Exercise 5.10 b.)

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b.) i.)

$$\mu(t) = \int_\Omega u(x,t)\,dx \overset{!}{=} \text{const.}$$

$$\Rightarrow \frac{\partial \mu(t)}{\partial t} \overset{!}{=} 0 = \frac{\partial}{\partial t}\int_\Omega u(x,t)\,dx$$

$$= \int_\Omega \underbrace{\frac{\partial}{\partial t} u(x,t)}_{=\partial_t u =}\,dx$$

$$= \int_\Omega \operatorname{div}(g(|\nabla u|)\cdot\nabla u)\,dx \qquad \Big|\; F := g(|\nabla u|)\cdot\nabla u$$

$$= \int_{\partial\Omega} \langle F,\nu\rangle\,ds$$

mit: $\langle F,\nu\rangle =$

$$= \langle g(|\nabla u|)\cdot\nabla u,\ \nu\rangle \qquad \Big|\; g \in \mathbb{R}$$

$$= g(|\nabla u|)\cdot\underbrace{\langle\nabla u,\nu\rangle}_{=\frac{\partial u}{\partial\nu} = 0}$$

$$\langle F,\nu\rangle = \underline{\underline{0}}$$

$$\Rightarrow \int_{\partial\Omega}\langle F,\nu\rangle\,dS = 0 \quad \square$$

b.) ii.)

$$\frac{\partial h(t)}{\partial t} \overset{!}{\leq} 0$$

$$\Rightarrow \frac{\partial}{\partial t}\int_\Omega u(x,t)^2\,dx = \int_\Omega \frac{\partial}{\partial t}\left((u(x,t))^2\right)dx$$

$$= \int_\Omega \frac{\partial}{\partial t}\left(u(x,t)\,u(x,t)\right)dx$$

$$= \int_\Omega 2\cdot\frac{\partial u(x,t)}{\partial t}\cdot u(x,t)\,dx$$

$$= 2\int_\Omega \frac{\partial u(x,t)}{\partial t}\cdot u(x,t)\,dx$$

$$= 2\int_\Omega \operatorname{div}(g(|\nabla u|)\cdot\nabla u)\cdot u(x,t)\,dx \qquad \Big|\; g\in\mathbb{R}$$

$$\nabla u = \begin{pmatrix}\frac{\partial u}{\partial x_1}\\ \vdots \\ \frac{\partial u}{\partial x_d}\end{pmatrix} \Rightarrow \operatorname{div}(\nabla u) = \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}$$

$$= 2\int_\Omega g(|\nabla u|)\cdot\sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}\,u(x,t)\,dx$$

$$\Rightarrow 0 \geq \int_\Omega g(|\nabla u|)\cdot\sum_{i=1}^d\frac{\partial^2 u}{\partial x_i^2}\,u(x,t)\,dx$$

mit: $\int_\Omega (\partial_i f)\cdot g\,dx = -\int_\Omega f\,\partial_i g\,dx + \int_{\partial\Omega} f g\,\nu_i\,dS$

$$\int_\Omega\left(\sum_{i=1}^d\frac{\partial}{\partial x_i}\cdot\frac{\partial u}{\partial x_i}\right)\cdot g(|\nabla u|)\cdot u(x,t)\,dx = -\int_\Omega\sum_{i=1}^d\frac{\partial u}{\partial x_i}\cdot\left(\sum_{i=1}^d\frac{\partial}{\partial x_i}\right)\left(g(|\nabla u|)\cdot u(x,t)\right)dx + \underbrace{\int_{\partial\Omega}\sum_{i=1}^d\left(\frac{\partial u}{\partial x_i}\right)\cdot\left(g(|\nabla u|)\cdot u(x,t)\right)\cdot\sum_{i=1}^d \nu_i\,dS}_{\underbrace{\int_{\partial\Omega}\underbrace{\sum_{i=1}^d\left(\frac{\partial u}{\partial x_i}\cdot\nu_i\right)}_{=\frac{\partial u}{\partial\nu}=0}\cdot g(|\nabla u|)\cdot u(x,t)\,ds}_{=0}}$$

$$\Rightarrow = -\int_\Omega \sum_{i=1}^d\frac{\partial u}{\partial x_i}\cdot\frac{\partial u}{\partial x_i}\cdot g(|\nabla u|)) + \underbrace{\sum_{i=1}^d\frac{\partial u}{\partial x_i}\cdot\frac{\partial g(|\nabla u|)}{\partial x_i}}_{=0}\cdot u(x,t)\,dx$$

$$= -\underbrace{\int_\Omega\left(\sum_{i=1}^d\frac{\partial u}{\partial x_i}\right)^2\cdot g(|\nabla u|)\,dx}_{\geq 0}$$

$$= - \int_\Omega \underbrace{\left(\sum_{i=1} \frac{\cdots}{\partial x_i} \right)^- \cdot g(|\nabla u|)\,dx}_{\geq 0}$$

$$\leq 0 \qquad \square$$