Exercise 1. Let R be a commutative unital ring. Show that

$$M_n = \{p \in R[x] \mid \deg p < n\}$$

is a submodule of $R[x]$.

Exercise 2. Let M be an R-module, and $I \subset R$ an ideal.

1. Show that $IM = \{\sum_{i=1}^{n} r_i m_i \mid n \in \mathbb{N}, r_i \in I, m_i \in M\}$ is a submodule of M.

2. Show that M/IM is an R/I module, with multiplication given by

$$(r + I)(m + IM) = rm + IM \text{ for all } r + I \in R/I, m + IM \in M/IM.$$

Exercise 3. Prove the Five Lemma: Let

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "\alpha_1"] & A_2 \arrow[r] \arrow[d, "\alpha_2"] & A_3 \arrow[r] \arrow[d, "\alpha_3"] & A_4 \arrow[r] \arrow[d, "\alpha_4"] & A_5 \arrow[d, "\alpha_5"] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \end{tikzcd}$$

be a commutative diagram of R-module homomorphisms.

(a) Show that if α_1 is surjective, and α_2 and α_4 are injective, then α_3 is also injective.

(b) Show that if α_5 is injective, and α_2 and α_4 are surjective, then α_3 is also surjective.

Exercise 4. Let $f : A \to A$ be an R-module homomorphism. Show that if $ff = f$, then $A \cong \ker f \oplus \operatorname{Im} f$.

Exercise 5. Let $f : A \to B$ and $g : B \to A$ be R-module homomorphisms. Show that if $gf = \mathrm{id}$, then $B \cong \operatorname{Im} f \oplus \ker g$.