Exam
DIT008: Discrete Mathematics

University of Gothenburg
Lecturer: Jan Gerken

7th January 2025, 8:30 – 12:30

No calculators or other aids are allowed.

You must show your calculations or describe your argument. Unless otherwise stated, if you provide just the final answer, you will not be awarded any points.

The grade boundaries are

- pass (3): at least 20 points (50%)
- pass with credit (4): at least 28 points (70%)
- distinction (5): at least 36 points (90%)

In case of questions, call Jan Gerken at 031-772-14-37.

Good luck!

Problem:	1	2	3	4	5	6	7	8	9	10	Total
Points:	5	2	4	4	5	4	4	5	5	2	40

Problem 1 (5 points)

Let $A = \{3, 5, 7\}$ and $B = \{15, 16, 17, 18\}$, and define a relation R from A to B as follows: $\forall\ (x, y) \in A \times B$,

$$(x, y) \in R \iff \frac{y}{x} \in \mathbb{Z}\,.$$

Apart from (e), no explanations are necessary.

(a) (1 point) Is $3\,R\,15$? Is $3\,R\,16$? Is $(7, 17) \in R$? Is $(3, 18) \in R$?

(b) (1 point) Write R as a set of ordered pairs.

(c) (1 point) Write the domain and co-domain of R.

(d) (1 point) Draw an arrow diagram for R.

(e) (1 point) Is R a function from A to B? Explain.

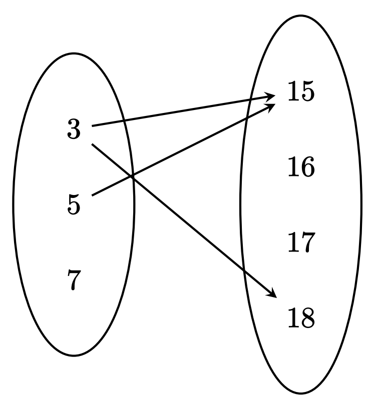
Solution:

(a) $3\ R\ 15$, $3\ \not R\ 16$, $(7, 17) \notin R$, $(3, 18) \in R$

(b) $R = \{(3, 15), (3, 18), (5, 15)\}$

(c) The domain of R is A and the co-domain of R is B

(d)



(e) R is not a function from A to B since (a) $7 \in A$ is not related to any element of B and (b) $3 \in A$ is related to both $15 \in B$ and $18 \in B$.

Problem 2 (2 points)
Write the form of the following argument. Is the argument valid or invalid? Justify your answer.

If Ann has the flu, then Ann has a fever.
Ann has a fever.
$\therefore$ Ann has the flu.

Solution: The argument has the form

$$\begin{array}{l} p \to q \\ q \\ \therefore p\,, \end{array}$$

which is invalid, it exhibits the converse error.

Problem 3 (4 points)
Prove the following statement by contradiction:

$\forall\, r, s \in \mathbb{R}$, if r is rational and s is irrational, then $r + 2s$ is irrational.

Solution:
Proof (by contradiction): Suppose not. That is, suppose $\exists\, r, s \in \mathbb{R}$ such that r is rational and s is irrational and $r+2s$ is rational. By definition of rational, $\exists\, a, b, c, d \in \mathbb{Z}$ with $b \neq 0$ and $d \neq 0$ such that

$$r = \frac{a}{b} \quad \text{and} \quad r + 2s = \frac{c}{d}\,.$$

Then, by substitution,

$$\frac{a}{b} + 2s = \frac{c}{d}.$$

Solve this equation for s to obtain

$$s = \frac{1}{2}\left(\frac{c}{d} - \frac{a}{b}\right) = \frac{1}{2}\left(\frac{bc - ad}{bd}\right) = \frac{bc - ad}{2bd}.$$

But both $bc - ad \in \mathbb{Z}$ and $2bd \in \mathbb{Z}$ because products and differences of integers are integers, and $2bd \neq 0$ by the zero product property.

Hence s is a ratio of integers with a nonzero denominator, and so s is rational by definition of rational. This contradicts the supposition that s is irrational.

Problem 4 (4 points)

Prove by induction that

$$\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1 \qquad \forall\, n \geq 1\,.$$

Solution:
Proof by induction.
Base case ($n = 1$): For $n = 1$, the LHS is

$$\sum_{i=1}^{n} i \cdot i! = 1 \cdot 1! = 1$$

and the RHS is also

$$(n+1)! - 1 = 2! - 1 = 2 - 1 = 1.$$

Hence the statement is true for $n = 1$.
Inductive step. Suppose that

$$\sum_{i=1}^{k} i \cdot i! = (k+1)! - 1$$

for some $k \in \mathbb{Z}$. Then,

$$\sum_{i=1}^{k+1} i \cdot i! = \sum_{i=1}^{k} i \cdot i! + (k+1) \cdot (k+1)! = (k+1)! - 1 + (k+1) \cdot (k+1)!$$

by the induction hypothesis. Hence,

$$\begin{aligned}\sum_{i=1}^{k+1} i \cdot i! &= (k+1+1) \cdot (k+1)! - 1 \\ &= (k+2)! - 1\,.\end{aligned}$$

This is the statement to be proven for $n = k + 1$.

