Homework 2: Due November 22

Reading: Read chapter 2 of course notes.

1. (a) Consider the complete graph on 6 vertices. Construct the adjacency matrix and the Laplace matrix for this graph. (b) Compute the spectrum of G.

2. Let G be a finite graph on vertex set $\{1, 2, \ldots, n\}$ and A be its adjacency matrix. Prove that the (i, j)-entry of A^k counts the number of walks of length k from vertex i to vertex j in G. This proves Lemma 2.8 of the course notes.

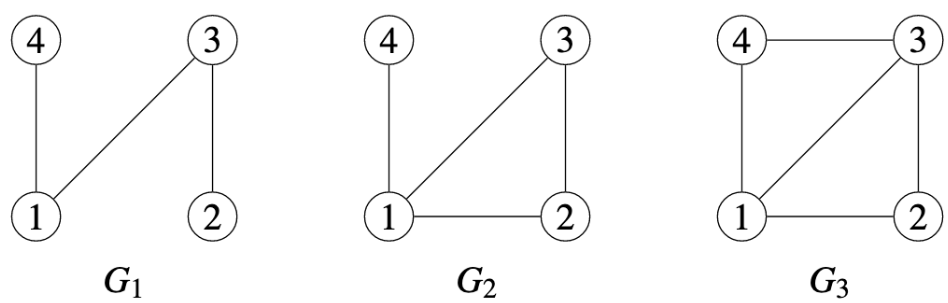
3. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Compute the $(1, 1)$-entry of A^k for any $k \geq 1$ without computing A^k explicitly.

4. **Hand In.** Exercise 2.10 in course notes: Let $G = (V, E)$ be a finite graph on n vertices. We call $H = (W, F)$ a subgraph of G when $W \subseteq V$ and $F \subseteq E$. If H is connected, contains exactly $n - 1$ edges, and $V = W$, we say that H is a spanning tree for G.

   Let G_1, G_2, G_3 be the graphs on vertex set $\{1, 2, 3, 4\}$ drawn in the picture below.

   (a) List all spanning trees for G_1, G_2 and G_3. (You can either draw them or list their edges.) How many do you get?

   (b) Compute the number of spanning trees for G_1, G_2, G_3 using Theorem 2.27.



5. Let G and H be graphs with

$$V(G) = V(H) = \{1, 2, 3, 4\} \quad E(G) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\} \quad E(H) = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}.$$

   Are G and H isomorphic? (One says that G and H are isomorphic if they are "the same up to relabeling", i.e. there exists a bijective function $f : V(G) \to V(H)$ such that $\{v, w\} \in E(G) \Leftrightarrow \{f(v), f(w)\} \in E(H)$.)

   Compute explicitly the spectrum of the Laplacian of G and of the Laplacian of H. If you had access to spectral information only, would you be able to tell G and H apart?

6. Exercise 2.11 of course notes. Prove Theorem 2.43 (The Metropolis Hastings Algorithm). Hint: Use Proposition 2.37.

7. Compute all of the centrality measures from class for the graph G_2 from exercise 4.

8. The file `users_hashtags.txt` contains Twitter data structured like this: the first entry in every block of lines is the author of a tweet, while the other entries are the hashtags used in that tweet. Different blocks are separated by empty lines. Using `Julia`, build a network in the following way:

   vertices are (unique) users and (unique) hashtags; the user U and the hashtag H are connected by an edge if and only if a tweet by U contains H. There are no edges between any two users or any two hashtags (in particular, the network is bipartite in a natural way).

   After removing vertices of degree zero, compute the spectrum of the Laplacian of the graph. Explain the meaining of the spectrum by referring to the various results on the spectrum from class.