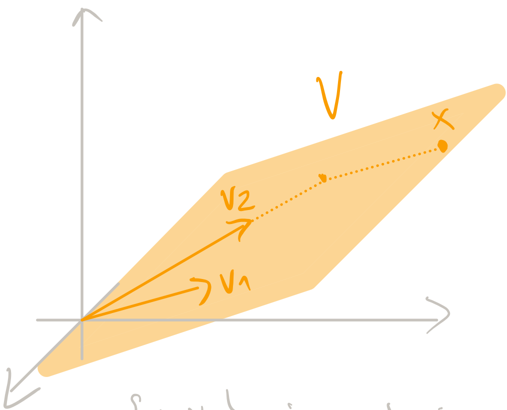
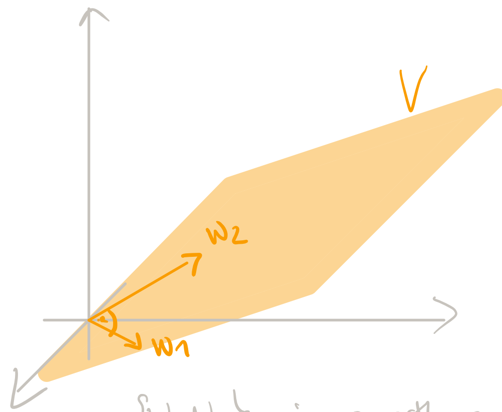



$\{v_1, v_2\}$ is a basis of $V$



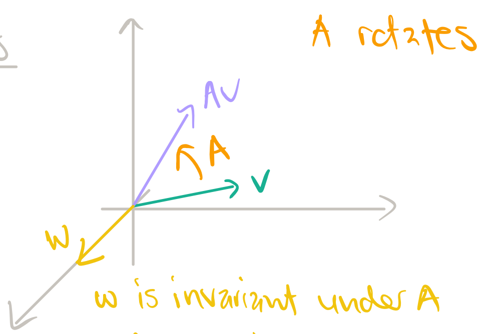

$\{w_1, w_2\}$ is an orthogonal basis of $V$

$\left\{\frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}\right\}$ is an orthonormal basis of $V$

$\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \frac{1}{\sqrt{8}}\begin{pmatrix}2\\-2\\0\end{pmatrix}, \frac{1}{5}\begin{pmatrix}0\\0\\5\end{pmatrix}\right\}$ orthonormal basis of $\mathbb{R}^3$.

## Matrices in $\mathbb{R}^{n\times n}$ $\quad m=n$

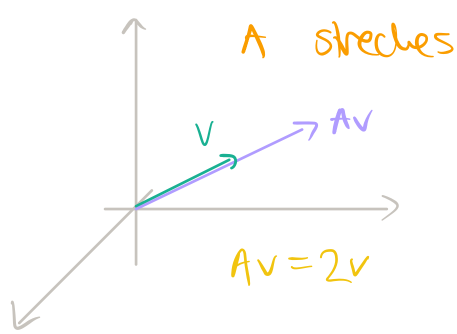
Examples




w is invariant under A

$Aw = w$

(A not sym, so only 1 EVe)




**Def.:** If a vector $w \in \mathbb{R}^n \setminus \{0\}$ fulfills $Aw = \lambda w$ for some $\lambda \in \mathbb{R}$, then $w$ is called an eigenvector and $\lambda$ an eigenvalue.

We can rewrite $Aw = \lambda w$

$\Leftrightarrow Aw - \lambda w = 0 \quad \Leftrightarrow (A - \lambda Id_n) w = 0$

$\Leftrightarrow (A - \lambda Id_n)$ is NOT invertible

$\Leftrightarrow \det(A - \lambda Id_n) = 0$

$f_A(\lambda)$ " characteristic polyn

Def: $A$ is symm. $\Leftrightarrow A = A^T$. $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} ✓$, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} ⚡$

"When choosing the correct system of coord., then a symm. matrix only stretches."

Thm (Spectral thm): Let $A \in \mathbb{R}^{n \times n}$ be symm. Then $A$ has only real EVa and $\exists$ basis $\{v_1, \dots, v_n\}$ of EVe s.t. $\langle v_i, v_j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}$. (orthonormal)

Moreover, if $Av_i = \lambda_i v_i$ and $V := [v_1 \cdots v_n]$, then

$$A = V \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} V^T.$$

e.g. $\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} = \mathrm{diag}(\lambda_1, \dots, \lambda_4)$

Proof: $f_A(t) := \det(A - t Id_n)$ charact. polyn. has at least one (complex) zero in $\mathbb{C}$.

