

KAPIL CHAUDHARY

MODULE THEORY

UNIVERSITY OF DELHI

Copyright © 2018 , All rights are reserved.

Kapil Chaudhary

UNIVERSITY OF DELHI

<https://contact.sirkapil.me/>

Licensed under the Apache License, Version 2.0 (the “License”); you may not use this file except in compliance with the License. You may obtain a copy of the License at <http://www.apache.org/licenses/LICENSE-2.0>. Unless required by applicable law or agreed to in writing, software distributed under the License is distributed on an “AS IS” BASIS, WITHOUT WARRANTIES OR CONDITIONS OF ANY KIND, either express or implied. See the License for the specific language governing permissions and limitations under the License.

First Print, January 2018

Contents

Introduction to Modules 9

Stay Tuned for next chapters! 19

There are few persons without whom this was impossible.

They deserve credit for it so i would love to thank them.

Special Thanks to :

- Dr. Anuj Bishnoi (Subject Teacher)

- Edward Tufte (L^AT_EX Tufte Template)

Dedicated to my family and my best friend

Neeraj K. Gaud

Introduction

Warning :

This is my first document created using latex so it may be possible that there are several errors. if you notice any error then you can report it here.

<https://github.com/sirkapil/module-theory/issues/new>¹

¹ (may require a github account)

About :

This sample book discusses the course "Module Theory" being taught to Post-Graduate (M.Sc. Mathematics) students in Department of Mathematics under University of Delhi, Delhi.

All my \LaTeX documents are free and open-source. Each document is hosted in a github repository and can be found pinned here.

<https://github.com/sirkapil>

Contribution :

If you find my work useful and want to contribute then you are welcome by heart.

Any suitable changes to document repository through pull requests are highly appreciated. You can create a new pull request here. Be sure to read *contribution file* in root/.github folder of repository before creating any pull-request.

<https://github.com/sirkapil/module-theory/compare>

If you don't have a github account or facing difficulty in creating a pull-request , then feel free to drop down a message here about that you are interested in contribution of this project.

<https://cont.netlify.com>

https://twitter.com/kapil_rc

Introduction to Modules

Defination of Module

Left Module:

Let R be a ring with identity and M be an abelian group with addition. We say M is a left R -module if there exists a mapping²

² often called as scalar multiplication.

$$R \times M \rightarrow M$$

defined by

$$(a, x) \rightarrow ax$$

$$\forall a \in R \text{ and } x \in M$$

satisfying following properties :

$$(a + b)x = ax + bx \quad (1.1)$$

$$a(x + y) = ax + ay \quad (1.2)$$

$$(ab)x = a(bx) \quad (1.3)$$

$$1x = x \quad (1.4)$$

$$\forall a, b \in R \text{ and } x, y \in M$$

and denoted by ${}_R M$

Right Module:

Let R be a ring with identity and M be an abelian group with addition. We say M is a right R -module if there exists a mapping

$$M \times R \rightarrow M$$

defined by

$$(x, a) \rightarrow xa$$

$$\forall a \in R \text{ and } x \in M$$

satisfying following properties :

$$x(a + b) = xa + xb \quad (1.5)$$

$$(x + y)a = xa + ya \quad (1.6)$$

$$x(ab) = (xa)b \quad (1.7)$$

$$x1 = x \quad (1.8)$$

$$\forall a, b \in R \text{ and } x, y \in M$$

and denoted by M_R .

Examples :

1. Let V be a vector space over a field F then V is a left as well as right F -Module.
2. Let G be any abelian group under addition , then G is a \mathbb{Z} -Module where \mathbb{Z} is set of integers.
3. Let R be ring and $M = R[x]$ where $R[x]$ is a group of all polynomials with coefficients in R then M is a left as well as a right R -Module with scalar multiplication being usual multiplication.
4. Let M be collection of all $m \times n$ matrices over ring R , then M is left R -Module where scalar multiplication being usual multiplication of a scalar to a matrix.

Suppose ring R is a field then R -Module $R[x]$ is a vector space over field R .

In particular, if M is a set of $1 \times n$ matrices over R or $M = R^n$ (set of n -tuples) then R^n is a left R -module.

Remark: 2. Let R be a commutative ring then every left R -module can be transformed to right R -module and vice-versa.

