# Homework 3: Due December 7

Reading: Read section 3.1 and 3.2 of course notes.

1. **Hand in** Let $G$ be a bipartite graph with bipartition $V_1 \sqcup V_2$ and without isolated vertices. Prove that, if $\lambda$ is an eigenvalue of $L(G)$, then so is $2-\lambda$.

   **Hint:** take an eigenvector $f$ for $\lambda$ and set $g = T^{-1/2}f$. Show that $f' = T^{1/2}g'$ is an eigenvector for $2-\lambda$, where $g'(u) = \begin{cases} g(u) & u \in V_1 \\ -g(u) & u \in V_2 \end{cases}$. Recall Equation 2.1.4 from the notes!

   **Alternative hint:** you can do this directly by using Lemma 2.14:

   *Proof.* Note $f(u) = f'(u)$ for $u \in V_1$ and $f(u) = -f'(u)$ for $u \in V_2$.

   Case 1: If $u \in V_1$, use Lemma 2.14 and equation 2.12 to show that

   $$Lf'(u) = (2-\lambda)f(u).$$

   Case 2: If $u \in V_2$, Use Lemma 2.14 and notice that you are working with the negative of case 1 to conclude the proof. □

2. Prove Theorem 2.43.

   Hint: use Proposition 2.37.

   **Accidental repeat: see Homework 2 problem 6**

3. Let $X$ be a uniform Markov process on a graph $G$ with at least two vertices.

   (a) Prove that $X$ is irreducible if and only if $G$ is connected.

   *Proof structure.* ($\Rightarrow$) For this direction I recommend using the contrapositive: If $G$ is not connected, then $X$ is not irredicible. Let $u, v$ be in two separate components, and what does that say about $(P^k)_{u,v}$?

   ($\Leftarrow$) This direction I think is okay going both ways, either argue directly or by contrapositive. If you argue directly, use that $G$ connected means you must have $(P^k)_{u,v} > 0$ for some $k$, using the fact that connected means there is a path from $u$ to $v$. □

   (b) Assume further that $G$ is connected. Prove that $X$ is aperiodic if and only if $G$ is not bipartite.

   Hint for (b): in one of the implications you might need the fact that a graph is bipartite if and only if it does not contain any odd cycles.

   *Proof structure.* ($\Rightarrow$) I recommend this by contrapositive: If $G$ is bipartite, by the hint $G$ does not contain any odd cycles. Show that the gcd is then at least 2 to show $X$ is not aperiodic.

   ($\Leftarrow$) Suppose $G$ is not bipartite. then $G$ does contain odd cycles. Use this to argue that that $X$ must be aperiodic. □

4. Implement in `Julia` the algorithm described in Exercise 2.13 of the notes for the graph with vertex set $\{1,2,3\}$ and edge set $\{\{1,2\},\{1,3\}\}$, using 4 colors. Use this to estimate the number of admissible 4-colorings of the graph (idea: start from a random - not necessarily admissible! - coloring, walk for a certain fixed number of steps, record the final coloring; run the process 1000 times. How many times do you see each coloring?).

   **See Julia ExercisesHomework3_Problem4**

5. How would you define Page-Rank for a directed graph? Analyze the data from Notebook 3 using your ideas. **Edit Notebook 3 with the new definition you come up with. Does it match what you find when looking up the definition for a directed graph?**

6. (a) Given a connected graph, write a `Julia` function that computes the distance between two vertices.

   (b) Write functions that compute betweenness, closeness, degree and harmonic centralities for a connected graph.

   (c) Write a function that takes as inputs a connected graph and a type of centrality measure and returns the most important vertices with respect to that measure.

(d) Use the function from (c) to investigate some connected graphs with 10 vertices and 18 edges. Then do the same with the Krackhardt kite graph (see the Wikipedia entry).

Hint: Some useful commands from the `Graphs` package are `degree`, `adjacency_matrix` and `SimpleGraph`.

**See Julia ExercisesHomework3_Problem6**