The curvature and dimension of a closed surface

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October 4, 2019

Abstract

The curvature of a closed surface can lead to fractional dimension. In this paper, the properties of the 2-sphere surface of a three-dimensional ball and the $2.x$-dimensional surface of a three-dimensional fractal set are considered. Tessellation is used to approximate each surface, primarily because the $2.x$-dimensional surface of a three-dimensional fractal set is otherwise non-differentiable (having no well-defined surface normals).

1 The tessellation of a closed surface

Approximating the surface of a three-dimensional shape via triangular tessellation (a mesh) allows us to calculate the surface's dimension $D \in (2.0, 3.0)$.

First we calculate, for each triangle, the average dot product of the triangle's face normal $\hat{n}_i$ and its three neighbouring triangles' face normals $\hat{o_1}$, $\hat{o_2}$, $\hat{o_3}$:

$$d_i = \frac{\hat{n}_i \cdot \hat{o_1} + \hat{n}_i \cdot \hat{o_2} + \hat{n}_i \cdot \hat{o_3}}{3} \in (-1.0, 1.0]. \tag{1}$$

Because we assume that there are three neighbours per triangle, the mesh must be *closed* (no cracks or holes, precisely two triangles per edge). The reasion why the value -1.0 is not achievable is because that would lead to intersecting triangles.

Then we calculate the normalized measure of curvature:

$$k_i = \frac{1 - d_i}{2} \in [0.0, 1.0). \tag{2}$$

Once k_i has been calculated for all triangles, we can then calculate the average normalized measure of curvature K, where t is the number of triangles in the mesh:

$$K = \frac{1}{t} \sum_{i=1}^{t} k_i = \frac{k_1 + k_2 + ... + k_t}{t} \in (0.0, 1.0). \tag{3}$$

The reason why the value 0.0 is not achievable is because we are dealing with a closed surface, and so there's bound to be *some* curvature.

The dimension of the closed surface is:

$$D = 2 + K \in (2.0, 3.0). \tag{4}$$

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2 Marching Cubes

In this paper, Marching Cubes [1, 2] is used to generate the $2.x$-dimensional closed meshes. The entire C++ code for generating a mesh can be found at [3]. The entire C++ code for calculating a mesh's dimension can be found at [4].

Where $r \in [2, \infty)$ is the *integer* sampling resolution, $g_{\max} \in (-\infty, \infty)$ is the sampling grid maximum extent, $g_{\min} \in (-\infty, \infty)$ is the sampling grid minimum extent, and $g_{\max} > g_{\min}$, the Marching Cubes step size is:

$$\ell = \frac{g_{\max} - g_{\min}}{r - 1} \in (0.0, \infty). \tag{5}$$

In this paper $g_{\max} = 1.5$, $g_{\min} = -1.5$, and r is variable.

Analogously, Marching Squares [5] can be used to generate $1.x$-dimensional closed line paths, where $D \in (1.0, 2.0)$. See Figures 1 - 3 for some examples of a line path.

3 Conclusions

For a 2-sphere, the *local* curvature all but vanishes as ℓ decreases (e.g. as r increases):

$$\lim_{\ell \to 0.0} K(\ell) = 0.0. \tag{6}$$

This results in a dimension of practically (but never quite) 2.0, which is to be expected from a non-fractal surface. See Figures 4 - 6.

On the other hand, for the $2.x$-dimensional surface of a three-dimensional fractal set, the local curvature does not vanish as ℓ decreases:

$$\lim_{\ell \to 0.0} K(\ell) \neq 0.0. \tag{7}$$

This results in a dimension considerably greater than 2.0, but not equal to or greater than 3.0, which is to be expected from a fractal surface. See Figures 7 - 10.

As far as we know, this method of calculating the dimension of a closed surface is new.

References

[1] Lorensen, W. E.; Cline, Harvey E. (1987). "Marching cubes: A high resolution 3d surface construction algorithm". ACM Computer Graphics. 21 (4): 163–169

[2] `http://paulbourke.net/geometry/polygonise/`

[3] `https://github.com/sjhalayka/marching_cubes`

[4] `https://github.com/sjhalayka/meshdim`

[5] `https://en.wikipedia.org/wiki/Marching_squares`

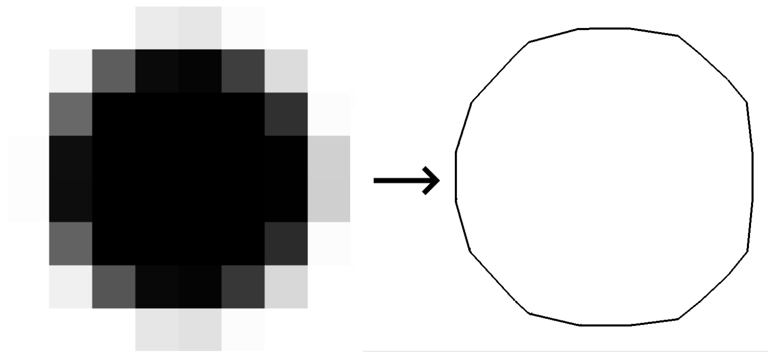

Figure 1: Example input and output of the Marching Squares algorithm, approximating a 1-sphere (a circle), where $r = 10$.

