Generalized Fiducial Inference on Differentiable Manifolds

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Contents

Generalized Fiducial Inference:

1. Introduction to Generalized Fiducial Inference (GFI) (Murph *et al.*, 2022b)
2. GFI on Differentiable Manifolds (Murph *et al.*, 2022a)
3. A Geometric Perspective on Bayesian and Generalized Fiducial Inference (Liu *et al.*, 2022)

Applied Computational Bayesian:

1. Bayesian Change-point Detection for Process Monitoring with Fault Detection

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These projects are connected!

Applied Computational Bayesian:

1. Bayesian Change-point Detection for Process Monitoring with Fault Detection

Fiducial Inference

[fiducial statistics was R.A. Fisher's] "bold attempt to make the Bayesian omelette without breaking the Bayesian eggs" -J.Savage, 1961

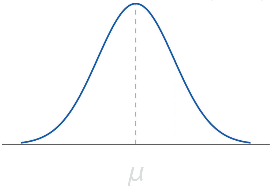
Fiducial Inference

FI Basic Idea:

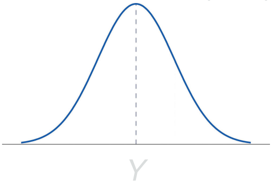
1. Define a structural equation. We will call this a *Data Generating Algorithm* (DGA).
   - Should express how the data were *generated*.
2. Fix the data and invert this equation
3. Get a distribution on the target parameter

Toy Example: a DGA for $Y \sim \mathcal{N}(\mu, 1)$ is $Y = \mu + Z$, $Z \sim \mathcal{N}(0, 1)$.

$$Y = \mu + Z \sim \mathcal{N}(\mu, 1)$$



$$\mu = Y - Z \sim \mathcal{N}(Y, 1)$$



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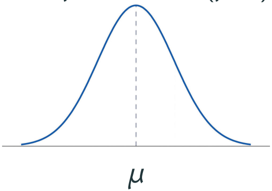

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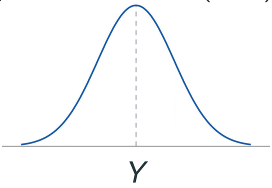

Generalized Fiducial Inference (GFI)

This quickly becomes more difficult...

Example: a DGA for $Y \sim$ Gamma(α, β) is $Y = F^{-1}_{\alpha,\beta}(U)$, $U \sim$ Unif(0,1).

Theorem (Hannig *et al.*, 2016)

For continuous data,

$$r_n(\theta) = \frac{f(\mathbf{y}|\theta)J(\mathbf{y},\theta)}{\int f(\mathbf{y}|\theta')\, J(\mathbf{y},\theta')\, d\theta'} \tag{1}$$

where $J(\mathbf{y},\theta) = D\left(\nabla_\theta A(u,\theta)|_{u=A^{-1}(\mathbf{y},\theta)}\right)$, $A(u,\theta)$ the DGA s.t. $Y = A(u,\theta)$, and $D(M) = (\det M'M)^{\frac{1}{2}}$.

Notes on Notation:

- $A(u,\theta)$ is the DGA, e.g. $A(Z,\mu) = \mu + Z$ from before.
- $f(\mathbf{y}|\theta)$ is the distribution of your observed data.

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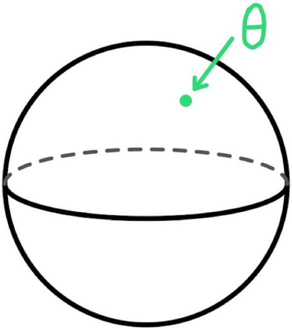
GFI on Constrained Parameter Spaces



$\leftrightarrow$ $g(\theta) = ||\theta|| - 1 = 0$

In general, we say $\theta \in \mathcal{M} \leftrightarrow g(\theta) = 0$ for $g : \mathbb{R}^d \to \mathbb{R}^t$.
Equivalently, $g^{-1}\{0\} = \mathcal{M}$.

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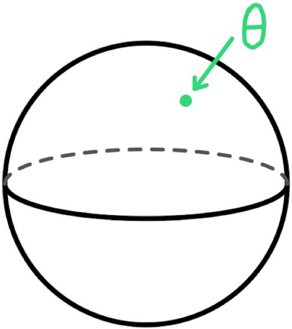

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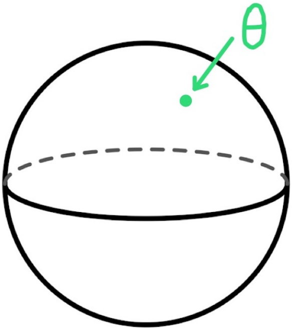

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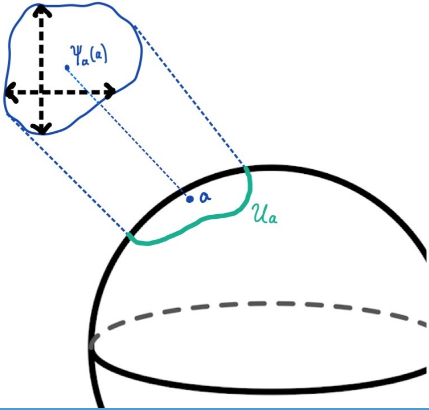
GFI on Constrained Parameter Spaces

Assuming $\nabla_\theta g \neq 0$ everywhere, the level set $\mathcal{M} := g^{-1}\{0\}$ is a **Riemannian Manifold**.

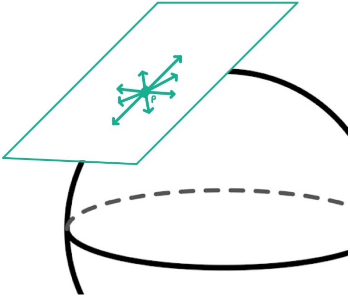


- Locally like $\mathbb{R}^{d-t}$ at every point a on the manifold;
- There exists an inner product on the tangent vectors at a;
- We can meaningfully define the derivative on a manifold.

GFI on Constrained Parameter Spaces

GFD Jacobian term

$$J(\mathbf{y}, \theta) = D\left(\nabla_\theta A(u, \theta)\big|_{u=A^{-1}(\mathbf{y},\theta)} \right).$$



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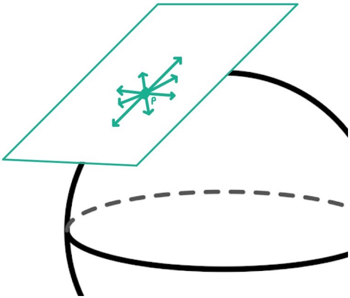
This is no longer a gradient on the *ambient* space and requires more thought.



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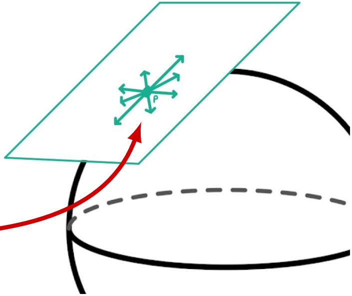
We take the derivative along the tangent directions.



GFI on Constrained Parameter Spaces

First Idea

Project this gradient onto the manifold using

$$P_\theta := I_d - (\nabla_\theta g)'(\nabla_\theta g(\nabla_\theta g)')^{-1}\nabla_\theta g.$$

to get

$$J(\mathbf{y},\theta) = D\left(\nabla_\theta A\,(u,\theta)|_{u=A^{-1}(\mathbf{y},\theta)}\,P_\theta\right).$$

Recall that $D(M) = (\det M'M)^{\frac{1}{2}}$.

But then $J(\mathbf{y},\theta) = 0$ since $\nabla_\theta A\,(u,\theta)|_{u=A^{-1}(\mathbf{y},\theta)}\,P_\theta$ is not full rank!

GFI on Constrained Parameter Spaces

Constrained GFD (CGFD)

$$r_{n,\mathcal{M}}(\theta|\mathbf{y}) = \frac{f(y|\theta)D^*\left(\nabla_\theta A(w,\theta)|_{w=A^{-1}(\mathbf{y},\theta)}P_\theta\right)}{\int_{\mathcal{M}} f(y|\theta^*)D^*\left(\nabla_{\theta^*} A(w,\theta^*)|_{w=A^{-1}(\mathbf{y},\theta^*)}P_{\theta^*}\right)d\lambda(\theta^*)},$$

for $\theta \in \mathcal{M}$, where $D^*(M) = (\operatorname{pdet} M'M)^{\frac{1}{2}}$, the pdet operator denotes the pseudodeterminant, and the integral in the denominator is taken over the manifold $\mathcal{M}$.

- Note that the CGFD is unnecessary when a parameterization to an unconstrained space is known.
- E.g. the CGFD $r_{n,\mathcal{M}}(x,y,z)$ can be defined on the unit sphere, but you could equally define the GFD on this sphere using polar coordinates, $r_n(\varphi,\theta)$.

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This is the Riemann–Lebesgue volume measure

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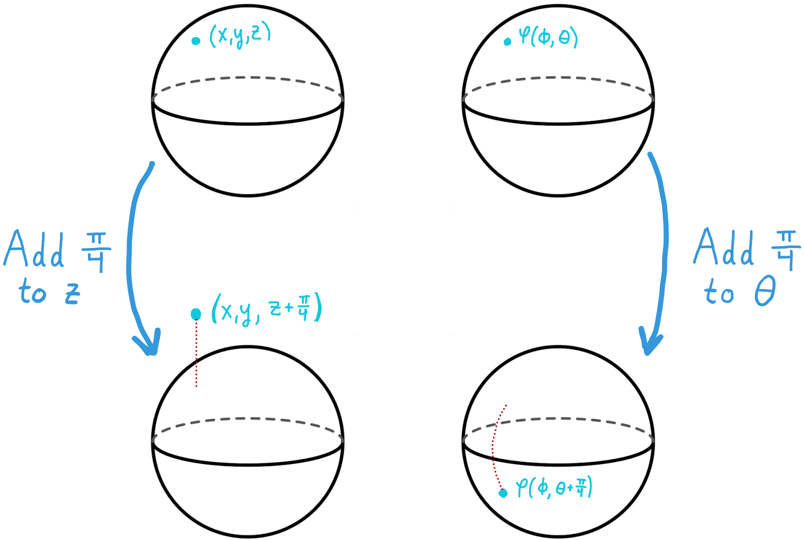

$$\varphi(\varphi, \theta) = (\cos\theta \sin\varphi \ ; \ \sin\theta \sin\varphi \ ; \ \cos\varphi)'$$

Theoretical Qualities of CGFD

"Theorem Idea:" The CGFD is equivalent to the CFD defined using a parameterization.