Proof. Let M be left R -module and R be a commutative ring.
so, \exists a mapping

$$R \times M \rightarrow M$$

defined by

$$(a, x) \rightarrow ax$$

for each $a \in R$ and $x \in M$ satisfying following properties :

$$\forall a, b \in R \text{ and } x, y \in M$$

$$(a + b)x = ax + bx$$

$$a(x + y) = ax + ay$$

$$(ab)x = a(bx)$$

$$1x = x$$

$\therefore R$ is a commutative ring.

Now, Define an another mapping

$$M \times R \rightarrow M$$

defined by

$$(x, a) \rightarrow x * a = ax$$

To check M is a right R -Module , we need to verify properties number ??-(1.8)

(i) Distributive Law

$$\begin{aligned} x * (a + b) &= (a + b)x \\ &= ax + bx \\ &= (x * a) + (x * b) \end{aligned}$$

(ii) Distributive Law

$$\begin{aligned} (x + y) * a &= a(x + y) \\ &= ax + ay \\ &= (x * a) + (y * a) \end{aligned}$$

(iii)

$$\begin{aligned} x * (ab) &= (ab)x \\ &= (ba)x \\ &= b(ax) \\ &= (ax) * b \end{aligned}$$

(iv)

$$\begin{aligned} x * 1 &= 1x \\ &= x \end{aligned}$$

Thus, ${}_R M$ is transformed to M_R .

Similarly, Converse statement can be verified.

■

Remark: 3. Let S be a subring of ring R then ${}_S M$ exists only if ${}_R M$ exists.

by existence means M is a valid left module over mentioned ring or subring. i.e. satisfying those four properties.

Remark: 4. Same Abelian group can have the structure of a Module for a number of different rings.

For Instance, The field \mathbb{R} is \mathbb{R} -module, \mathbb{Q} -module and \mathbb{Z} -module.

Remark: 5. Let I be left ideal of R then quotient ring R/I is a left R -module.

Here scalar multiplication is

verification: 'left to reader'

Hint: you need to verify those four properties: (1.1)-(1.4)

■

$R \times R/I \rightarrow R/I$
defined as

$$\begin{aligned} (a, x + I) &\mapsto ax + I \\ \forall a \in R \text{ and } \forall x + I \in R/I \end{aligned}$$

Theorem 1.1. (Elementary Properties:)

Let M be a left R -module. Suppose 0_m and 0_r denotes additive identities of M and R respectively. Then, for each $x \in M$ and $r \in R$

(i)

$$0_m = 0_r x = r 0_m$$

(ii)

$$r(-x) = (-r)x = -rx$$

Proof. (i) As 0_m is the additive identity of M . so, $0_m = 0_m + 0_m$

$$\text{Consider } r(0_m + 0_m) = r 0_m = r 0_m + r 0_m$$

$$\text{but, } r(0_m + 0_m) = (r 0_m) + (r 0_m)$$

so, we have

$$r 0_m + r 0_m = r 0_m + 0_m$$

as $(M, +)$ is an abelian group so left and right cancellation law holds.

$$\begin{aligned} \cancel{r 0_m} + r 0_m &= \cancel{r 0_m} + 0_m \\ r 0_m &= 0_m \end{aligned}$$

a similar argument can be used to prove $0_m = 0_r x$.

(ii) as M is a left R -module so $(r, x) \mapsto rx \in M$

Now, Consider $(-r)x + rx$

using distributive law

$$\begin{aligned} (-r)x + rx &= (-r + r)x \\ &= 0_r x \\ &= 0_m \end{aligned}$$

i.e. $(-r)x$ is additive inverse of (rx) but additive inverse of (rx) is $-rx$ and it is unique for an abelian group (M here)

$$\therefore (-r)x = -rx$$

a similar argument can be used to prove that $r(-x) = -rx$. ■

Definition 1.1 (Ring Homomorphism). Let R and S be two rings with identities $1_r, 1_s$ respectively then a map (say f)

$$f : R \rightarrow S$$

is said to be a ring homomorphism or ring linear map if for every $a, b \in R$ following properties holds

$\therefore (r, 0_m) \mapsto r 0_m \in M$
so, $r 0_m = r 0_m + 0_m$
 $\therefore M$ is a left R -module.
(using distributive property)

often called as ring homo

if $R = S$ then we call ring homo as ring endomorphism. For instance, let f be ring homo from R to R . we say f is endomorphism of R and denoted by $\text{End } R$

(i) *Preserves Addition*

$$(a + b)f = (a)f + (b)f$$

(ii) *Preservers Multiplication*

$$(ab)f = (a)f.(b)f$$

(iii) *Maps identity to identity*

$$(1_r)f = 1_s$$

Remark: 6. Such a mapping need not to be bijective. if it is bijective then we say it is a ring isomorphism or rings are isomorphic.