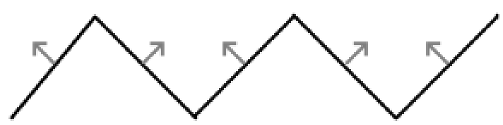

Figure 2: Illustrated is a section of a closed line path, with surface normals. The average dot product of neighbouring line segments is $d_i = 0.0$. This leads to a normalized measure of $k_i = (1 - d_i)/2 = 0.5$, which in turn leads to an average normalized measure of $K = 0.5$. The dimension is $D = 1 + K = 1.5$.

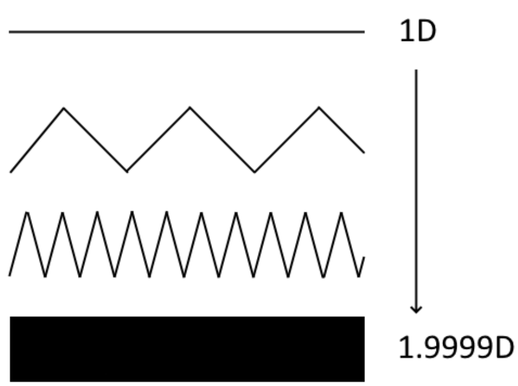

Figure 3: A section of a closed line path as it goes from dimension 1.0 (at top) to 1.9999 (at bottom). In the end, where the dimension is 1.9999, the result is practically a rectangle. The reason why the dimension cannot be 2.0 is because that would lead to intersecting line segments.

Figure 4: Low resolution ($r = 10$) surface for the iterative equation is $Z = Z^2$. The surface's dimension is 2.02.

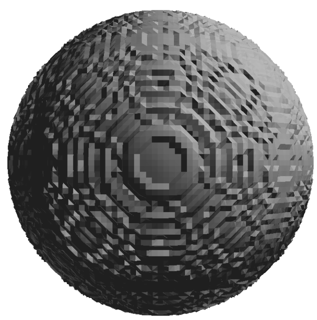

Figure 5: Medium resolution ($r = 100$) surface for the iterative equation is $Z = Z^2$. The surface's dimension is 2.06.

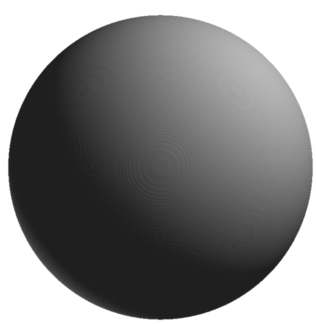

Figure 6: High resolution ($r = 1000$) surface for the iterative equation is $Z = Z^2$. The surface's dimension is practically 2.0.

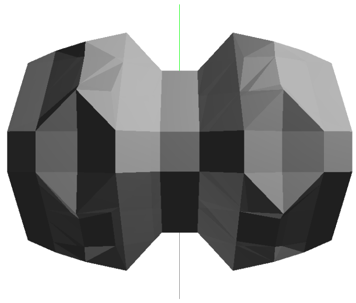

Figure 7: Low resolution ($r = 10$) surface for the iterative equation is $Z = Z\cos(Z)$. The surface's dimension is 2.05.

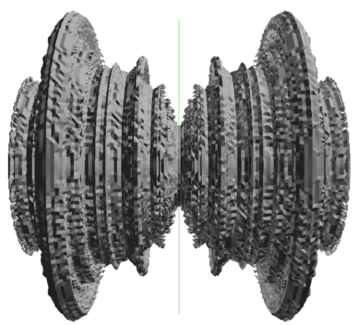

Figure 8: Medium resolution ($r = 100$) surface for the iterative equation is $Z = Z\cos(Z)$. The surface's dimension is 2.11.

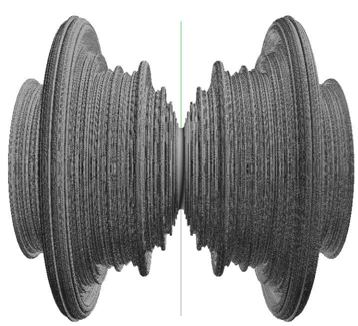

Figure 9: High resolution ($r = 1000$) surface for the iterative equation is $Z = Z\cos(Z)$. The surface's dimension is 2.08.

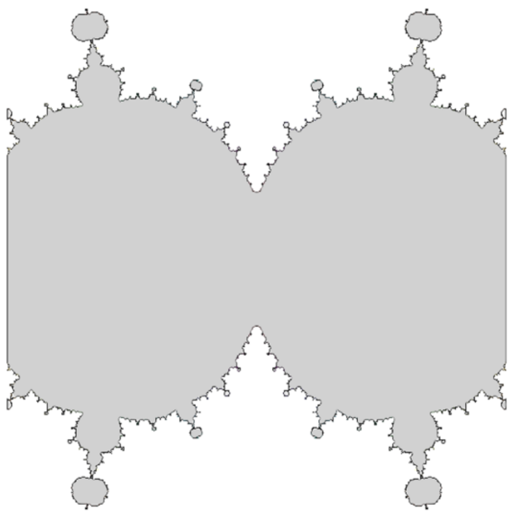

Figure 10: A two-dimensional slice of $Z = Z\cos(Z)$, showing the fractal nature of the set.