Homework 5: Due January 24

Reading: Read section 4 of course notes.

1. Let $z \in \mathbb{R}^M$ be a random variable with $\mu := \mathbb{E}z \in \mathbb{R}^M$. Show that the covariance matrix of z is given by $\Sigma = \mathbb{E}(z-\mu)(z-\mu)^T$. Use this to show that Σ is positive semi-definite.

   Hint for solution. Recall the Covariance matrix is given by

   $$\Sigma_{ij} = \mathrm{Cov}(z^{(i)}, z^{(j)}) = \mathbb{E}\left[z^{(i)}z^{(j)} - \mathbb{E}(z^{(i)})\mathbb{E}(z^{(j)})\right].$$

   Compare this to when you expand $\mathbb{E}(z-\mu)(z-\mu)^T$ in coordinates. Then using the fact that AA^T is always positive semi-definite for square matrices combined with linearity of the expected value give the result.

2. Take again the `MNIST` dataset from the `MLDatasets.jl` package and load the training data for pictures of ones and zeros. Use PCA to reduce the number of parameters representing these pictures. Then, load a point x from the test data set and compute the posterior distribution for $(\zeta \mid x)$ in Theorem 3.44 using x. Use the posterior distribution to generate synthetic data.

   Follow Notebook 6 with this different data set

3. Consider the function $f(\Sigma) = \Sigma^{-1}$, where $\Sigma \in \mathbb{R}^{n\times n}$ is invertible.

   Prove that f is differentiable at Σ. Hint: Formulate f as a rational function in the entries of Σ. Show that $\frac{\partial f}{\partial \Sigma_{ij}} = -\Sigma^{-1}e_i e_j^T \Sigma^{-1}$,

   where e_k is the k-th standard basis vector in $\mathbb{R}^n$. Hint: Differentiate both sides of $\Sigma\Sigma^{-1} = \mathbf{1}_n$.

   Proof idea: By induction on n, ignoring the second part of the hint

   First consider an 1×1 matrix. In this case, Σ invertible implies $\Sigma = [\sigma]$ for some $\sigma \in \mathbb{R}_{\neq 0}$. Then $f(\Sigma) = [\sigma^{-1}]$, and

   $$f'(\sigma) = -\frac{1}{\sigma^2} = -\Sigma^{-1}\Sigma^{-1}.$$

   Now suppose that f is differentiable at every $(n-1)\times(n-1)$ matrix. We can write

   $$f(\Sigma) = \frac{1}{\det(\Sigma)}C$$

   where C is the cofactor matrix given by $C_{ij} = (-1)^{i+j}M_{ij}$ and M_{ij} is the determinant of the $(n-1)\times(n-1)$ submatrix formed by deleting the ith row and the jth column. So now the cofactor matrix is formed by polynomials, and the determinant is also a polynomial. So when taking the derivative in any component, we can just use the quotient rule, hence this is differentiable.

4. (**Hand in**) Let $A = (0,0), B = (1,1), C = (2,1), D = (1,-2)$. Draw all the possible Vietoris-Rips complexes as r ranges in $(0, +\infty)$.

   Hint There are 5 total, which come from the 5 unique distances between pairs of points in A, B, C, D.

5. We are given a simplex P in $\mathbb{R}^3$ obtained as the convex hull of $(0,0,0), (0,1,2), (1,0,-1), (2,1,1)$.

   Find $A \in \mathbb{R}^{4\times 3}$ and $b \in \mathbb{R}^4$ such that $P = \{x \in \mathbb{R}^3 \mid Ax \leq b\}$. Explain the procedure you used.

   (Here we use "$u \leq v$" as a shortcut for "$u_i \leq v_i$ for every i".)

   Hint: The shape is a tetrahedron. It can be defined as a union of four half-spaces. What this means in terms of the problem is that the ith row of A, a_i satisfies $a_i x \leq b_i$. Recall $a_i x = b$ is the equation for a plane in $\mathbb{R}^3$, so a half-space just includes the inequality to decide what side of the plane we are sitting on.

   This description reduces now to (1) taking the 4 eqautions of planes on the boundary of the tetrahedron [This is a quick google if you don't remember how to determine a plane equation from 3 points]. (2) testing a point to determine the sign of b_i so that the tetrahedron lies on the correct side of the plane.

6. Let $P_0 = (0, 0, 0, \ldots, 0)$, $P_1 = (1, 1, 1, \ldots, 1)$, $P_2 = (-1, 1, 1, \ldots, 1)$ in $\mathbb{R}^n$. (aside from the first one, all the entries of P_1 and P_2 are "1"s). Let $VR(r)$ be the associated Vietoris-Rips complex for some $r > 0$.

   a) Prove that for any $r > 0$ one has that $\{P_0, P_1\} \in VR(r)$ if and only if $\{P_0, P_2\} \in VR(r)$. b) Compute all the possible $VR(r)$ (for $r > 0$) when n is 3, 4, or 5.

   Hints: For (a) use the fact that

   $$\|P_0 - P_1\| = \sqrt{n} = \|P_0 - P_2\|.$$

   For (b), we only need to compute the last distance which is

   $$\|P_1 - P_2\| = 2$$

   .

   So when n is 3, $\sqrt{n} < 2$, so the radii of importance are 0, $\frac{\sqrt{n}}{2}$, $\frac{1}{2}$ giving 3 distinct complexes.

   When $n = 4$, there are only two distinct complexes for $r = 0$ and $r = 2$.

   When $n = 5$, there are again 3 distinct complexes, but now in the order 0, $\frac{1}{2}$, $\frac{\sqrt{n}}{2}$.