Theorem (Invariant to parameterization)

- Fix a point $\theta \in \mathcal{M}$ and coordinate chart $(\psi_\theta, \mathcal{U}_\theta)$, such that $\theta \in \mathcal{U}_\theta$.
- Let $V \times W \subset \mathbb{R}^d$ be the open set around the point θ on which the implicit function $\varphi : V \to W$ exists and is diffeomorphic.

Then for any open set $M_\theta \subseteq \mathcal{U}_\theta \cap (V \times W)$,

$$\int_{M_\theta} f(y|\theta^*) D^* \left((\nabla_\theta A(w, \theta^*)) P_\theta \right) d\lambda(\theta^*) = \int_{\psi_\theta(M_\theta)} f(y|\psi_\theta^{-1}(u)) D \left(\nabla_u A(w, \psi_\theta^{-1}(u)) \right) du$$

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Sphere Example

- $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_m \sim \mathcal{N}_3(\mu, I)$ with DGA:

$$\mathbf{Y}_i = \mu + \mathbf{Z}_i, \quad i = 1, \ldots, m,$$

  where $Z_i \overset{i.i.d.}{\sim} \mathcal{N}_3(0, I)$.
- Consider constraint function $g(\mu) = ||\mu|| - 1$, the mean vector on the unit sphere, and $\mu := \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \end{pmatrix}'$.
- We can also parameterize this problem directly with $\mu := \begin{pmatrix} \cos\theta \sin\varphi \\ \sin\theta \sin\varphi \\ \cos\varphi \end{pmatrix}$ and define a fiducial density on θ, φ.

Sphere Example

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix};$$

explicit

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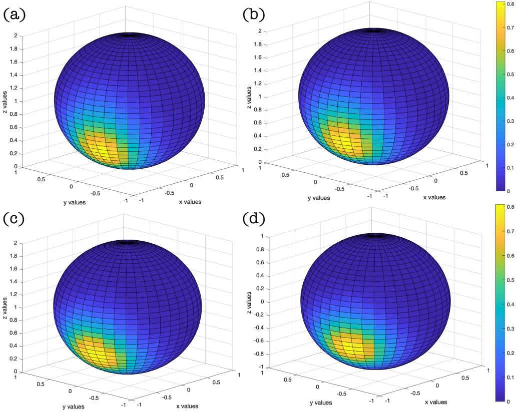

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sampled with HMC

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sampled with MH

Theoretical Qualities of CGFD

Lemma (Invariance of g)

Let $g^{-1}(\{0\}) = h^{-1}(\{0\}) = \mathcal{M}$. Then, for all $\theta \in \mathcal{M}$,

$$r_{n,g^{-1}(\{0\})}(\theta) = r_{n,h^{-1}(\{0\})}(\theta).$$

- E.g. $g(x) = 1 - ||x||$ and $h(x) = c(1 - ||x||)$ for $c \in \mathbb{R} \backslash \{0\}$.
- *Proof idea.* Write P_θ in terms of the implicit function.

Bernstein–von Mises Inheritance

“Theorem Idea:" Local Asymptotic Normality for the ambient GFD is inherited by the CGFD.

Theorem: Local Asymptotic Normality Inheritance

Recall the local parameter: $s = \sqrt{n}(\theta - \theta_0)$. Under technical assumptions,

$$\int_{\mathbb{R}^d} \left| r^*_{\mathbf{n}}(s) - \varphi^*_{0,I(\theta_0)}(s) \right| ds \overset{n}{\to} 0 \ \text{ implies that}$$

$$\int_{\mathcal{M}} \left| r^*_{\mathbf{n},\mathcal{M}}(s) - \varphi^*_{0,I(\theta_0),\mathcal{M}}(s) \right| d\lambda(s) \overset{n}{\to} 0,$$

where $q^*(s) := n^{-d/2} q(n^{-1/2} s + \theta_0)$ for any function q.

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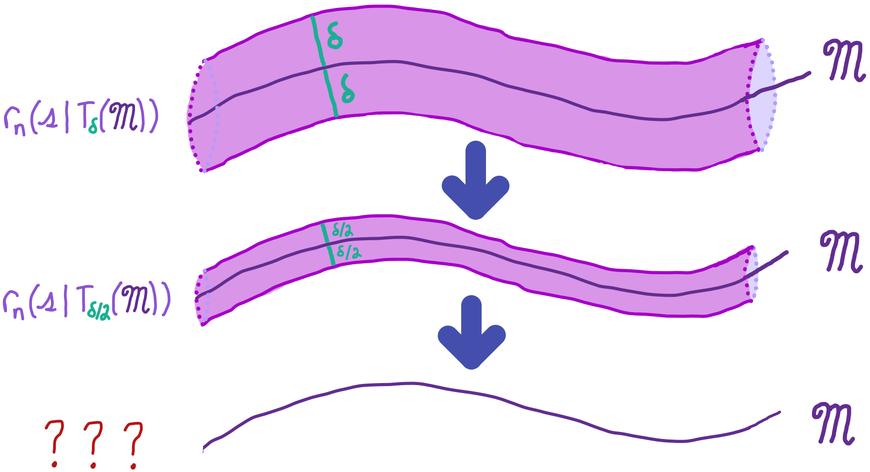
One Idea



$\Gamma_n(s \mid T_\delta(\mathcal{M}))$
δ
δ
$\mathcal{M}$
$\Gamma_n(s \mid T_{\delta/2}(\mathcal{M}))$
$\delta/2$
$\delta/2$
$\mathcal{M}$
???
$\mathcal{M}$

Bernstein–von Mises Inheritance

Theorem (Hwang, 1980)

Under the technical assumptions from Hwang (1980), the limiting density is

$$\frac{dP}{d\lambda}(u) = \frac{r_n(\theta|\mathbf{y}) \det\left(\nabla_\theta g (\nabla_\theta g)^T\right)^{-1/2}}{\int_{\mathcal{M}} r_n(\theta|\mathbf{y}) \det\left(\nabla_\theta g (\nabla_\theta g)^T\right)^{-1/2} d\lambda}.$$

We'll call this the *extrinsic* constrained GFD.

The extrinsic constrained GFD is **not** the same as the CGFD (which, in contrast, is *intrinsic*).

This result requires that the manifold be compact.

Bernstein–von Mises Inheritance

E.g.: Consider the equal means problem: two independent observations X_1, X_2 with DGA,

$$X_i = A(Z_i, \theta) := \mu + \Sigma^{1/2} Z_i, \tag{2}$$

where $\mu = (\mu_1, \mu_2)'$, $\Sigma^{1/2} = \texttt{diag}(\sigma_1, \sigma_2)$, $\theta := (\mu', \sigma_1, \sigma_2)'$, and $Z_i \overset{iid}{\sim} \mathcal{N}_2(0, I_2)$ for $i \in \{1, 2\}$.

1. For $g(\theta) := \mu_2 - \mu_1$, $\det\left(\nabla_\theta g (\nabla_\theta g)^T\right)^{-1/2} = 1$,
2. For $h(\theta) = \mu_2^3 - \mu_1^3$, $\det\left(\nabla_\theta h (\nabla_\theta h)^T\right)^{-1/2} = (9\mu_1^4 + 9\mu_2^4)^{-1/2}$.

Bernstein–von Mises Inheritance

The extrinsic constrained GFD is useful for understanding how ambient qualities are inherited by submanifold densities.

Proof outline:

- Prove that the extrinsic constrained GFD "inherits“ the Bernstein-von Mises property when satisfied by the ambient GFD;
- Show that the extrinsic constrained GFD and the intrinsic CGFD are indistinguishable in the local limit on a compact subset around the truth;
- Assume the intrinsic CGFD decays off of the compact subset of the manifold (satisfied for i.i.d. with bounded DGA Jacobian).

