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Some visually interesting non-standard quaternion fractal sets

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ABSTRACT

The quaternion Julia set's standard iterative function is replaced with several others, most of which seem to be previously undocumented. The results are reproduced here mainly for the sake of visual interest.

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1. Introduction

The quaternion Julia set [1] can be produced by marching along a finite lattice of regularly spaced points in **R**3, testing at each one to see if the iterative function

$$Z = Z^2 + C \tag{1}$$

causes the magnitude $|Z| = \sqrt{Z_x^2 + Z_y^2 + Z_z^2 + Z_w^2}$ to exceed a threshold value while undergoing a limited number of iterations. Only when the magnitude remains below or equal to the threshold is a point within the set. For all sets given here, the magnitude threshold value is 4, the iteration limit is 8, and the lattice extends from -1.5 to 1.5 along all three spatial dimensions.

For each lattice point x, y, z, the initial value of Z is:

$$Z_x = x, \tag{2}$$
$$Z_y = y, \tag{3}$$
$$Z_z = z, \tag{4}$$

and Z_w is some arbitrary value. For all sets given here, initially $Z_w = 0$. All four components of the quaternion C are also arbitrary.

Using the standard product operation described in [2,3], the squaring of Z produces a quaternion in the form of a scalar and a 3-vector:

$$t = t' = Z_x, \tag{5}$$
$$\vec{V} = \vec{V}' = Z_y, Z_z, \quad Z_w, \tag{6}$$
$$Z \cdot Z = tt' - \vec{V} \cdot \vec{V}', \quad t\vec{V}' + t'\vec{V} + \vec{V} \times \vec{V}', \tag{7}$$
$$(Z \cdot Z)_x = Z_x Z_x - (Z_y Z_y + Z_z Z_z + Z_w Z_w), \tag{8}$$
$$(Z \cdot Z)_y = Z_x Z_y + Z_x Z_y + (Z_z Z_w - Z_w Z_z), \tag{9}$$
$$(Z \cdot Z)_z = Z_x Z_z + Z_x Z_z + (Z_w Z_y - Z_y Z_w), \tag{10}$$
$$(Z \cdot Z)_w = Z_x Z_w + Z_x Z_w + (Z_y Z_z - Z_z Z_y). \tag{11}$$

The non-commutativity of the product operation is important in the case of inequal operands, ex: if $A \neq B$, then $A \cdot B \neq B \cdot A$.

Quaternion addition is performed piecewise, ex: $Z^2 + C = (Z \cdot Z)_x + C_x, \ldots$.

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2. Standard and non-standard sets visualized

Where $Z = Z^2 + C$, and $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$, all points within the standard quaternion Julia set are visualized in Fig. 1 using a crude surface approximation.

As mentioned in [4], altering the standard iterative function by increasing the exponent beyond 2 provides interesting results. See Fig. 2 for $Z = Z^4 + C$, and $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$.

Visually interesting sets can also be produced through the use of functions such as $\sin Z$ [2]:

$$t = Z_x, \tag{12}$$

$$\vec{V} = Z_y,\ Z_z, \quad Z_w, \tag{13}$$

$$|\vec{V}| = \sqrt{Z_y^2 + Z_z^2 + Z_w^2}, \tag{14}$$

$$\mathrm{norm}(Z) = |Z|^2 = t^2 + \vec{V} \cdot \vec{V}, \tag{15}$$

$$\mathrm{inv}(Z) = t/\mathrm{norm}(Z), \quad -\vec{V}/\mathrm{norm}(Z), \tag{16}$$

$$\sin Z = \sin t \cosh |\vec{V}|, \quad \cos t \sinh |\vec{V}|\vec{V}/|\vec{V}|, \tag{17}$$

$$\cos Z = \cos t \cosh |\vec{V}|, \quad -\sin t \sinh |\vec{V}|\vec{V}/|\vec{V}|, \tag{18}$$

$$\exp Z = \exp t \cos |\vec{V}|, \quad \exp t \sin |\vec{V}|\vec{V}/|\vec{V}|, \tag{19}$$

$$\sinh Z = \sinh t \cos |\vec{V}|, \quad \cosh t \sin |\vec{V}|\vec{V}/|\vec{V}|, \tag{20}$$

$$\cosh Z = \cosh t \cos |\vec{V}|, \quad \sinh t \sin |\vec{V}|\vec{V}/|\vec{V}|. \tag{21}$$

