Homework 3: Due December 7

Reading: Read section 3.1 and 3.2 of course notes.

1. **Hand in** Let G be a bipartite graph with bipartition $V_1 \sqcup V_2$ and without isolated vertices. Prove that, if λ is an eigenvalue of $L(G)$, then so is $2 - \lambda$.

   Hint: take an eigenvector f for λ and set $g = T^{-1/2}f$. Show that $f' = T^{1/2}g'$ is an eigenvector for $2 - \lambda$, where $g'(u) = \begin{cases} g(u) & u \in V_1 \\ -g(u) & u \in V_2 \end{cases}$. Recall Equation 2.1.4 from the notes!

2. Prove Theorem 2.43.

   Hint: use Proposition 2.37.

3. Let X be a uniform Markov process on a graph G with at least two vertices.

   (a) Prove that X is irreducible if and only if G is connected.

   (b) Assume further that G is connected. Prove that X is aperiodic if and only if G is not bipartite.

   Hint for (b): in one of the implications you might need the fact that a graph is bipartite if and only if it does not contain any odd cycles.

4. Implement in `Julia` the algorithm described in Exercise 2.13 of the notes for the graph with vertex set $\{1, 2, 3\}$ and edge set $\{\{1, 2\}, \{1, 3\}\}$, using 4 colors. Use this to estimate the number of admissible 4-colorings of the graph (idea: start from a random - not necessarily admissible! - coloring, walk for a certain fixed number of steps, record the final coloring; run the process 1000 times. How many times do you see each coloring?).

5. How would you define Page-Rank for a directed graph? Analyze the data from Notebook 3 using your ideas.

6. (a) Given a connected graph, write a `Julia` function that computes the distance between two vertices.

   (b) Write functions that compute betweenness, closeness, degree and harmonic centralities for a connected graph.

   (c) Write a function that takes as inputs a connected graph and a type of centrality measure and returns the most important vertices with respect to that measure.

   (d) Use the function from (c) to investigate some connected graphs with 10 vertices and 18 edges. Then do the same with the Krackhardt kite graph (see the Wikipedia entry).

   Hint: Some useful commands from the `Graphs` package are `degree`, `adjacency_matrix` and `SimpleGraph`.