Problem 5 (5 points)
Consider the sequence $b_0, b_1, b_2, \ldots$ defined recursively by

$$\begin{aligned} b_0 &= 1 \\ b_k &= 2b_{k-1} + 3 \quad \forall\, k \geq 1\,. \end{aligned}$$

(a) (2 points) Use iteration to guess an explicit formula for the sequence. Write out the iteration, not just the guess you arrive at.

(b) (3 points) Simplify your answer using the closed-form expression for the sum of the geometric or arithmetic sequence.

Solution:

(a)

$$\begin{aligned} b_0 &= 1 \\ b_1 &= 2b_0 + 3 = 2 \cdot 1 + 3 = 2 + 3 \\ b_2 &= 2b_1 + 3 = 2 \cdot (2+3) + 3 = 2^2 + 2 \cdot 3 + 3 \\ b_3 &= 2b_2 + 3 = 2 \cdot (2^2 + 2 \cdot 3 + 3) + 3 = 2^3 + 2^2 \cdot 3 + 2 \cdot 3 + 3 \\ b_4 &= 2b_3 + 3 = 2 \cdot (2^3 + 2^2 \cdot 3 + 2 \cdot 3 + 3) + 3 = 2^4 + 2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 3 \end{aligned}$$

Hence, one can guess that

$$b_k = 2^k + 3 \cdot 2^{k-1} + 3 \cdot 2^{k-2} + \cdots + 3 \cdot 2 + 3\,.$$

(b) The sum of the geometric sequence is given by

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1} \quad \forall\, r \in \mathbb{R},\ n \in \mathbb{Z},\ r \neq 1,\ n \geq 0\,.$$

Hence, the guess from (a) can be simplified to

$$\begin{aligned} b_k &= 2^k + 3 \cdot 2^{k-1} + 3 \cdot 2^{k-2} + \cdots + 3 \cdot 2 + 3 \\ &= (3-2) \cdot 2^k + 3 \cdot 2^{k-1} + 3 \cdot 2^{k-2} + \cdots + 3 \cdot 2 + 3 \\ &= 3 \sum_{i=1}^{k} 2^i - 2 \cdot 2^k \\ &= 3 \cdot \frac{2^{k+1} - 1}{2 - 1} - 2^{k+1} \\ &= 3(2^{k+1} - 1) - 2^{k+1} \\ &= 2^{k+2} - 3\,. \end{aligned}$$

Problem 6 (4 points)

In a certain discrete math class, three quizzes were given. Out of the 30 students in the class:

- 15 scored 12 or above on quiz #1,
- 12 scored 12 or above on quiz #2,
- 18 scored 12 or above on quiz #3,
- 7 scored 12 or above on quizzes #1 and #2,
- 11 scored 12 or above on quizzes #1 and #3,
- 8 scored 12 or above on quizzes #2 and #3,
- 4 scored 12 or above on quizzes #1, #2, and #3.

(a) (2 points) How many students scored 12 or above on at least one quiz? Explain.

(b) (2 points) How many students scored 12 or above on quizzes 1 and 2 but not 3? Explain.

Solution:

(a) Let A, B, and C be the sets of all students who scored 12 or above on quizzes #1, #2, and #3, respectively. Then $N(A) = 15$, $N(B) = 12$, $N(C) = 18$, $N(A \cap B) = 7$, $N(A \cap C) = 11$, $N(B \cap C) = 8$, and $N(A \cap B \cap C) = 4$. By the inclusion/exclusion principle,

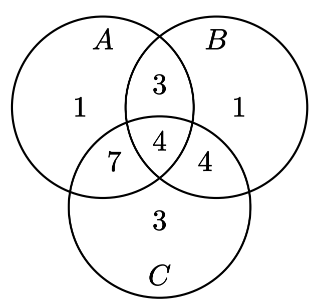
$$N(A \cup B \cup C) = 15 + 12 + 18 - 7 - 11 - 8 + 4 = 23\,.$$

So 23 students scored 12 or above on at least one quiz.

(b) The set of students who scored 12 or above on quizzes 1 and 2 but not 3 is $(A \cap B) - C = (A \cap B) - (A \cap B \cap C)$. Since $(A \cap B \cap C) \subseteq (A \cap B)$, we have

$$N((A \cap B) - C) = N(A \cap B) - N(A \cap B \cap C) = 7 - 4 = 3.$$

Venn-diagram for this problem:



Problem 7 (4 points)

A large pile of coins consists of 1 euro coins from Belgium, Finland, Luxembourg and Slovakia (at least 10 of each). Explain your answers.