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MAYO CLINIC

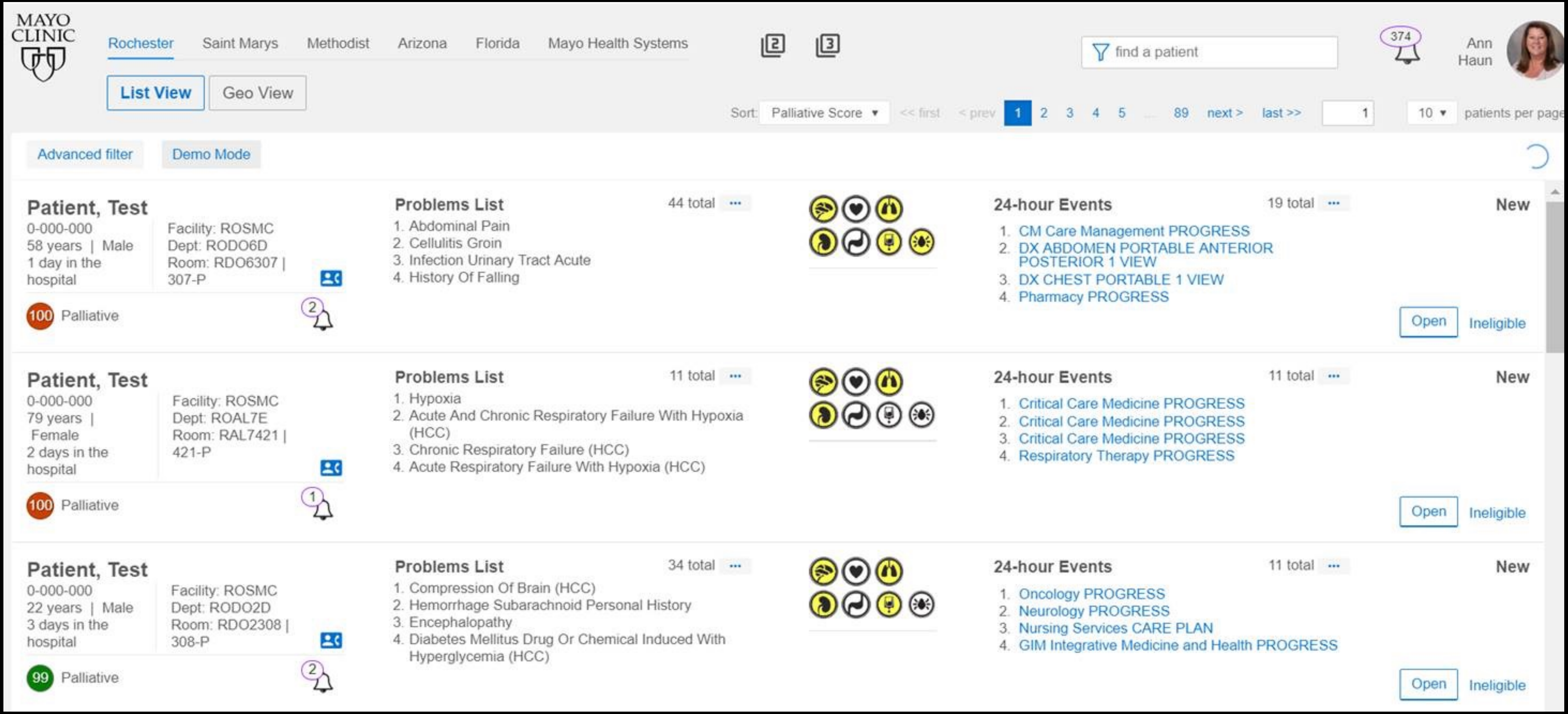
BAYESIAN CHANGE-POINT DETECTION FOR PROCESS MONITORING WITH FAULT DETECTION

KERN CENTER FOR THE SCIENCE OF HEALTH CARE DELIVERY

Alexander C. Murph

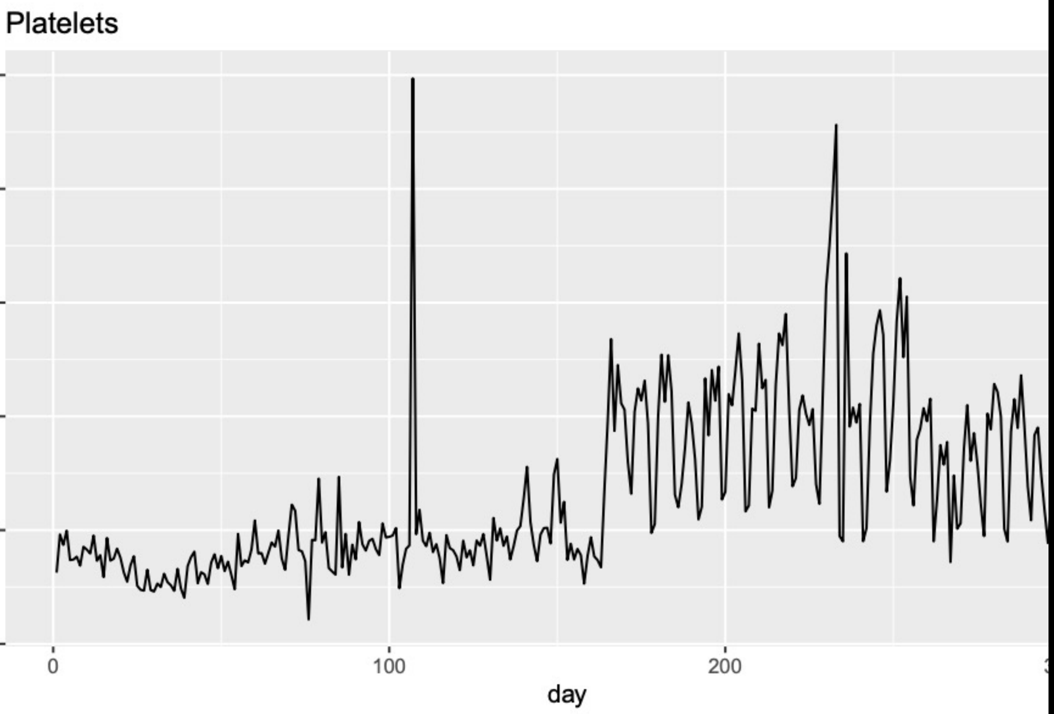
April 21st, 2023

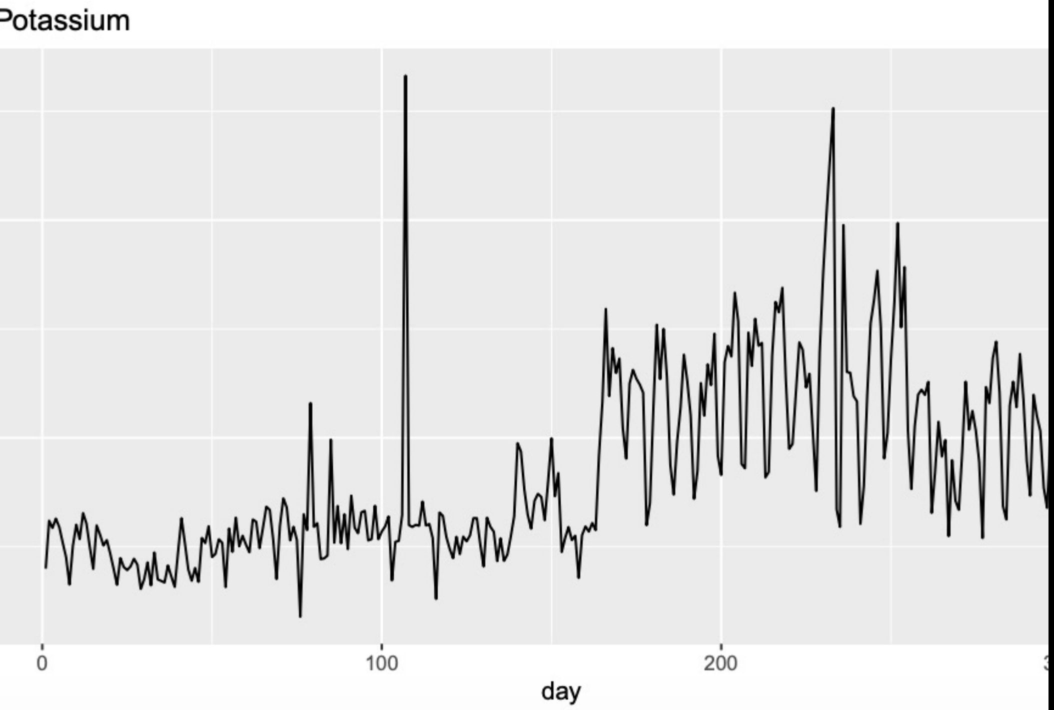
CONTROL TOWER INTERFACE



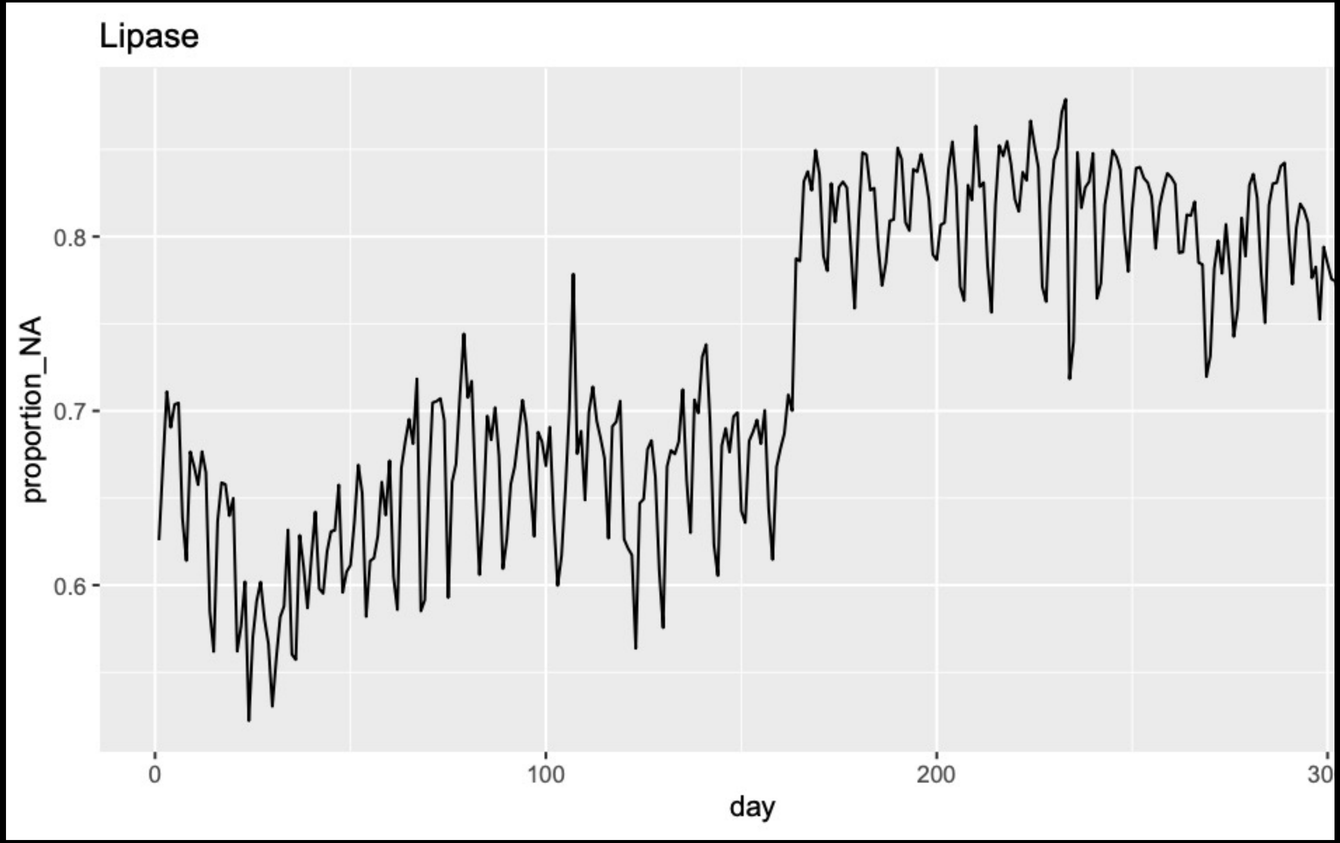
Number in the circle:
"The probability of getting an inpatient palliative care consultation in the next 7 days."

