Homology – i.e. detecting holes

We almost have all the details to write our first algorithm from TDA.

Algorithm [Simple version of Persistent Homology]

Input Data points $\{x_0, \dots, x_n\} \subseteq \mathbb{R}^D$ and a sequence of positive real numbers

$$0 < r_1 < \dots < r_m .$$

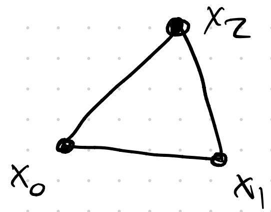
$\mathcal{P} = \{x_0, \dots, x_n\}$

For each $i = 1, \dots, m$

1. Compute $C_{r_i}(\mathcal{P})$ or $VR_{r_i}(\mathcal{P})$
2. Compute homology of $C_{r_i}(\mathcal{P})$ or $VR_{r_i}(\mathcal{P})$

Output homology information for each $i = 1, \dots, m$.

$\boxed{\text{Today}}$ Homology for simplices in $\mathbb{R}^2$.

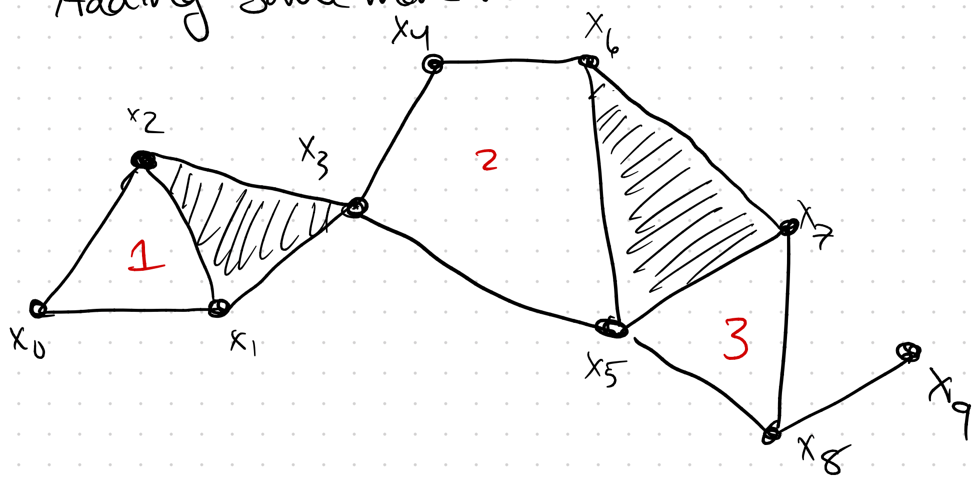


Consider complex

$K = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}\}$.

<u>Idea</u> K has one hole

Adding some more vertices



I claim here we should have 3 holes.

$\boxed{\text{Theorem}}$ Let K be a connected simplicial comlex in $\mathbb{R}^2$, that is $\dim(K) \leq 2$. Then,

$$1 - \#\{\text{holes}\} = \#\{\text{vertices}\} - \#\left\{\begin{matrix}\text{edges}\\ \text{of } K\end{matrix}\right\} + \#\{\text{triangles}\}.$$

From example $\quad 1 - \#\text{holes} = 10 - 14 + 2$

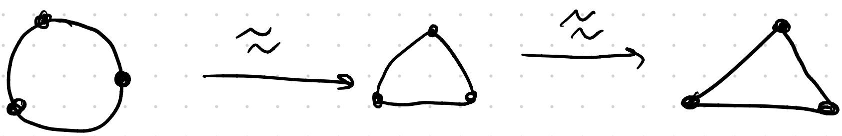
<u>Exercise</u> $\to$ try it with your own example!

$\#\text{holes} = 1 + 2 = 3$ ✓

$\boxed{\text{Definition}}$ Let $\sigma \subseteq \{0, \dots, n\}$, and K a simplicial complex on $\{x_0, \dots, x_n\}$. We say σ is a <u>hole of K</u> if $\sigma \notin K$, but $K' = \{\alpha \in K \mid \alpha \subsetneq \sigma\}$ builds a unique circle.

