

# The curvature and dimension of a closed surface

S. Halayka\*

October 8, 2019

## Abstract

The curvature of a closed surface can lead to fractional dimension. In this paper, the properties of the 2-sphere surface of a three-dimensional ball and the  $2.x$ -dimensional surface of a three-dimensional fractal set are considered. Tessellation is used to approximate each surface, primarily because the  $2.x$ -dimensional surface of a three-dimensional fractal set is otherwise non-differentiable (having no well-defined surface normals).

## 1 Overview

Unlike in traditional geometry where dimension is an integer, fractional (non-integer) dimension occurs in *fractal* geometry. In fractal geometry, there are currently many ways to calculate the dimension of a surface [1, 2]. This paper uses a new method of calculating the fractional dimension of a surface – it is *curvature* that leads to this fractional dimension.

In this paper we will focus on the tessellation of closed surfaces. For instance, Marching Cubes [3, 4] can be used to generate  $2.x$ -dimensional triangular tessellations (meshes), where dimension  $D \in (2.0, 3.0)$ .

We will focus on the difference between the curvature and dimension of a 2-sphere and the  $2.x$ -dimensional surface of a three-dimensional fractal set. We will generate both a 2-sphere and the  $2.x$ -dimensional surface of a three-dimensional fractal set by using iterative quaternion equations. For example, a 2-sphere is generated from the iterative quaternion Julia set equation  $Z = Z^2 + C$ , but where  $C = 0.0, 0.0, 0.0, 0.0$ , and the  $2.x$ -dimensional surface of a three-dimensional fractal set is generated from the iterative quaternion equation  $Z = Z \cos(Z)$ . See [5] for information on how to perform quaternion multiplication, addition, cos, etc.

## 2 The tessellation of a closed surface

Approximating the surface of a three-dimensional shape as a mesh allows us to calculate the surface's dimension  $D \in (2.0, 3.0)$ . This includes approximation of both a 2-sphere and the  $2.x$ -dimensional surface of a three-dimensional fractal set.

---

\*sjhalayka@gmail.com

First we calculate, for each triangle, the average dot product of the triangle's face normal  $\hat{n}_i$  and its three neighbouring triangles' face normals  $\hat{o}_1, \hat{o}_2, \hat{o}_3$ :

$$d_i = \frac{\hat{n}_i \cdot \hat{o}_1 + \hat{n}_i \cdot \hat{o}_2 + \hat{n}_i \cdot \hat{o}_3}{3} \in (-1.0, 1.0]. \quad (1)$$

Because we assume that there are three neighbours per triangle, the mesh must be *closed* (no cracks or holes, precisely two triangles per edge). The reason why the value  $-1.0$  is not achievable is because that would lead to intersecting triangles.

Then we calculate the normalized measure of curvature:

$$k_i = \frac{1 - d_i}{2} \in [0.0, 1.0). \quad (2)$$

Once  $k_i$  has been calculated for all triangles, we can then calculate the average normalized measure of curvature  $K$ , where  $t$  is the number of triangles in the mesh:

$$K = \frac{1}{t} \sum_{i=1}^t k_i = \frac{k_1 + k_2 + \dots + k_t}{t} \in (0.0, 1.0). \quad (3)$$

The reason why the value  $0.0$  is not achievable is because we are dealing with a closed surface, and so there's bound to be *some* curvature.

The dimension of the closed surface is:

$$D = 2 + K \in (2.0, 3.0). \quad (4)$$

As far as we know, this method of calculating the dimension of a closed surface is new [6, 7]. The entire C++ code for generating a mesh can be found at [8]. The entire C++ code for calculating a mesh's dimension can be found at [9].

### 3 Vanishing versus non-vanishing curvature

Where  $r \in [2, \infty)$  is the *integer* sampling resolution,  $g_{\max} \in (-\infty, \infty)$  is the sampling grid maximum extent,  $g_{\min} \in (-\infty, \infty)$  is the sampling grid minimum extent, and  $g_{\max} > g_{\min}$ , the Marching Cubes step size is:

$$\ell = \frac{g_{\max} - g_{\min}}{r - 1} \in (0.0, \infty). \quad (5)$$

In this paper  $g_{\max} = 1.5$ ,  $g_{\min} = -1.5$ , and  $r$  is variable.

