$$1x = x \quad (4)$$

and denoted by  ${}_R M$

### Right Module:

Let  $R$  be a ring with identity and  $M$  be an abelian group with addition. We say  $M$  is a right  $R$ -module if there exists a mapping

$$M \times R \rightarrow M$$

defined by

$$(x, a) \rightarrow xa$$

for each  $a \in R$  and  $x \in M$  satisfying following properties :

$$\forall a, b \in R \text{ and } x, y \in M$$

$$x(a + b) = xa + xb \quad (5)$$

$$(x + y)a = xa + ya \quad (6)$$

$$x(ab) = (xa)b \quad (7)$$

$$x1 = x \quad (8)$$

and denoted by  $M_R$

Examples :

1. Let  $V$  be a vector space over a field  $F$  then  $V$  is a left as well as right  $F$ -Module.
2. Let  $G$  be any abelian group under addition , then  $G$  is a  $\mathbb{Z}$ -Module where  $\mathbb{Z}$  is set of integers.
3. Let  $R$  be ring and  $M = R[x]$  where  $R[x]$  is a group of all polynomials with coefficients in  $R$  then  $M$  is a left as well as a right  $R$ -Module with scalar multiplication being usual multiplication.
4. Let  $M$  be collection of all  $m \times n$  matrices over ring  $R$  , then  $M$  is left  $R$ -Module where scalar multiplication being usual multiplication of a scalar to a matrix.

Suppose ring  $R$  is a field then  $R$ -Module  $R[x]$  is a vector space over field  $R$ .

In particular, if  $M$  is a set of  $1 \times n$  matrices over  $R$  or  $M = R^n$  (set of  $n$ -tuples) then  $R^n$  is a left  $R$ -module.

**Remark: 1.** Let  $R$  be a commutative ring then every left  $R$ -module can be transformed to right  $R$ -module and vice-versa.

*Proof.* Let  $M$  be left  $R$ -module and  $R$  be a commutative ring.  
so,  $\exists$  a mapping

$$R \times M \rightarrow M$$

defined by

$$(a, x) \rightarrow ax$$

for each  $a \in R$  and  $x \in M$  satisfying following properties :

$$\forall a, b \in R \text{ and } x, y \in M$$

$$(a + b)x = ax + bx$$

$$a(x + y) = ax + ay$$

$$(ab)x = a(bx)$$

$$1x = x$$

$\therefore R$  is a commutative ring.

Now, Define an another mapping

$$M \times R \rightarrow M$$

defined by

$$(x, a) \rightarrow x * a = ax$$

To check  $M$  is a right  $R$ -Module , we need to verify properties number (5)-(8)

1.

$$\begin{aligned}
 x * (a + b) &= (a + b)x \\
 &= ax + bx \\
 &= (x * a) + (x * b)
 \end{aligned}$$

2.

$$\begin{aligned}
 (x + y) * a &= a(x + y) \\
 &= ax + ay \\
 &= (x * a) + (y * a)
 \end{aligned}$$

3.

$$\begin{aligned}
 x * (ab) &= (ab)x \\
 &= (ba)x \\
 &= b(ax) \\
 &= (ax) * b
 \end{aligned}$$

4.

$$\begin{aligned}
 x * 1 &= 1x \\
 &= x
 \end{aligned}$$

Thus,  ${}_R M$  is transformed to  $M_R$ .

Similarly, Converse statement can be verified.

■

**Remark: 2.** Let  $S$  be a subring of ring  $R$  then  ${}_S M$  exists only if  ${}_R M$  exists.

by existence means  $M$  is a valid left module over mentioned ring or subring. i.e. satisfying those four properties.

**Remark: 3.** Same Abelian group can have the structure of a Module for a number of different rings.

For Instance, The field  $\mathbb{R}$  is  $\mathbb{R}$ -module,  $\mathbb{Q}$ -module and  $\mathbb{Z}$ -module.

