The curvature and dimension of a closed surface

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Abstract

The curvature of a closed surface can lead to fractional dimension. In this paper, the properties of the 2-sphere surface of a three-dimensional ball and the $2.x$-dimensional surface of a three-dimensional fractal set are considered. Tessellation is used to approximate each surface, primarily because the $2.x$-dimensional surface of a three-dimensional fractal set is otherwise non-differentiable (having no well-defined surface normals).

1 Introduction

Unlike in traditional geometry where dimension is an integer, fractional (non-integer) dimension occurs in fractal geometry.

In fractal geometry, there are currently many ways to calculate the dimension of a surface [1, 2]. This paper uses a new method of calculating the fractional dimension of a surface – it is *curvature* that leads to this fractional dimension.

In this paper we will focus on closed surfaces. For instance, Marching Cubes [3, 4] can be used to generate $2.x$-dimensional triangular tessellations (meshes), where dimension $D \in (2.0, 3.0)$.

For the remainder of this paper we will focus on the difference between a 2-sphere and the $2.x$-dimensional surface of a three-dimensional fractal set.

2 The tessellation of a closed surface

Approximating the surface of a three-dimensional shape as a mesh allows us to calculate the surface's dimension $D \in (2.0, 3.0)$. This includes approximation of both a 2-sphere, as well as the $2.x$-dimensional surface of a three-dimensional fractal set.

First we calculate, for each triangle, the average dot product of the triangle's face normal $\hat{n}_i$ and its three neighbouring triangles' face normals $\hat{o}_1$, $\hat{o}_2$, $\hat{o}_3$:

$$d_i = \frac{\hat{n}_i \cdot \hat{o}_1 + \hat{n}_i \cdot \hat{o}_2 + \hat{n}_i \cdot \hat{o}_3}{3} \in (-1.0, 1.0]. \quad (1)$$

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Because we assume that there are three neighbours per triangle, the mesh must be *closed* (no cracks or holes, precisely two triangles per edge). The reasion why the value -1.0 is not achievable is because that would lead to intersecting triangles.

Then we calculate the normalized measure of curvature:

$$k_i = \frac{1 - d_i}{2} \in [0.0, 1.0). \tag{2}$$

Once k_i has been calculated for all triangles, we can then calculate the average normalized measure of curvature K, where t is the number of triangles in the mesh:

$$K = \frac{1}{t} \sum_{i=1}^{t} k_i = \frac{k_1 + k_2 + ... + k_t}{t} \in (0.0, 1.0). \tag{3}$$

The reason why the value 0.0 is not achievable is because we are dealing with a closed surface, and so there's bound to be *some* curvature.

The dimension of the closed surface is:

$$D = 2 + K \in (2.0, 3.0). \tag{4}$$

As far as we know, this method of calculating the dimension of a closed surface is new [5, 6]. The entire C++ code for generating a mesh can be found at [7]. The entire C++ code for calculating a mesh's dimension can be found at [8].

3 Vanishing and non-vanishing curvature

Where $r \in [2, \infty)$ is the *integer* sampling resolution, $g_{\max} \in (-\infty, \infty)$ is the sampling grid maximum extent, $g_{\min} \in (-\infty, \infty)$ is the sampling grid minimum extent, and $g_{\max} > g_{\min}$, the Marching Cubes step size is:

$$\ell = \frac{g_{\max} - g_{\min}}{r - 1} \in (0.0, \infty). \tag{5}$$

In this paper $g_{\max} = 1.5$, $g_{\min} = -1.5$, and r is variable.

For a 2-sphere, the *local* curvature all but vanishes as ℓ decreases (as r increases):

$$\lim_{\ell \to 0.0} K(\ell) = 0.0. \tag{6}$$

This results in a dimension of practically (but never quite) 2.0, which is to be expected from a non-fractal surface. See Figures 1 - 3.

On the other hand, for the 2.x-dimensional surface of a three-dimensional fractal set, the local curvature does not vanish as ℓ decreases:

$$\lim_{\ell \to 0.0} K(\ell) \neq 0.0. \tag{7}$$

This results in a dimension considerably greater than 2.0, but not equal to or greater than 3.0, which is to be expected from a fractal surface. See Figures 4 - 7.

4 Notes

Analogously, Marching Squares [9, 10, 11] can be used to generate 1.x-dimensional closed line paths, where dimension $D \in (1.0, 2.0)$. See Figures 8 - 10 for some examples of a line path. These figures might be helpful if there is difficullty envisioning the curvature in the case of Marching Cubes.

In the case of Marching Squares, the edge length is $E \in (0.0, \sqrt{2}\ell)$. In the case of Marching Cubes, the edge length is $E \in (0.0, \sqrt{3}\ell)$. In both cases, they can achieve neither the minumum nor the maximum edge length, because that would lead to degenerate triangles.