For a 2-sphere, the *local* curvature all but vanishes as  $\ell$  decreases (as  $r$  increases):

$$\lim_{\ell \rightarrow 0.0} K(\ell) = 0.0. \quad (6)$$

This results in a dimension of practically (but never quite)  $2.0$ , which is to be expected from a non-fractal surface. See Figures 1 - 3.

On the other hand, for the  $2.x$ -dimensional surface of a three-dimensional fractal set, the local curvature does not vanish as  $\ell$  decreases:

$$\lim_{\ell \rightarrow 0.0} K(\ell) \neq 0.0. \quad (7)$$

This results in a dimension considerably greater than  $2.0$ , but not equal to or greater than  $3.0$ , which is to be expected from a fractal surface. See Figures 4 - 7.

## 4 Notes

The minimum Marching Cubes step size, in real life, is the Planck length  $\ell_P$ .

Marching *Squares* [10, 11, 12] can be used to generate 1. $x$ -dimensional closed line paths, where dimension  $D \in (1.0, 2.0)$ . See Figures 8 - 10 for some examples of a line path. These figures might be helpful if there is difficulty envisioning the curvature in the case of Marching Cubes.

## References

- [1] <http://paulbourke.net/fractals/fracdim/>
- [2] [https://en.wikipedia.org/wiki/Fractal\\_dimension](https://en.wikipedia.org/wiki/Fractal_dimension)
- [3] Lorensen, W. E.; Cline, Harvey E. (1987). "Marching cubes: A high resolution 3d surface construction algorithm". *ACM Computer Graphics*. 21 (4): 163–169
- [4] <http://paulbourke.net/geometry/polygonise/>
- [5] <http://www.theworld.com/~sweetser/quaternions/intro/tools/tools.html>
- [6] Mandelbrot, B. (1967). "How Long is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension". *Science*. 156 (3775): 636–8.
- [7] Mandelbrot, B. (1982). "The Fractal Geometry of Nature". ISBN 978-0716711865.
- [8] [https://github.com/sjhalayka/marching\\_cubes](https://github.com/sjhalayka/marching_cubes)
- [9] <https://github.com/sjhalayka/meshdim>
- [10] Maple, C. (2003). Geometric design and space planning using the marching squares and marching cube algorithms. *Proc. 2003 Intl. Conf. Geometric Modeling and Graphics*. pp. 90–95
- [11] [https://en.wikipedia.org/wiki/Marching\\_squares](https://en.wikipedia.org/wiki/Marching_squares)
- [12] <https://github.com/sjhalayka/Marching-Squares>



Figure 1: Low resolution ( $r = 10$ ) surface for the iterative quaternion equation is  $Z = Z^2$ . The surface's dimension is 2.02.

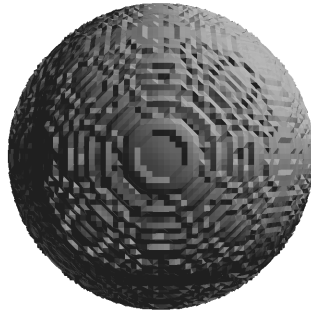


Figure 2: Medium resolution ( $r = 100$ ) surface for the iterative quaternion equation is  $Z = Z^2$ . The surface's dimension is 2.06.

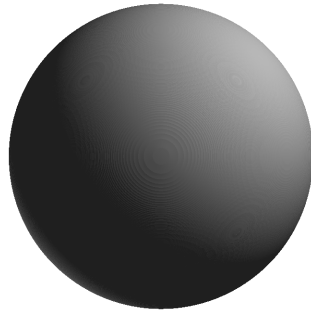


Figure 3: High resolution ( $r = 1000$ ) surface for the iterative quaternion equation is  $Z = Z^2$ . The surface's dimension is practically 2.0.