Thank you to Patrick Wilson for sharing this slide!





WELL, WHAT COULD GO WRONG?



A (NOT SO) ELEGANT PROBLEM

(Sub) problems:

- Missing Data
- Mixed Data (discrete, binary, continuous)
- Censored Values

- HUGE data (n in millions, p ~= 250)

(Sub) solutions:

- Bayesian Latent Variables
- Gaussian Graphical Models (GGMs)

(Sub-Sub) problems:

- Lack of information for some priors
- Learning parameters for a GGM (especially the conditional independence graph structure)

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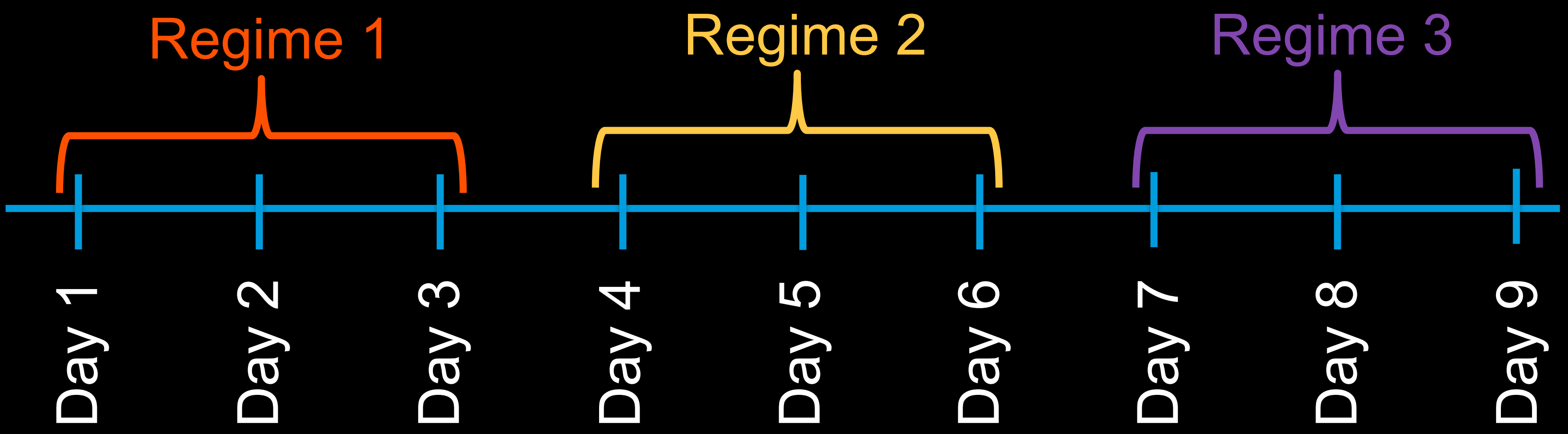
HOW DO WE FIND A CHANGE POINT?

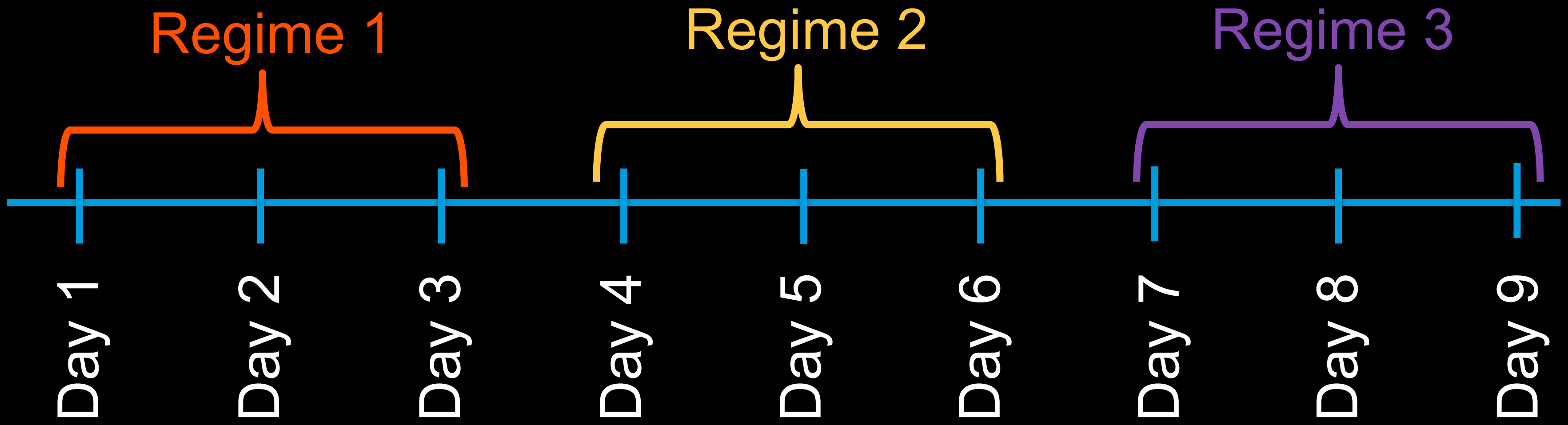
LET'S CLASSIFY OBSERVATIONS OVER TIME INTO "REGIMES"



Data we observed on _____

SUPPOSE WE OBSERVED 9 DAYS OF DATA…

We can *encode* each day's regime assignment as a vector of length 9:



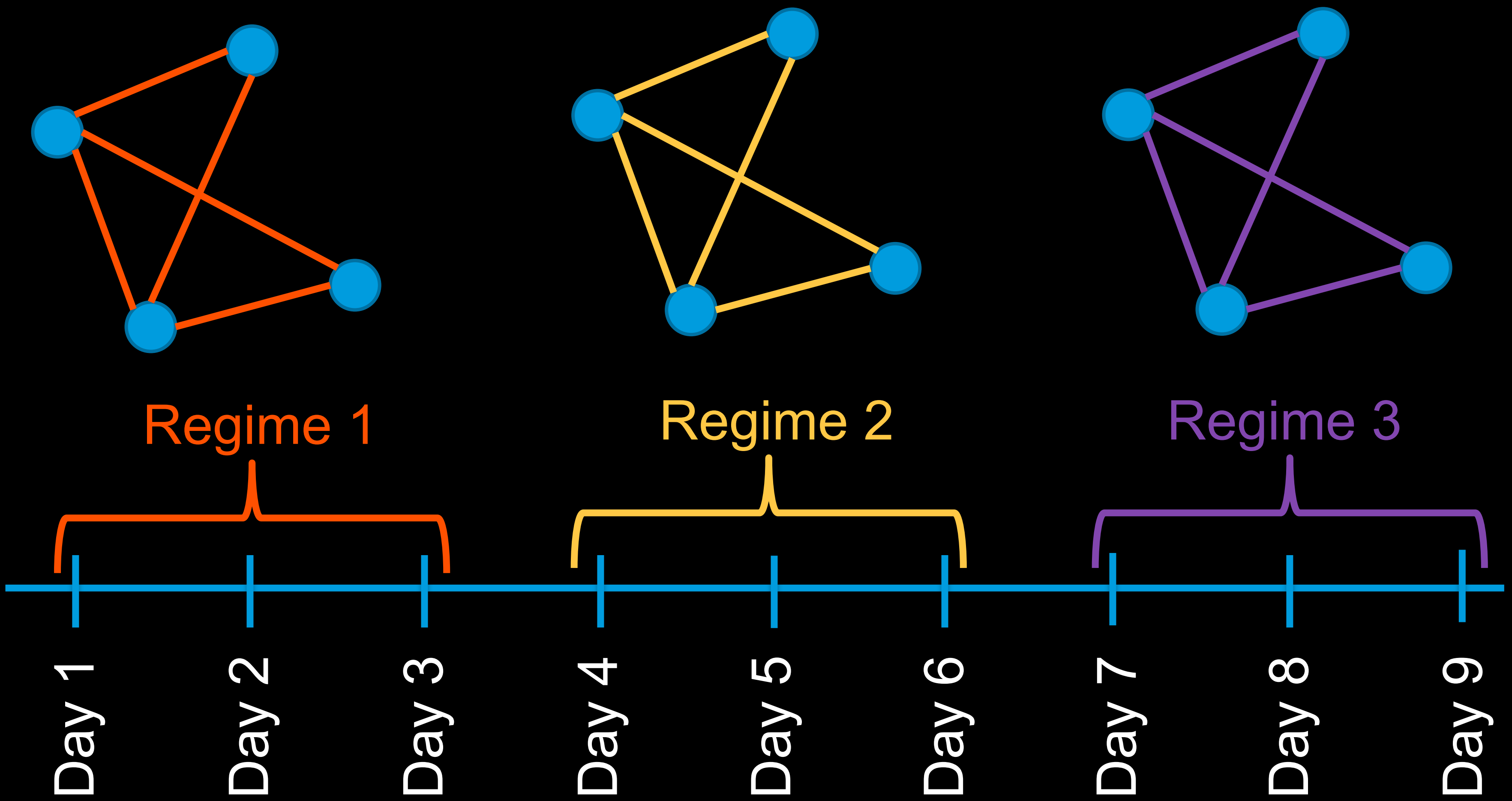
Data we observed on ______

becomes

$$\phi = (1, 1, 1, 2, 2, 2, 3, 3, 3)$$

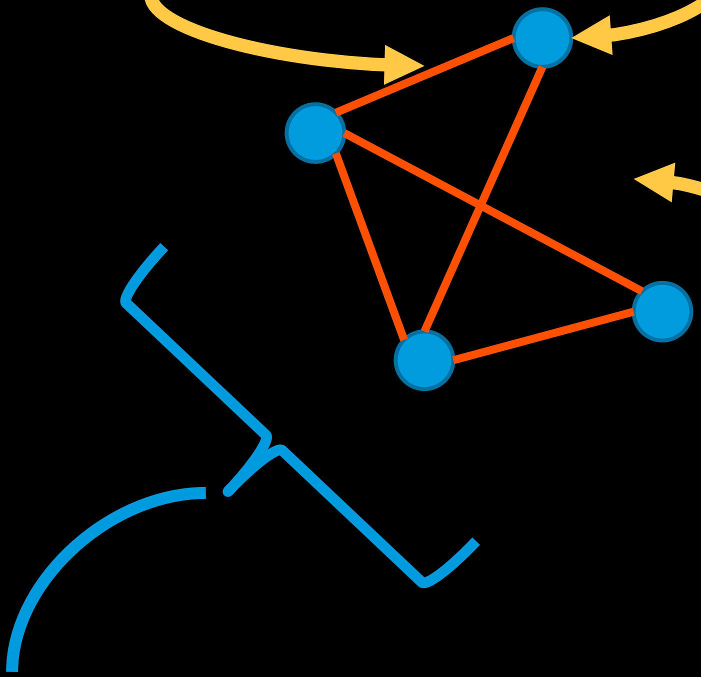
WHAT'S THE LIKELIHOOD?