References

[1] http://paulbourke.net/fractals/fracdim/

[2] https://en.wikipedia.org/wiki/Fractal_dimension

[3] Lorensen, W. E.; Cline, Harvey E. (1987). "Marching cubes: A high resolution 3d surface construction algorithm". ACM Computer Graphics. 21 (4): 163–169

[4] http://paulbourke.net/geometry/polygonise/

[5] Mandelbrot, B. (1967). "How Long is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension". Science. 156 (3775): 636–8.

[6] Mandelbrot, B. (1982). "The Fractal Geometry of Nature". ISBN 978-0716711865.

[7] https://github.com/sjhalayka/marching_cubes

[8] https://github.com/sjhalayka/meshdim

[9] Maple, C. (2003). Geometric design and space planning using the marching squares and marching cube algorithms. Proc. 2003 Intl. Conf. Geometric Modeling and Graphics. pp. 90–95

[10] https://en.wikipedia.org/wiki/Marching_squares

[11] https://github.com/sjhalayka/Marching-Squares

Figure 1: Low resolution ($r = 10$) surface for the iterative equation is $Z = Z^2$. The surface's dimension is 2.02.

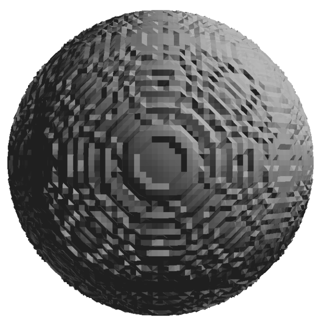

Figure 2: Medium resolution ($r = 100$) surface for the iterative equation is $Z = Z^2$. The surface's dimension is 2.06.

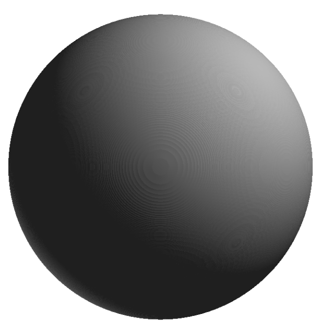

Figure 3: High resolution ($r = 1000$) surface for the iterative equation is $Z = Z^2$. The surface's dimension is practically 2.0.

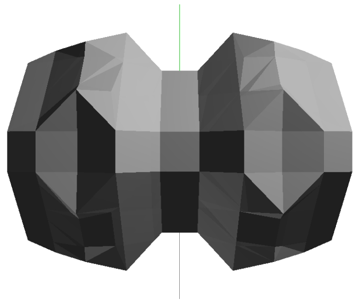

Figure 4: Low resolution ($r = 10$) surface for the iterative equation is $Z = Z\cos(Z)$. The surface's dimension is 2.05.

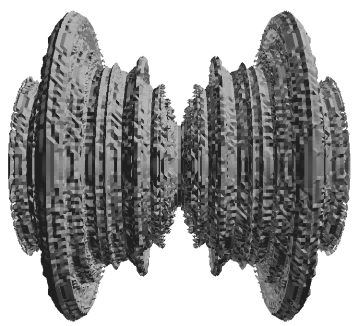

Figure 5: Medium resolution ($r = 100$) surface for the iterative equation is $Z = Z\cos(Z)$. The surface's dimension is 2.11.

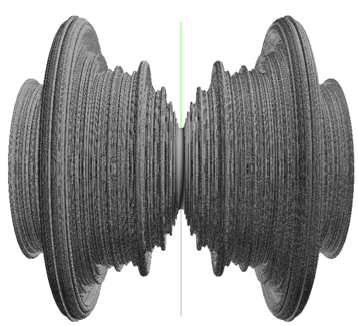

Figure 6: High resolution ($r = 1000$) surface for the iterative equation is $Z = Z\cos(Z)$. The surface's dimension is 2.08.

