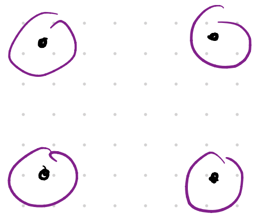
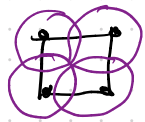
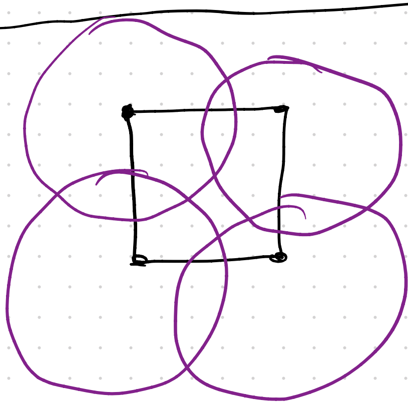
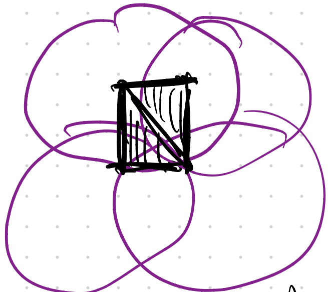
Motivating Example



K_{r_1} First chosen r detects hole on small square







K_{r_2} detects hole on large square, but doesn't see small square.

Idea Small square hole shows up for r_1 (birth)
Small square hole gone for r_2 (death).

Definition A chain of simplicial complexes of the form $K_1 \subseteq \cdots \subseteq K_m$ is a __filtration__ of __length__ m.

$\boxed{\text{example}}$ Given $r_1 < \cdots < r_m$

$K_i = C_{r_i}$ or $K_i = VR_{r_i}$ gives filtration.

$\boxed{\text{Algorithm}}$ [The real deal of persistent homology]

- $\boxed{\text{Input}}$ $\mathcal{P} = \{x_0, \ldots, x_n\} \subseteq \mathbb{R}^D$

  $0 < r_1 < \cdots < r_m$

- Compute filtration $K_1 \subseteq \cdots \subseteq K_m$

$$K_i = \left\{ \begin{matrix} C_{r_i} \\ VR_{r_i} \end{matrix} \right. \text{ for } i = 1, \ldots, m$$

- For $1 \leq i < j \leq m$ and $0 \leq n \leq \dim(K_i) - 1$

  $\mu_n^{i,j}$ = # n-dim'l holes that appear at index i and vanish at j.

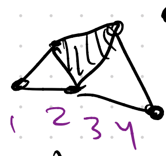
- $\boxed{\text{Output}}$ All information $\mu_n^{i,j}$

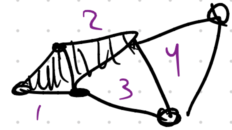
A sketch for defining $\mu_n^{i,j}$ rigorously. [see §4.4 for details]

We define the inclusion map

$$\iota_{i,j} : K_i \to K_j \quad \text{for } i < j$$

$\iota_{i,j}(\Delta) = \Delta.$

For example $K_1 =$ 

$K_2 =$ 

$\text{Image}(\iota_{i,j}) =$ 

$\boxed{\text{Def 4.30 + Lemma 4.31}}$ $\Rightarrow$ we have also a natural inclusion

$$(\iota_{i,j})_* : H_n(K_i) \longrightarrow H_n(K_j)$$

$\text{Image}(\iota_{1,2})_* \to$ sees hole 3

$\text{Kernel}(\iota_{1,2})_* \to$ contains hole 1 since the hole becomes boundary of a simplex.

triangle 2 is never a hole, so not seen by $H_1(K_i)$

triangle 4 is a hole in K_2, but not in image of $(\iota_{i,j})_*$ since not a 1-dim'l hole in K_1.

$\boxed{\text{Definition}}$ The n^{th} persistent Betti numbers are

$$\beta_n^{i,j} = \dim(\text{Im}(\iota_{i,j})_*)$$

= # n-dim'l holes in K_i still present in K_j

$\boxed{\text{Proposition}}$ The number of n-dimensional holes that appear at index i and vanish at j is

$$\mu_n^{i,j} := \left(\beta_n^{i,j-1} - \beta_n^{i,j}\right) - \left(\beta_n^{i-1,j-1} - \beta_n^{i-1,j}\right)$$

for $i < j$.

