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MODULE THEORY

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Contents

There are few persons without whom this was impossible.

They deserve credit for it so i would love to thank them.

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Dedicated to my family and my best friend

“Neeraj K. Gaud„

Introduction

Warning :

This is my first document created using latex so it may be possible that there are several errors. if you notice any error then you can report it here.

https://github.com/sirkapil/module-theory/issues/new[1]

[1] (may require a github account)

About :

This sample book discusses the course "Module Theory" being taught to Post-Graduate (M.Sc. Mathematics) students in Department of Mathematics under University of Delhi, Delhi.

All my LaTeX documents are free and open-source. Each document is hosted in a github repository and can be found pinned here.

https://github.com/sirkapil

Contribution :

If you find my work useful and want to contribute then you are welcome by heart.

Any suitable changes to document repository through pull requests are highly appreciated. You can create a new pull request here. Be sure to read *contribution file* in root/.github folder of repository before creating any pull-request.

https://github.com/sirkapil/module-theory/compare

If you don't have a github account or facing difficulty in creating a pull-request , then feel free to drop down a message here about that you are interested in contribution of this project.

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Introduction to Modules

Definition of Module

Definition 1.1 (Left Module). *Let R be a ring with identity and M be an abelian group with addition. We say M is a left $R-$module if there exists a mapping*[2]

$$R \times M \to M$$

defined by

$$(a\ ,\ x) \to ax$$

$\forall\ a \in\ R$ and $x \in\ M$

satisfying following properties :

$$(a+b)x = ax + bx \tag{1.1}$$
$$a(x+y) = ax + ay \tag{1.2}$$
$$(ab)x = a(bx) \tag{1.3}$$
$$1x = x \tag{1.4}$$

$\forall\ a\ ,\ b \in R$
$x\ , y\ \in M$

and denoted by $_RM$

[2] often called as scaler multiplication.

Definition 1.2 (Right Module). *Let R be a ring with identity and M be an abelian group with addition. We say M is a right $R-$module if there exists a mapping*

$$M \times R \to M$$

defined by

$$(x\ ,\ a) \to xa$$

$\forall\ a \in R$ and $x \in M$

satisfying following properties :

$$x(a+b) = xa + xb \tag{1.5}$$
$$(x+y)a = xa + ya \tag{1.6}$$
$$x(ab) = (xa)b \tag{1.7}$$
$$x1 = x \tag{1.8}$$

$\forall\ a\ ,\ b \in R$
$x\ , y\ \in M$

and denoted by M_R.

Examples :

1. Let V be a vector space over a field F then V is a left as well as right $F-$Module.

2. Let G be any abelian group under addition , then G is a $\mathbb{Z}-$Module where $\mathbb{Z}$ is set of integers.

3. Let R be ring and $M = R[x]$ where $R[x]$ is a group of all polynomials with coefficents in R then M is a left as well as a right $R-$Module with scaler multiplication being usual multiplication.

Suppose ring R is a field then $R-$Module $R[x]$ is a vector space over field R.

4. Let M be collection of all $m \times n$ matrices over ring R , then M is left $R-$Module where scaler multiplication being usual multiplication of a scaler to a matrix.

   In particular, if M is a set of $1 \times n$ matrices over R or $M = R^n$(set of $n-$tuples) then R^n is a left $R-$module.

Remark: 1. *Let R be a commutative ring then every left $R-$module can be transformed to right $R-$module and vice-versa.*

Proof. Let M be left $R-$module and R be a commutative ring.
so, $\exists$ a mapping

$$R \times M \to M$$

defined by

$$(a\ ,\ x) \to ax$$

for each $a \in R$ and $x \in M$ satisfying following properties :

$\forall\ a\ ,\ b \in R$ and $x\ , y\ \in M$

$$\begin{aligned} (a+b)x &= ax+bx \\ a(x+y) &= ax+ay \\ (ab)x &= a(bx) \\ 1x &= x \end{aligned}$$

$\because$ R is a commutative ring.
Now, Define an another mapping

$$M \times R \to M$$

defined by

$$(x\ ,\ a) \to x * a\ =\ ax$$

To check M is a right $R-$Module , we need to verify properties number **??**-(1.8) *(i) Distribuitive Law*

$$\begin{aligned} x * (a+b) &= (a+b)x \\ &= ax + bx \\ &= (x*a) + (x*b) \end{aligned}$$

(ii) Distributive Law

$$\begin{aligned} (x+y) * a &= a(x+y) \\ &= ax + ay \\ &= (x*a) + (y*a) \end{aligned}$$

(iii)

$$\begin{aligned} x * (ab) &= (ab)x \\ &= (ba)x \\ &= b(ax) \\ &= (ax) * b \end{aligned}$$

(iv)

$$\begin{aligned} x * 1 &= 1x \\ &= x \end{aligned}$$

Thus, ${}_RM$ is transformed to M_R.

