# Exam
# DIT008: Discrete Mathematics

University of Gothenburg
Lecturer: Jan Gerken

26 August 2025, 14:00 – 18:00

No calculators or other aids are allowed.

You must show your calculations or describe your argument. Unless otherwise stated, if you provide just the final answer, you will not be awarded any points.

The grade boundaries are

- pass (3): at least 20 points (50%)
- pass with credit (4): at least 28 points (70%)
- distinction (5): at least 36 points (90%)

In case of questions, call Jan Gerken at 031-772-14-37.

Good luck!

| Problem: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Points: | 2 | 3 | 6 | 3 | 4 | 3 | 4 | 3 | 3 | 3 | 4 | 2 | 40 |

**Problem 1** (2 points)
Use a truth table to establish the truth of the following statement: "The converse and inverse of a conditional statement are logically equivalent to each other."

**Problem 2** (3 points)
For this question, you do not need to justify your answers.

(a) (2 points) Which of the following statements are true?

- $\{5\} \in \{1, 3, 5\}$
- $\{5\} \subseteq \{1, 3, 5\}$
- $\{5\} \in \{\{1\}, \{3\}, \{5\}\}$
- $\{5\} \subseteq \{\{1\}, \{3\}, \{5\}\}$

(b) (1 point) Let $A = \{a, b, c\}$ and $B = \{u, v\}$. Write $A \times B$ and $B \times A$.

**Problem 3** (6 points)
Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the *composition* of $g$ and $f$, denoted by $g \circ f$, is a function $g \circ f : X \to Z$ defined by $(g \circ f)(x) = g(f(x))\ \forall x \in X$. Prove the following statements

(a) (3 points) If $f$ and $g$ are injective, then $g \circ f$ is also injective.

(b) (3 points) If $f$ and $g$ are surjective, then $g \circ f$ is also surjective.

**Problem 4** (3 points)
Prove that if one solution for a quadratic equation of the form $x^2+bx+c = 0$ is rational (where $b$ and $c$ are rational), then the other solution is also rational. Use the fact that if the solutions of the equation are $r$ and $s$, then $x^2 + bx + c = (x - r)(x - s)$, $\forall x \in \mathbb{R}$.

**Problem 5** (4 points)
Use mathematical induction to prove that $\forall n \in \mathbb{Z}$ with $n \geq 2$, $2^n < (n + 1)!$.

**Problem 6** (3 points)
In a *Triple Tower of Hanoi*, there are three poles in a row and $3n$ disks, three of each of $n$ different sizes, where $n$ is any positive integer. Initially, one of the poles contains all the disks placed on top of each other in triples of decreasing size. Disks are transferred one by one from one pole to another, but at no time may a larger disk be placed on top of a smaller disk. However, a disk may be placed on top of one of the same size. Let $t_n$ be the minimum number of moves needed to transfer a tower of $3n$ disks from one pole to another.

(a) (1 point) Find $t_1$ and $t_2$. Justify your answer.

(b) (2 points) Find a recursion relation for $t_n$, $n \leq 2$. Justify your answer.

**Problem 7** (4 points)
A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions: (a) Rabbit pairs are not fertile during their first two months of life, but thereafter they give birth to four new male/female pairs at the end of every month; (b) No deaths occur. Let $s_n$ be the number of pairs of rabbits alive at the end of month $n$, for each integer $n \leq 1$, and let $s_0 = 1$. Find a recursion relation for $s_n$ with suitable initial conditions. Justify your answer.

**Problem 8** (3 points)
Given any set of 30 integers, must there be two that have the same remainder when they are divided by 25? Write an answer that would convince a good but skeptical fellow student who has learned the statement of the pigeonhole principle but not seen an application like this one. Either describe the pigeons, the pigeonholes, and how the pigeons get to the pigeonholes, or describe a function by giving its domain, co-domain, and how elements of the domain are related to elements of the co-domain.

**Problem 9** (3 points)
Two new drugs are to be tested using a group of 9 laboratory mice, each tagged with a number for identification purposes. Drug $A$ is to be given to 3 mice, drug $B$ is to be given to another 3 mice, and the remaining 3 mice are to be used as controls. How many ways can the assignment of treatments to mice be made? (A single assignment involves specifying the treatment for each mouse – whether drug $A$, $B$, or no drug.)

**Problem 10** (3 points)
A password consists of 4 characters, where each character is either a letter in $\{A, B\}$ or a digit in $\{1, 2\}$.

(a) (1 point) How many different passwords are possible?

(b) (1 point) How many different passwords with exactly 2 letters and 2 digits are possible?

(c) (1 point) How many different passwords with at least one letter and at least one digit are possible?

**Problem 11** (4 points)

(a) (2 points) A certain connected graph has 68 vertices and 72 edges. Does it have a circuit? Explain.

(b) (2 points) A certain graph has 19 vertices, 16 edges, and no circuits. Is it connected? Explain.

**Problem 12** (2 points)
Using the definition of the $O$-notation, prove that $15x^3 + 8x + 4$ is $O(x^3)$. Do not use the theorem on polynomial orders.