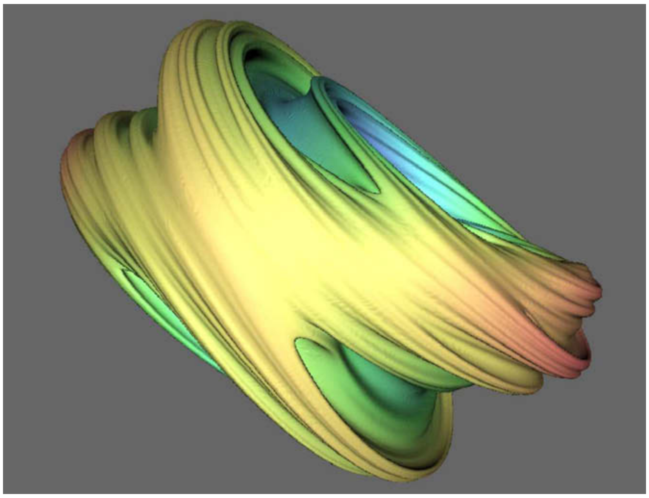

Fig. 1. $Z = Z^2 + C$, and $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$.

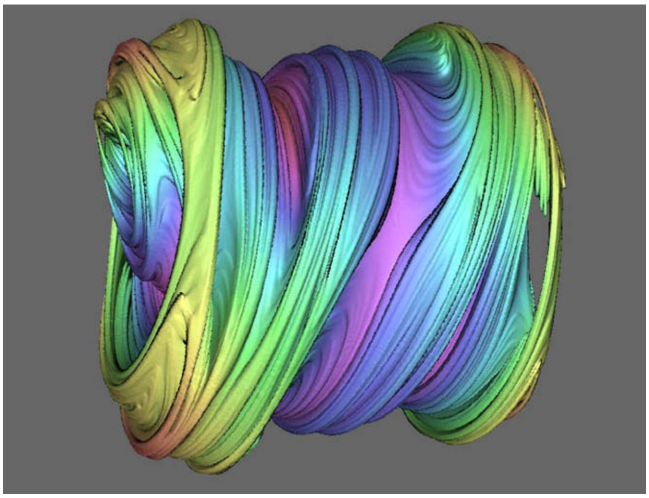

Fig. 2. $Z = Z^4 + C$, and $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$.

See Fig. 3 for $Z = \sin Z + C \cdot \sin Z$, and $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$.
See Fig. 4 for $Z = \sin Z + C \cdot \sin Z$, and $C_{xyzw} = -0.3, 0.5, 0.7, 0.2$.
See Fig. 5 for $Z = \cos Z + C \cdot \cos Z$, and $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$.
See Fig. 6 for $Z = \exp(Z^2) + C$, and $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$, where the complement of the set is used for visualization.
See Fig. 7 for $Z = \mathrm{inv}(\sinh Z) + C \cdot \mathrm{inv}(\sinh Z)$, and $C_{xyzw} = 1, 1, 1, 1$.
See Fig. 8 for $Z = C \cdot (\mathrm{inv}(\sinh Z) \cdot \cosh Z)$, and $C_{xyzw} = 1, 1, 1, 1$, where the complement of the set is used for visualization.

3. Comments

Without straying too far from the present subject of fractal sets, it should be noted that the quaternions also have a wide range of practical applications.