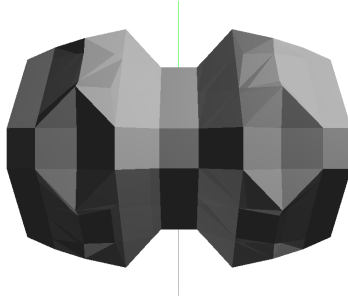


Figure 4: Low resolution ( $r = 10$ ) surface for the iterative quaternion equation is  $Z = Z \cos(Z)$ . The surface's dimension is 2.05.

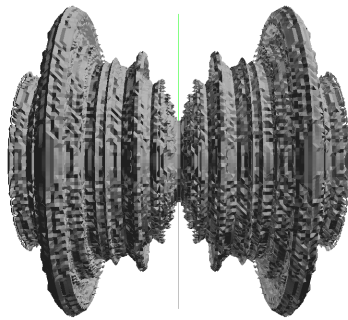


Figure 5: Medium resolution ( $r = 100$ ) surface for the iterative quaternion equation is  $Z = Z \cos(Z)$ . The surface's dimension is 2.11.

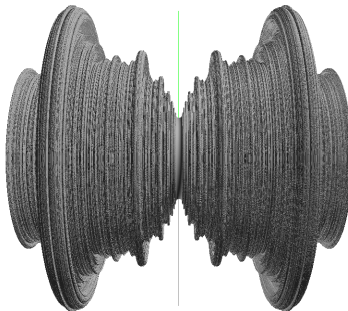


Figure 6: High resolution ( $r = 1000$ ) surface for the iterative quaternion equation is  $Z = Z \cos(Z)$ . The surface's dimension is 2.08.

