Generalized Fiducial Inference on Differentiable Manifolds

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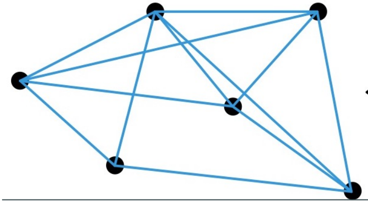
Gaussian Graphical Models

- $X \sim \mathcal{N}_d(\mu, \Lambda)$ where $\Lambda := \Sigma^{-1}$;
- $G = (V, E)$ be an undirected graph, where V and E are the sets of nodes and edges, respectively. Nodes $i, j \in V$ are connected $\leftrightarrow (i, j) \in E$.

X is said to *satisfy the undirected Gaussian Graphical Model with graph G* if $\Lambda_{i,j} = 0$ for all $E_{i,j} = 0$.



$$\Leftrightarrow \begin{pmatrix} 5.61 & 0.80 & 2.33 & 0.03 & 2.18 & 0.00 \\ 0.80 & 6.66 & 0.15 & -0.06 & 1.78 & 3.04 \\ 2.33 & 0.15 & 3.16 & 0.00 & 0.90 & -0.74 \\ 0.03 & -0.06 & 0.00 & 3.93 & 0.00 & 2.58 \\ 2.18 & 1.78 & 0.90 & 0.00 & 4.57 & 2.39 \\ 0.00 & 3.04 & -0.74 & 2.58 & 2.39 & 5.07 \end{pmatrix}^{1}$$

[1] example numbers from Kundu *et al.* (2019)

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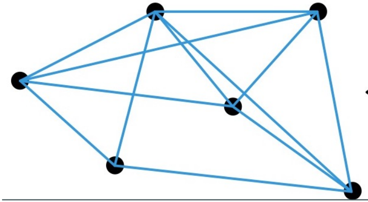

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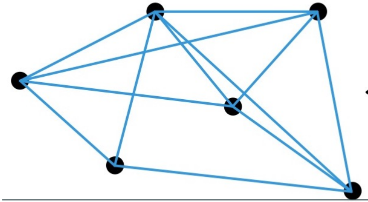

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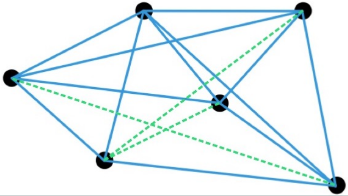

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[2]

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Gaussian Graphical Models

Our work thus far allows us to define a GFD on X with a fixed graph G. We can express

$$\Sigma_{i,j}^{-1} = 0 \leftrightarrow (i,j) \notin E,$$

as the zeros of the function

$$g_{i,j}(\Sigma) = \Sigma_{i,j}^{-1}.$$

Gaussian Graphical Models

Combining this with our previous work on the Multivariate Normal (Murph *et al.*, 2022), we can use the Cayley Transform to decompose

$$g_{i,j}(\Sigma) = \Sigma_{i,j}^{-1}.$$

into

$$g_{i,j}(A, \Lambda) = \Big((I - A)(I + A)^{-1} \Lambda^{-2} (I - A)^{-1} (I + A) \Big)_{i,j}$$

(The full DGA is: $\mathbf{Y}_i = \mu + (I_d + A)(I_d - A)^{-1} \Lambda \mathbf{Z}_i, \quad i = 1, \ldots, m.$)

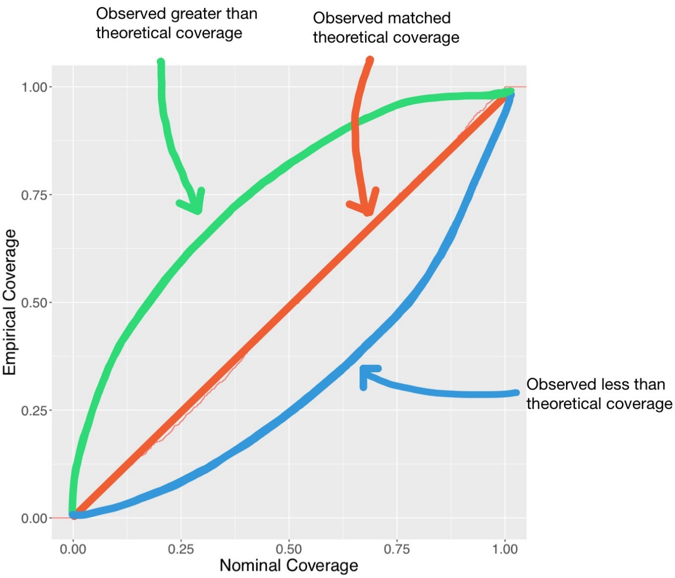
Gaussian Graphical Models

Ex: Let's sample GGMs from the Bayesian posterior and the CGFD for fixed graph structure G (assuming a zero mean).

Bayesian: MVN model with G-Wishart prior; we can sample values directly from the posterior;

Generalized Fiducial: Propose values in a Metropolis-Hastings MCMC set-up using a G-Wishart with a change-of-variables to (A, Λ). Accepted samples are from the CGFD.

Gaussian Graphical Models



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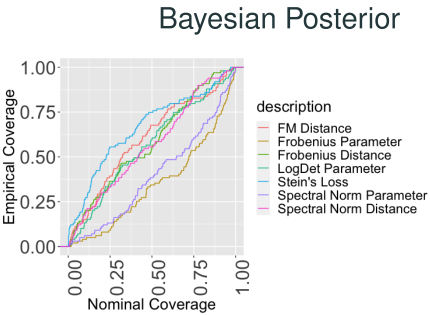

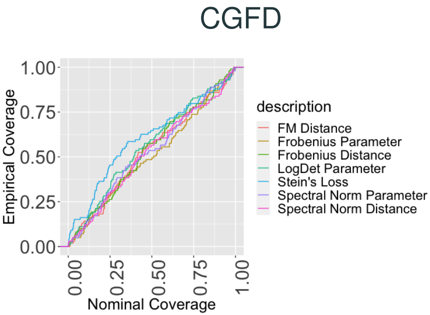

Thanks for your attention!

Questions?

References i

Kundu, S., Mallick, B. K. and Baladandayuthapani, V. (2019) Efficient Bayesian Regularization for Graphical Model Selection. *Bayesian Analysis*, **14**, 449 – 476.

Murph, A. C., Hannig, J. and Williams, J. P. (2022) Introduction to generalized fiducial inference. In *Handbook of Bayesian, Fiducial, and Frequentist Inference*. Chapman & Hall.