$\boxed{\text{Chapter 2}}$ Network Analysis

A network is a collection of entities with connections.

~ Social networks, airports

Mathematically networks are graphs.

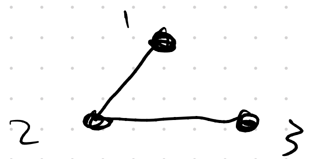
$\boxed{\text{Def}}$ A graph $G = (V, E)$ is a pair consisting of a finite number of vertices

$$V = \{1, \dots, n\}$$

and a finite number of edges

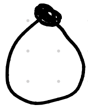
$$E \subseteq \{ \{i,j\} \mid i, j \in V \text{ and } i \neq j \}.$$

$\boxed{\text{examples}}$ $V = \{1, 2, 3\}$ $E = \{\{1 2\}, \{2 3\}\}$



~ Note there are other notions of graphs and relationships that arise in networks which we will not cover in this class.

① edges connect vertex to itself.



(comment on own post)

② directed graphs

(following on instagram)

- Requiring $i \neq j \Rightarrow$ ① cannot happen. this is sometimes called a <u>simple</u> graph

- Notation $\{i, j\}$ is unordered set $\Rightarrow$ <u>undirected</u> graph.

3 important data structures associated to a graph:

<u>Adjacency matrix $A(G)$</u>

$$A(G) = (a_{ij}) \in \mathbb{R}^{n \times n}$$

$$a_{ij} = \begin{cases} 1 & \{i,j\} \in E \\ 0 & \text{else} \end{cases}$$

$\{i, j\} \in E \Rightarrow$ vertices i & j are <u>adjacent</u>

<u>Laplace Matrix $L(G)$</u>

$$L(G) = (l_{ij}) \in \mathbb{R}^{n \times n}$$

$$l_{ij} = \begin{cases} 1 & i = j \\ \dfrac{-1}{\sqrt{\deg(i)\deg(j)}} & \{i,j\} \in E \\ 0 & \text{else} \end{cases}$$

$\deg(i)$ = degree of i $= \#\{v \in V \mid \{i, v\} \in E\}$

Degree matrix $T(G)$

$$T(G) = (t_{ij}) \in \mathbb{R}^{n \times n}$$

$$t_{ij} = \begin{cases} \deg(i) & i=j \\ 0 & \text{else} \end{cases}$$

$\boxed{\text{Lemma}}$

$$L = I_n - T^{-1/2} A T^{-1/2}$$

where I_n = $n \times n$ identity matrix

$\boxed{\text{exercise}}$ Compute the Laplace matrix in given examples, then in general examples if time.

①

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 1 \end{bmatrix} \end{array}$$

$\deg(1) = \deg(6) = 1$

$\deg(2,3,4,5) = 2$

②

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

$\deg(i) = 2$

③

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

$\deg(i) = 5$

$\Rightarrow$ all of-diagonal entries are $-\frac{1}{5}$

④

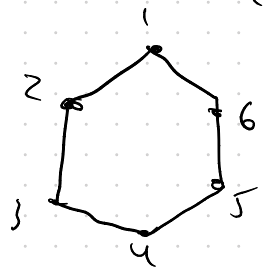
$$\begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{bmatrix}$$

$\deg(i) = 3 \Rightarrow$ all other entries $-\frac{1}{9}$

1. G path if $E=\{\{i,i+1\} \mid 1\le i<n\}$

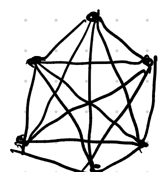
~ path on 6 vertices

2. G cycle if $E=\{\{i,i+1\} \mid 1\le i<n\} \cup \{1,n\}$.



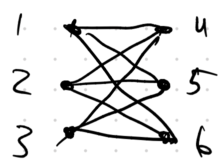
3. G is a complete graph on n vertices if
$E=\{\{i,j\} : i,j\in V,\ i\neq j\}$

~ Complete graph on 6 vertices



4. G is a bipartite graph if $V=V_1\cup V_2$, $V_1\cap V_2=\emptyset$
so that $E=\{\{i,j\} \mid i\in V_1, j\in V_2\}$

~ Complete bipartite graph on 3+3 vertices



L is symmetric with real entries

$\Rightarrow$ By spectral theorem, all the eigenvalues of L are real.

$\boxed{\text{Definition}}$ The eigenvalues of L $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ are called the <u>spectrum</u> of G. We define $\lambda_G = \lambda_1$.

Prop 2.20 of notes

① $\lambda_k = 1 - \cos\left(\frac{\pi k}{5}\right) \rightsquigarrow \lambda_0 = 0 \quad \lambda_G = 1 - \cos\left(\frac{\pi}{5}\right)$

② $\lambda_k = 1 - \cos\left(\frac{\pi k}{3}\right) \rightsquigarrow \lambda_0 = 0 \quad \lambda_G = 1 - \cos(\pi/3)$

③ $\lambda_0 = 0$ and for $k \geq 1$ $\lambda_k = \frac{6}{5}$

④ $\lambda_0 = 0$, $\quad 1 < k \leq n-2$ $\lambda_k = 1$, $\quad \lambda_{n-1} = 2.$

$\boxed{\text{Spectrum of Graph Theorem}}$ $0 = \lambda_0 \leq \underbrace{\boxed{\lambda_1}}_{\lambda_G} \leq \cdots \leq \lambda_{n-1}$.

