Given $\{x_0, \dots, x_n\} \subseteq \mathbb{R}^D$ collection of data points

For $r > 0$

(1) The <u>Čech Complex of level</u> r is:

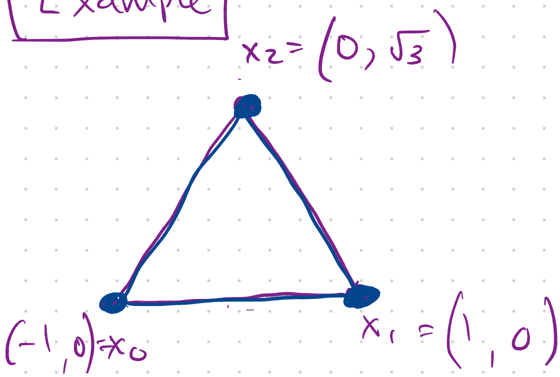
$$C_r(\{x_0, \dots, x_n\}) := \left\{ \alpha \subseteq \{0, \dots, n\} \;\middle|\; \bigcap_{p \in \alpha} B_r(x_p) \neq \emptyset \right\}$$

where $B_r(x) = \{ z \in \mathbb{R}^D \mid \|x - z\| \leq r \}$

(2) The <u>Victoris–Rips complex</u> of level r is:

$$VR_r(\{x_0, \dots, x_n\}) := \left\{ \alpha \subseteq \{0, \dots, n\} \;\middle|\; \max_{p,q \in \alpha} \|x_p - x_q\| \leq 2r \right\}$$

$\boxed{\text{Example}}$ $r = 1$



$\alpha = \{0\}$ $\quad B_1(x_0) \neq \emptyset$ ✓

$\alpha = \{1\}$ $\quad B_1(x_1) \neq \emptyset$

$\alpha = \{2\}$ $\quad B_1(x_2) \neq \emptyset$ ✓

$\alpha = \{0, 1\}$ $\quad B_1(x_0) \cap B_1(x_1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ✓

$\alpha = \{1, 2\}$ $\quad \left\{ \text{ similar } \right\}$

$\alpha = \{0, 2\}$

X $\alpha = \{0, 1, 2\}$ $\quad B_1(x_0) \cap B_1(x_1) \cap B_1(x_2) = \emptyset$

The Čech complex is

$$C_r(\{x_0, x_1, x_2\}) = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}\}.$$

i.e. $C_r(\{x_0, x_1, x_2\})$ is the complete graph on 3 vertices

Simplicial complexes generalize graphs by keeping track of higher dimensional sets.

$\boxed{\text{Definition}}$ An abstract simplicial complex A on vertices $\{0, \dots, n\}$ is a collection of subsets $\alpha \subseteq \{0, \dots, n\}$ so that $\forall\, \alpha \in A$ and $\beta \subseteq \alpha$, $\beta \in A$.

The dimension of α is $\dim(\alpha) := |\alpha| - 1$.

The dimension of A is $\dim(A) := \max_{\alpha \in A} \dim(\alpha)$.

$\dim C_R = \dim(\text{graph}) = 1$.

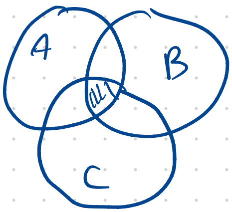
$\boxed{\text{example}}$ C_r is an abstract simplicial complex.

$\boxed{\text{Pf}}$ By def'n C_r is a collection of subsets of $\{0, \dots, n\}$.

Let $\alpha \in A$ and $\beta \subseteq \alpha$. Then

$$\bigcap_{p\in\beta} B_r(x_p) \supseteq \bigcap_{p\in\alpha} B_r(x_p) \neq \emptyset$$

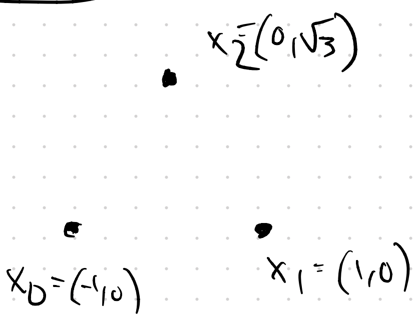
$$\Rightarrow \bigcap_{p\in\beta} B_r(x_p) \neq \emptyset.$$



Since $\boxed{A\cap B\cap C \subseteq A\cap B}$

Hence $\beta \in C_r(\{0,\dots,n\})$.

$\boxed{\text{Exercise}}$ Comput $V_r(\{x_0,x_1,x_2\})$, $\dim(V_R\{x_0,x_1,\dots,x_2\})$ and prove V_R is an abstract simplicial complex.



$$V_R(\{x_0,x_1,x_2\}) = \{\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

$$\dim V_R(\{x_0,x_1,x_2\}) = 2.$$

$\boxed{\text{Pf}}$ We need to check $\alpha\in V_r$ with $\beta\subseteq\alpha \Rightarrow \beta\in V_r$

$\boxed{\text{Since } A\subseteq B \Rightarrow \max(A)\leq\max(B)}$

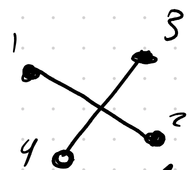
$$\max_{p,q\in\beta}\|x_p-x_q\| \leq \max_{p,q\in\alpha}\|x_p-x_q\| \leq 2r$$

$$\Rightarrow \beta\in V_r$$

$\boxed{\text{Theorem}}$ [All abstract simplicial complexes are geometric]

Let A abstract simplicial complex of dimension d.
Then A has a geometric realization in $\mathbb{R}^{2d+1}$.

