

# The curvature and dimension of a closed surface

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## Abstract

In this short memorandum, the curvature and dimension properties of the 2-sphere surface of a 3-dimensional ball and the  $2.x$ -dimensional surface of a 3-dimensional fractal set are considered. Tessellation is used to approximate each surface, primarily because the  $2.x$ -dimensional surface of a 3-dimensional fractal set is otherwise non-differentiable (having no well-defined surface normals). It is found that the curvature of a closed surface *must* lead to fractional dimension.

## 1 Overview

Unlike in traditional geometry where dimension is an integer (e.g.  $(3+1)$ -dimensional space-time), fractional (non-integer) dimension occurs in *fractal* geometry. In fractal geometry, there are currently many ways to calculate the dimension of a surface [1, 2]. This memo uses a new method of calculating the fractional dimension of a surface – it is *curvature* that leads to this fractional dimension.

Our main focus will be on the curvature and dimension of tessellated closed surfaces. For example, Marching Cubes [3, 4] can be used to generate triangular tessellations (meshes), where dimension  $D \in (2.0, 3.0)$ .

We will draw attention to the difference in curvature and dimension between a 2-sphere and the  $2.x$ -dimensional surface of a 3-dimensional fractal set. We will generate both a 2-sphere and the  $2.x$ -dimensional surface of a 3-dimensional fractal set by using iterative quaternion equations.

Some notes are given at the end of this memo.

## 2 The tessellation of a closed surface

Approximating the surface of a 3-dimensional shape as a mesh allows us to calculate the surface's dimension  $D \in (2.0, 3.0)$ . This includes approximation of both a 2-sphere and the  $2.x$ -dimensional surface of a 3-dimensional fractal set.

First we calculate, for each triangle, the average dot product of the triangle's face normal  $\hat{n}_i$  and its 3 neighbouring triangles' face normals  $\hat{o}_1, \hat{o}_2, \hat{o}_3$ :

$$d_i = \frac{\hat{n}_i \cdot \hat{o}_1 + \hat{n}_i \cdot \hat{o}_2 + \hat{n}_i \cdot \hat{o}_3}{3} \in (-1.0, 1.0]. \quad (1)$$

Because we assume that there are 3 neighbours per triangle, the mesh must be *closed* (no cracks or holes, precisely two triangles per edge). The reason why the value  $-1.0$  is not achievable is because that would lead to intersecting triangles.

Then we calculate the normalized measure of curvature:

$$k_i = \frac{1 - d_i}{2} \in [0.0, 1.0). \quad (2)$$

Once  $k_i$  has been calculated for all triangles, we can then calculate the average normalized measure of curvature  $K$ , where  $t$  is the number of triangles in the mesh:

$$K = \frac{1}{t} \sum_{i=1}^t k_i = \frac{k_1 + k_2 + \dots + k_t}{t} \in (0.0, 1.0). \quad (3)$$

The reason why the value  $0.0$  is not achievable is because we are dealing with a closed surface, and so there's bound to be *some* curvature.

The dimension of the closed surface is:

$$D = 2 + K \in (2.0, 3.0). \quad (4)$$

As far as we know, this method of calculating the dimension of a closed surface is new [5, 6]. The entire C++ code for generating a mesh can be found at [7]. The entire C++ code for calculating a mesh's dimension can be found at [8].

### 3 Vanishing versus non-vanishing curvature

Where  $r \in [2, \infty)$  is the *integer* sampling resolution,  $g_{\max} \in (-\infty, \infty)$  is the sampling grid maximum extent,  $g_{\min} \in (-\infty, \infty)$  is the sampling grid minimum extent, and  $g_{\max} > g_{\min}$ , the Marching Cubes step size is:

$$\ell = \frac{g_{\max} - g_{\min}}{r - 1} \in (0.0, \infty). \quad (5)$$

In this memo  $g_{\max} = 1.5$ ,  $g_{\min} = -1.5$ , and  $r$  is variable.

On one hand, a 2-sphere can be generated by the iterative quaternion Julia set equation

$$Z = Z^2 + C, \quad (6)$$

where the translation constant is  $C = 0.0, 0.0, 0.0, 0.0$ . For a 2-sphere, the *local* curvature all but vanishes as  $\ell$  decreases (as  $r$  increases):

$$\lim_{\ell \rightarrow 0.0} K(\ell) = 0.0. \quad (7)$$

This results in a dimension of practically (but never quite)  $2.0$ , which is to be expected from a non-fractal surface. See Figures 1 - 3.