**Remark: 4.** Let  $I$  be left ideal of  $R$  then quotient ring  $R/I$  is a left  $R$ -module.

Here scalar multiplication is

verification: 'left to reader'

$$R \times R/I \rightarrow R/I$$

defined as

$$(a, x + I) \mapsto ax + I$$

$$\forall a \in R \text{ and } \forall x + I \in R/I$$

**Hint:** you need to verify those four properties: (1)-(4)

■

**Theorem 0.0.1.** (Elementary Properties:)

Let  $M$  be a left  $R$ -module. Suppose  $0_m$  and  $0_r$  denotes additive identities of  $M$  and  $R$  respectively. Then, for each  $x \in M$  and  $r \in R$

(i)

$$0_m = 0_r x = r 0_m$$

(ii)

$$r(-x) = (-r)x = -rx$$

*Proof.* (i) As  $0_m$  is the additive identity of  $M$ . so,  $0_m = 0_m + 0_m$

$$\text{Consider } r(0_m + 0_m) = r 0_m = r 0_m + 0_m$$

$$\text{but, } r(0_m + 0_m) = (r 0_m) + (r 0_m)$$

so, we have

$$r 0_m + r 0_m = r 0_m + 0_m$$

as  $(M, +)$  is an abelian group so left and right cancellation law holds.

$$\begin{aligned} \cancel{r 0_m} + r 0_m &= \cancel{r 0_m} + 0_m \\ r 0_m &= 0_m \end{aligned}$$

a similar argument can be used to prove  $0_m = 0_r x$ .

(ii) as  $M$  is a left  $R$ -module so  $(r, x) \mapsto rx \in M$

Now, Consider  $(-r)x + rx$

using distributive law

$$\begin{aligned} (-r)x + rx &= (-r + r)x \\ &= 0_r x \\ &= 0_m \end{aligned}$$

i.e.  $(-r)x$  is additive inverse of  $(rx)$  but additive inverse of  $(rx)$  is  $-rx$  and it is unique for an abelian group ( $M$  here)

$$\therefore (-r)x = -rx$$

a similar argument can be used to prove that  $r(-x) = -rx$ . ■

**Definition 0.0.1** (Ring Homomorphism). Let  $R$  and  $S$  be two rings with identities  $1_r, 1_s$  respectively then a map (say  $f$ )

$$f : R \rightarrow S$$

is said to be a ring homomorphism or ring linear map if for every  $a, b \in R$  following properties holds

$\therefore (r, 0_m) \mapsto r 0_m \in M$   
so,  $r 0_m = r 0_m + 0_m$   
 $\therefore M$  is a left  $R$ -module.  
(using distributive property)

often called as ring homo

if  $R = S$  then we call ring homo as ring endomorphism. For instance, let  $f$  be ring homo from  $R$  to  $R$ . we say  $f$  is endomorphism of  $R$  and denoted by  $\text{End } R$

(i) *Preserves Addition*

$$(a + b)f = (a)f + (b)f$$

(ii) *Preservers Multiplication*

$$(ab)f = (a)f.(b)f$$

(iii) *Maps identity to identity*

$$(1_r)f = 1_s$$

**Remark: 5.** Such a mapping need not to be bijective. if it is bijective then we say it is a ring isomorphism or rings are isomorphic.

**Theorem 0.0.2.** Let  $R$  be a ring and  $M$  be any abelian group with addition. then  $M$  is a right  $R$ -module if and only if there exists a map which is ring homomorphism from  $R$  to  $\text{End } M$

$M$  is a right  $R$ -module

$$\begin{array}{c} \Downarrow \\ \exists f : R \xrightarrow[\text{Homo}]{\text{Ring}} \text{End } M \end{array}$$

*Proof. (Forward Part)* Let us suppose that  $M$  is a right  $R$ -module.

**Claim:** there exists a map which is ring homomorphism from  $R$  to  $\text{End } M$

from hypothesis, there exist a map

■