WE MUST DEFINE THE DISTRIBUTION OF THE DATA *GIVEN* A REGIME VECTOR



Weighted edges correspond to values in the *precision matrix* of a multivariate normal model.

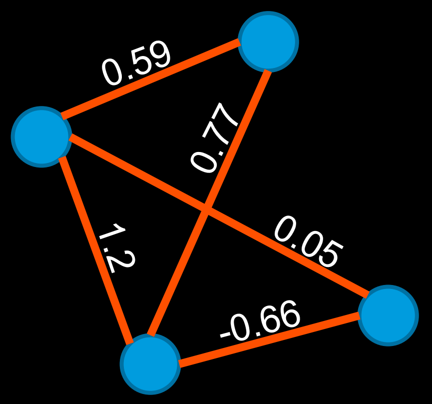
The nodes correspond to the variables in our dataset.



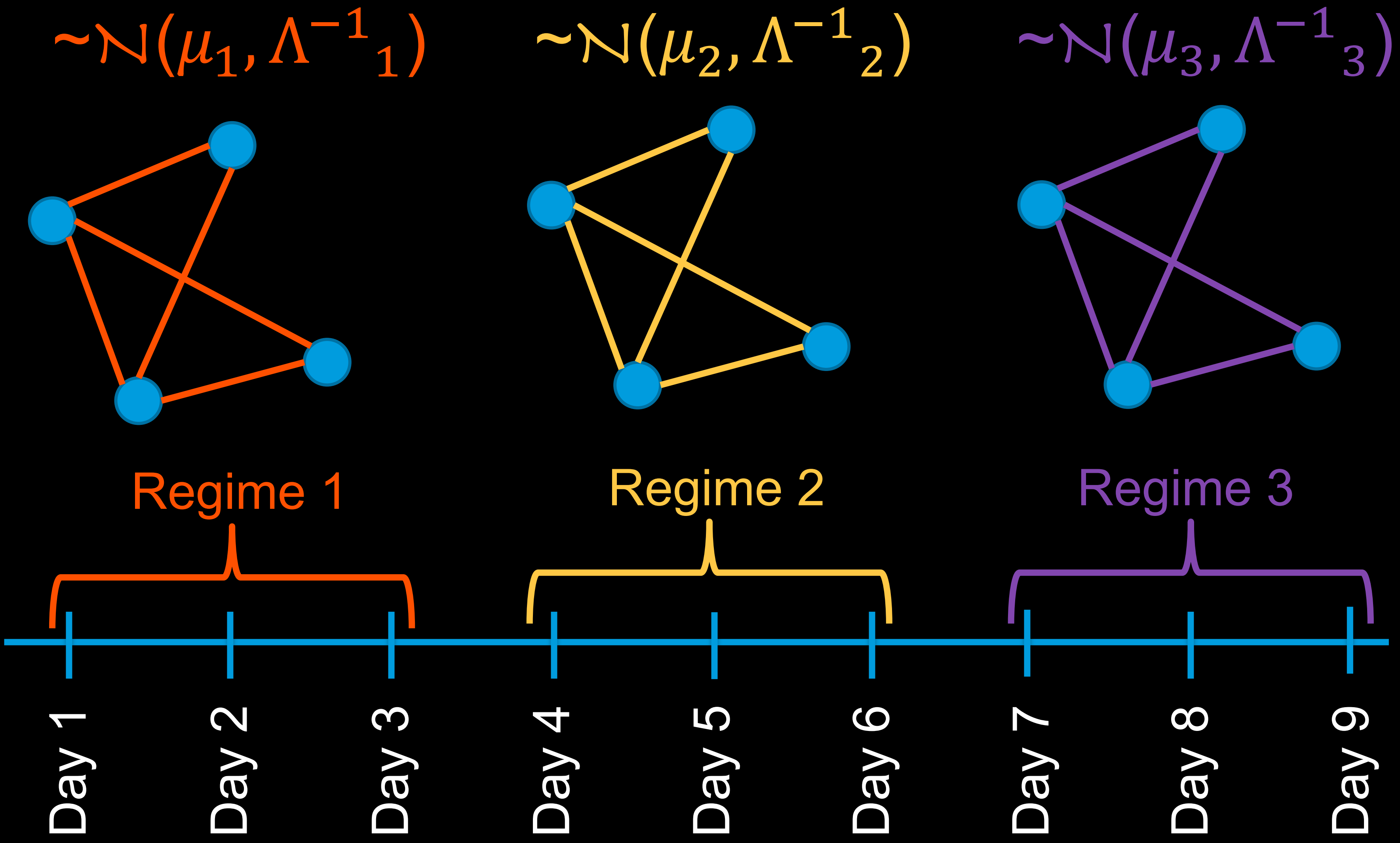
The *absence* of an edge corresponds to a zero in the precision matrix

$$\Lambda, \text{ where } \quad Y \sim \mathcal{N}_4(\mu, \Lambda^{-1})$$

FOR INSTANCE,



$$\Lambda = \begin{bmatrix} 1 & 0.59 & 0.05 & 1.2 \\ 0.59 & 1 & 0 & 0.77 \\ 0.05 & 0 & 1 & -0.66 \\ 1.2 & 0.77 & -0.66 & 1 \end{bmatrix}$$



$\phi = (1, 1, 1, 2, 2, 2, 3, 3, 3)$

INFERENCE ON *THREE* PARAMETERS OF INTEREST

- The set-up on the previous slides admits a likelihood for our data (a product of normal mixture models).
- To get good posterior samples on ϕ, μ, Λ, G, we require:
  - Priors on each parameter;
  - Sampling procedures for proposal values.

SPARSITY ENCODED USING THE G-WISHART

Prior Distributions on the Parameters

$$\Lambda \sim \text{G-Wishart}(\delta, I)$$

(δ, degrees of freedom)

$$\mu | \Lambda \sim \mathcal{N}(\mu_0, \Lambda^{-1})$$

$$(\mu, \Lambda) \sim f_{\mu|\Lambda} f_{\Lambda}$$

$$p(\Lambda := \Sigma^{-1} | G) = \frac{1}{I_G(\delta, D)} (\det \Lambda)^{(\delta-2)/2} \exp\left\{ -\frac{1}{2} \langle \Lambda, D \rangle \right\}$$

★ is a normalizing constant gotten by integrating over a space of precision matrices with fixed zero structure

TO EVALUATE

Laplace Approximation:

(Dobra and Lenkowski 2011)

$$h_{\delta,D}(\Lambda) = -\frac{1}{2}[\mathrm{tr}(\Lambda^T D) - (\delta - 2)\log(\det K)]$$

$$I_G(\hat{\delta}, D) = \exp\left\{h_{\delta,D}(\hat{\Lambda})\right\}(2\pi)^{|V|/2}[\det H_{\delta,D}(\hat{\Lambda})]^{-1/2}$$

(V is the set of free elements in Λ, H is the hessian of h)

When we must calculate the normalizing constant, there exists a fast approximation!

TO SAMPLE

Algorithm 1 Exact sampling from the precision matrix.

Input: A graph $G = (V, E)$ with precision matrix K and $\Sigma = K^{-1}$

Output: An exact sample from the precision matrix.

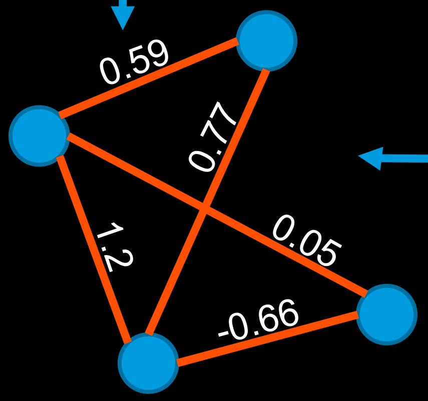
1: Set $\Omega = \Sigma$
2: **repeat**
3: **for** $i = 1, \ldots, p$ **do**
4: Let $N_i \subset V$ be the neighbor set of node i in G. Form Ω_{N_i} and $\Sigma_{N_i,i}$ and solve $\hat{\beta}_i^* = \Omega_{N_i}^{-1}\Sigma_{N_i,i}$.
5: Form $\hat{\beta}_i \in R^{p-1}$ by padding the elements of $\hat{\beta}_i^*$ to the appropriate locations and zeros in those locations not connected to i in G.
6: Update $\Omega_{i,-i}$ and $\Omega_{-i,i}$ with $\Omega_{-i,-i}\hat{\beta}_i$.
7: **end for**
8: **until** convergence
9: **return** $K = \Omega^{-1}$

(algorithm Mohammadi 2019)

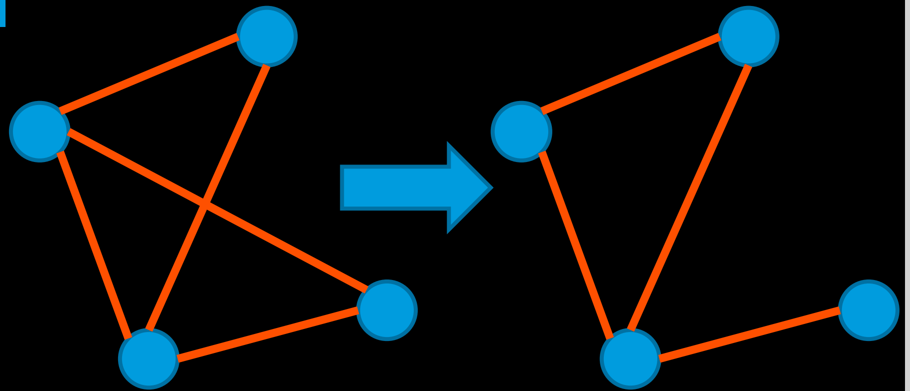
There exists algorithms to sample from a G-Wishart *without* having to calculate the normalizing constant!

