Homework 5: Due January 24

Reading: Read section 4 of course notes.

1. Let $z \in \mathbb{R}^M$ be a random variable with $\mu := \mathbb{E}z \in \mathbb{R}^M$. Show that the covariance matrix of z is given by $\Sigma = \mathbb{E}(z-\mu)(z-\mu)^T$. Use this to show that Σ is positive semi-definite.

2. Take again the `MNIST` dataset from the `MLDatasets.jl` package and load the training data for pictures of ones and zeros. Use PCA to reduce the number of parameters representing these pictures. Then, load a point x from the test data set and compute the posterior distribution for $(\zeta \mid x)$ in Theorem 3.44 using x. Use the posterior distribution to generate synthetic data.

3. Consider the function $f(\Sigma) = \Sigma^{-1}$, where $\Sigma \in \mathbb{R}^{n\times n}$ is invertible.

   Prove that f is differentiable at Σ. Hint: Formulate f as a rational function in the entries of Σ. Show that $\frac{\partial f}{\partial \Sigma_{ij}} = -\Sigma^{-1}e_i e_j^T \Sigma^{-1}$,

   where e_k is the k-th standard basis vector in $\mathbb{R}^n$. Hint: Differentiate both sides of $\Sigma\Sigma^{-1} = \mathbf{1}_n$.

4. Let $A = (0,0), B = (1,1), C = (2,1), D = (1,-2)$. Draw all the possible Vietoris-Rips complexes as r ranges in $(0,+\infty)$.

5. We are given a simplex P in $\mathbb{R}^3$ obtained as the convex hull of $(0,0,0), (0,1,2), (1,0,-1), (2,1,1)$.

   Find $A \in \mathbb{R}^{4\times 3}$ and $b \in \mathbb{R}^4$ such that $P = \{x \in \mathbb{R}^3 \mid Ax \le b\}$. Explain the procedure you used.

   (Here we use "$u \le v$" as a shortcut for "$u_i \le v_i$ for every i".)

6. Let $P_0 = (0,0,0,\dots,0)$, $P_1 = (1,1,1,\dots,1)$, $P_2 = (-1,1,1,\dots,1)$ in $\mathbb{R}^n$. (Aside from the first one, all the entries of P_2 are "1"s). Let $VR(r)$ be the associated Vietoris-Rips complex for some $r > 0$.

   a) Prove that for any $r > 0$ one has that $\{P_0, P_1\} \in VR(r)$ if and only if $\{P_0, P_2\} \in VR(r)$. b) Compute all the possible $VR(r)$ (for $r > 0$) when n is 3, 4, or 5.