Exam
DIT008: Discrete Mathematics

University of Gothenburg
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27 October 2025, 8:30 – 12:30

No calculators or other aids are allowed.

You must show your calculations or describe your argument. Unless otherwise stated, if you provide just the final answer, you will not be awarded any points.

The grade boundaries are

- pass (3): at least 20 points (50%)
- pass with credit (4): at least 28 points (70%)
- distinction (5): at least 36 points (90%)

In case of questions, call Jan Gerken at 031-772-14-37.

Good luck!

Problem:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	3	3	4	3	7	3	3	3	3	4	4	40

Problem 1 (3 points)

Consider the argument form:

$$\begin{array}{rl} & p \to \sim q \\ & q \to \sim p \\ \therefore & p \vee q \end{array}$$

Use a truth table to determine whether this form of argument is valid or invalid. Annotate the table as appropriate and include a few words explaining how the truth table supports your answer.

Solution:

p	q	$\sim p$	$\sim q$	$p \to \sim q$	$q \to \sim p$	$p \vee q$
T	T	F	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	T	F

Columns 5 and 6 of the truth table show the premises, column 7 shows the conclusion. Rows 2-4 are the critical rows because all premises are true in these rows. Row 4 shows that it is possible for an argument of this form to have true premises and a false conclusion. Therefore, the argument form is invalid.

Problem 2 (3 points)

The following is the definition for $\lim_{x\to a} f(x) = L$:

> For every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for every real number x, if $a - \delta < x < a + \delta$ and $x \neq a$ then
>
> $$L - \varepsilon < f(x) < L + \varepsilon\,.$$

Write what it means for $\lim_{x\to a} f(x) \neq L$. In other words, write the negation of the statement above. Do not use negations of elementary operators and quantifiers such as $\not\forall$, $\not\exists$, $\not<$ etc. You can use $\neq$. No further justification necessary.

Solution: There exists a real number $\varepsilon > 0$ such that for all real numbers $\delta > 0$, there exists a real number x such that $a - \delta < x < a + \delta$ and $x \neq a$, and either $L - \varepsilon \geq f(x)$ or $f(x) \geq L + \varepsilon$.

Problem 3 (4 points)

Prove by induction:

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2 \quad \forall\, n \in \mathbb{Z}^{\geq 0}\,.$$

Solution: Let the property $P(n)$ be the equation

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2\,.$$

We will show that $P(n)$ is true for every integer $n \geq 0$.

Base step ($n = 0$): $P(0)$ is true because the left-hand side equals

$$\sum_{i=1}^{0+1} i \cdot 2^i = 1 \cdot 2^1 = 2\,,$$

and the right-hand side equals $0 \cdot 2^{0+2} + 2 = 2$ also.

Inductive hypothesis: Let k be any integer with $k \geq 0$, and suppose $P(k)$, i.e.

$$\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2\,.$$

Induction step: We must show $P(k+1)$, i.e.

$$\sum_{i=1}^{k+2} i \cdot 2^i = (k+1) \cdot 2^{k+3} + 2\,.$$

The left-hand side of $P(k+1)$ is

$$\begin{aligned}
\sum_{i=1}^{k+2} i \cdot 2^i &= \sum_{i=1}^{k+1} i \cdot 2^i + (k+2) \cdot 2^{k+2} \\
&= (k \cdot 2^{k+2} + 2) + (k+2) \cdot 2^{k+2} \quad \text{by inductive hypothesis} \\
&= (k + (k+2)) \cdot 2^{k+2} + 2 \\
&= (2k+2) \cdot 2^{k+2} + 2 \\
&= (k+1) \cdot 2^{k+3} + 2 \quad \text{by algebra,}
\end{aligned}$$

and this is the right-hand side of $P(k+1)$ as was to be shown.

Problem 4 (3 points)

Prove the following statement by contradiction:

$\forall n \in \mathbb{Z}$, if n^3 is even, then n is even.

Solution: Suppose not. That is, suppose there is an integer n such that n^3 is even and n is odd.

By definition of odd, $n = 2a+1$ for some integer a. Thus, by substitution and algebra,

$$\begin{aligned} n^3 &= (2a+1)^3 \\ &= (2a+1)^2(2a+1) \\ &= (4a^2+4a+1)(2a+1) \\ &= 8a^3+12a^2+6a+1 \\ &= 2(4a^3+6a^2+3a)+1\,. \end{aligned}$$

Let $t = 4a^3+6a^2+3a$. Then $n^3 = 2t+1$, and $t \in \mathbb{Z}$ because it is a sum of products of integers. It follows that n^3 is odd, which contradicts the supposition that n^3 is even.