Similarly, Converse statement can be verified.

■

Remark: 2. *Let S be a subring of ring R then ${}_SM$ exists only if ${}_RM$ exists.*

by existance means M is a valid left module over mentioned ring or subring. i.e. satisfying those four properties.

Remark: 3. *Same Abelian group can have the structure of a Module for a number of different rings.*

For Instance, The field $\mathbb{R}$ is $\mathbb{R}-$module,$\mathbb{Q}-$module and $\mathbb{Z}-$module.

Remark: 4. *Let I be left ideal of R then quotient ring R/I is a left R-module.*

verification: Left to reader

Hint: you need to verify those four properties: (1.1)-(1.4) ■

Here scaler multiplication is

$$R \times R/I \rightarrowtail R/I$$

defined as

$$(a\ ,\ x+I) \rightarrowtail ax + I$$

$\forall\ a \in R$ and $\forall\ x + I \in R/I$

Theorem 1.1. ***(Elementry Properties:)***

Let M be a left R-module . Suppose 0_m and 0_r denotes additive identities of M and R respectively. Then, for each $x \in M$ and $r \in R$

(i)

$$0_m = 0_r\ x = r\ 0_m$$

(ii)

$$r(-x) = (-r)x = -rx$$

Proof. *(i)* As 0_m is the additive identity of M. so, $0_m = 0_m + 0_m$

Consider $r(0_m + 0_m) = r\ 0_m = r\ 0_m + 0_m$

but, $r(0_m + 0_m) = (r\ 0_m) + (r\ 0_m)$

so, we have

$$r\ 0_m + r\ 0_m = r\ 0_m + 0_m$$

$\because\ (r\ ,\ 0_m) \rightarrowtail r\ 0_m \in M$
so, $r\ 0_m = r\ 0_m + 0_m$
$\because\ M$ is a left R-module.
(using distribuitive property)

as $(M, +)$ is an abelian group so left and right cancellation law holds.

$$\begin{aligned} \cancel{r\ 0_m} + r\ 0_m &= \cancel{r\ 0_m} + 0_m \\ r\ 0_m &= 0_m \end{aligned}$$

a similiar argument can be used to prove $0_m = 0_r\ x$.

(ii) as M is a left R-module so $(r\ , x) \rightarrowtail rx \in M$

Now, Consider $(-r)x + rx$

using distribuitive law

$$\begin{aligned} (-r)x + rx &= (-r + r)x \\ &= 0_r\ x \\ &= 0_m \end{aligned}$$

i.e. $(-r)x$ is additive inverse of (rx) but additive inverse of (rx) is $-rx$ and it is unique for an abelian group(M here)

$$\therefore\ (-r)x = -rx$$

a similar argument can be used to prove that $r(-x) = -rx$.

∎

Definition 1.3 (Ring Homomorphism). *Let R and S be two rings with identities 1_r , 1_s respectively then a map(say f)*

often called as ring homo

$$f : R \to S$$

is said to be a ring homomorphism or ring linear map if for every a , $b \in R$ following properties holds

if R = S then we call ring homo as ring endomorphism. For instance , let f be ring homo from R to R . we say f is endomorphism of R and denoted by *End R* *(i) Preserves Addition*

$$(a+b)f = (a)f + (b)f$$

(ii) Preservers Multiplication

$$(ab)f = (a)f.(b)f$$

(iii) Maps identity to identity

$$(1_r)f = 1_s$$

Remark: 5. *Such a mapping need not to be bijective. if it is bijective then we say it is a ring isomorphism or rings are isomorphic.*

Theorem 1.2. *Let R be a ring and M be any abelian group with addition. then M is a right R-module if and only if there exists a map which is ring homomorphism from R to $End\ M$*

M is a right R-module

$\Updownarrow$

$\exists\ f : R \xrightarrow[\text{Homo}]{\text{Ring}} End\ M$

Proof. (Forward Part) Let us suppose that M is a right R-module.