$x^2+1$ has zeros $\pm i$

Inner product in $\mathbb{C}^n$: $\langle a, b \rangle_{\mathbb{C}} := \bar{a}^T \cdot b$

$a = (a_1, \dots, a_n)$, $a_i = a_i^1 + i a_i^2$

$b = (b_1, \dots, b_n)$, $b_i = b_i^1 + i b_i^2$

$$\Longrightarrow \langle a, b \rangle = \sum_{i=1}^{n} \bar{a}_i b_i = \sum_{i=1}^{n} (a_i^1 - i a_i^2)(b_i^1 + i b_i^2)$$

Step 1 We show: all EVa are real.

Case 1 $0$ is the ONLY zero of $f_A$

$\{\text{EVe of } A\} = \ker(A) \leftarrow$ find o.n.b. for $\ker(A)$.

Case 2 there is $\lambda \neq 0$ with $f_A(\lambda) = 0$. $\lambda \in \mathbb{C}$

Then $\exists\, \tilde{v} \neq 0$ with $A\tilde{v} = \lambda \tilde{v}$

Define $v := \frac{\tilde{v}}{\|\tilde{v}\|}$. Then still $Av = \lambda v$ & $\|v\|^2 = 1^2$.

In particular

$$\lambda = \lambda \underbrace{\langle v, v \rangle}_{=1} = \langle v, \lambda v \rangle = \langle v, Av \rangle$$

$$= \bar{v}^T (Av) = \overline{v^T \overline{(Av)}} = \overline{\langle Av, v \rangle} = \overline{\lambda \underbrace{\langle v, v \rangle}_{=1}} = \bar{\lambda}$$

As $\lambda = \bar{\lambda}$, it follows $\lambda \in \mathbb{R}$.

Hence, all EVa of $A$ are real.

Step 2 Find $v_1$: Either $v$ or $i \cdot v$ is real-valued, call it $v_1$.

Step 3 Find $v_2$:

Define $\mathcal{U} := (\mathbb{R} \cdot v_1)^\perp = \{ w \in \mathbb{R}^n : \langle w, t v_1 \rangle = 0 \ \forall t \in \mathbb{R} \}$.

In particular, for $t = \lambda$: $0 = \langle w, \lambda v_1 \rangle = \langle w, A v_1 \rangle$

$A = A^T$ & no complex numbers $= \langle Aw, v_1 \rangle$

Hence, $(Aw) \perp v_1 \implies (Aw) \in \mathcal{U}$.

We can define $A|_{\mathcal{U}} : \mathcal{U} \longrightarrow \mathcal{U}$, $w \longmapsto Aw$ and do the

Same argument as before: find EVa $v_2$ of $A|_{\mathcal{U}}$ with $\|v_2\| = 1$ and observe as

$$v_2 \in \mathcal{U} = (\mathbb{R} v_1)^\perp : \quad \langle v_1, v_2 \rangle = 0$$

Step 4 Conclude.

Do this $n$ times, i.e. define $\mathcal{U}_2 := (\mathbb{R} v_1 + \mathbb{R} v_2)^\perp$

$\rightsquigarrow$ get $v_3$ EVa orthogonal to both $v_1, v_2$

$\rightsquigarrow$ get $\{v_1, \dots, v_n\}$ EVa of $A$ and $\langle v_i, v_j \rangle = \delta_{ij}$. $\square$

Now: what if $m \neq n$? $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ symm does not make sense

Still we want

$$A = U \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix} V^T$$

What are $U, V, \sigma_i$? How to compute them?

Idea: $A \sim$ "$\sqrt{A^2}$" (whatever that means)

Observe $A^T \cdot A \in \mathbb{R}^{n \times n}$ always symm $\boxed{A} \cdot \boxed{A^T} = \boxed{A^TA}$

Spectral thm $\Rightarrow$ find an. EVe $\{v_1, \dots, v_n\}$ of $A^TA$ with EVa $\{\lambda_1, \dots, \lambda_n\}$.