Problem 5 (7 points)

Define $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows: $F(x, y) = (3y - 1, 1 - x)\ \forall\, (x, y) \in \mathbb{R} \times \mathbb{R}$.

(a) (1 point) Compute $F(0, 0)$ and $F(1, 4)$.

(b) (3 points) Is F injective? Prove or give a counterexample.

(c) (2 points) Is F surjective? Prove or give a counterexample.

(d) (1 point) Is F bijective? If not, explain why not. If yes, find F^{-1}.

Solution:

(a)

$$\begin{aligned} F(0,0) &= (3 \cdot 0 - 1, 1 - 0) = (-1, 1) \\ F(1,4) &= (3 \cdot 4 - 1, 1 - 1) = (11, 0) \end{aligned}$$

(b) F is injective.

Proof: Let $(x_1, y_1) \in \mathbb{R} \times \mathbb{R}$ and $(x_2, y_2) \in \mathbb{R} \times \mathbb{R}$ such that $F(x_1, y_1) = F(x_2, y_2)$. We need to show that $(x_1, y_1) = (x_2, y_2)$.

By the definition of F, $(3y_1 - 1, 1 - x_1) = (3y_2 - 1, 1 - x_2)$, and so by definition of equality of ordered pairs,

$$3y_1 - 1 = 3y_2 - 1 \quad \text{and} \quad 1 - x_1 = 1 - x_2\,.$$

Adding 1 to both sides of the left-hand equation and subtracting 1 from both sides of the right-hand equation gives that

$$3y_1 = 3y_2 \quad \text{and} \quad -x_1 = -x_2\,,$$

and dividing both sides of the left-hand equation by 3 and of the right-hand equation by -1 gives that

$$y_1 = y_2 \quad \text{and} \quad x_1 = x_2\,.$$

Thus, by definition of equality of ordered pairs,

$$(x_1, y_1) = (x_2, y_2)\,.$$

As was to be shown.

(c) F is surjective.

Proof: Let $(u, v) \in \mathbb{R} \times \mathbb{R}$. We must show that $\exists\ (x, y) \in \mathbb{R} \times \mathbb{R}$ such that $F(x, y) = (u, v)$.

Let $x = 1 - v$ and let $y = \frac{u+1}{3}$. Then (x, y) is in $\mathbb{R} \times \mathbb{R}$, and

$$\begin{aligned} F(x, y) &= (3y - 1, 1 - x) \\ &= \left(3\left(\frac{u+1}{3}\right) - 1, 1 - (1 - v)\right) \\ &= (u, v) \end{aligned}$$

as was to be shown.

Note: You can use scratch work to find the values of x and y which reproduce u and v under the assumption that they exist. But this scratch work must not be part of the proof because the proof itself establishes the existence.

(d) F is bijective because F is both injective and surjective.

F^{-1} is given by: $\forall\,(u, v) \in \mathbb{R} \times \mathbb{R}$,

$$F^{-1}(u, v) = \left(1 - v, \frac{u+1}{3}\right)\,.$$

Problem 6 (3 points)

Consider the set S of all strings of a's and b's. For each integer $n \geq 0$, let

$$a_n = \text{the number of strings of length } n \text{ that do not contain the pattern } bb\,.$$

Find a recurrence relation with appropriate initial conditions for $a_0, a_1, a_2, \ldots$ and explain your answer carefully.

Solution: There is one string of length zero that does not contain the pattern bb, namely the empty string; there are two strings of length one that do not contain the pattern bb, namely a and b. Thus the initial conditions for the recurrence relation are

$$a_0 = 1 \text{ and } a_1 = 2\,.$$

Let $k \in \mathbb{Z}$ with $k \geq 2$. Any string of length k that does not contain the pattern bb starts either with an a or with a b. If it starts with an a, this can be followed by any string of length $k-1$ that does not contain the pattern bb. There are a_{k-1} of these. If the string starts with a b, then it must be followed by an a and that can be followed by any string of length $k-2$ that does not contain the pattern bb. There are a_{k-2} of these. Therefore, for all integers $k \geq 2$,

$$a_k = a_{k-1} + a_{k-2}\,.$$

Problem 7 (3 points)

The binomial theorem states that $\forall\, a, b \in \mathbb{R}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \quad \forall\, n \in \mathbb{Z}^{\geq 0}.$$

(a) (1 point) Use this theorem to compute $(2x+1)^4$. Write out the intermediate steps of your calculation.

