

Homology - i.e. detecting holes

We almost have all the details to write our first algorithm from TDA.

Algorithm [Simple version of Persistent Homology]

Input Data points $\{x_0, \dots, x_n\} \subseteq \mathbb{R}^d$ and a sequence of positive real numbers

$$0 < r_1 < \dots < r_m.$$

$$P = \{x_0, \dots, x_n\}$$

For each $i = 1, \dots, m$

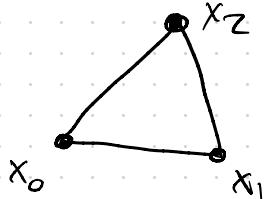
1. Compute $C_{r_i}(P)$ or $VR_{r_i}(P)$

2. Compute homology of $C_{r_i}(P)$ or $VR_{r_i}(P)$

Output homology information for each $i = 1, \dots, m$.

Today

Homology for simplices in \mathbb{R}^2 .

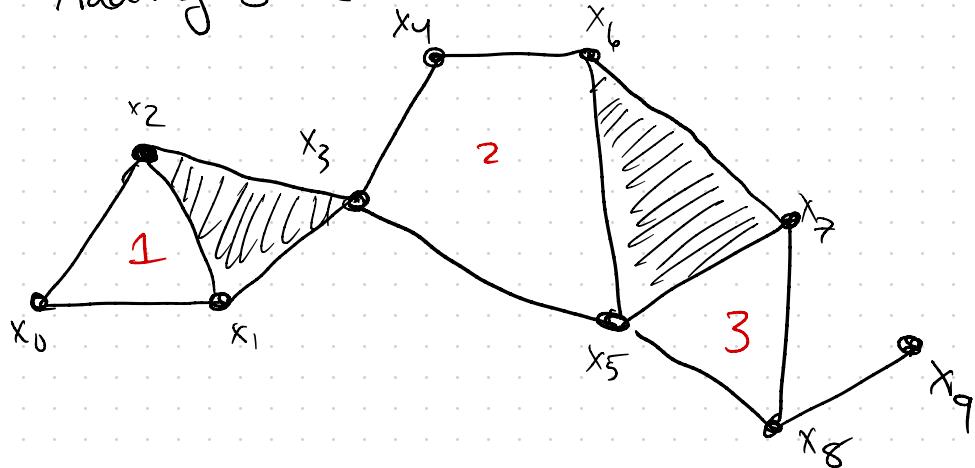


Consider complex

$$K = \{\{03, 123, 23\}, \{0, 13\}, \{0, 123\}, \{1, 23\}\}$$

Idea K has one hole

Adding some more vertices



I claim here we should have 3 holes.

Theorem Let K be a simplicial complex in \mathbb{R}^2 , that is connected

$\text{dim}(K) \leq 2$. Then,

$$2 - \#\{\text{holes}\} = \#\{\text{vertices}\} - \#\{\text{edges}\} + \#\{\text{triangles}\}$$

From example $2 - \#\text{holes} = 10 - 14 + 2$

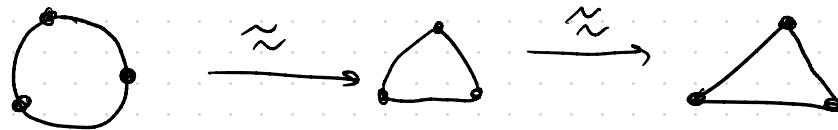
Exercise → try it with your own example! $\# \text{holes} = 1 + 2 = 3$ ✓

Definition Let $\sigma \subseteq \{0, \dots, n\}$, and K a simplicial complex

on $\{x_0, \dots, x_n\}$. We say σ is a hole of K if
 σ is a graph contained in K [i.e. subset of 1-skeleton]
s.t. $\sigma = (V, E)$ is a cycle with any $\{x_p, x_q, x_r\} \subseteq V$
has $\{x_p, x_q, x_r\} \notin K$.

* What do we mean by "circle"?

The "topological" part of TDA means we consider shapes up to continuous deformations. i.e.



So a triangle is a circle topologically. [not geometrically]

* Why is the theorem useful?

In persistent homology, we want to return # holes for each r_i in C_{r_i} or VR_{r_i} .

$$\# \text{holes} = 1 - \left(\underbrace{\#\text{vertices}}_{n+1} - \underbrace{\#\text{edges}}_{\#\text{subsets of } C_p \text{ of length 2}} + \underbrace{\#\text{triangles}}_{\#\text{subsets of } C_p \text{ of length 3}} \right)$$

Very easy to compute!

Proof of Theorem

Proof by induction.

Base case

0 edges: $K = \{\emptyset\}$. No subsets

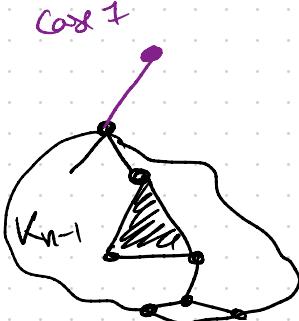
$\sigma \subseteq \{\emptyset\}$ with $\sigma \neq K \Rightarrow$ no holes.

$$\# \text{holes} = 0 = 1 - (\underset{\substack{\uparrow \\ \text{vertex}}}{1} - \underset{\substack{\uparrow \\ \text{edges}}}{0} + \underset{\substack{\uparrow \\ \text{triangles}}}{0})$$

Induction step

Suppose the formula holds for a simplicial complex on $n-1$ edges, K_{n-1} .