In terms of group theory, the unit quaternions represent the special unitary Lie group SU(2) [5], which is an integral part of the standard model of particle physics [6,7]. As well, the algebras related to the exceptional Lie groups E_7 are constructed using the quaternions (and the 8-component octonions) [5], which have been used by El Naschie to investigate $E^{(\infty)}$ theory [8,9].

The unit quaternion representation of rotation in 3-dimensional space is also commonly used in computer science [10], including virtual camera control [11], inverse kinematics [12], and even encryption [13].

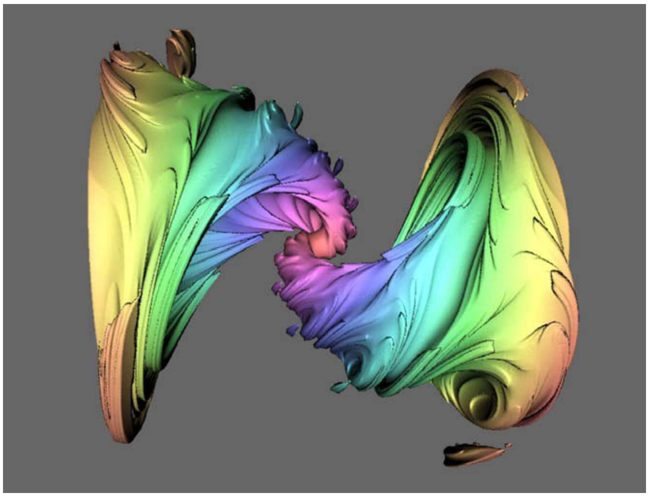

Fig. 3. $Z = \sin Z + C \cdot \sin Z$, and $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$.

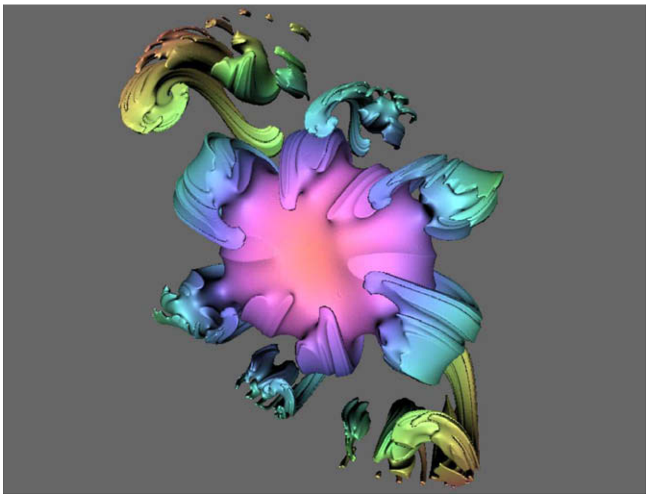

Fig. 4. $Z = \sin Z + C \cdot \sin Z$, and $C_{xyzw} = -0.3, 0.5, 0.7, 0.2$.

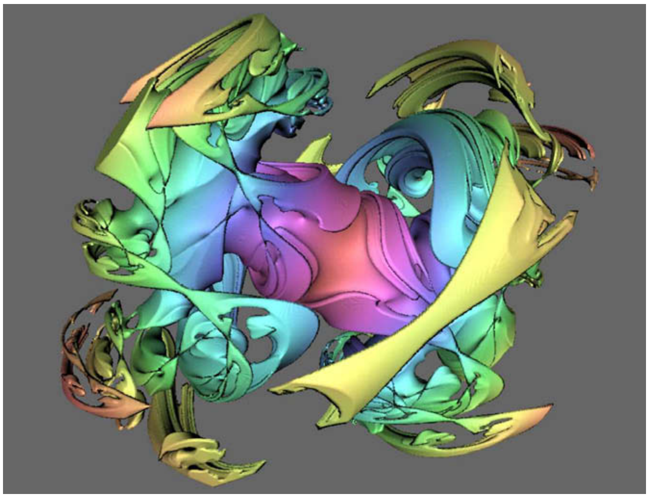

Fig. 5. $Z = \cos Z + C \cdot \cos Z$, and $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$.

