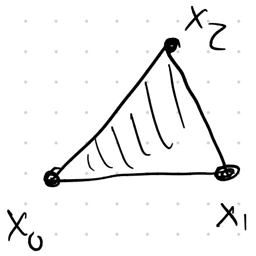
Recall from last time

*Note new definition of hole in lecture notes from last time.



∂_1 : Edges $\to$ Vertices

∂_2 : triangles $\to$ edges

And $\partial_1(\partial_2(\{x_0, x_1, x_2\})) = 2\{x_0\} + 2\{x_1\} + 2\{x_2\}$

We formalize as follows

$\boxed{\text{Def}}$ $S = \{s_1, \dots, s_m\}$ a finite set. The free vector space of S over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ is

$$F(S) := \left\{ \sum_{i=1}^{m} a_i s_i \;\middle|\; a_i \in \mathbb{F}_2 \right\}.$$

The s_i are linearly independent, $\sum a_i s_i + \sum b_i s_i = \sum (a_i + b_i) s_i$

$\lambda \sum a_i s_i = \sum (\lambda a_i) s_i$.

$F(S) \cong \mathbb{F}_2^m$ so in particular $\dim(F(S)) = m$.

$\boxed{\text{Def}}$ Let $n \geq 0$ and K a simplicial complex.

Recall $K^{(p)} = p$-skeleton of $K = \{\alpha \in K \mid \dim(\alpha) \leq p\}$

The <u>vector space of n-chains</u> in K is

$$C_n(K) = F\left(K^{(n)} \setminus K^{(n-1)}\right)$$

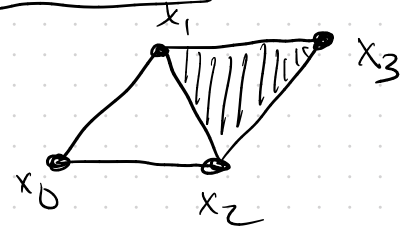
* i.e. $C_n(K) =$ free vector space of all n-simplices in K.
* $C_{-1}(K) = \{0\}$.

$\boxed{\text{Definition}}$ The <u>boundary operator</u> is the map

$$\partial_n : C_n(K) \longrightarrow C_{n-1}(K)$$

$$\partial_n(\{x_0, \ldots, x_n\}) = \sum_{i=0}^{n} \{x_0, \ldots, x_n\} \setminus \{x_i\}$$

$\boxed{\text{Exercise}}$



Compute kernel & image for all ∂_n with $n \geq 1$ for the given K.

1. $C_n(K) = ?$
2. images & kernels
3. Determine dimension of each kernel & image.

$C_{-1}(K) = \{0\}$

$C_0(K) = \text{Span}\{x_0, x_1, x_2, x_3\}$

$C_1(K) = \text{span}\{\{x_0,x_1\}, \{x_0,x_2\}, \{x_1,x_2\}, \{x_1,x_3\}, \{x_2,x_3\}\}$

$C_2(K) = \text{Span}\{\{x_1, x_2, x_3\}\}$

$C_n(K) = \{0\}$ for $n \geq 3$.

$\partial_0 : C_0 \to C_{-1}$

$\ker(\partial_0) = \text{Span}\{x_0, x_1, x_2, x_3\}$, dim $= 4$

$\text{im}(\partial_0) = 0$, dim $= 0$.

$\partial_1 : C_1 \to C_0$

$\text{im}(\partial_1) = \text{Span}\{\underbrace{x_0 + x_1, x_0 + x_2}_{x_1 + x_2 \text{ in this span}}, \underbrace{x_1 + x_3}_{x_2 + x_3 \text{ in span of all } 3.}\}$

$\dim(\text{im}) = 3$

$$\ker(\partial_1) = \text{Span}\left\{ \begin{array}{l} \{x_0,x_2\} + \{x_0,x_1\} + \{x_1,x_2\}, \\ \{x_1,x_2\} + \{x_1,x_3\} + \{x_2,x_3\} \end{array} \right\}$$

$\partial_2 : C_2 \to C_1$

$\text{im}(\partial_2) = \text{Span}\{\{x_1,x_2\} + \{x_1,x_3\} + \{x_2,x_3\}\}$

$\ker(\partial_2) = \{0\}$.

∂_n trivial for all $n \geq 3$., $\text{Im}(\partial_3) = \{0\}$.

$\boxed{\text{Def'n}}$ Let K be a simplicial complex. The <u>n-th homology vector space</u> is

$$H_n(K) := \ker(\partial_n) / \mathrm{Im}(\partial_{n+1})$$

The <u>n^{th} Betti number</u> (# of n-dim'l holes)

$$\beta_n(K) = \dim(H_n(K)) = \dim(\ker(\partial_n)) - \dim(\mathrm{Im}(\partial_{n+1})).$$

$\boxed{\text{Exercise}}$ ① Compute Betti #'s in example from above

② Show $\mathrm{Im}(\partial_{n+1}) \subseteq \ker(\partial_n)$ by showing for all $n \geq 1$, $\partial_{n-1} \circ \partial_n = 0$.

① $\beta_0(K) = 4 - 3 = 1$ (1 connected component)
$\beta_1(K) = 2 - 1 = 1$ (1 one-dim'l hole)
$\beta_n(K) = 0$ for all $n \geq 2$.

② $$\partial_{n-1}\partial_n(\{x_0, \dots, x_n\}) = \sum_{i=0}^{n} \partial_{n-1}(\{x_0, \dots, x_n\} \setminus \{x_i\})$$

$$= \sum_{i=0}^{n} \sum_{\substack{j=0 \\ i \neq j}}^{n} \{x_0, \dots, x_n\} \setminus \{x_i, x_j\} = 0$$

same removing combination $\{i,j\}$ occurs twice.