(a) (2 points) How many different collections of 10 coins can be chosen?

(b) (2 points) How many different collections of 10 coins chosen at random will contain at least one coin from each country?

Solution:

(a) We are counting the number of multisets of size 10 chosen from a set of 4. Hence, there are

$$\binom{10+4-1}{10} = \binom{13}{10} = \frac{13!}{3! \cdot 10!} = \frac{13 \cdot 12 \cdot 11}{6} = 2 \cdot 13 \cdot 11 = 2 \cdot 143 = 286$$

different collections.

(b) If each country must be present at least once, there are 6 coins left to choose from the pile. Hence, we are counting 6-combinations of a set of 4. Therefore there are

$$\binom{6+4-1}{6} = \binom{9}{6} = \frac{9!}{3! \cdot 6!} = \frac{9 \cdot 8 \cdot 7}{6} = 12 \cdot 7 = 84$$

possible collections of this type.

Problem 8 (5 points)

The adjacency matrix of a graph with vertices v_1, v_2 and v_3 is given by

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

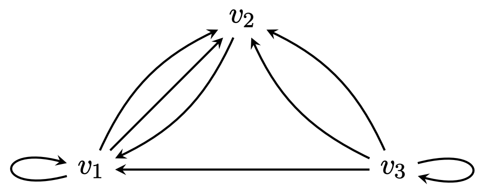
(a) (2 points) Draw a directed graph whose adjacency matrix is A. No explanation necessary.

(b) (2 points) Compute A^3.

(c) (1 point) How many walks of length 3 are there from vertex v_3 to vertex v_2? How many walks of length 3 are there from vertex v_1 to vertex v_3? Explain.

Solution:

(a)

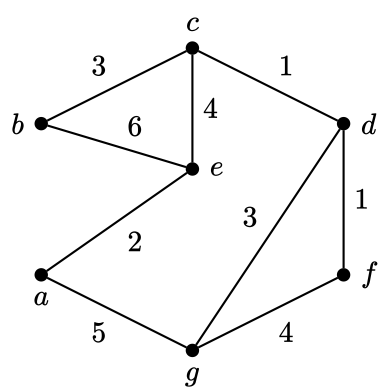


(b)

$$A^3 = A \cdot A^2 = A \cdot \begin{pmatrix} 3 & 2 & 0 \\ 1 & 2 & 0 \\ 4 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 6 & 0 \\ 3 & 2 & 0 \\ 9 & 10 & 1 \end{pmatrix}$$

(c) The number of walks of length 3 from v_3 to v_2 is $(A^3)_{32} = 10$ and the number of walks of length 3 from v_1 to v_3 is $(A^3)_{13} = 0$.

Problem 9 (5 points)
Consider the following weighted graph:



Use Dijkstra's algorithm to find the shortest path from a to d and its length. To do this, create a table with the following columns:

Step	F	$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	$V(T)$	$E(T)$	v

Solution:

Step	F	$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	$V(T)$	$E(T)$	v
0	$\{a\}$	0	∞	∞	∞	∞	∞	∞	$\{a\}$	$\emptyset$	a
1	$\{e, g\}$	0	∞	∞	∞	2	∞	5	$\{a, e\}$	$\{\{a, e\}\}$	e
2	$\{b, c, g\}$	0	8	6	∞	2	∞	5	$\{a, e, g\}$	$\{\{a, e\},$ $\{a, g\}\}$	g
3	$\{b, c, d, f\}$	0	8	6	8	2	9	5	$\{a, c, e, g\}$	$\{\{a, e\},$ $\{a, g\}$ $\{c, e\}\}$	c
4	$\{b, d, f\}$	0	8	6	7	2	9	5	$\{a, c, d, e, g\}$	$\{\{a, e\},$ $\{a, g\}$ $\{c, e\}$ $\{c, d\}\}$	d

Hence, the shortest path found by the algorithm is a, e, c, d of length 7.

Problem 10 (2 points)

Use the definition of the O-notation to prove that $2x^2 + 3x + 4$ is $O(x^2)$. Do not use the theorem on polynomial orders.

Solution: $\forall\, x > 1$,

$$0 \leq 2x^2 + 3x + 4$$

because $2x^2$, $3x$ and 4 are all positive for $x > 1$. Furthermore,

$$2x^2 + 3x + 4 \leq 2x^2 + 3x^2 + 4x^2 = 9x^2$$

because $x < x^2$ and $1 < x^2$ for $x > 1$. Therefore, we have

$$0 \leq 2x^2 + 3x + 4 \leq 9x^2 \quad \forall\, x > 1\,.$$

Hence, by definition of O-notation, $2x^2 + 3x + 4$ is $O(x^2)$.