(b) (2 points) Use this theorem to prove that,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k} 2^k = 1 \quad \forall\, n \in \mathbb{Z}^{\geq 0}\,.$$

Solution:

(a) Use the binomial theorem with $a = 2x$, $b = 1$, $n = 4$. Then

$$\begin{aligned}(2x+1)^4 &= \sum_{k=0}^{4} \binom{4}{k} (2x)^{4-k} 1^k \\ &= \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3 + \binom{4}{2}(2x)^2 + \binom{4}{3}(2x)^1 + \binom{4}{4}(2x)^0 \\ &= 16x^4 + 4 \cdot 8x^3 + 6 \cdot 4x^2 + 4 \cdot 2x + 1 \\ &= 16x^4 + 32x^3 + 24x^2 + 8x + 1\,.\end{aligned}$$

(b) **Proof:** According to the binomial theorem with $a = 3$ and $b = -2$,

$$(3 + (-2))^n = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} (-2)^k\,.$$

Now $3 + (-2) = 1$ and $(-2)^k = (-1)^k \cdot 2^k$. Thus

$$\sum_{k=0}^{n} \binom{n}{k} 3^{n-k} (-2)^k = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 3^{n-k} 2^k\,,$$

and so

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k 3^{n-k} 2^k = 1\,.$$

Problem 8 (3 points)

If six integers are chosen from the set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, must there be at least two integers with the property that the sum of the smaller plus the larger is 11? Why or why not? Use the pigeonhole principle. Describe the function appearing in the pigeonhole principle by giving its domain, co-domain, and how elements of the domain are related to elements of the co-domain.

Solution: Let X be the set of the six integers chosen from S, and let Y be the set of all pairs of integers from S with the property that the sum of the smaller plus the larger is 11. Then,

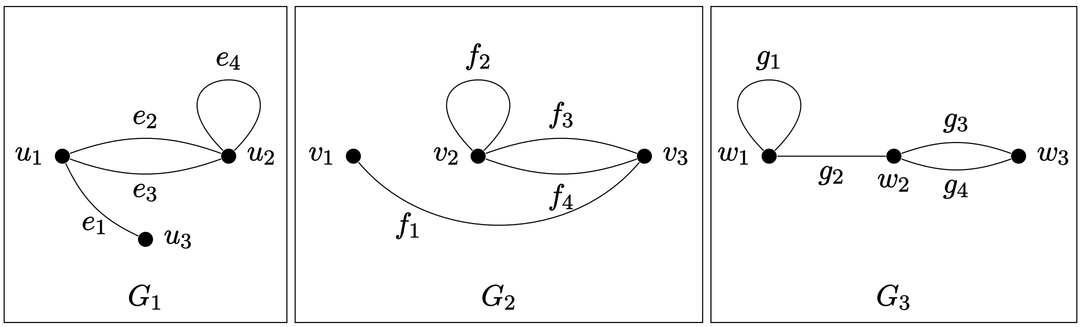
$$Y = \{\{1, 10\}, \{2, 9\}, \{3, 8\}, \{4, 7\}, \{5, 6\}\}\,.$$

Hence, $N(Y) = 5$ and each integer in S occurs in exactly one of the pairs. Consider the function f from X to Y defined by the rule: $f(x) =$ the pair in Y to which x

belongs. Since X has 6 elements and Y has 5 elements and $6 > 5$, by the pigeonhole principle, f is not injective. Thus $\exists\, x_1, x_2 \in X$ with $x_1 < x_2$ such that $f(x_1) = f(x_2)$. This means that x_1 and x_2 are distinct integers in the same pair, which implies that $x_1 + x_2 = 11$.

Problem 9 (3 points)

Determine for each of the pairs of graphs given below if they are isomorphic. If they are, give vertex and edge functions that define the isomorphism. If they are not, explain why not.