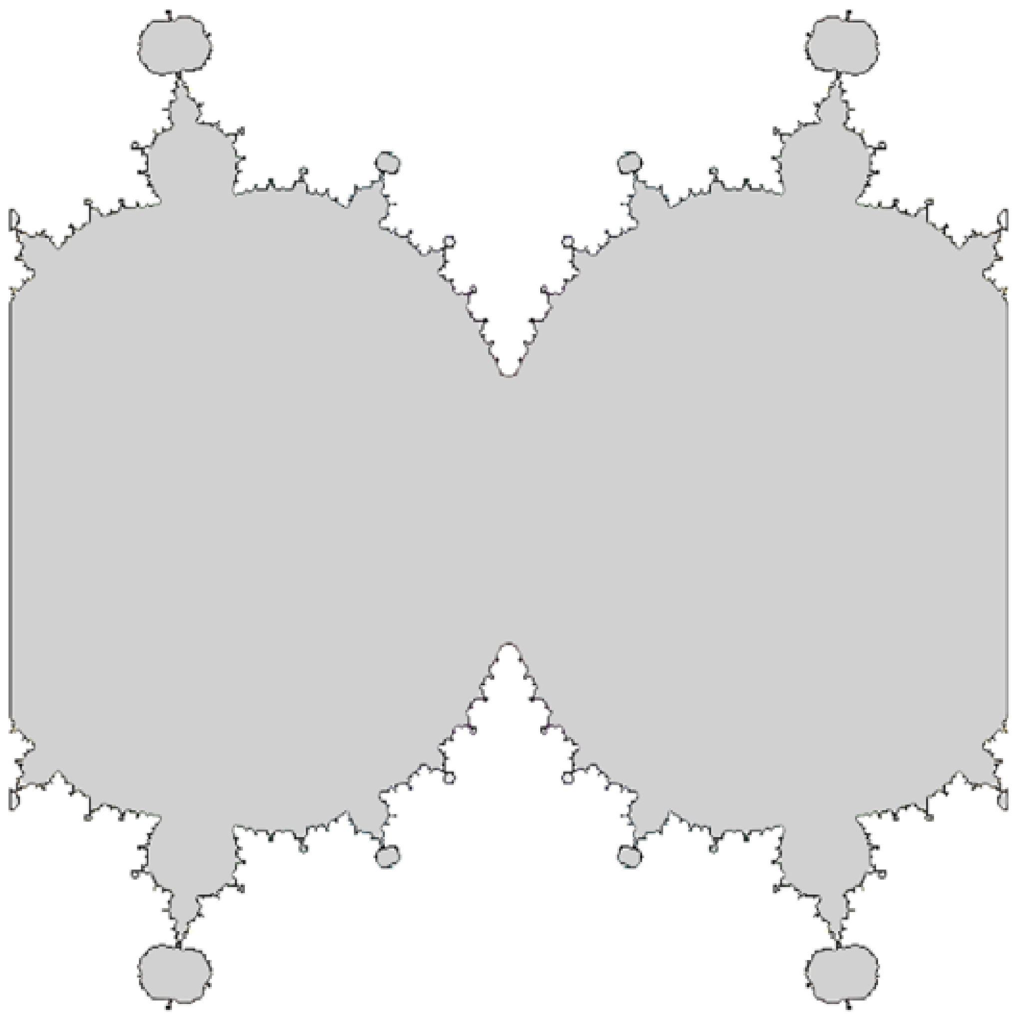

Figure 7: A two-dimensional slice of $Z = Z\cos(Z)$, showing the fractal nature of the set.

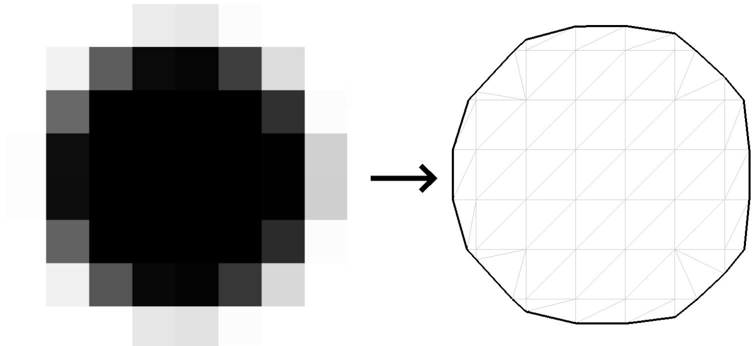

Figure 8: Example input (a two-dimensional greyscale image) and output (a $2.x$-dimensional closed set of line segments) of the Marching Squares algorithm, approximating a 1-sphere (a circle), where sampling resolution is $r = 8$.

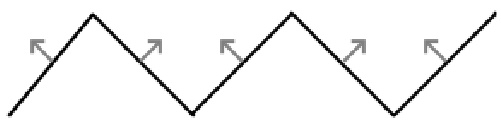

Figure 9: Illustrated is a section of a closed line path, with surface normals. The average dot product of neighbouring line segments is $d_i = 0.0$. This leads to a normalized measure of curvature $k_i = (1 - d_i)/2 = 0.5$, which in turn leads to an average normalized measure of curvature $K = 0.5$. The dimension is $D = 1 + K = 1.5$.

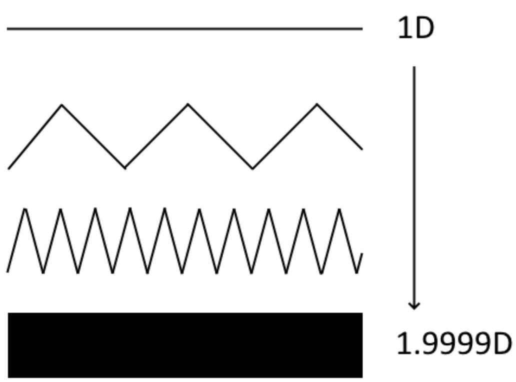

Figure 10: A section of a closed line path as it goes from dimension 1.0 (at top) to 1.9999 (at bottom). In the end, where the dimension is 1.9999, the result is practically a rectangle. The reason why the dimension cannot be 2.0 is because that would lead to intersecting line segments.