$\boxed{\text{Mathematical tool}}$ "eigenvectors are functions"

$$\mathbb{R}^n \longleftrightarrow \mathcal{F}(V) = \{f : V \to \mathbb{R}\}$$

$$x = (x_1, \dots, x_n) \longleftrightarrow f : V \longrightarrow \mathbb{R}$$

$$f(i) = x_i$$

$$Lx = L\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} l_{1j} x_j \\ \sum_{j=1}^{n} l_{2j} x_j \\ \vdots \end{bmatrix} = \begin{bmatrix} Lf(1) \\ Lf(2) \\ \vdots \\ Lf(n) \end{bmatrix}$$

Inner product

$$\langle x, y \rangle = x^T y = \sum_{i=1}^{n} x_i y_i \qquad \langle f, g \rangle = \sum_{u \in V} f(u) g(u)$$

$\boxed{\text{Lemma}}$ For any $u \in V$,

$$Lf(u) = \frac{1}{\sqrt{\deg(u)}} \sum_{\substack{v \in V \\ \{u,v\} \in E}} \frac{f(u)}{\sqrt{\deg(u)}} - \frac{f(v)}{\sqrt{\deg(v)}}$$

$\boxed{\text{Definition}}$ The <u>Rayleigh quotient</u> of L for $f \in \mathcal{F}(V)$ is $\dfrac{\langle f, Lf \rangle}{\langle f, f \rangle}$.

$\boxed{\text{Thm}}$ (Rayleigh quotient is non-negative)

$$\frac{\langle f, Lf \rangle}{\langle f, f \rangle} = \frac{1}{\sum_{u \in V} f(u)^2} \sum_{\{u,v\} \in E} \left(\frac{f(u)}{\sqrt{\deg(u)}} - \frac{f(v)}{\sqrt{\deg(v)}} \right)^2$$

$\boxed{\text{Proof of Spectrum of graph Theorem}}$

Let λ_k be an eigenvalue for $k = 0, \dots, n-1$.

Let $f_k \neq 0$ be an eigenvector. Then

Since the Rayleigh quotient is non negative,

$$\Rightarrow \quad 0 \leq \frac{\langle f_k, L f_k \rangle}{\langle f_k, f_k \rangle} = \frac{\langle f_k, \lambda_k f_k \rangle}{\langle f_k, f_k \rangle} = \lambda_k \frac{\langle f_k, f_k \rangle}{\langle f_k, f_k \rangle}$$

$$= \lambda_k.$$

We now have to show $\lambda_0 = 0$.

Consider vector $f(u) = \sqrt{\deg(u)}$.

By Rayleigh quotient non-negative theorem,

$$\langle f, Lf \rangle = \sum_{\{u,v\} \in E} \left(\frac{f(u)}{\sqrt{\deg(u)}} - \frac{f(v)}{\sqrt{\deg(v)}} \right)^2 = 0.$$

If λ eigenvector for f,

$$0 = \langle f, Lf \rangle = \langle f, \lambda f \rangle = \lambda \langle f, f \rangle$$

$\Rightarrow \lambda = 0$.

By spectral Theorem, $\exists$ basis of eigenvectors w/ eigenvalue λ_j $f_1, \dots, f_n$ so that $f = \sum_{j=1}^{n} c_j f_j$

$$0 = \langle f, Lf \rangle = \left\langle \sum_{j=1}^{n} c_j f_j, \sum_{j=1}^{n} c_j \lambda_j f_j \right\rangle$$

$$= \sum_{u \in V} \lambda_j c_j^2 f_j(u)^2$$

Since $f_j \neq 0$ and c_j cannot all be 0

$\Rightarrow$ at least one $\lambda_j = 0$.

$\blacksquare$

$\boxed{\text{Proof if time}}$ How do we isolate an entry of a matrix so we can still do computations?

Want to show $L = I_n - T^{-1/2} A T^{-1/2}$.

Set e_j = jth standard basis vector

$e_1 = (1, 0, \ldots, 0)$
$e_2 = (0, 1, \ldots, 0)$

$$l_{ij} = e_i^T L e_j$$

$\boxed{\text{exercise}}$ finish proof.

$$\left(T^{-1/2} A T^{-1/2}\right)_{ij} = e_i^T T^{-1/2} A T^{-1/2} e_j$$

$$(T = T^T) \quad = \left(T^{-1/2} e_i\right)^T A \left(T^{-1/2} e_j\right)$$

$$= \frac{1}{\sqrt{\deg(i)\deg(j)}} \boxed{e_i^T A e_j}$$

$= 1$ if $\{i, j\} \in E$
0 otherwise.