Homework 2: Hints

1. (a) Consider the complete graph on 6 vertices. Construct the adjacency matrix and the Laplace matrix for this graph. (b) Compute the spectrum of G.

   Note in the adjacency matrix, every pair of edges is in G, so there are 0s on the diagonal, and 1 on the off diagonal. Similarly for the Laplace matrix, the degree of every vertex is 5:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \quad L = (\ell_{ij}) \in \mathbb{R}^{6\times 6} \text{ where } \ell_{ij} = \begin{cases} 1 & i = j \\ -\frac{1}{5} & i \neq j \end{cases}$$

   The plugging in L into for example Juila, we see that the eigenvalues are $\lambda_0 = 0$, and $\lambda_k = 6/5$ for $k = 1, \ldots, 5$.

2. Let G be a finite graph on vertex set $\{1, 2, \ldots, n\}$ and A be its adjacency matrix. Prove that the (i, j)-entry of A^k counts the number of walks of length k from vertex i to vertex j in G. This proves Lemma 2.8 of the course notes.

   Proof. We proceed by induction on k. For the base case, when $k = 1$, there are no walks from a vertex to itself of length 1, and the only walks of length 1 occurs when there is an edge connecting the two vertices. This is exactly the definition of A.

   For the induction step, suppose $(A^{k-1})_{ij}$ counts the number of walks of length $k-1$ from vertex i to vertex j. Then

$$(A^k)_{ij} = \sum_{\ell=1}^{|V|} (A^{k-1})_{i\ell} A_{\ell j}.$$

   When $A_{\ell j} = 0$ it does not contribute to the sum, and when $A_{\ell j} \neq 0$, then the ℓ vertex is a neighbor of j. So to travel from i to j, we take all walks from i to ℓ and then each of those gives a path to j by connecting to its neighbor. Adding these possibilities for $\ell = 1, \ldots, |V|$ gives all possible walks. □

3. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Compute the $(1, 1)$-entry of A^k for any $k \geq 1$ without computing A^k explicitly.

   Using problem 2 above, the $(1, 1)$ entry of A^k is the number of walks of length k from vertex 1 to vertex 1. Since the entire graph is given by the edges $\{1, 2\}$ and $\{1, 3\}$, any path from 1 to 1 must cross to edge 2 or 3 and back, giving an even path length. Hence $(A^k)_{11} = 0$ when k is odd. When k is even, there are first two paths of length 2, since we can either do $1-2-1$ or $1-3-1$. Any longer paths are taken by making combinations of these two options. Hence $A^{2k} = 2^k$.

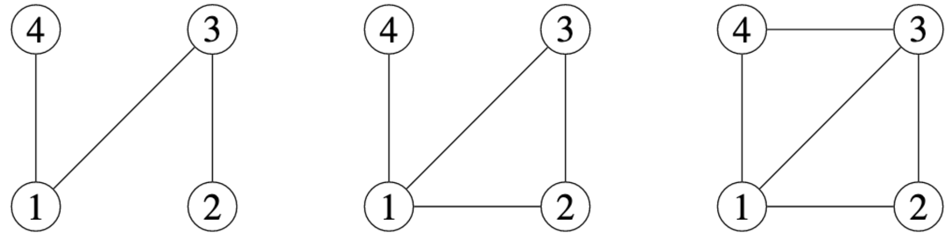
4. **Hand In.** Exercise 2.10 in course notes: Let $G = (V, E)$ be a finite graph on n vertices. We call $H = (W, F)$ a subgraph of G when $W \subseteq V$ and $F \subseteq E$. If H is connected, contains exactly $n-1$ edges, and $V = W$, we say that H is a spanning tree for G.

   Let G_1, G_2, G_3 be the graphs on vertex set $\{1, 2, 3, 4\}$ drawn in the picture below.

   (a) List all spanning trees for G_1, G_2 and G_3. (You can either draw them or list their edges.) How many do you get?

   (b) Compute the number of spanning trees for G_1, G_2, G_3 using Theorem 2.27.

   Answers are: 1,3, and 8, respectively, (a) and (b) gives two ways to confirm this.



G_1 G_2 G_3

5. Let G and H be graphs with

$$V(G) = V(H) = \{1,2,3,4\} \quad E(G) = \{\{1,2\},\{1,3\},\{1,4\}\} \quad E(H) = \{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}.$$

Are G and H isomorphic? (One says that G and H are isomorphic if they are "the same up to relabeling", i.e. there exists a bijective function $f\colon V(G) \to V(H)$ such that $\{v,w\} \in E(G) \Leftrightarrow \{f(v), f(w)\} \in E(H)$.)

Compute explicitly the spectrum of the Laplacian of G and of the Laplacian of H. If you had access to spectral information only, would you be able to tell G and H apart?