$\boxed{\text{Lemma}}$ Graphs containing at least 1 vertex is a 1-dim'l abstract complex. By Theorem above, every graph can be realized in $\mathbb{R}^3$.



not geometric realization

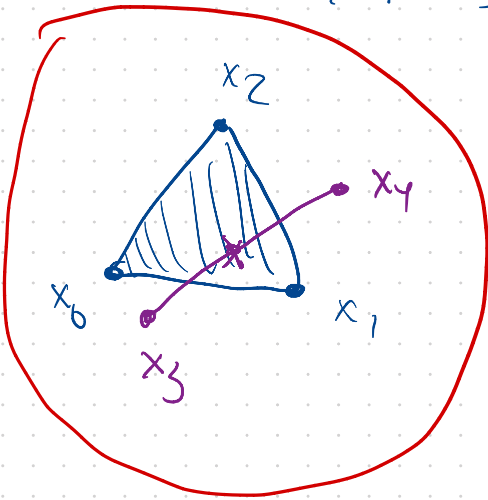
$\boxed{\text{Exercise}}$ give an example of a graph that cannot be realized in $\mathbb{R}^2$? (called non-planar)

Fun fact: every non-planar graph contains a K_5 or $K_{3,3}$ subgraph.

K_5: complete on 5 vertices

$K_{3,3}$: complete bipartite graph.

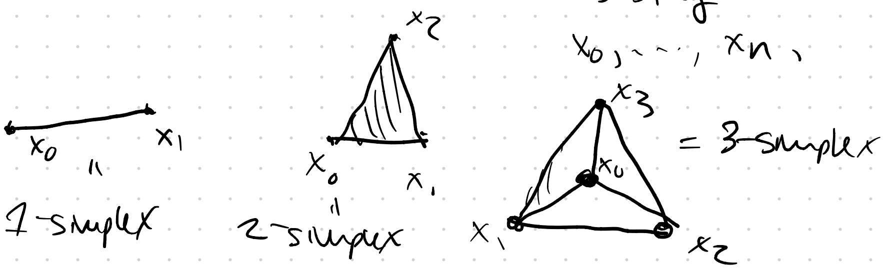
$\boxed{\text{example}}$ Consider the simplicial complex formed by $\{0,1,2\}$ and $\{3,4\}$.



__Not__ a __geometric simplicial complex.__

__Recall__ n-dim'l simplex is convex hull of $\{x_0, \cdots, x_n\}$.

where convex hull = set of all points whose lines stay between $x_0, \cdots, x_n$.



__Faces__ are lower-dimensional simplices on boundary.

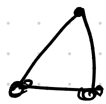
$\boxed{\text{Def}}$ A __geometric simplicial complex__ K = finite collection of simplices

(a) $\forall \Delta \in K$, and faces $\sigma \in \Delta$, we have $\sigma \in K$.

(b) $\Delta, \sigma \in K$ so that $\bar{\Delta} \cap \bar{\sigma} \neq \emptyset \Rightarrow \bar{\Delta} \cap \bar{\sigma}$ is a face of both Δ and σ.

C_r vs. VR_r as simplicial complexes

$VR_r = \begin{cases} + \text{ computationally easier } \binom{|X|}{2} \text{ distances} \\ - \; VR_r \text{ automatically fills in simplices} \end{cases}$



So cannot have unfilled triangle

$C_r =$ + can represent 

− computationally difficult: have to determine if system of polynomial equations has a solution.

Proposition For $r>0$, $P=\{x_1,\dots,x_n\}\subseteq \mathbb{R}^D$,

$$C_r(P)\subseteq VR_r(P)\subseteq C_{2r}(P)$$

Exercise verify for triangle example + fast challenge

Pf (a) $C_r(P)\subseteq VR_r(P)$

(b) $\alpha \in VR_r(P)$, $d := \dim(\alpha)$

$$z := \frac{1}{d+1} \sum_{p \in \alpha} x_p. \quad \text{Claim } z \in \bigcap_{p \in \alpha} B_{2r}(x_p).$$

For $q \in \alpha$

$$\|x_q - z\| = \frac{1}{d+1} \Big\| \sum_{\substack{p \in \alpha \\ p \neq q}} x_q - x_p \Big\|$$

$$\leq \frac{1}{d+1} \sum_{p \in \alpha} \|x_q - x_p\|$$

$$\leq 2r \cdot \frac{1}{d+1} \cdot |\alpha| = 2r \quad \blacksquare$$