On the other hand, the  $2.x$ -dimensional surface of a 3-dimensional fractal set can be generated by the iterative quaternion equation

$$Z = Z \cos(Z). \quad (8)$$

For the  $2.x$ -dimensional surface of a 3-dimensional fractal set, the local curvature does not necessarily vanish as  $\ell$  decreases:

$$\lim_{\ell \rightarrow 0.0} K(\ell) \neq 0.0. \quad (9)$$

This results in a dimension considerably greater than 2.0 (but not equal to or greater than 3.0), which is to be expected from a fractal surface. See Figures 4 - 7.

## 4 Notes

1. See [9] for information on how to perform quaternion multiplication, cos, etc.
2. See [10] for information on iterative quaternion equations.
3. Marching *Squares* [11, 12, 13] can be used to generate closed line paths, where dimension  $D \in (1.0, 2.0)$ . See Figures 8 - 10 for some examples of a line path. These figures might be helpful if there is difficulty envisioning the curvature and dimension in the case of Marching Cubes.
4. The triangle edge length produced by Marching Cubes is related to the space diagonal of a cube:  $\ell_E \in (0, \sqrt{3}\ell]$ . For Marching Squares, it is related to the face diagonal of a square:  $\ell_E \in (0, \sqrt{2}\ell]$ .
5. There is research on Marching Hypercubes at [14, 15]: where dimension  $D \in (3.0, 4.0)$ , the output is a closed set of tetrahedra. As local curvature all but vanishes, the tetrahedra become as close to regular as possible. For instance, where  $P_{tc}$  is the tetrahedron centre,  $P_{fc}$  is a face centre, and  $P_{ntc}$  is a neighbouring tetrahedron centre:

$$\hat{A}_0 = \text{normalize}(P_{tc} - P_{fc}), \quad (10)$$

$$\hat{B}_0 = -\text{normalize}(P_{ntc} - P_{fc}), \quad (11)$$

and so on and so forth for the other three neighbouring tetrahedra. The value of  $d_i$  is:

$$d_i = \frac{\hat{A}_0 \cdot \hat{B}_0 + \hat{A}_1 \cdot \hat{B}_1 + \hat{A}_2 \cdot \hat{B}_2 + \hat{A}_3 \cdot \hat{B}_3}{4} \in (-1.0, 1.0]. \quad (12)$$

## References

- [1] <http://paulbourke.net/fractals/fracdim/>
- [2] [https://en.wikipedia.org/wiki/Fractal\\_dimension](https://en.wikipedia.org/wiki/Fractal_dimension)
- [3] Lorensen, W. E.; Cline, Harvey E. (1987). "Marching cubes: A high resolution 3d surface construction algorithm". *ACM Computer Graphics*. 21 (4): 163–169
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- [6] Mandelbrot, B. (1982). "The Fractal Geometry of Nature". ISBN 978-0716711865.
- [7] [https://github.com/sjhalayka/marching\\_cubes](https://github.com/sjhalayka/marching_cubes)
- [8] <https://github.com/sjhalayka/meshdim>
- [9] <http://www.theworld.com/~sweetser/quaternions/intro/tools/tools.html>
- [10] <http://paulbourke.net/fractals/quatjulia/>
- [11] Maple, C. (2003). Geometric design and space planning using the marching squares and marching cube algorithms. *Proc. 2003 Intl. Conf. Geometric Modeling and Graphics*. pp. 90–95
- [12] [https://en.wikipedia.org/wiki/Marching\\_squares](https://en.wikipedia.org/wiki/Marching_squares)
- [13] <https://github.com/sjhalayka/Marching-Squares>
- [14] <http://web.cse.ohio-state.edu/~wenger.4/papers/isosurfaceD.v3.pdf>
- [15] <http://web.cse.ohio-state.edu/~wenger.4/papers/isosurface.pdf>



Figure 1: Low resolution ( $r = 10$ ) surface for the iterative quaternion equation  $Z = Z^2 + C$ , where  $C = 0.0, 0.0, 0.0, 0.0$ . The surface's dimension is 2.02.

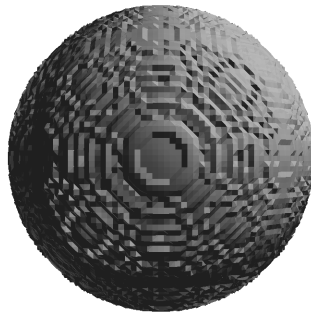


Figure 2: Medium resolution ( $r = 100$ ) surface for the iterative quaternion equation  $Z = Z^2 + C$ , where  $C = 0.0, 0.0, 0.0, 0.0$ . The surface's dimension is 2.06.