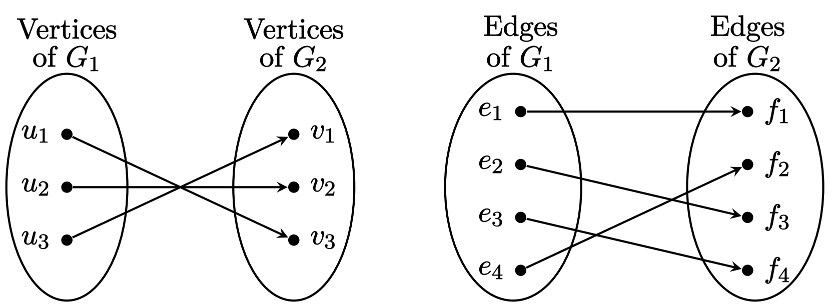
(a) (1 point) G_1 and G_2

(b) (1 point) G_1 and G_3

(c) (1 point) G_2 and G_3

Solution:

(a) G_1 is isomorphic to G_2. One way to define the isomorphism is as follows:



Alternatively, e_2 could be sent to f_4 and e_3 could be sent to f_3.

(b) G_1 is not isomorphic to G_3 because G_1 has one vertex of degree 1, one vertex of degree 3, and one vertex of degree 4 whereas G_3 has two vertices of degree 3 and one vertices of degree 2. Also G_1 has a circuit of length 3 $(u_1, e_2, u_2, e_4, u_2, e_2, u_1)$

whereas G_3 does not. Showing that any of these graph isomorphism invariants differ between the two graphs is sufficient.

(c) G_2 is not isomorphic to G_3 because G_2 has one vertex of degree 1, one vertex of degree 3, and one vertex of degree 4 whereas G_3 has two vertices of degree 3 and one vertices of degree 2. Also G_2 has a circuit of length 3 $(v_3, f_3, v_2, f_2, v_2, f_4, v_3)$ whereas G_3 does not. Showing that any of these graph isomorphism invariants differ between the two graphs is sufficient.

Alternatively, one can argue using the facts that graph isomorphisms are symmetric and transitive. Thus, in this case, because G_1 is not isomorphic to G_3 and G_1 is isomorphic to G_2, it is impossible for G_2 to be isomorphic to G_3.

Problem 10 (4 points)
Prove that $\lceil \log_2(n) \rceil$ is $\Theta(\log_2(n))$.

Solution: If n is any positive integer, then $\log_2 n$ is defined and by definition of ceiling,

$$\lceil \log_2 n \rceil - 1 < \log_2 n \leq \lceil \log_2 n \rceil \,. \tag{$*$}$$

Adding 1 to both sides of the left-hand inequality gives

$$\lceil \log_2 n \rceil < \log_2 n + 1 \,.$$

If, in addition, n is greater than 2, then, since the logarithmic function with base 2 is increasing,

$$\log_2 n > \log_2 2 = 1 \,.$$

Thus,

$$\lceil \log_2 n \rceil < \log_2 n + 1 < \log_2 n + \log_2 n = 2 \log_2 n \,.$$

Together with the right-hand side of $(*)$, we obtain

$$\log_2 n \leq \lceil \log_2 n \rceil \leq 2 \log_2 n \,.$$

Let $A = 1$, $B = 2$, and $k = 2$. Then

$$A \log_2 n \leq \lceil \log_2 n \rceil \leq B \log_2 n \quad \forall n \geq k \,.$$

Therefore, by definition of Θ-notation, $\lceil \log_2 n \rceil$ is $\Theta(\log_2 n)$.

Problem 11 (4 points)

(a) (3 points) Find the total number of additions and multiplications that must be performed when the following algorithm is executed. Show your work carefully.

```
for i := 1 to n
    for j := i to n
        a := 2 · (5 · i + j + 1)
    next j
next i
```

(b) (1 point) Find the order for the number of additions and multiplications in the algorithm segment of part (a). Justify your answer.

Solution:

(a) For each iteration of the inner loop there are 2 multiplications and 2 additions for a total of 4 elementary operations. The number of iterations of the inner loop can be deduced from the following table, which shows the values of i and j for which the inner loop is executed.

i	1				2				$\cdots$	$n-1$		n
j	1	2	$\cdots$	n	2	3	$\cdots$	n	$\cdots$	$n-1$	n	n
			n				$n-1$				2	1

Therefore, the number of iterations of the inner loop is

$$n + (n-1) + \cdots + 2 + 1 = \frac{n(n+1)}{2}.$$

The total number of elementary operations that must be performed when the algorithm is executed is the number performed during each iteration of the inner loop times the number of iterations of the inner loop:

$$4 \cdot \left(\frac{n(n+1)}{2}\right) = 2n^2 + 2n.$$

(b) By the theorem on polynomial orders, $2n^2 + 2n$ is $\Theta(n^2)$, and so the algorithm segment has order n^2.