Learning these edge weights is a relatively easy problem

Learning when edges are present/absent is **not** easy.



We would like to be able to…



DOUBLE REVERSIBLE JUMP METROPOLIS-HASTINGS MONTE CARLO

LEARNING THE *NUMBER* OF PARAMETERS, WHILE ALSO LEARNING THE PARAMETERS THEMSELVES.

This process combines two neat ideas:

1. Put a sampling distribution on the *Cholesky decomposition* Λ=Ψ`Ψ of the precision matrix Λ;
2. Learn an ancillary grid of values along with the observed graph and precision matrix. Then, propose a new graph and matrix by *swapping.*

$$(\Lambda^d, G^d, \hat{\Lambda}^d, \hat{G}^d)$$

$$(\Lambda^{d+1}, G^{d+1}, \hat{\Lambda}^{d+1}, \hat{G}^{d+1})$$

SO WHERE ARE THE CHANGEPOINTS?

- (1, 1, 1, 2, 2, 2, 3, 3, 3)
- (1, 1, 2, 2, 2, 2, 2, 3, 3)
- (1, 1, 2, 2, 2, 2, 2, 2, 2)
- (1, 1, 2, 2, 3, 3, 3, 3, 3)
- (1, 1, 1, 1, 2, 2, 2, 2, 2)
- (1, 1, 1, 1, 2, 2, 2, 2, 3)
- (1, 1, 1, 1, 2, 2, 2, 2, 2)
- (1, 1, 1, 1, 1, 1, 1, 1, 1)

⋮

SO WHERE ARE THE CHANGEPOINTS?

- (1, 1, 1, 2, 2, 2, 3, 3, 3)
- (1, 1, 2, 2, 2, 2, 2, 3, 3)
- (1, 1, 2, 2, 2, 2, 2, 2, 2)
- (1, 1, 2, 2, 3, 3, 3, 3, 3)
- (1, 1, 1, 1, 2, 2, 2, 2, 2)
- (1, 1, 1, 1, 2, 2, 2, 2, 3)
- (1, 1, 1, 1, 2, 2, 2, 2, 2)
- (1, 1, 1, 1, 1, 1, 1, 1, 1)

⋮

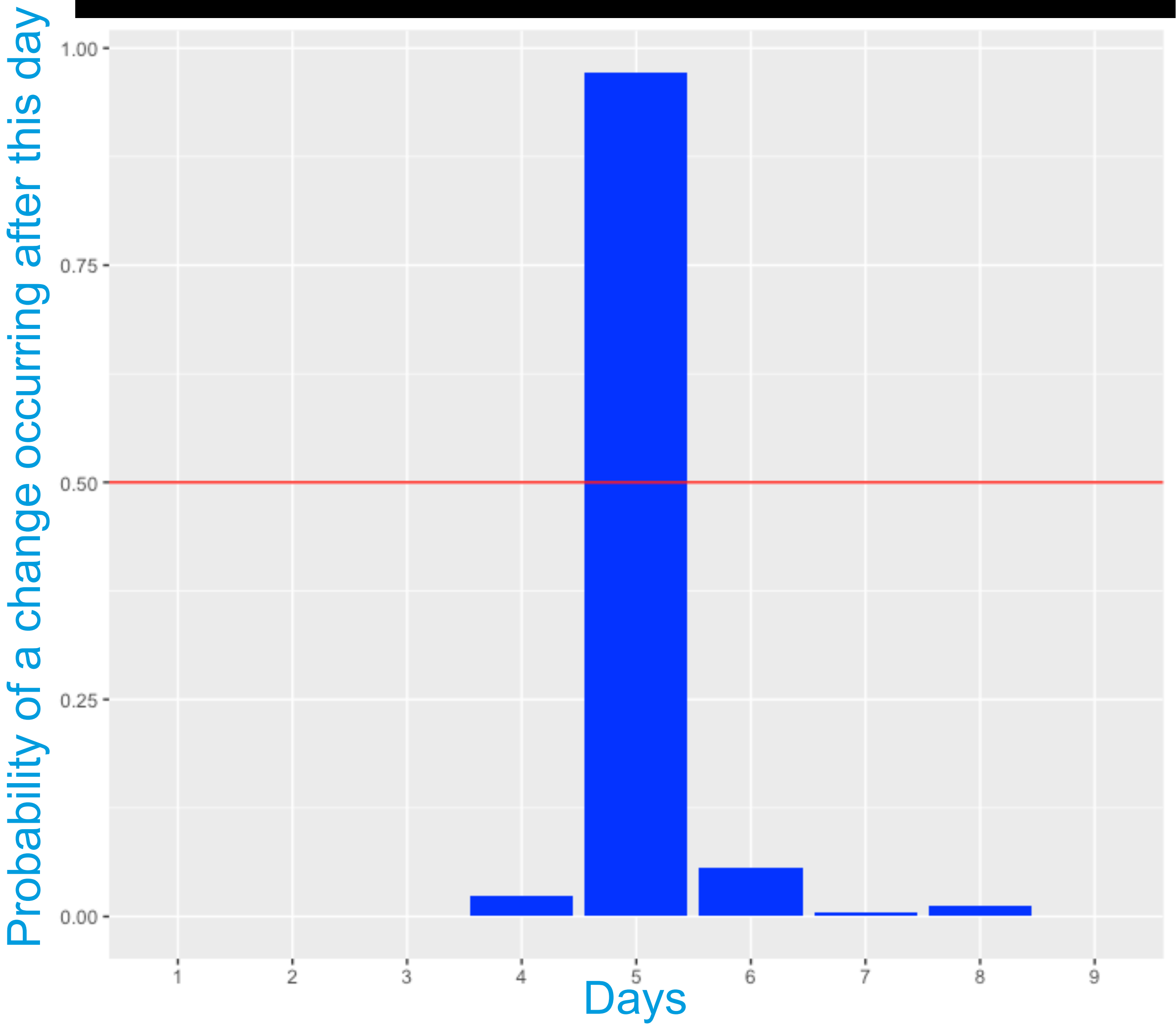
$\Rightarrow$

- (0, 0, 1, 0, 0, 1, 0, 0)
- (0, 1, 0, 0, 0, 0, 1, 0)
- (0, 1, 0, 0, 0, 0, 0, 0)
- (0, 1, 0, 1, 0, 0, 0, 0)
- (0, 0, 0, 1, 0, 0, 0, 0)
- (0, 0, 0, 1, 0, 0, 0, 1)
- (0, 0, 0, 1, 0, 0, 0, 0)
- (0, 0, 0, 0, 0, 0, 0, 0)

⋮

- (0, 0, 1, 0, 0, 1, 0, 0)
- (0, 1, 0, 0, 0, 0, 1, 0)
- (0, 1, 0, 0, 0, 0, 0, 0)
- (0, 1, 0, 1, 0, 0, 0, 0)
- (0, 0, 0, 1, 0, 0, 0, 0)
- (0, 0, 0, 1, 0, 0, 0, 1)
- (0, 0, 0, 1, 0, 0, 0, 0)
- (0, 0, 0, 0, 0, 0, 0, 0)

⋮



COMPETING METHODS

Hotelling T^2 (HT2) Sliding Window

Given four days of data, days 1,2, & 3 are compared against day 4.

$$T*2 = (\bar{y}_1 - \bar{y}_2)' \left(\frac{S_1}{n_1} + \frac{S_2}{n_2} \right)^{-1} (\bar{y}_1 - \bar{y}_2) \sim T^2_{p,v}$$

$\bar{y}_1$ Mean value for days 1, 2, & 3

$\bar{y}_2$ Mean value for day 4

S_1 Sample variance for days 1, 2, & 3

S_2 Sample variance for day 4

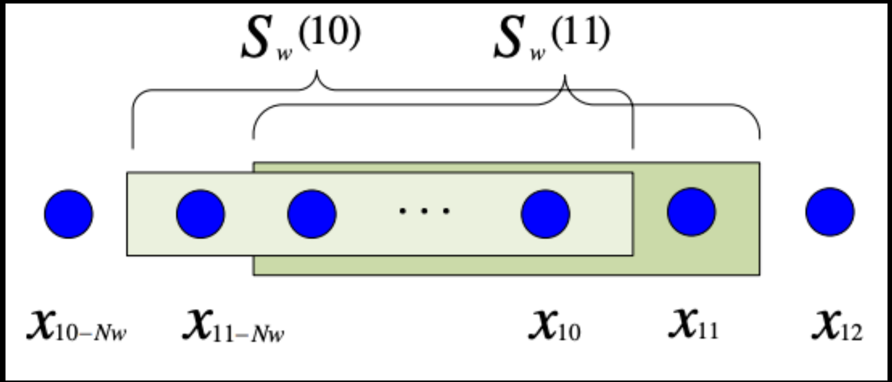
$T^2_{p,v}$ Hotelling T^2 statistic, with pooled degrees of freedom v and number of dimensions p

n_1 Number of observations for days 1, 2, & 3

n_2 Number of observations for day 4

Real Time Contrasts (RTC) with a Gradient Boosting Machine (GBM) Classifier

- Given a classifier $c(x)$ learned to a hold-out sample, does this classifier have significant predictive power?
- We simulate a distribution of AUC values on $c(x)$'s ability to distinguish day 4 from days 1, 2, & 3.
- If 0.5 is NOT in the center 95% confidence region, a changepoint is flagged.