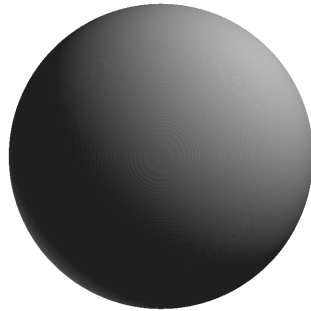


Figure 3: High resolution ( $r = 1000$ ) surface for the iterative quaternion equation  $Z = Z^2 + C$ , where  $C = 0.0, 0.0, 0.0, 0.0$ . The surface's dimension is practically 2.0.

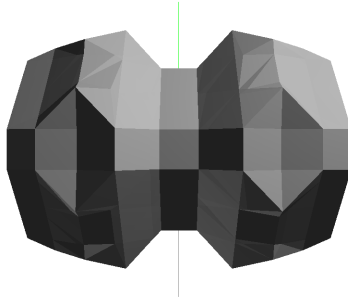


Figure 4: Low resolution ( $r = 10$ ) surface for the iterative quaternion equation  $Z = Z \cos(Z)$ . The surface's dimension is 2.05.

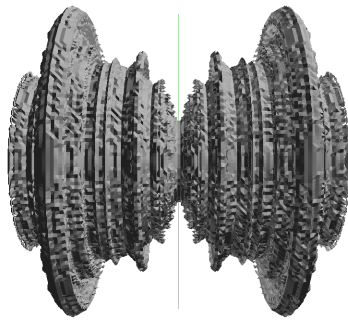


Figure 5: Medium resolution ( $r = 100$ ) surface for the iterative quaternion equation  $Z = Z \cos(Z)$ . The surface's dimension is 2.11.

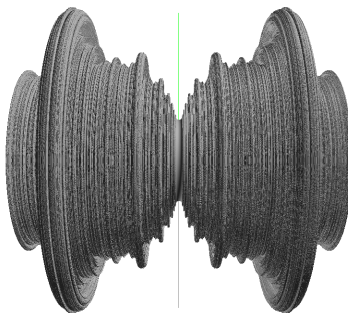


Figure 6: High resolution ( $r = 1000$ ) surface for the iterative quaternion equation  $Z = Z \cos(Z)$ . The surface's dimension is 2.08.