Theorem 1.2. Let R be a ring and M be any abelian group with addition. then M is a right R -module if and only if there exists a map which is ring homomorphism from R to $\text{End } M$

M is a right R -module

$$\begin{array}{c} \Downarrow \\ \exists f : R \xrightarrow[\text{Homo}]{\text{Ring}} \text{End } M \end{array}$$

Proof. (Forward Part) Let us suppose that M is a right R -module.

Claim: there exists a map which is ring homomorphism from R to $\text{End } M$

$\therefore M$ is a left R -module , so there exist a map

$$f : M \times R \rightarrow M$$

defined by

$$(x, a) \mapsto ax$$

satisfying following properties:

$$(x + y)a = (x)a + (y)a$$

$$x(a + b) = xa + xb$$

$$x(ab) = (xa)b$$

$$x1 = x$$

$$\forall x, y \in M \text{ \& } a, b \in R$$

for each $a \in R$, define a map(say ϕ_a)

$$\phi_a : M \rightarrow M$$

such that for each $x \in M$

$$(x)\phi_a = xa \in M$$

Now, we'll show that $\phi_a \in \text{End } M$

Let $x, y \in M$

Consider $(x + y)\phi_a$

$$\begin{aligned} &= (x + y)a && \text{using definition of } \phi_a \\ &= xa + ya && \text{using (1.9)} \\ &= (x)\phi_a + (y)\phi_a \end{aligned}$$

so, ϕ_a preserves addition and is a group homo from M to M .

i.e. $\phi_a \in \text{End } M$

Now, we can define a map (say f)

$$f : R \rightarrow \text{End } M$$

defined as

$$(a)f \mapsto \phi_a \quad \forall a \in R \text{ and } \phi_a \in \text{End } M$$

Now, We'll show that f is a ring homomorphism.

(A)

$$\begin{aligned} (a + b)f &= \phi_{a+b} \\ &= \phi_a + \phi_b \\ &= (a)f + (b)f \end{aligned} \quad \begin{aligned} &\text{for each } x \in M \text{ we have,} \\ (x)\phi_{a+b} &= x(a + b) = xa + xb \\ &= (x)\phi_a + (x)\phi_b \\ \therefore \phi_{a+b} &= \phi_a + \phi_b \end{aligned}$$

(B)

$$\begin{aligned} (ab)f &= \phi_{ab} \\ &= \phi_a \cdot \phi_b \\ &= (a)f (b)f \end{aligned} \quad \begin{aligned} &\text{for each } x \in M \text{ we have,} \\ (x)\phi_{ab} &= x(ab) = (xa)b \\ &= (xa)\phi_b = (x)\phi_a \cdot \phi_b \\ \therefore \phi_{ab} &= \phi_a \cdot \phi_b \end{aligned}$$

(C)

$$(1)f = \phi_1 \quad \text{for each } x \in M \text{ we have,}$$

Thus, Forward Part is proved.

$$\begin{aligned} (x)\phi_1 &= x(1) \\ &= x \end{aligned}$$

$\therefore \phi_1$ is identity of $\text{End } M$

(Converse Part) Assume that \exists a ring homo. (say f)

$$f : R \xrightarrow[\text{Homo}]{\text{Ring}} \text{End } M$$

for any $a \in R$, we denote the $(a)f$ by $f_a \in \text{End } M$

Claim: M is a right R -module.

so let's define a map

$$R \times M \longrightarrow M$$

defined by

$$(a, x) \mapsto x * a = (x)f_a$$

to prove M is a right R -module, we need to verify four properties (1.5)- (1.9) of right R -module.

(i)

$$\begin{aligned} (x + y) * a &= (x + y)f_a \\ &= (x)f_a + (y)f_a \\ &= x * a + y * a \end{aligned} \quad \because f_a \in \text{End } M$$

(ii)

$$\begin{aligned} x * (a + b) &= (x)f_{a+b} \\ &= (x)(f_a + f_b) \\ &= (x)f_a + (x)f_b \\ &= (x * a) + (x * b) \end{aligned} \quad \because f_a, f_b \in \text{End } M$$

(iii)

$$\begin{aligned} x * (ab) &= (x)f_{ab} \\ &= (x)f_a \circ f_b \\ &= (xf_a)f_b \\ &= (x * a) * b \end{aligned}$$

(iv)

$$x * 1 = (x)f_1 = x$$

Thus, M is a right R -module. ■

$\because f_1$ is identity in $\text{End } M$

Definition 1.2 (Anti-Ring Homomorphism). Let R and S be two rings with identities 1_r and 1_s respectively. Define a map f

$$f : R \rightarrow S$$

satisfying following properties, for each $a, b \in R$

(i)

$$(a + b)f = (a)f + (b)f$$

(ii)

$$(ab)f = (b)f (a)f$$

(iii)

$$(1_r)f = 1_s$$

Then, f is called anti-ring homomorphism.