$\boxed{\text{Pf}}$ $\underbrace{\beta_n^{i,j-1}}_{\text{Holes in } K_i \text{, in } j-1} - \underbrace{\beta_n^{i,j}}_{\text{subtract holes still in at level } j} = \#$ holes in K_i vanish at index j.

$\beta_n^{i-1,j-1} - \beta_n^{i-1,j} = \#$ holes in K_{i-1} vanish at index j

subtract holes that were in K_{i-1} (i.e. weren't born at K_i).

$$K_{i-1} \subseteq K_i \subseteq K_j$$

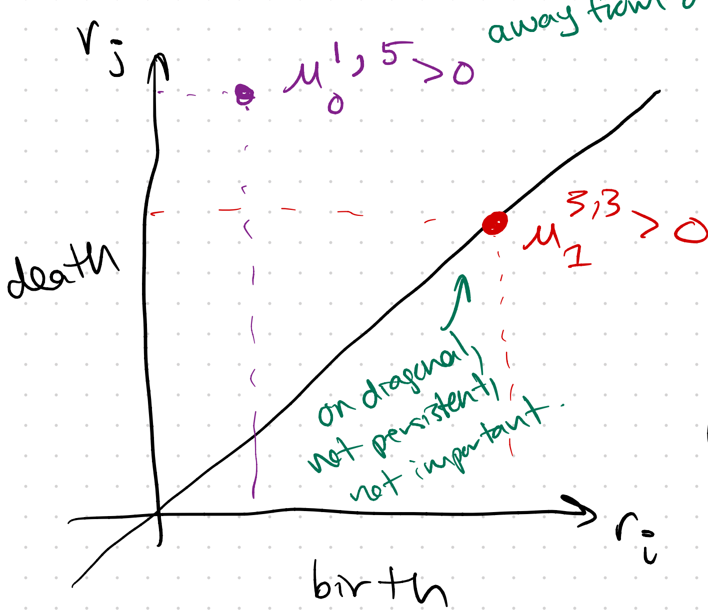
holes that died at K_j

$\boxed{\text{XL}}$

Notebook 7 | Output of ripser.jl:

Persistane Diagram

away from diagonal ⇒ persistent ⇒ significant



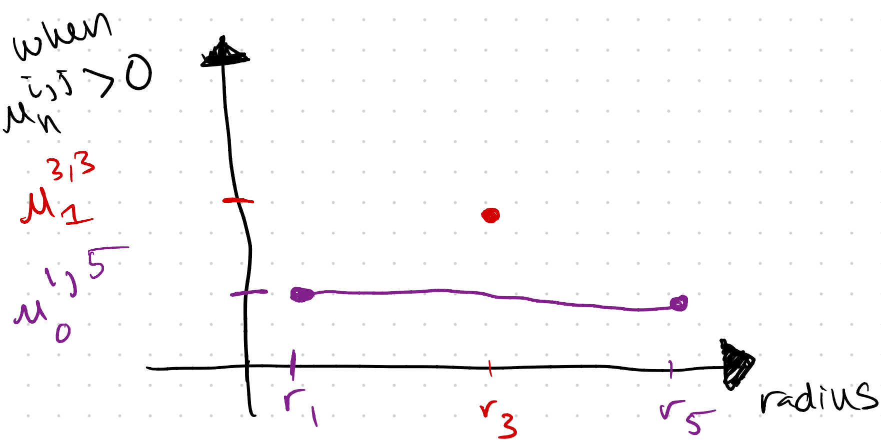
• $n = 0$

• $n = 1$

place a point with fixed n if $\mu_n^{i,j} > 0$.

OR

Barcode Plot



• $n = 0$

• $n = 1$

Notebook 7 exercises

- Ellipse → what happens as you change ellipse, or number of points or amount of noise?

- Cyclooctane model – What would you change or work with to compare our method w/ $\beta_1 = 2$ and $\beta_2 = 2$ to get the paper's result of $\beta_1 = 1$, $\beta_2 = 2$.

- Go to ripser.jl github: Examples
  - ✯ Stability → circles
  - ✯ Cubical persistant homology: Black holes
  - ✯ Image classification: malaria infected cells