A: Continuous, Has Missing Values, Unimodal, Covariance Change
B: Continuous, No Missing Values, Unimodal, MeanChange
C: Continuous, No Missing Values, Bimodal, Mean Change
D: Missingness Amount Change
Proportion of Simulations Flagged
1.00
0.75
0.50
0.25
0.00
10
14
20
Day Before Possible Regime Change
Method Avg. FP %
A: T2 Scan Stat 0.0592; RTM GBM 0.0485; Mixture Model 0.0592; MVN Model 0.0577
B: T2 Scan Stat 0.0508; RTM GBM 0.0515; Mixture Model 0.0577; MVN Model 0.0546
C: T2 Scan Stat 0.0385; RTM GBM 0.0592; Mixture Model 0.0538; MVN Model 0.0631
D: T2 Scan Stat 0.0615; RTM GBM 0.0546; Mixture Model 0.0608; MVN Model 0.0592

FAULT DETECTION ANALYSIS

Once a changepoint is found, how do we determine what *caused* it?

Let

Dist'n Before: X

Dist'n After: Y

Hellinger Distance: H(X,Y)

and

Marginal Dist'n **w/o** Variable i: X \ i

Marginal Dist'n **of** Variable i: X : i

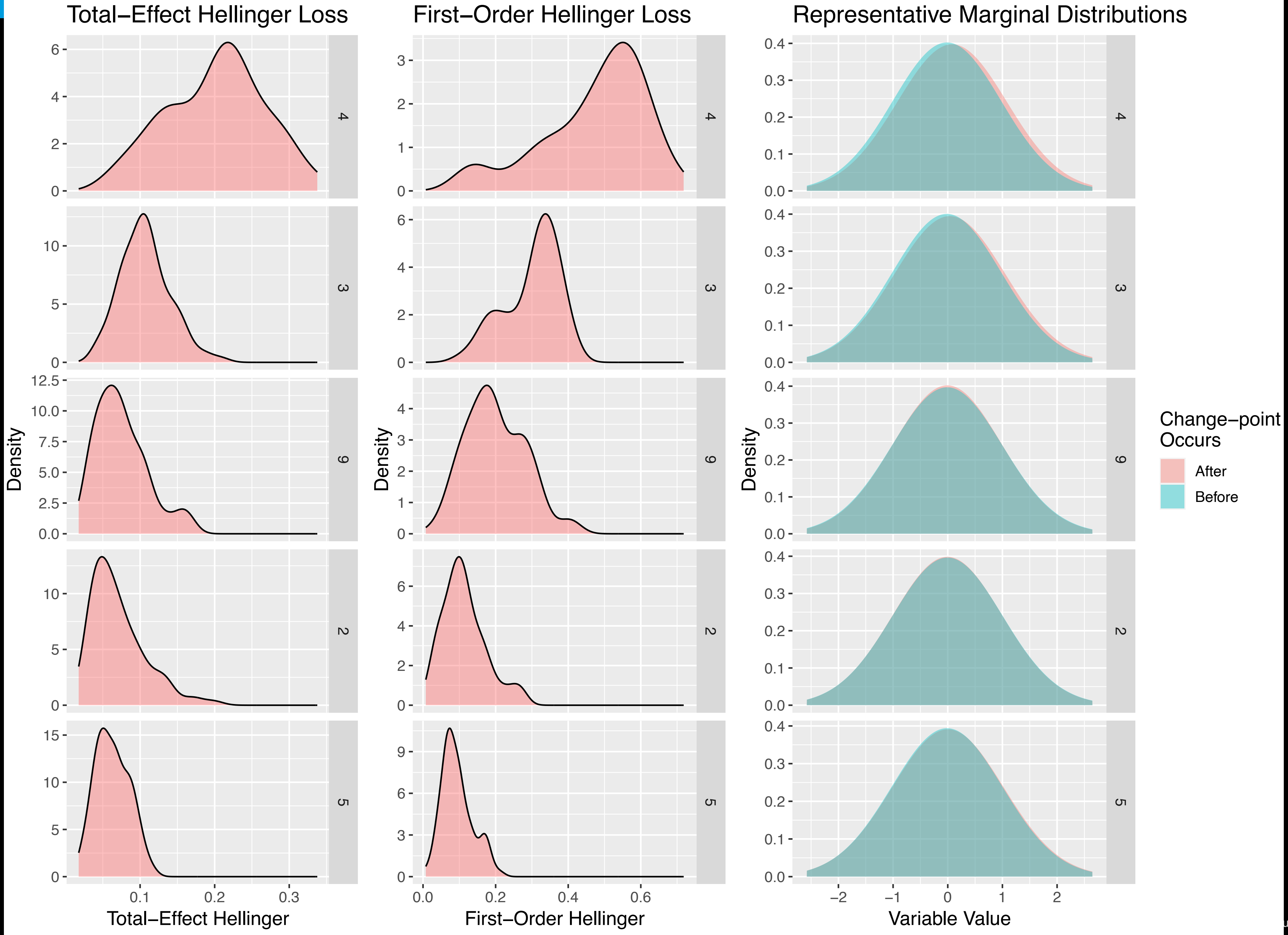
then we Define:

Total-Effect Loss of i: 1 – H(X \ i, Y \ i)/H(X,Y)

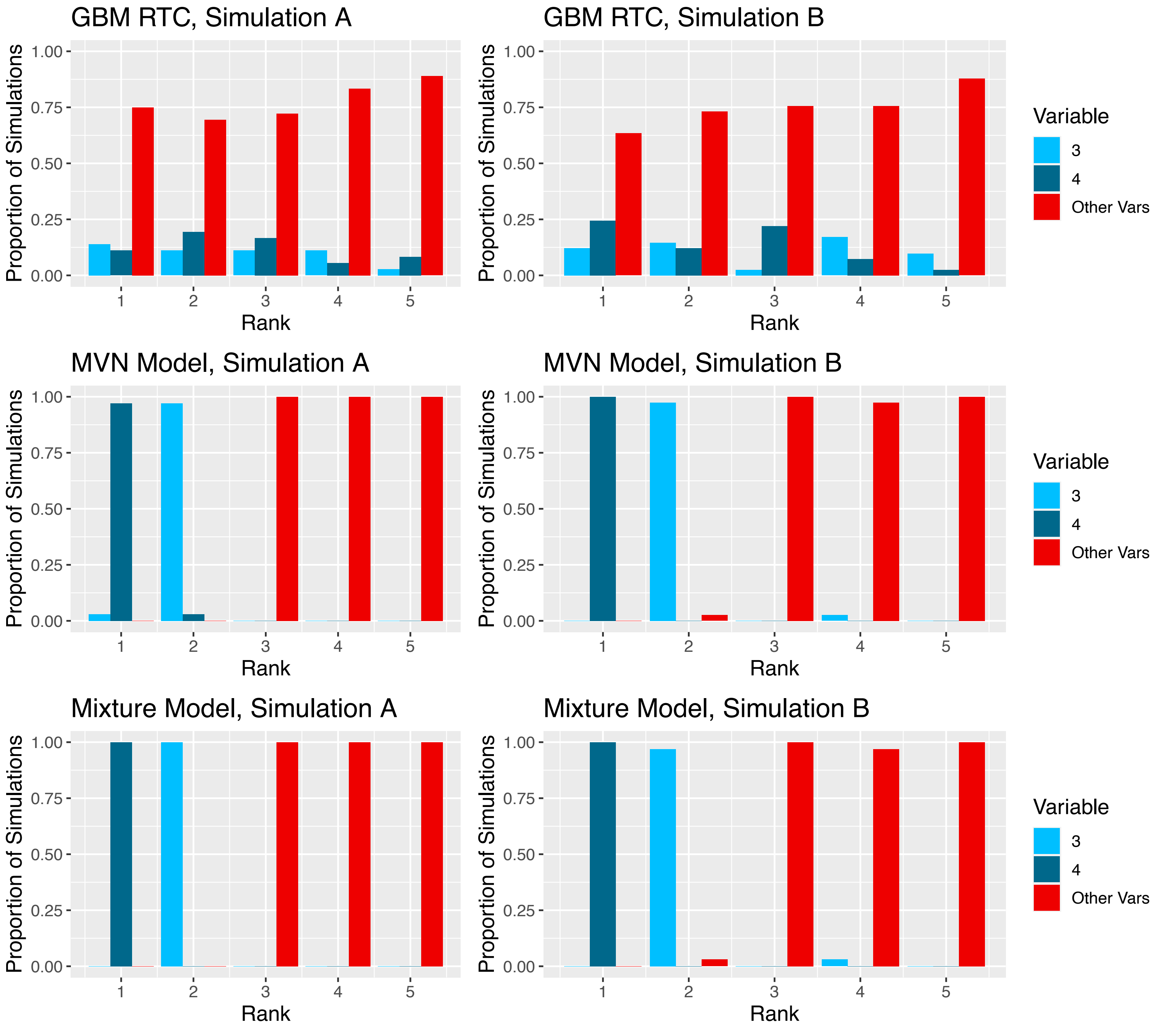
First-Order Loss of i: H(X : i, Y : i)/H(X,Y)

These metrics *decompose* the effects of each individual variable on the changepoint

Simulation



Total−Effect Hellinger Loss
First−Order Hellinger Loss
Representative Marginal Distributions
Density
Total−Effect Hellinger
First−Order Hellinger
Variable Value
Change−point Occurs
After
Before
4
3
6
2
5


GBM RTC, Simulation A
GBM RTC, Simulation B
MVN Model, Simulation A
MVN Model, Simulation B
Mixture Model, Simulation A
Mixture Model, Simulation B
Proportion of Simulations
Rank
Variable
3
4
Other Vars