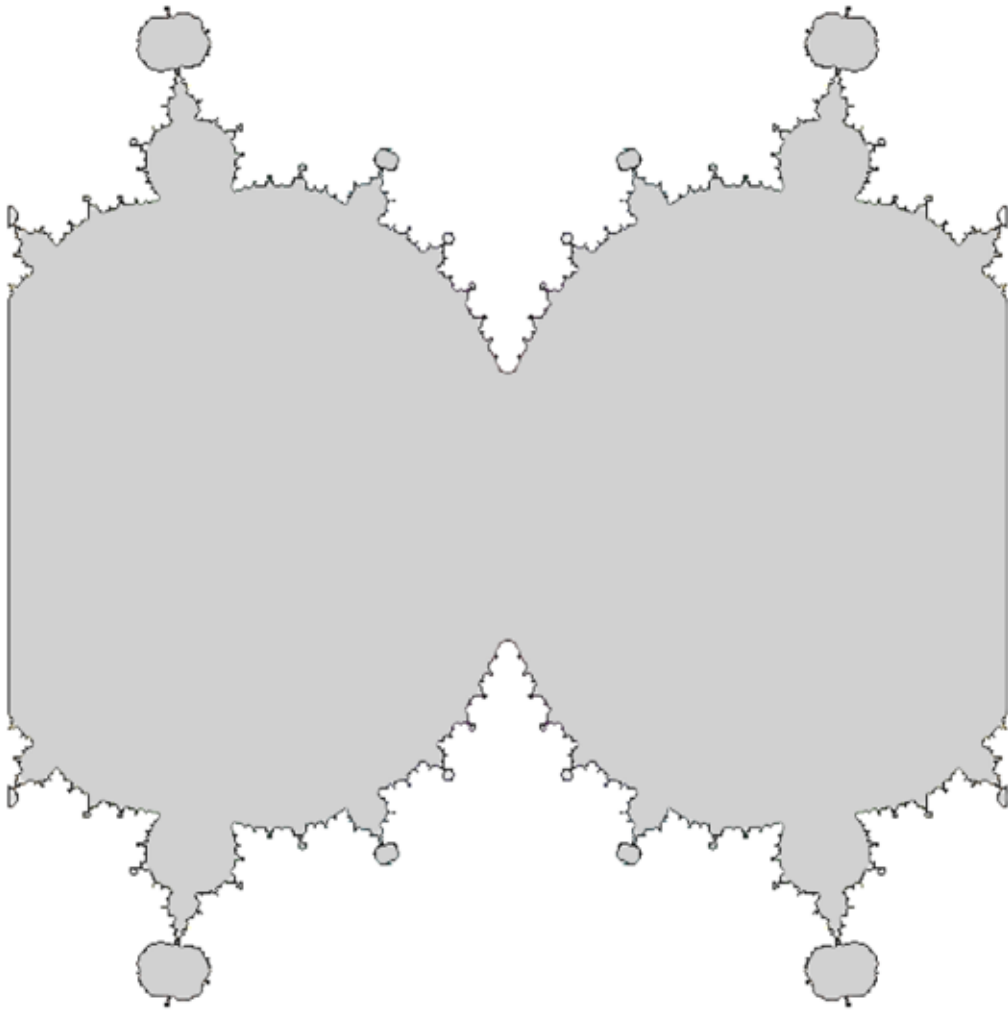


Figure 7: A 2-dimensional slice of the iterative quaternion equation  $Z = Z \cos(Z)$ , showing the self-similar nature of the set at all scales.

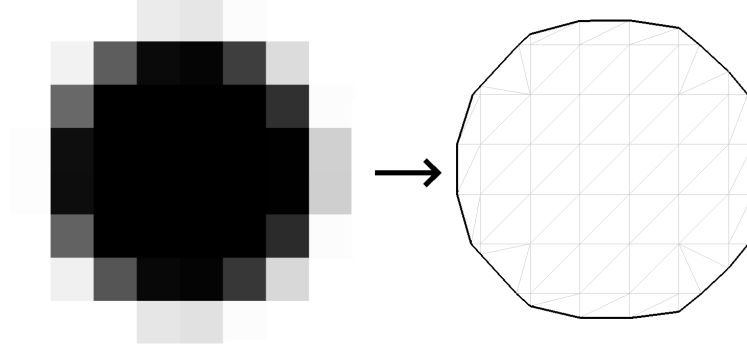


Figure 8: Example input (a 2-dimensional greyscale image, consisting of 8x8 pixels) and output (a 1.x-dimensional closed set of line segments) of the Marching Squares algorithm, approximating a 1-sphere (a circle), where sampling resolution is  $r = 8$ . Note that for Marching Cubes, the input is a 3-dimensional ‘greyscale image’, consisting of voxels, and the output is a 2.x-dimensional closed set of triangles.



Figure 9: Illustrated is a section of a closed line path, with surface normals. The average dot product of neighbouring line segments is  $d_i = 0.0$ . This leads to a normalized measure of curvature  $k_i = (1 - d_i)/2 = 0.5$ , which in turn leads to an average normalized measure of curvature  $K = 0.5$ . The dimension is  $D = 1 + K = 1.5$ . Note that for Marching Cubes, the dimension is  $D = 2 + K$ .

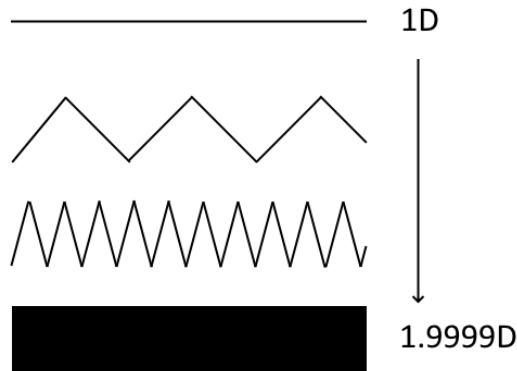


Figure 10: A section of a closed line path as it goes from dimension 1.0 (at top) to 1.9999 (at bottom). In the end, where the dimension is 1.9999, the result is practically a rectangle. The reason why the dimension cannot be 2.0 is because that would lead to intersecting line segments. Note that for Marching Cubes, the dimension cannot be 3.0 because that would lead to intersecting triangles.