Exam
DIT008: Discrete Mathematics

University of Gothenburg
Lecturer: Jan Gerken

7^{th} January 2025, 8:30 – 12:30

No calculators or other aids are allowed.

You must show your calculations or describe your argument. Unless otherwise stated, if you provide just the final answer, you will not be awarded any points.

The grade boundaries are

- pass (3): at least 20 points (50%)
- pass with credit (4): at least 28 points (70%)
- distinction (5): at least 36 points (90%)

In case of questions, call Jan Gerken at 031-772-14-37.

Good luck!

Problem:	1	2	3	4	5	6	7	8	9	10	Total
Points:	5	2	4	4	5	4	4	5	5	2	40

Problem 1 (5 points)
Let $A = \{3, 5, 7\}$ and $B = \{15, 16, 17, 18\}$, and define a relation R from A to B as follows: $\forall\ (x, y) \in A \times B$,

$$(x, y) \in R \iff \frac{y}{x} \in \mathbb{Z}\,.$$

Apart from (e), no explanations are necessary.

(a) (1 point) Is $3\,R\,15$? Is $3\,R\,16$? Is $(7, 17) \in R$? Is $(3, 18) \in R$?

(b) (1 point) Write R as a set of ordered pairs.

(c) (1 point) Write the domain and co-domain of R.

(d) (1 point) Draw an arrow diagram for R.

(e) (1 point) Is R a function from A to B? Explain.

Problem 2 (2 points)
Write the form of the following argument. Is the argument valid or invalid? Justify your answer.

If Ann has the flu, then Ann has a fever.
Ann has a fever.
$\therefore$ Ann has the flu.

Problem 3 (4 points)
Prove the following statement by contradiction:

$\forall\ r, s \in \mathbb{R}$, if r is rational and s is irrational, then $r + 2s$ is irrational.

Problem 4 (4 points)
Prove by induction that

$$\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1 \qquad \forall\ n \geq 1\,.$$

Problem 5 (5 points)

Consider the sequence $b_0, b_1, b_2, \ldots$ defined recursively by

$$\begin{aligned} b_0 &= 1 \\ b_k &= 2b_{k-1} + 3 \quad \forall\, k \geq 1\,. \end{aligned}$$

(a) (2 points) Use iteration to guess an explicit formula for the sequence. Write out the iteration, not just the guess you arrive at.

(b) (3 points) Simplify your answer using the closed-form expression for the sum of the geometric or arithmetic sequence.

Problem 6 (4 points)

In a certain discrete math class, three quizzes were given. Out of the 30 students in the class:

- 15 scored 12 or above on quiz #1,
- 12 scored 12 or above on quiz #2,
- 18 scored 12 or above on quiz #3,
- 7 scored 12 or above on quizzes #1 and #2,
- 11 scored 12 or above on quizzes #1 and #3,
- 8 scored 12 or above on quizzes #2 and #3,
- 4 scored 12 or above on quizzes #1, #2, and #3.

(a) (2 points) How many students scored 12 or above on at least one quiz? Explain.

(b) (2 points) How many students scored 12 or above on quizzes 1 and 2 but not 3? Explain.

Problem 7 (4 points)

A large pile of coins consists of 1 euro coins from Belgium, Finland, Luxembourg and Slovakia (at least 10 of each). Explain your answers.

(a) (2 points) How many different collections of 10 coins can be chosen?

(b) (2 points) How many different collections of 10 coins chosen at random will contain at least one coin from each country?

Problem 8 (5 points)

The adjacency matrix of a graph with vertices v_1, v_2 and v_3 is given by

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

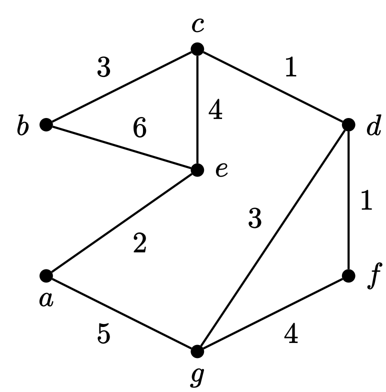
(a) (2 points) Draw a directed graph whose adjacency matrix is A. No explanation necessary.

(b) (2 points) Compute A^3.

(c) (1 point) How many walks of length 3 are there from vertex v_3 to vertex v_2? How many walks of length 3 are there from vertex v_1 to vertex v_3? Explain.

Problem 9 (5 points)

Consider the following weighted graph:



Use Dijkstra's algorithm to find the shortest path from a to d and its length. To do this, create a table with the following columns:

Step	F	$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	$V(T)$	$E(T)$	v

Problem 10 (2 points)

Use the definition of the O-notation to prove that $2x^2 + 3x + 4$ is $O(x^2)$. Do not use the theorem on polynomial orders.