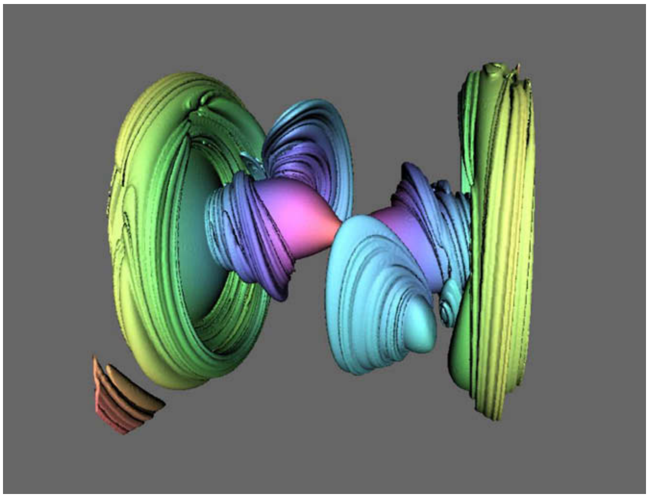

Fig. 6. The complement of $Z = \exp(Z^2) + C$, and $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$.

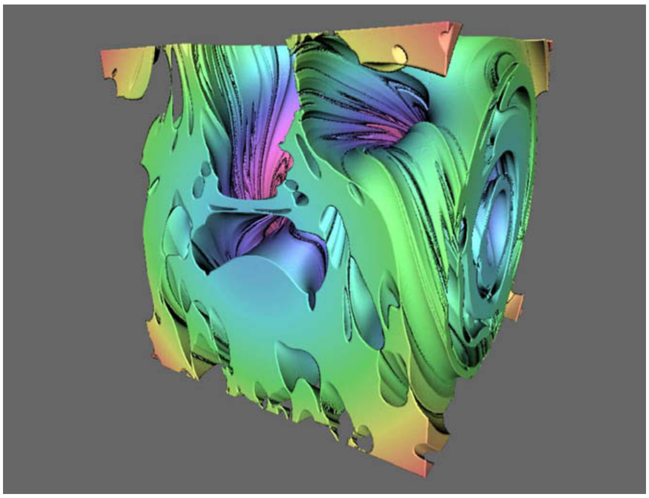

Fig. 7. $Z = \mathrm{inv}(\sinh Z) + C \cdot \mathrm{inv}(\sinh Z)$, and $C_{xyzw} = 1, 1, 1, 1$.

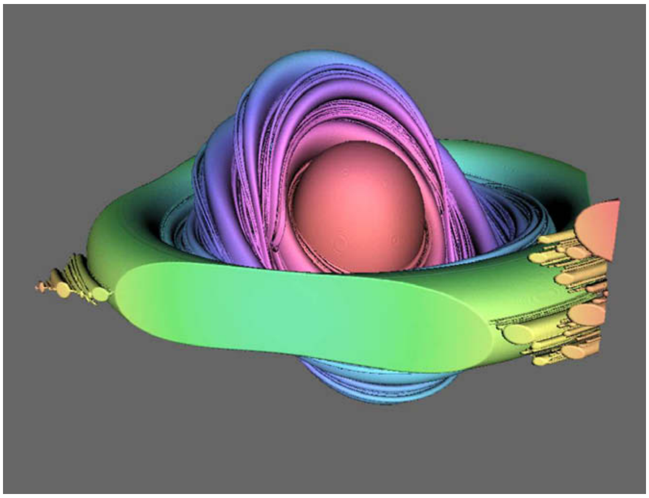

Fig. 8. The complement of $Z = C \cdot (\mathrm{inv}(\sinh Z) \cdot \cosh Z)$, and $C_{xyzw} = 1, 1, 1, 1$.

The work of Griffin and Joshi [14,15] is recommended to those wishing to explore the octonion Julia set.

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