★ What do we mean by "circle"?

the "topological" part of TDA means we consider shapes up to continuous deformations. i.e.



So a triangle is a circle topologically. [not geometrically]

★ Why is the therem useful?

In persistent homology, we want to return # holes for each r_i in C_{r_i} or VR_{r_i}.

$$\#\text{holes} = 1 - \{\underbrace{\#\text{vertices}}_{\substack{n+1 \\ \{x_0, \dots, x_n\}}} - \underbrace{\#\text{edges}}_{\substack{\#\text{subsets} \\ \text{of } C_r \text{ of} \\ \text{length } 2}} + \underbrace{\#\text{triangles}}_{\substack{\#\text{subsets of} \\ C_r \text{ of length } 3.}}\}$$

very easy to compute!

Proof of Theorem

Proof by induction.

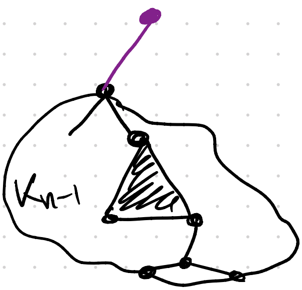
Base case 0 edges: $K = \{\{x_0\}\}$. No subsets

$\sigma \subseteq \{0\}$ with $\sigma \notin K$. $\Rightarrow$ no holes.

$$\#\text{holes} = 0 = 1 - (\underset{\text{vertex}}{\underset{\uparrow}{1}} - \underset{\text{edges}}{\underset{\uparrow}{0}} + \underset{\text{triangles}}{\underset{\uparrow}{0}})$$

Induction step Suppose the formula holds for a simplicial complex on $n-1$ edges, K_{n-1}.

Case 1



Say K_{n-1} has V_{n-1} vertices

E_{n-1} edges $\quad H_{n-1}$ holes

T_{n-1} triangles

Case 1 Add a vertex, so doesn't add any holes or triangles.

$$\#\text{holes} = 1 - (\#\text{vertices} - \#\text{edges} + \#\text{triangles})$$

$$= 1 - \left[(V_{n-1}+1) - (E_{n-1}+1) + T_{n-1} \right]$$

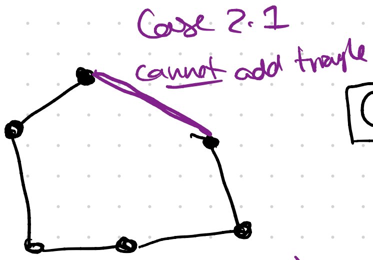
$$= 1 - (V_{n-1} - E_{n-1} + T_{n-1})$$

$$= H_{n-1}. \checkmark$$

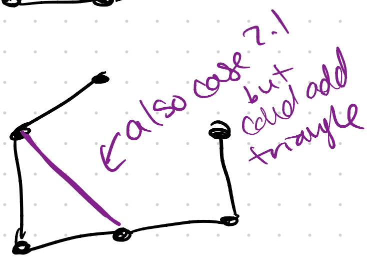
So formula still holds in this case

Case 2 — Don't add vertices when adding 1 edge

This always creates a circle.



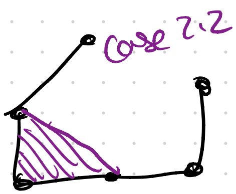
Case 2.1 either the circle has ≥ 3 vertices and we cannot add a triangle, or we choose not to add triangle.



$$\# \text{holes} = H_{n-1} + 1.$$

$$1-(\#\text{vertices} - \#\text{Edges} + \#\text{triangles})$$
$$= 1-(V_{n-1} - E_{n-1} - 1 + T_{n-1})$$
$$= 1-(V_{n-1} - E_{n-1} + T_{n-1}) + 1$$
$$= H_{n-1} + 1 \quad \checkmark$$