Lemma $\beta_0(K) = \#$ connected components.

Let $V = \{\text{vertices on } K\}$.

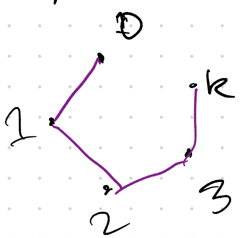
$$\beta_0(K) = \dim\left(\ker(\partial_0) / \operatorname{Im}(\partial_1)\right) = |V| - \dim(\operatorname{Im}(\partial_1))$$

Divide V into m connected components $V_1 \sqcup \cdots \sqcup V_m$.

Then $|V| = |V_1| + \cdots + |V_m|$.

$$\dim(\operatorname{Im}(\partial_1)) = \sum_{i=1}^{m} \dim\left(\operatorname{Im}(\partial_1|_{V_i})\right) = \sum_{i=1}^{m} |V_i| - 1$$

So $\beta_0(K) = \#$ connected components once we show for any connected simplex on V_i, $\dim(\partial_1) = |V_i| - 1$.



If $|V_i| = k$, start with x_0, $\exists$ path to any other vertex x_i, all of which are independent

$$\partial_1(\{x_0, \dots, x_k\}) = \left\{ \begin{array}{l} x_0 + \cdots + x_1,\ x_0 + \cdots + x_2 \\ x_0 + \cdots + x_3, \dots, x_0 + \cdots + x_k \end{array} \right\}$$

This has $k-1$ elements.

Making Betti numbers easy to compute!

Def K simplicial complex, $k_i = \#\, i$-dim'l simplices

The <u>Euler characteristic</u> (chi $= \chi$)

$$\chi(K) = \sum_{i \geq 0} (-1)^i k_i$$

From last time: for K connected $\underbrace{1 - \#\,1\text{-holes}}_{\beta_0(K) - \beta_1(K)} = \underbrace{(\text{vertices} - \text{edges} + \text{triangles})}_{\text{Euler characteristic}}$

$\boxed{\text{Proposition}}$

$$\sum_{i \geq 0} (-1)^i k_i = \chi(K) = \sum_{i \geq 0} (-1)^i \beta_i(K).$$

$\boxed{\text{Pf}}$ By rank-nullity theorem

$$k_i = \dim(C_i(K)) = \underbrace{\dim(\mathrm{Im}(\partial_i))}_{\text{rank}} + \underbrace{\dim(\ker(\partial_i))}_{\text{nullity}}$$

So

$$\sum_{i \geq 0} (-1)^i k_i = \sum_{i \geq 0} (-1)^i \left[\dim(\mathrm{Im}(\partial_i)) + \dim(\ker(\partial_i))\right]$$

$$= \underbrace{\dim(\mathrm{Im}(\partial_0))}_{=0} + \underbrace{\dim(\ker(\partial_0)) - \dim(\mathrm{Im}(\partial_1))}_{\text{new pairing}} - \cdots$$

$$= \sum_{i \geq 0} (-1)^i \dim(\ker(\partial_i)) + (-1)^{i+1} \dim(\mathrm{Im}(\partial_{i+1}))$$

$$= \sum_{i \geq 0} (-1)^i \left[\dim(\ker(\partial_i)) - \dim(\mathrm{Im}(\partial_{i+1}))\right]$$

$$= \sum_{i \geq 0} (-1)^i \beta_i(K).$$

$\boxed{\text{Algorithm}}$ (Simple version persistent homology)

$\boxed{\text{Input}}$ Data $P=\{x_0,\dots,x_n\}\subseteq \mathbb{R}^D$

Radii $\quad 0<r_1<\dots<r_m$

For $i=1,\dots,m$

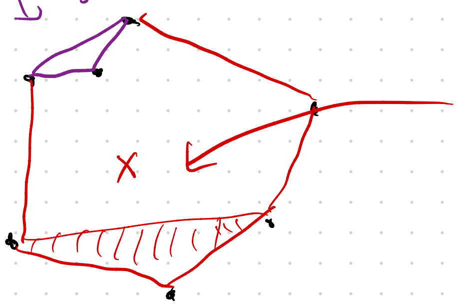
compute $C_{r_i}(P)$ or $VR_{r_i}(P)$

Return $\left(\beta_j\left[C_{r_i}(P)\right] \text{ for } j\geq 0\right)$

$\boxed{\text{Output}}$ matrix of Betti numbers $\left(\beta_j(C_{r_i})\right)_{\substack{i=1,\dots,m\\ j\geq 0}}$

Interpretation The values that persist for a long sequence of r_i gives geometry of data set.

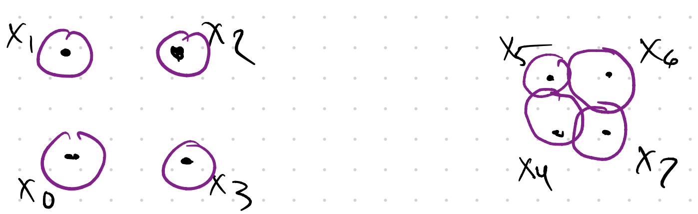
hole for only a small range of r



this hole will hold for a much larger range of r, until radius encompasses full data set.

$\boxed{\text{Problem}}$ Our choice of r_i might miss some geometry of the set:

Suppose we have two circles as data set:



detect "hole" on right

detect hole on left

Two circles should have 2 holes, but the choice of r_i was unfortunate and we completely missed "seeing" both holes at the same time.

☆ For next time we define persistent Betti numbers, which is last piece of information we need for persistent homology. Bring computer on wednesday