Claim: there exists a map which is ring homomorphism from R to $End\ M$

$\because$ M is a left R-module , so there exist a map

$$f : M \times R \rightarrowtail M$$

defined by

$$(x\ , a) \rightarrowtail ax$$

satisfying following properties:

$$\begin{aligned} (x+y)a &= (x)a + (y)a \\ x(a+b) &= xa + xb \\ x(ab) &= (xa)b \\ x1 &= x \end{aligned}$$

$\forall\ x, y\ \in M$ & $a, b \in R$

for each $a \in R$, define a map(say ϕ_a)

$$\phi_a : M \rightarrowtail\ M$$

such that for each $x \in M$

$$(x)\phi_a = xa \in\ M$$

Now, we'll show that $\phi_a \in\ End\ M$

Let $x, y \in M$

Consider $(x+y)\phi_a$

$$
\begin{aligned}
&= (x+y)a && \text{using defination of } \phi_a \\
&= xa + ya && \text{using (1.9)} \\
&= (x)\phi_a + (y)\phi_a
\end{aligned}
$$

so, ϕ_a preserves addition and is a group homo from M to M.

i.e. $\phi_a \in End\ M$

Now, we can define a map (say f)

$$f : R \rightarrowtail End\ M$$

defined as

$$(a)f \rightarrowtail \phi_a \qquad \forall a \in R \text{ and } \phi_a \in End\ M$$

Now, We'll show that f is a ring homomorphism.

(A)

$$
\begin{aligned}
(a+b)f &= \phi_{a+b} \\
&= \phi_a + \phi_b \\
&= (a)f + (b)f
\end{aligned}
$$

for each $x \in M$ we have,

$$
\begin{aligned}
(x)\phi_{a+b} &= x(a+b) \ = xa + xb \\
&= (x)\phi_a + (y)\phi_b
\end{aligned}
$$

$\therefore\ \phi_{a+b} = \phi_a + \phi_b$

(B)

$$
\begin{aligned}
(ab)f &= \phi_{ab} \\
&= \phi_a \cdot \phi_b \\
&= (a)f\ (b)f
\end{aligned}
$$

for each $x \in M$ we have,

$$
\begin{aligned}
(x)\phi_{ab} &= x(ab) = (xa)b \\
&= (xa)\phi_b = (x)\phi_a \cdot \phi_b
\end{aligned}
$$

$\therefore\ \phi_{ab} = \phi_a \cdot \phi_b$

(C)

$$(1)f = \phi_1$$

for each $x \in M$ we have,

$$
\begin{aligned}
(x)\phi_1 &= x(1) \\
&= x
\end{aligned}
$$

$\therefore\ \phi_1$ is identity of$End\ M$

Thus, Forward Part is proved.

(Converse Part) Assume that $\exists$ a ring homo.(say f)

$$f : R \xrightarrow[\text{Homo}]{\text{Ring}} End\ M$$

for any $a \in R$,we denote the $(a)f$ by $f_a \in End\ M$

Claim: M is a right R-module.

so let's define a map

$$R \times M \longrightarrow M$$

defined by

$$(a, x) \rightarrowtail x * a = (x)f_a$$

to prove M is a right R-module , we need to verify four properties (1.5)- (1.9) of right R-module.

(i)

$$\begin{aligned} (x+y) * a &= (x+y)f_a \\ &= (x)f_a + (y)f_a \\ &= x * a + y * a \end{aligned}$$

$\because\ f_a \ \in End\ M$

(ii)

$$\begin{aligned} x * (a+b) &= (x)f_{a+b} \\ &= (x)(f_a + f_b) \\ &= (x)f_a + (x)f_b \\ &= (x * a) + (x * b) \end{aligned}$$

$\because\ f_a\ ,\ f_b \in End\ M$

(iii)

$$\begin{aligned} x * (ab) &= (x)f_{ab} \\ &= (x)f_a \centerdot f_b \\ &= (xf_a)f_b \\ &= (x * a) * b \end{aligned}$$

(iv)

$$x * 1 = (x)f_1 = x$$

Thus, M is a right R-module.

$\because\ f_1$is identity in$End\ M$

■

Definition 1.4 (Anti-Ring Homomorphism). *Let R and S be two rings with identities 1_r and 1_s respectively. Define a map f*

$$f : R \rightarrowtail S$$

satisfying following properties, for each $a, b \in R$

(i)

$$(a+b)f = (a)f + (b)f$$

(ii)

$$(ab)f = (b)f\ (a)f$$

(iii)

$$(1_r)f = 1_s$$

Then, f is called anti-ring homomorphism.