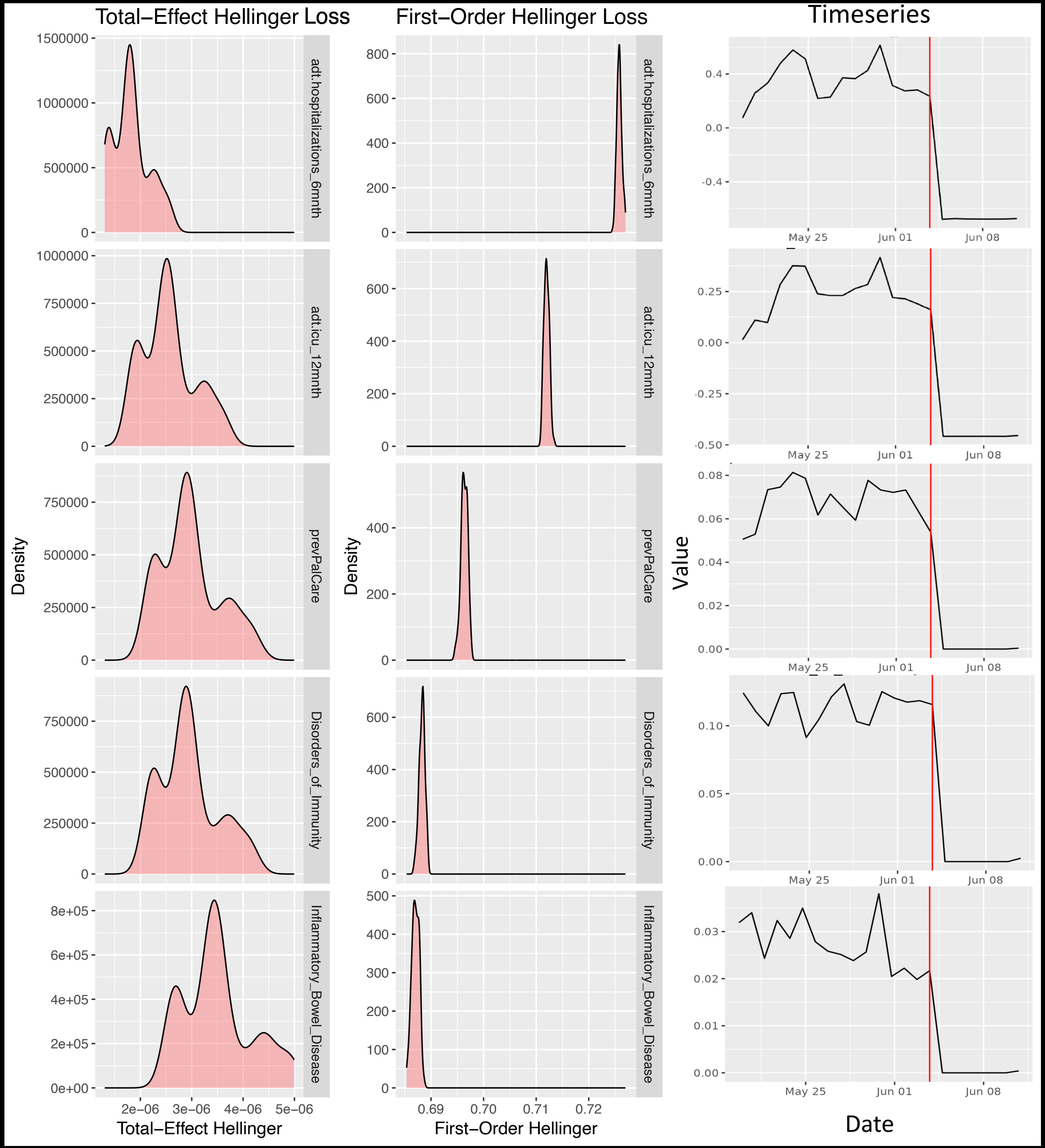
DATA APPLICATION

Let’s investigate a month (mid-May to mid-June 2020) of the data from the Palliative Care Model

- Each observation in the data is a palliative care prediction that occurs for a patient at the Mayo Clinic in Rochester, MN.
- Variables include patient labs, the presence/absence of specific chronic diseases, basic demographics, and information about a patient's movement within the hospital.
- There are **123 variables**, 35 of which are continuous, with 27 variables having some amount missing values.
- During the month investigated, there were **298039 data observations**.

This is a *big* dataset with *many* data challenges; there are missing values, and various data types (censored & discrete variables).

Application

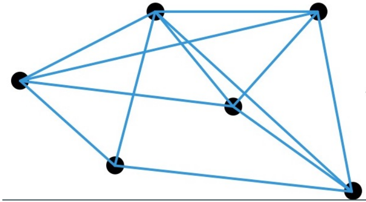


QUESTIONS?

Gaussian Graphical Models

- $X \sim \mathcal{N}_d\left(\mu, \Lambda\right)$ where $\Lambda := \Sigma^{-1}$;
- $G = (V, E)$ be an undirected graph, where V and E are the sets of nodes and edges, respectively. Nodes $i, j \in V$ are connected $\leftrightarrow (i, j) \in E$.

X is said to *satisfy the undirected Gaussian Graphical Model with graph* G if $\Lambda_{i,j} = 0$ for all $E_{i,j} = 0$.



$$\Leftrightarrow \begin{pmatrix} 5.61 & 0.80 & 2.33 & 0.03 & 2.18 & 0.00 \\ 0.80 & 6.66 & 0.15 & -0.06 & 1.78 & 3.04 \\ 2.33 & 0.15 & 3.16 & 0.00 & 0.90 & -0.74 \\ 0.03 & -0.06 & 0.00 & 3.93 & 0.00 & 2.58 \\ 2.18 & 1.78 & 0.90 & 0.00 & 4.57 & 2.39 \\ 0.00 & 3.04 & -0.74 & 2.58 & 2.39 & 5.07 \end{pmatrix}$$ [1]

[1] example numbers from Kundu *et al.* (2019)

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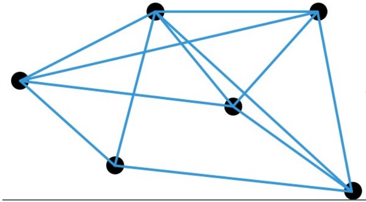

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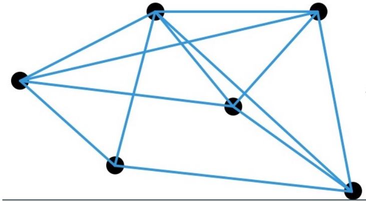

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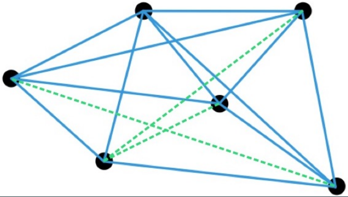

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[2] example numbers from Kundu *et al.* (2019)

Gaussian Graphical Models

Our work thus far allows us to define a GFD on X with a fixed graph G. We can express

$$\Sigma^{-1}_{i,j} = 0 \leftrightarrow (i,j) \notin E,$$

as the zeros of the function

$$g_{i,j}(\Sigma) = \Sigma^{-1}_{i,j}.$$

Gaussian Graphical Models

Combining this with our previous work on the Multivariate Normal (Murph *et al.*, 2022), we can use the Cayley Transform to decompose

$$g_{i,j}(\Sigma) = \Sigma_{i,j}^{-1}.$$

into

$$g_{i,j}(A, \Lambda) = \Big((I - A)(I + A)^{-1}\Lambda^{-2}(I - A)^{-1}(I + A)\Big)_{i,j}$$

(The full DGA is: $\mathbf{Y}_i = \mu + (I_d + A)(I_d - A)^{-1}\Lambda\mathbf{Z}_i, \quad i = 1, \dots, m.$)

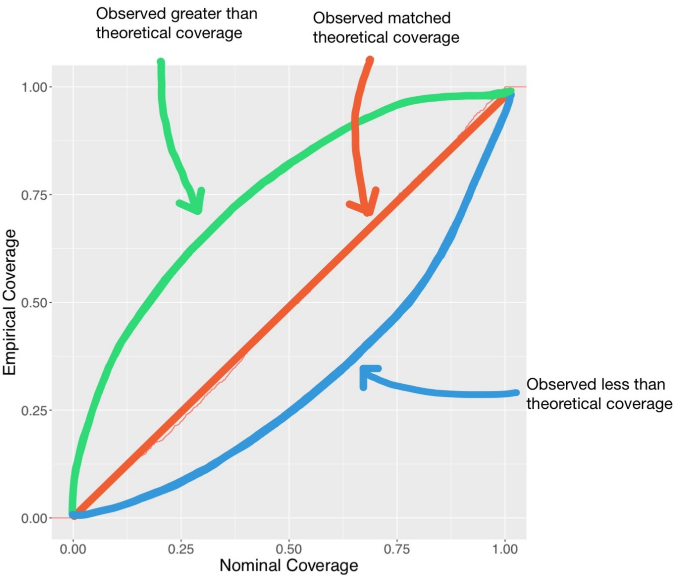
Gaussian Graphical Models

Ex: Let's sample GGMs from the Bayesian posterior and the CGFD for fixed graph structure G (assuming a zero mean).

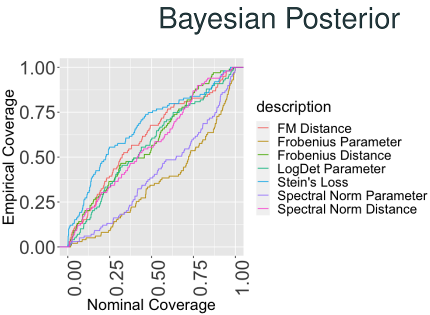
Bayesian: MVN model with G-Wishart prior; we can sample values directly from the posterior;

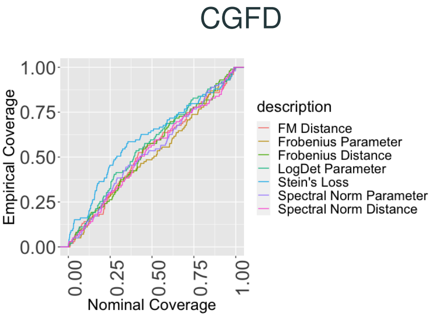
Generalized Fiducial: Propose values in a Metropolis-Hastings MCMC set-up using a G-Wishart with a change-of-variables to (A, Λ). Accepted samples are from the CGFD.

Gaussian Graphical Models



Gaussian Graphical Models





Thanks for your attention!

Questions?

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