Theorem 1.3. Let R be a ring and M be any abelian group with addition. then M is a left R -module if and only if there exists a map which is anti-ring homomorphism from R to $\text{End } M$.

Proof. Left to reader. ■

$$\begin{array}{c} M \text{ is a left } R\text{-module} \\ \Downarrow \\ \exists f : R \xrightarrow[\text{Homo}]{\text{Anti-Ring}} \text{End } M \end{array}$$

Submodules

Definition 1.3 (SubModule). Let M be a left (right) R -module then a subset N of M is called a submodule of M if N is a left (right) R -module under the operation induced from M .

In other words, A subset N of M is called submodule of M if

- (i) N is subgroup of M .
- (ii) N is closed under induced scalar multiplication from M .

Theorem 1.4 (Criterion for Checking Modules). Let M be a left (right) R -module and N be a subset of M then N is a submodule of M if and only if

(i)

$$x - y \in N$$

$$\forall x, y \in N$$

(ii)

$$ax \in N$$

$$\forall a \in R \ \& \ x \in N$$

Proof. Left to reader. ■

Examples:

1. As every Vector Space V over a Field F is a F -module. So, submodules of V are subspaces of V .
2. As every abelian group G is a \mathbb{Z} -module. So, all subgroups of G are submodules.
3. Let R be a ring then R is a left as well as right R -module then left (right) ideals of R are left (right) submodules of R .
4. $\{0\}$ and M are trivial submodules of any left (right) R -module M .

Remark: 7.

1. Union of two submodules need not to be a submodule.
2. Intersection of any number of submodules is again a submodule.

Think an example !

Hint: Verify using criterion for checking modules.

Remark: 8. (Smallest Submodule containing a set)

Let M be any left (right) R -module and S be any subset of M . Suppose \mathcal{F} be the family of all submodules of M containing S .

$$\text{Let } P = \bigcap_{N \in \mathcal{F}} N$$

then P is a submodule of M containing S as being intersection of an indexed family of submodules containing S .

Moreover, P is the smallest submodule of M containing S . i.e. for any arbitrary submodule $K \in \mathcal{F}$, we have $P \subseteq K$. Such submodule P of M is said to be generated by set S and is denoted by

$$P = \langle S \rangle = (S)$$

Remark: 9.

Let S be any subset of left R -module M and $\langle S \rangle$ is the smallest submodule of M containing S .

1. if S is non-empty and finite, $S = \{x_1, x_2, x_3, \dots, x_n\}$

$$\langle S \rangle = \langle \{x_1, x_2, x_3, \dots, x_n\} \rangle = \langle x_1, x_2, x_3, \dots, x_n \rangle$$

is said to be a finitely generated by S and is smallest submodule of M containing S .

2. if $S = \phi$ i.e. S is an empty set

$$\langle S \rangle = \langle \phi \rangle = \{0\}$$

3. if $S = \{a\}$ i.e. S is singleton then $\langle S \rangle = \langle a \rangle$ is said to be a cyclic submodule.

Definition 1.4 (Cyclic module). A module M is said to be a cyclic module if it can be generated by a single element.

For Example: A ring R over itself is a module and can be generated by identity element $\{1\}$ so is a cyclic module.

Theorem 1.5. *Let M be left R module and S being any subset of M .*

$$\langle S \rangle = \begin{cases} \{0\} & \text{if } S = \phi \\ \left\{ \sum_{i \in J_n} a_i x_i \mid a_i \in R, x_i \in S \right\} & \text{otherwise} \end{cases}$$

Proof. Case-I Let us suppose that $S = \phi$, as $\langle S \rangle$ is the intersection of all the submodules of M containing S .

In particular, $\{0\}$ also contains S i.e.

$$\{0\} \in \mathcal{F}$$

so,

$$\begin{aligned} \langle S \rangle &= \bigcap_{N \in \mathcal{F}} N \\ &= \{0\} \end{aligned}$$

■

Stay Tuned for next chapters!