Theorem 1.3. *Let R be a ring and M be any abelian group with addition. then M is a left R-module if and only if there exists a map which is anti-ring homomorphism from R to $End\ M$.*

M is a left R-module

$\Updownarrow$

$\exists\ f : R \xrightarrow[\text{Homo}]{\text{Anti-Ring}} End\ M$

Proof. Left to reader. ∎

Definition 1.5 (SubModule). *Let M be a left (right) R-module then a subset N of M is called a submodule of M if N is a left (right) R-module under the operation induced from M.*

In other words, A subset N of M is called submodule of M if

(i) N is subgroup of M.

(ii) N is closed under induced scaler multiplication from M.

Theorem 1.4 (Criterion for Checking Modules). *Let M be a left (right) R-module and N be a subset of M then N is a submodule of M if and only if*

(i)

$$x - y \in N$$

$\forall\ x, y \in N$

(ii)

$$ax \in N$$

$\forall\ a \in R\ \&\ x \in N$

Proof. Left to reader. ∎

Examples:

1. As every Vector Space V over a Field F is a F-module. So, submodules of V are subspaces of V.
2. As every abelian group G is a $\mathbb{Z}$-module. So, all subgroups of G are submodules.
3. Let R be a ring then R is a left as well as right R-module then left (right) ideals of R are left (right) submodules of R.
4. $\{0\}$ and M are trivial submodules of any left (right) R-module M.

Remark: 6.

1. *Union of two submodules need not to be a submodule.*

Think an example !

2. *Intersection of any number of submodules is again a submodule.*

Hint: Verify using criterion for checking modules.

Remark: 7. ***(Smallest Submodule containing a set)***

Let M be any left (right) R-module and S be any subset of M. Suppose $\mathcal{F}$ be the family of all submodules of M containing S.

$$\text{Let } P = \bigcap_{N \in \mathcal{F}} N$$

then P is a submodule of M containing S as being intersection of an indexed family of submodules containing S.

Moreover, P is the smallest submodule of M containing S. i.e. for any arbitrary submodule $K \in \mathcal{F}$, we have $P \subseteq K$. Such submodule P of M is said to be generated by set S and is denoted by

$$P = \langle S \rangle = (S)$$

Remark: 8.

Let S be any subset of left R-module M and $\langle S \rangle$ is the smallest submodule of M containing S.

1. *if S is non-empty and finite , $S = \{x_1, x_2, x_3, \cdots, x_n\}$*

$$\langle S \rangle = \langle \{x_1, x_2, x_3, \cdots, x_n\} \rangle = \langle x_1, x_2, x_3, \cdots, x_n \rangle$$

is said to be a finitely generated by S and is smallest submodule of M containing S.

2. *if $S = \phi$ i.e. S is an empty set*

$$\langle S \rangle = \langle \phi \rangle = \{0\}$$

3. *if $S = \{a\}$ i.e. S is singleton then $\langle S \rangle = \langle a \rangle$ is said to be a cyclic submodule.*

Definition 1.6 (Cyclic module). [3] *A module M is said to be a cyclic module if it can be generated by a single element.*

[3] P. M. Cohn. *Basic Algebra.* Springer, 2 edition, 2005. ISBN 978-1-4471-1060-6

For Example: A ring R over itself is a module and can be generated by identity element $\{1\}$ so is a cyclic module.

Theorem 1.5. *Let M be left R module and S being any subset of M.*

$$\langle S \rangle = \begin{cases} \{0\} & \text{if } S = \phi \\ \left\{ \sum_{i \in J_n} a_i x_i \;\middle|\; a_i \in R \,, x_i \in S \right\} & \text{otherwise} \end{cases}$$

Proof. *Case-I* Let us suppose that $S = \phi$,as $\langle S \rangle$ is the intersection of all the submodules of M containing S.

i.e. Every submodule of M will contain S

In particular, $\{0\}$ also contains S i.e.

$$\{0\} \in \mathcal{F}$$

$\because$ $\mathcal{F}$ is a collection of all submodules of M containing S

so,

$$\begin{aligned} \langle S \rangle &= \bigcap_{N \in \mathcal{F}} N \\ &= \{0\} \end{aligned}$$

Case-II Suppose S is non-empty and let

$$P = \left\{ \sum_{i \in J_n} a_i x_i \;\middle|\; a_i \in R \text{ , } x_i \in S \right\}$$

First , we'll show that $S \subseteq P$

Let $x \in \ S$ then it can be expressed in following form:

$$x = 1.x = \sum_{i \in J_1} a_i x_i$$

with $a_1 = 1$ and $x_1 = x$

$$\therefore x \in P \Rightarrow S \subseteq P$$

$\because$ x was chosen arbitirary.

Now ,we'll show that P is a submodule of M using submodule criterion.

