Exercise 1. Let R be a ring, and M an abelian group. Define

$$\mathrm{Hom}_{\mathbb{Z}}(R, M) = \{f : R \to M \mid f \text{ is a } \mathbb{Z}\text{-module homomorphism}\}.$$

Show that $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ is an R-module with multiplication $(rf)(x) = rf(x)$ for any $r \in R$, $f \in \mathrm{Hom}_{\mathbb{Z}}(R, M)$, and $x \in R$.

Exercise 2. Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

Exercise 3. Show that every projective abelian group is free.

Exercise 4. Show that a direct product of R-modules $\prod_{i \in I} J_i$ is injective if and only if each J_i is injective.

Exercise 5. Let R be a commutative, unital ring. Show that the following are equivalent.

(i) Every R-module is projective.

(ii) Every R-module is injective.

(iii) Every short exact sequence of R-modules is split exact.