

Exercise 1. Let R be a commutative unital ring. Show that

$$M_n = \{p \in R[x] \mid \deg p < n\}$$

is a submodule of $R[x]$.

Exercise 2. Prove the Five Lemma: Let

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

be a commutative diagram of R -module homomorphisms.

- (a) Show that if α_1 is surjective, and α_2 and α_4 are injective, then α_3 is also injective.
- (b) Show that if α_5 is injective, and α_2 and α_4 are surjective, then α_3 is also surjective.

Exercise 3. Let $f : A \rightarrow A$ be an R -module homomorphism. Show that if $ff = f$, then $A \cong \ker f \oplus \operatorname{Im} f$.

Exercise 4. Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be R -module homomorphisms. Show that if $gf = \operatorname{id}$, then $B \cong \operatorname{Im} f \oplus \ker g$.

Exercise 5.