Observe $\lambda_i = \lambda_i \langle v_i, v_i \rangle = \langle v_i, (A^TA) v_i \rangle$

$= \langle A v_i, A v_i \rangle = \|A v_i\|^2 \geq 0$

Enumerate them s.t. $\begin{cases} \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0 \\ \lambda_{r+1} = \dots = \lambda_n = 0 \end{cases}$ $rk(A)$

Then $0 = \lambda_{r+1} = \langle A v_{r+1}, A v_{r+1} \rangle \Rightarrow A v_{r+1} = 0$

$\Rightarrow v_{r+1} \in \ker(A)$

Hence $\operatorname{span}\{\underbrace{v_{r+1}, \dots, v_n}_{\text{o.n. basis for}}\} = \ker(A)$

$\operatorname{span}\{\underbrace{v_1, \dots, v_r}_{\text{o.n. basis for}}\}^\perp$ $\quad = \quad \underbrace{\operatorname{Im}(A^T)^\perp}$

Define $\quad u_i := \lambda_i^{-1/2} (A v_i) \quad$ for $i \le r$

They are orthogonal as

$$\langle u_i, u_j \rangle = (\lambda_i \lambda_j)^{-1/2} \langle A v_i, A v_j \rangle = (\lambda_i \lambda_j)^{-1/2} \langle v_i, (A^T A) v_j \rangle$$
$$= \left(\frac{\lambda_j}{\lambda_i}\right)^{1/2} \langle v_i, v_j \rangle = \underbrace{(\lambda_j/\lambda_i)^{1/2}}_{=1 \text{ if } i=j} \delta_{ij}$$

As $\operatorname{span}\{u_1, \dots, u_r\} \subseteq \operatorname{Im}(A)$ & same dim, it is an o.n.b.

Denote $\sigma_i := \sqrt{\lambda_i}$. Then

$$A v_i = \sigma_i u_i \quad \forall i \le r$$

and thus, as matrices $\underset{\in \mathbb{R}^{m \times r}}{U := [u_1, \dots, u_r]}$, $\underset{\in \mathbb{R}^{n \times r}}{V := [v_1, \dots, v_r]}$

we have $\quad A = U \Sigma V^T$, $\quad \Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}$ singular values

singular value decomposition

**Thm:** (Sing decom) Let $A \in \mathbb{R}^{m \times n}$ and $r := \operatorname{rk}(A)$. Then $\exists\, U \in \mathbb{R}^{m \times r}$, $V \in \mathbb{R}^{n \times r}$ with $\underbrace{U^T U = \mathrm{id}_r = V^T V}_{\text{orthog.}}$ s.t.

$$A = U \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix} V^T \quad \text{for \underline{unique} numbers } \sigma_1, \dots, \sigma_n > 0.$$

Moreover, $\begin{cases} \operatorname{Im}(A) = \operatorname{Im}(U) \\ \operatorname{Im}(A^T) = \operatorname{Im}(V) \end{cases}$ and if $\{\sigma_i\}_i$ are pairwise distinct,

we can order them $\sigma_1 > \dots > \sigma_r > 0$, then $U$ & $V$ are unique up to the sign of there columns.

END of the LECTURE

**Proof:** Existence & $\begin{cases} \mathrm{Im}(A) = \mathrm{Im}(U) \\ \mathrm{Im}(A^T) = \mathrm{Im}(V) \end{cases}$ is done above.

Assume it is not unique : $A = U \underbrace{\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}}_{=:\Sigma} V^T$

$= \tilde{U} \begin{pmatrix} \tilde{\sigma}_1 & & \\ & \ddots & \\ & & \tilde{\sigma}_r \end{pmatrix} \tilde{V}^T$

Then $AA^T = U \Sigma \underbrace{V^T V}_{id} \Sigma^T U^T = U \Sigma^2 U^T$,

$AA^T = \tilde{U} \Sigma \tilde{V}^T \tilde{V} \tilde{\Sigma} \tilde{U}^T = \tilde{U} \tilde{\Sigma}^2 \tilde{U}^T$.

Denote $U = [u_1, \dots, u_n]$

$\tilde{U} = [\tilde{u}_1, \dots, \tilde{u}_n]$

Then $(AA^T) u_i = \sigma_i^2 u_i$

$(AA^T) \tilde{u}_i = \tilde{\sigma}_i^2 \tilde{u}_i$ $\Rightarrow$ $\begin{cases} \sigma_i^2 \\ \tilde{\sigma}_i^2 \end{cases}$ are EV of $(AA^T)$

As the $u_i$, $\tilde{u}_i$ are orthonormal, $\{\sigma_1^2, \dots, \sigma_r^2\}$, $\{\tilde{\sigma}_1^2, \dots, \tilde{\sigma}_r^2\}$ are <u>all</u> EV of $(AA^T)$, so they are the same set.

Moreover, if they are distinct, they are all simple and thus the eigenspace is 1D & the EVe is unique $(\pm)$.

$\boxtimes$