Contents

Problem Set 1

Exercise 1.1. Let R be an integral domain, and let T be an integral domain such that $R \subset T \subset \operatorname{Frac} R$. Show that $\operatorname{Frac} R = \operatorname{Frac} T$.

Exercise 1.2. Let R be an integral domain, and $S \subset R$ a multiplicative subset that does not contain 0. Show that if R is a PID, then so is $S^{-1}R$.

Exercise 1.3. Let R be a commutative unital ring, $S \subset R$ a multiplicative subset, and $I \subset R$ an ideal. Show that $S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$ (Recall that $\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\}$).

Exercise 1.4. Let R be a commutative unital ring. Show that R is local if and only if whenever $r + s = 1$, then either $r \in R^*$ or $s \in R^*$

Exercise 1.5. Show that every nonzero homomorphic image of a local ring is local.

Problem Set 2

Exercise 2.1. Let R be a commutative unital ring. Show that

$$M_n = \{p \in R[x] \mid \deg p < n\}$$

is a submodule of $R[x]$.

Exercise 2.2. Let M be an R-module, and $I \subset R$ an ideal.

1. Show that $IM = \{\sum_{i=1}^{n} r_i m_i \mid n \in \mathbb{N}, r_i \in I, m_i \in M\}$ is a submodule of M.

2. Show that M/IM is an R/I module, with multiplication given by

$$(r + I)(m + IM) = rm + IM \text{ for all } r + I \in R/I, m + IM \in M/IM.$$

Exercise 2.3. Prove the Five Lemma: Let

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "\alpha_1"] & A_2 \arrow[r] \arrow[d, "\alpha_2"] & A_3 \arrow[r] \arrow[d, "\alpha_3"] & A_4 \arrow[r] \arrow[d, "\alpha_4"] & A_5 \arrow[d, "\alpha_5"] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \end{tikzcd}$$

be a commutative diagram of R-module homomorphisms with each row exact.

(a) Show that if α_1 is surjective, and α_2 and α_4 are injective, then α_3 is also injective.

(b) Show that if α_5 is injective, and α_2 and α_4 are surjective, then α_3 is also surjective.

Exercise 2.4. Let $f : A \to A$ be an R-module homomorphism. Show that if $ff = f$, then $A \cong \ker f \oplus \operatorname{Im} f$.

Exercise 2.5. Let $f : A \to B$ and $g : B \to A$ be R-module homomorphisms. Show that if $gf = \mathrm{id}$, then $B \cong \operatorname{Im} f \oplus \ker g$.

Problem Set 3

Exercise 3.1. Let R be a ring, and M an abelian group. Define

$$\mathrm{Hom}_{\mathbb{Z}}(R, M) = \{f : R \to M \mid f \text{ is a } \mathbb{Z}\text{-module homomorphism}\}.$$

Show that $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ is an R-module with multiplication $(rf)(x) = rf(x)$ for any $r \in R$, $f \in \mathrm{Hom}_{\mathbb{Z}}(R, M)$, and $x \in R$.

Exercise 3.2. Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

Exercise 3.3. Show that every projective abelian group is free.

Exercise 3.4. Show that a direct product of R-modules $\prod_{i \in I} J_i$ is injective if and only if each J_i is injective.

Exercise 3.5. Let R be a commutative, unital ring. Show that the following are equivalent.

(i) Every R-module is projective.

(ii) Every R-module is injective.

(iii) Every short exact sequence of R-modules is split exact.

Problem Set 4

Exercise 4.1. If A is a finite abelian group, show that $A \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

Exercise 4.2. Show that $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \cong \mathbb{Z}_d$, where $d = (m, n)$. **Hint:** Write $d = am + bn$ for some integers a, b.

Exercise 4.3. Let M be an R-module, and $I \subset R$ an ideal. Show that $R/I) \otimes_R M \cong M/IM$.

Exercise 4.4. Let R be commutative, and $I, J \subset R$ ideals. Show that $R/I \otimes_R R/J \cong R/(I + J)$.

Exercise 4.5. Let R be commutative. An R-module is called *flat* if tensoring with that module is left exact. Show that every projective R-module is flat.

Problem Set 5

Exercise 5.1. Let R be a PID, and $I \subset R$ an ideal. Show that R/I is both Noetherian and Artinian.

Exercise 5.2. Let R be Noetherian, and $P \subset R$ a prime ideal. Show that R_P is Noetherian.

Exercise 5.3. Let R be an Artinian ring. Show that every prime ideal of R is maximal.

Exercise 5.4. Let R be a ring, $S \subset R$ a multiplicative set, and $I \subset R$ an ideal. Show that $S^{-1}(\operatorname{rad} I) = \operatorname{rad}(S^{-1}I)$.

Exercise 5.5. Let R be Noetherian, and $I, J \subset R$ ideals with $J \subset \operatorname{rad} I$. Show that there exists $n \in \mathbb{N}$ with $J^n \subset I$.

Problem Set 6

Exercise 6.1. Let R be a ring in which every maximal ideal is of the form cR for some $c \in R$ satisfying $c^2 = c$. Show that R is Noetherian (**Hint:** Show that every prime ideal is maximal).

Exercise 6.2. Let $(R, \mathfrak{M})$ be a Noetherian local ring. Suppose that $\mathfrak{M}/\mathfrak{M}^2$ is generated by the set $\{a_1+\mathfrak{M}^2, \ldots, a_n+\mathfrak{M}^2\}$. Show that $\mathfrak{M} = a_1R+\cdots+a_nR$.

Exercise 6.3. Let $R \subset S$ be an integral extension, and suppose that R and S are both integral domains. Show that R is a field if and only if S is a field.

Exercise 6.4. Show that if $R \subset S$ is an integral extension, then $S[x_1, \ldots, x_n]$ is integral over $R[x_1, \ldots, x_n]$.

Exercise 6.5. Let R be an integral domain with fractional field k. Show that if R is integrally closed and t is transcendental over k, then $R[t]$ is integrally closed.

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