Say K_{n-1} has V_{n-1} vertices



E_{n-1} edges H_{n-1} holes
 T_{n-1} triangles

Case 1

Add a vertex, so doesn't add any holes or triangles.

$$\begin{aligned}\# \text{holes} &= 1 - (\# \text{vertices} - \# \text{edges} + \# \text{triangles}) \\ &= 1 - [(V_{n-1} + 1) - (E_{n-1} + 1) + T_{n-1}] \\ &= 1 - (V_{n-1} - E_{n-1} + T_{n-1}) \\ &= H_{n-1}. \quad \checkmark\end{aligned}$$

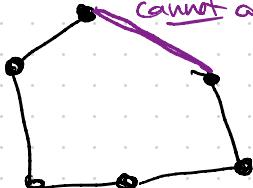
So formula still holds in this case

Case 2

Don't add vertices when adding 1 edge

Case 2.1

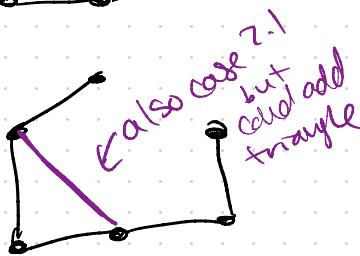
cannot add triangle



This always creates a circle.

Case 2.1

either the circle has ≥ 3 vertices and we cannot add a triangle, or we choose not to add triangle.



$$\# \text{holes} = f_{n-1} + 1.$$

$$1 - (\# \text{vertices} - \# \text{edges} + \# \text{triangles})$$

$$= 1 - (V_{n-1} - E_{n-1} - 1 + T_{n-1})$$

$$= 1 - (V_{n-1} - E_{n-1} + T_{n-1}) + 1$$

$$= f_{n-1} + 1 \quad \checkmark$$

Case 2.2

circle added has 3 edges and we do add triangle.

$$\# \text{holes} = f_{n-1}$$

$$1 - (\# \text{vertices} - \# \text{edges} + \# \text{triangles})$$

$$= 1 - (V_{n-1} - (E_{n-1} + 1) + (T_{n-1} + 1))$$

$$= 1 - (V_{n-1} - E_{n-1} + T_{n-1})$$

$$= f_{n-1} \quad \checkmark$$



Next time we'll consider higher dimensional holes. For now, the current definition of a "hole" is a 1-dimensional hole, and we'll discuss an alternative way to count the holes with linear algebra.

Definition The $0, 1, 2$ linear boundary operators mod 2

$$\partial_0 : \{ \text{Vertices} \} \longrightarrow \{ 0 \}.$$

$$\partial_0(\{x_p\}) = 0.$$

$$\partial_1 : \{ \text{edges} \} \longrightarrow \{ \text{vertices} \}$$

$$\partial_1(\{x_p, x_q\}) = \{x_p + x_q\}$$

$$\partial_2 : \{ \text{triangles} \} \longrightarrow \{ \text{edges} \}$$

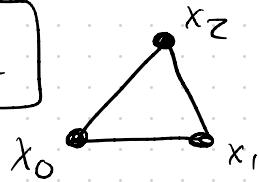
$$\partial_2 \left(\begin{array}{c} x_1 \\ \backslash \diagup \diagdown / \\ x_0 \quad x_2 \end{array} \right) = \begin{array}{c} x_p \\ \cdot \\ x_0 \quad x_2 \end{array} + \begin{array}{c} x_q \\ \cdot \\ x_0 \quad x_2 \end{array} = \{x_0, x_1\} + \{x_0, x_2\} + \{x_1, x_2\}$$

basis elements.

Fact

$$\begin{aligned} \partial_1(\partial_2(\triangle)) &= \partial_1(\{x_0, x_1\}) + \partial_1(\{x_0, x_2\}) + \partial_1(\{x_1, x_2\}) \\ &= 2\{x_0\} + 2\{x_1\} + 2\{x_2\} \\ &= 0 \quad (\text{since we look mod 2}). \end{aligned}$$

(Example)



$$= k = \left\{ \begin{array}{l} \{x_0\}, \{x_1\}, \{x_2\}, \\ \{x_0, x_1\}, \{x_1, x_2\}, \{x_0, x_2\} \end{array} \right\}$$

The first homology vector space is

$$H_1(K) = \ker(\partial_1) / \text{Im}(\partial_2)$$

The first Betti number $\beta_1(K) = \dim H_1(K)$ counts
the # holes in K .

Since K contains no triangle, $\text{Im}(\partial_2) = \emptyset$

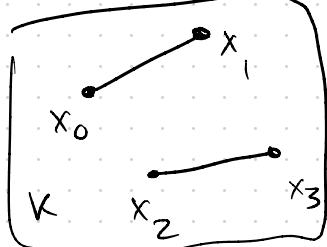
$$\Rightarrow H_1(K) = \ker(\partial_1)$$

As from the example, we need even #s of all 3 vertices
to be in the kernel. So

$\ker(\partial_1)$ is 1-dim'l space over
 $\{x_0, x_1\} + \{x_1, x_2\} + \{x_0, x_2\}$.

$$\Rightarrow \beta_1(K) = \dim(\ker(\partial_1)) = 1.$$

Exercise



$\beta_0(K) = ?$ What does it mean?

$$H_0(K) = \ker(\partial_0) / \text{Im}(\partial_1)$$

$$\text{Im}(\partial_1) = \left\{ \{x_0\} + \{x_1\} \right\}$$

and

$$\left. \left\{ \{x_2\} + \{x_3\} \right\} \right\} \quad \begin{matrix} 2 \text{ basis} \\ \text{elements} \end{matrix}$$

$$\ker(\partial_0) = \left\{ \{x_0\} + \{x_1\} + \{x_3\} + \{x_4\} \right\}$$

4 basis elements

So $\beta_0(K) = \dim H_0(K) = 4 - 2 = 2$. = # connected components.