Case 2.2 circle added has 3 edges and we do add triangle.



$$\# \text{holes} = H_{n-1}$$

$$1-(\#\text{vertices} - \#\text{edges} + \#\text{triangles})$$
$$= 1-(V_{n-1} - (E_{n-1}+1) + (T_{n-1}+1))$$
$$= 1-(V_{n-1} - E_{n-1} + T_{n-1})$$
$$= H_{n-1} \quad \checkmark$$

Next time we'll consider higher dimensional holes. For now, the current definition of a "hole" is a 1-dimensional hole, and we'll discuss an alternative way to count # holes with linear algebra.

<u>**Definition**</u> <u>The 0, 1, 2 linear boundary operators mod 2</u>

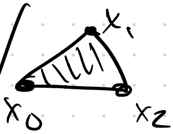
$$\partial_0 : \{\text{Vertices}\} \longrightarrow \{0\}.$$

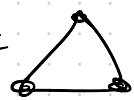
$$\partial_0(\{x_p\}) = 0.$$

$$\partial_1 : \{\text{edges}\} \longrightarrow \{\text{vertices}\}$$

$$\partial_1(\{x_p, x_q\}) = \{x_p\} + \{x_q\}$$

$$\partial_2 : \{\text{triangles}\} \longrightarrow \{\text{edges}\}$$

$\partial_2\Big($ 

$\Big) =$  $= \{x_0, x_1\} + \{x_0, x_2\} + \{x_1, x_2\}$

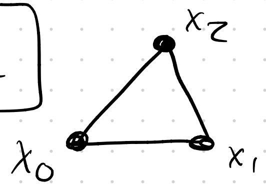
basis elements.

$\boxed{\text{Fact}}$ $\partial_1\big(\partial_2($  $)\big) = \partial_1(\{x_0, x_1\}) + \partial_1(\{x_0, x_1\}) + \partial_1\{x_0, x_2\}$

$= 2\{x_0\} + 2\{x_1\} + 2\{x_2\}$

$= 0$ (since we look mod 2).

$\boxed{\text{Example}}$



$= K = \left\{ \begin{array}{l} \{x_0\}, \{x_1\}, \{x_2\}, \\ \{x_0, x_1\}, \{x_1, x_2\}, \{x_0, x_2\} \end{array} \right\}$

The first homology vector space is

$$H_1(K) = \ker(\partial_1) / \operatorname{Im}(\partial_2)$$

The first betti number $\beta_1(K) = \dim H_1(K)$ counts the # holes in K.

Since K contains no triangle, $\operatorname{Im}(\partial_2) = \emptyset$

$$\Rightarrow \quad H_1(K) = \ker(\partial_1)$$

As from the example, we need even #s of all 3 vertices to be in the kernel. So

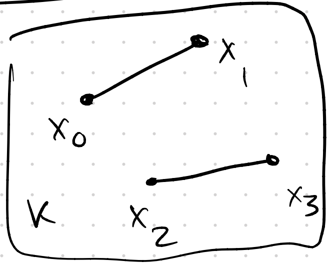
$\ker(\partial_1)$ is 1-dim'l space over

$$\{x_0, x_1\} + \{x_1, x_2\} + \{x_0, x_2\}.$$

$$\Rightarrow \quad \beta_1(K) = \dim(\ker(\partial_1)) = 1.$$

Exercise

$\beta_0(K) = ?$ what does it mean?



$$H_0(K) = \ker(\partial_0) / \mathrm{Im}(\partial_1)$$

$$\mathrm{Im}(\partial_1) = \{\{x_0\}+\{x_1\}\} \text{ and } \{\{x_2\}+\{x_3\}\}.$$

2 basis elements

$$\ker(\partial_0) = \underbrace{\{x_0\}+\{x_1\}+\{X_3\}+\{x_4\}}_{4 \text{ basis elements}}$$

So $\beta_0(K) = \dim H_0(K) = 4-2 = 2.$ = # connected components.