Let u , $v \in P$. so, we need to show $u + \alpha v \in P$

for any $\alpha \in R$

$$u = \sum_{i \in J_n} a_i x_i$$

$\forall x_i \in S$ & $a_i \in R$

$$v = \sum_{j \in J_m} b_j y_j$$

$\forall y_j \in S$ & $b_j \in R$

define, for any $\alpha \in R$

$$z_k = x_k \qquad , \qquad c_k = a_k$$

$k \in J_n$

$$z_{k+j} = y_j \qquad , \qquad c_{k+j} = \alpha b_j$$

$j \in J_m$

Thus, we have

$$\begin{aligned} u + \alpha v &= \sum_{i \in J_n} a_i x_i + \alpha \sum_{j \in J_m} b_j y_j \\ &= \sum_{i \in J_n} a_i x_i + \sum_{j \in J_m} \alpha b_j y_j \\ &= \sum_{k \in J_n} c_k z_k + \sum_{k=n+1}^{n+m} c_k z_k \\ &= \sum_{k \in J_{n+m}} c_k z_k \end{aligned}$$

so, P is a submmodule of M containing S.

Now, we'll show that P is smallest submmodule of M containing S.

Let K be any arbitirary submodule of M containing S

$$\text{i.e. } K \in \mathcal{F}$$

$\because$ K is a submodule and $S \subseteq K$

$\therefore$ K is closed under scaler multiplication and addition.

$$\text{i.e} \quad \sum_{i \in J_n} a_i x_i \in K$$

$\forall\, a_i \in R \;\&\; x_i \in S$

so ,

$$P = \langle S \rangle \subseteq K$$

Hence, P is smallest submmodule of M containing S.

■

Definition 1.7 (Generating Set / Set of Generators). *A set of generators for a left (right) R-module M is a subet S of M such that*

$$M = \langle S \rangle$$

if no proper submodule of M contains S then S generates M (verify ?)

Examples

1. A ring R considered as left (right) R-module is generated by identity element $\{1\}$
2. $\mathbb{Z} \times \mathbb{Z}$ over $\mathbb{Z}$ can be generated by
$$S = \{(0,1),(1,0)\}$$
3. All finite dimensional vector space can be generated by it's basis (finite), so is finitely generated submodule.
4. Let R be a ring, I be left(right) ideal of R then it is a left(right) R-module. So, every finitely generated left(right) ideals of R are finitely generated submodule.
5. A submodule of left R-module is cyclic iff it is prinicipal ideal of ${}_RR$.

Definition 1.8. *Let M be a left R-module and $\{N_\alpha\}_{\alpha\in\Omega}$ be family of submodules of M then sum $\sum\limits_{\alpha\in\Omega} N_\alpha$ is defined to be a submodule of M generated by $\bigcup\limits_{\alpha\in\Omega} N_\alpha$* where Ω is indexing set.

$$\left\langle \bigcup_{\alpha\in\Omega} N_\alpha \right\rangle = \sum_{\alpha\in\Omega} N_\alpha$$

Moreover, $\sum\limits_{\alpha\in\Omega} N_\alpha$ is smallest submodule of M containing N_α for each $\alpha \in \Omega$

Proposition 1.6. *Let M be a left R-module and $\{N_\alpha\}_{\alpha\in\Omega}$ be family of submodules of M then sum* where Ω is indexing set.

$$\sum_{\alpha\in\Omega} N_\alpha = \left\{ \sum_{\alpha\in\Omega} x_\alpha \,\middle|\, x_\alpha \in N_\alpha \quad , \; x_\alpha = 0 \textit{ for almost all } \alpha \right\}$$

for each $\alpha \in \Omega$

Proof. Let

$$P = \left\{ \sum_{\alpha\in\Omega} x_\alpha \,\middle|\, x_\alpha \in N_\alpha \quad , \; x_\alpha = 0 \text{ for almost all } \alpha \right\}$$

We need to show that P is smallest submodule of M containing each N_α

Claim 1 :. P is submodule of M
Clearly, P is non-emptpy. Taking $x_\alpha = 0$ for each $\alpha \in \Omega$, we have

$$\Rightarrow 0 \in P$$

Also, for a fixed but arbitirary $\alpha \in \Omega$ taking

$$x_i = \begin{cases} x, & \text{if } i = \alpha \\ 0, & \text{otherwise.} \end{cases}$$

so $x = \sum x_\alpha \in P$ ∎

Stay Tuned for next chapters!

Bibliography

P. M. Cohn. *Basic Algebra*. Springer, 2 edition, 2005. ISBN 978-1-4471-1060-6.