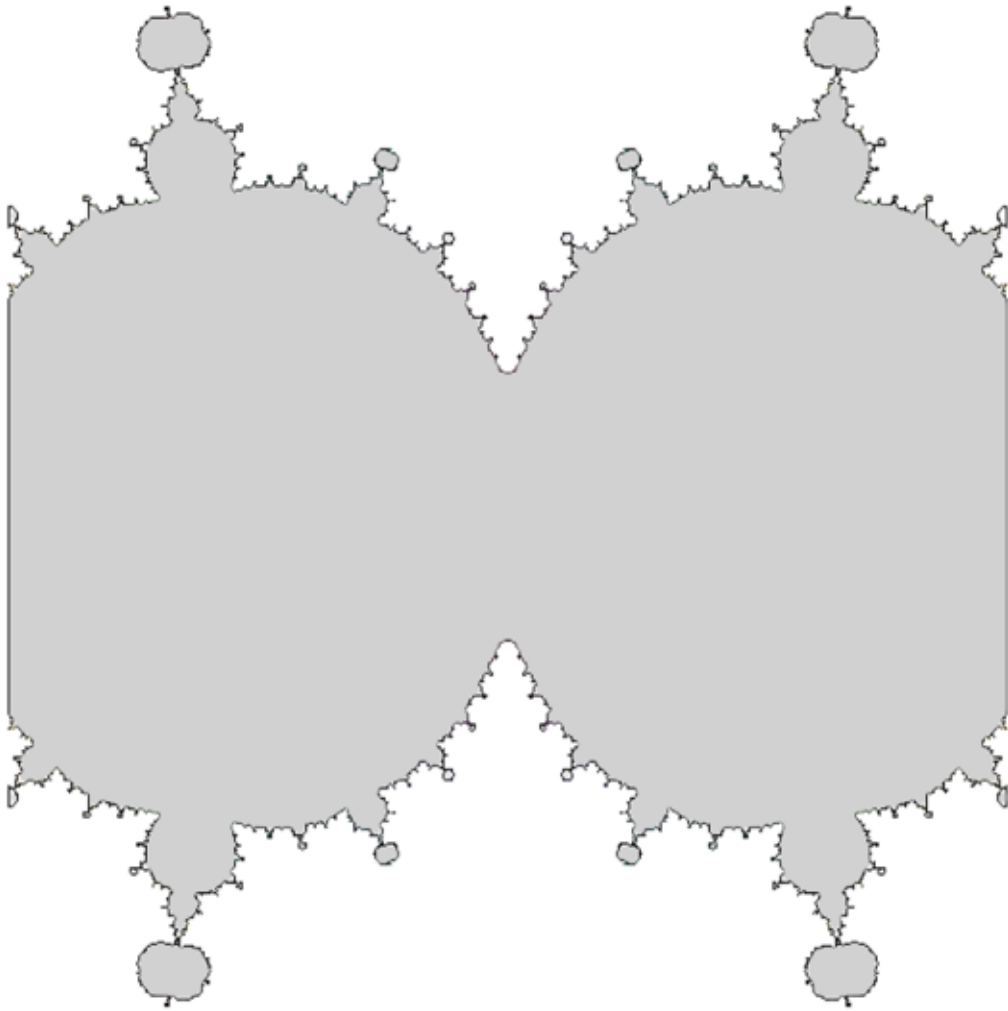


Figure 7: A two-dimensional slice of the iterative quaternion equation  $Z = Z \cos(Z)$ , showing the fractal nature of the set.

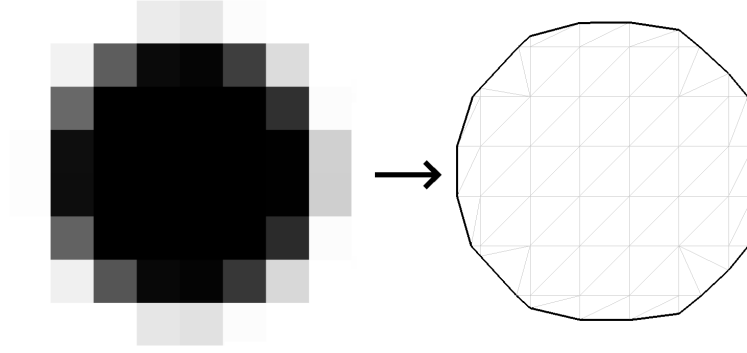


Figure 8: Example input (a two-dimensional greyscale image, consisting of pixels) and output (a 1. $x$ -dimensional closed set of line segments) of the Marching Squares algorithm, approximating a 1-sphere (a circle), where sampling resolution is  $r = 8$ . Note that for Marching *Cubes*, the input is a three-dimensional ‘greyscale image’, consisting of voxels, and the output is a 2. $x$ -dimensional closed set of triangles.



Figure 9: Illustrated is a section of a closed line path, with surface normals. The average dot product of neighbouring line segments is  $d_i = 0.0$ . This leads to a normalized measure of curvature  $k_i = (1 - d_i)/2 = 0.5$ , which in turn leads to an average normalized measure of curvature  $K = 0.5$ . The dimension is  $D = 1 + K = 1.5$ .

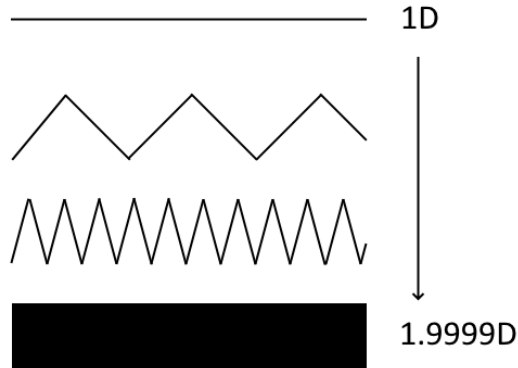


Figure 10: A section of a closed line path as it goes from dimension 1.0 (at top) to 1.9999 (at bottom). In the end, where the dimension is 1.9999, the result is practically a rectangle. The reason why the dimension cannot be 2.0 is because that would lead to intersecting line segments.