Hint Note that they cannot be isomorphic since G has 3 edges and H has 4 edges, so there's no bijective function of the edges. However H and G have the same spectrum with the 4 eigenvalues given by $(2,1,1,0)$.

6. Exercise 2.11 of course notes. Prove Theorem 2.43 (The Metropolis Hastings Algorithm). Hint: Use Proposition 2.37.

proof steps

(a) Show P defines a Markov process by verifying 1-3 of the definition (Definition 2.29)
(b) Verify aperiodicity and irreducibility by using Definition 3.39, and looking at what the powers of P can be.
(c) Lastly to see that π is stationary, by Proposition 2.37 we only need to show that π is reversible. Plug in using the definition of P to check reversibility.

7. Compute all of the centrality measures from class for the graph G_2 from exercise 4.

Answers: Note we write (a,b,c,d) to mean the measure at vertex 1 is a, at vertex 2 is b, and so on.

- $c_R = (1, \frac{2}{3}, \frac{2}{3}, \frac{1}{3})$
- $c_D = (3,2,2,1)$
- $c_C = (\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{5})$.
- $c_H = (3, \frac{5}{2}, \frac{5}{2}, 2)$.
- $c_B = (2,0,0,0)$.
- We will go through the example for finding

$$c_M(4) = \frac{1}{\mathbb{E}\tau(1,4) + \mathbb{E}\tau(2,4) + \mathbb{E}\tau(3,4)}.$$

First by symmetry we have $\mathbb{E}(\tau(3,4)) = \mathbb{E}(\tau(2,4))$. Also, we always arrive at 1 after the first step, and so can never arrive at 1 for the first time in later steps either, giving $\mathbb{E}\tau(1,4) = 1$.

To compute $\mathbb{E}(2,4)$, when $t=1$, the probability is zero. Then when $t = 2k$ for $k \geq 1$, then we must arrive at 2 for the first time from vertex 1. So the probability is $1/3$ times the probability we end up at 1 in $2k-1$ steps starting at vertex 4, only going between vertices $1,3,4$ We set $\tilde{P}$ to be the restriction of the Markov process to $1,3,4$. Similarly in the odd case when $t = 2k+1$ case, the probability that we go from 3 to 2 is $1/2(\tilde{P}^{2k})_{3,4}$. So we have

$$\mathbb{E}\tau(2,4) = \sum_{k=1}^{\infty} \frac{1}{2}(\tilde{P}^{2k})_{3,4} + \frac{1}{3}(\tilde{P}^{2k-1})_{1,4}.$$

We compute (via computer)

$$\tilde{P}^k = \begin{bmatrix} 0 & 1/2 & 1 \\ 1/3 & 0 & 0 \\ 1/3 & 0 & 0 \end{bmatrix}^k = \frac{2^{-k/2-1}}{3} \begin{bmatrix} 3((-1)^k+1) & -\frac{3(-1)^k-3}{\sqrt{2}} & -\sqrt{2}(3(-1)^k-3) \\ -\sqrt{2}((-1)^k-1) & ((-1)^{k+1}) & -2((-1)^{k+1}-1) \\ -\sqrt{2}((-1)^k-1) & (-1)^{k+2}+1 & 2((-1)^{k+1}+1) \end{bmatrix}$$

In the odd case, the $1,4$ position is

$$\frac{2^{-(2k-1)/2-1}}{3}(6\sqrt{2}) = 2^{-k}$$

and in the even case, the $3,4$ position is

$$\frac{2^{-k-1}}{3} \cdot 4 = \frac{2^{-k+1}}{3}.$$

Thus

$$\mathbb{E}\tau(2,4) = \sum_{k=1}^{\infty} \frac{2^{-k+1}}{3} = \frac{2}{3}\sum_{k=1}^{\infty} 2^{-k} = \frac{2}{3}$$

Putting it together,

$$c_M(4) = \frac{1}{1+2\mathbb{E}\tau(2,4)} = \frac{1}{1+4/3} = \frac{3}{7}.$$

8. The file `users_hashtags.txt` contains Twitter data structured like this: the first entry in every block of lines is the author of a tweet, while the other entries are the hashtags used in that tweet. Different blocks are separated by empty lines. Using `Julia`, build a network in the following way:

   vertices are (unique) users and (unique) hashtags; the user U and the hashtag H are connected by an edge if and only if a tweet by U contains H. There are no edges between any two users or any two hashtags (in particular, the network is bipartite in a natural way).

   After removing vertices of degree zero, compute the spectrum of the Laplacian of the graph. Explain the meaining of the spectrum by referring to the various results on the spectrum from class.

   See JuliaExercises/Homework2_Problem8.ipynb