CHALMERS
EXAMINATION / TENTAMEN

Course code/kurskod	Course name/kursnamn		
DAT060	Logic in Computer Science		
Anonymous code Anonym kod		Examination date Tentamensdatum	Number of pages Antal blad
DAT060-0055-DHW		2023-10-26	14

Grade
Betyg

* I confirm that I've no mobile or other similar electronic equipment available during the examination.
Jag intygar att jag inte har mobiltelefon eller annan liknande elektronisk utrustning tillgänglig under eximinationen.

Solved task Behandlade uppgifter No/nr		Points per task Poäng på uppgiften	Observe: Areas with bold contour are to completed by the teacher. Anmärkning: Rutor inom bred kontur ifylles av lärare.
1	✓	6	
2	✓	2.75	
3	✓	14	
4	✓	10.5.	
5	✓	8	
6	✓	5	
7	✓	7,5	
8	✓	6	
9		59.75.	
10			
11			Excellent!
12			
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Bonus poäng			
Total examination points Summa poäng på tentamen			

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	DAT060-0055-DHW	6	1	1

1. a)

1.	$p \lor \neg p$	LEM
2.	p	assumption 1
3.	$p \lor (p \to q)$	$\lor i_1$ 2
4.	$\neg p$	assumption 2
5.	p	assumption
6.	$\bot$	$\neg e$ 4, 5
7.	q	$\bot e$ 6
8.	$p \to q$	$\to i$ 5-7
9.	$p \lor (p \to q)$	$\lor i_2$ 8
10.	$p \lor (p \to q)$	$\lor e$ 1, 2-3, 4-9

b)

1.	$\neg(p \land s)$	premise
2.	$\neg(\neg s \land q)$	premise
3.	$(p \land r) \land q$	assumption
4.	q	$\land e_2$ 3
5.	$p \land r$	$\land e_1$ 3
6.	p	$\land e_1$ 5
7.	r	$\land e_2$ 5
8.	$s \lor \neg s$	LEM
9.	s	assumption 1
10.	$p \land s$	$\land i$ 6, 9
11.	$\bot$	$\neg e$ 1, 10
12.	$\neg s$	assumption 2
13.	$\neg s \land q$	$\land i$ 12, 4
14.	$\bot$	$\neg e$ 2, 13
15.	$\bot$	$\lor e$ 8, 9-11, 12-14
16.	$\neg((p \land r) \land q)$	$\neg i$ 3-15

Answer only one question on this page. Do not write on the back of this paper
Behandla endast en uppgift på detta blad. Skriv ej på baksidan

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a) $[\![p]\!]_v = T$ $[\![q]\!]_v = T$ $[\![r]\!]_v = F$ $[\![s]\!]_v = T$

v(..) = ... -0.25

b) The given formula is a conjunction. To satisfy the left conjunct, we have to satisfy one of its two disjuncts. We attempt this by evaluating p, q, s as true, so that $[\![s \to p \wedge q]\!]_v = T$. Since the evaluation for all atoms must not be the same we evaluate r as false, and now check if the right conjunct is satisfied. $[\![r \vee q]\!]_v = T$ since $[\![q]\!]_v = T$ so we must have $[\![s \vee p]\!] = T$. And we do, since $[\![s]\!]_v = T$, so this valuation satisfies the formula.

✓

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a)

1.	$\neg \forall x\, P(x)$	premise
2.	$\neg \exists x\, \neg P(x)$	assumption
3.	x_0	fresh
4.	$\neg P(x_0)$	assumption
5.	$\exists x \neg P(x)$	$\exists i$ 4, $x_0 \to x$
6.	$\bot$	$\neg e$ 2, 5
7.	$P(x_0)$	PBC 4-6
8.	$\forall x\, P(x)$	$\forall i$ 3-7
9.	$\bot$	$\neg e$ 1, 8
10.	$\exists x\, \neg P(x)$	PBC 2-9

LOOKS LIKE $\neg$

b) Invalid, consider

$\mathcal{M} :=$ $A = \{0, 1\}$

$P^{\mathcal{M}} = \{0\}$ $Q^{\mathcal{M}} = \emptyset$ $R^{\mathcal{M}} = \{0\}$

$\mathcal{M} \models \exists x (P(x) \land \neg Q(x))$ since there's an element, $0 \in A$, for which $0 \in P^{\mathcal{M}}$ and $0 \notin Q^{\mathcal{M}}$. Also $\mathcal{M} \models \exists x (\neg Q(x) \land \neg R(x))$ since there's an element, $1 \in A$, for which $1 \notin Q^{\mathcal{M}}$ and $1 \notin R^{\mathcal{M}}$. But $\mathcal{M} \not\models \exists x (P(x) \land \neg R(x))$ since for both elements in A we have either 1) the case of 0 where $0 \in P(x)$ and $0 \in R(x)$ so $\mathcal{M} \not\models_{[x \mapsto 0]} P(x) \land \neg R(x)$ or 2) the case of 1 where $1 \notin P(x)$ so $\mathcal{M} \not\models_{[x \mapsto 1]} P(x) \land \neg R(x)$.

So the semantic entailment doesn't hold; by contraposition of soundness, the sequent is invalid.

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c)

1.	$\exists x \forall y R(y,x)$	premise
2.	$\forall y R(y, y_0)$	fresh y_0, assumption
3.	x_0	fresh x_0
4.	$R(x_0, y_0)$	$\forall e$ 2, x_0
5.	$\exists y R(x_0, y)$	$\exists i$ 4, $y_0 \to y$
6.	$\forall x \exists y R(x,y)$	$\forall x i$ 3-5
7.	$\forall x \exists y R(x,y)$	$\exists_x e$ 1, 2-6
8.	$\forall x \exists y R(x,y) \to \neg \exists x R(x,x)$	premise?
9.	$\neg \exists x R(x,x)$	$\to e$ 7, 8
10.	$\forall y R(y, x_0)$	fresh x_0, assumption
11.	$R(x_0, x_0)$	$\forall_y e$ 10, x_0
12.	$\exists x R(x,x)$	$\exists i$ 12, $x_0 \to x$
13.	$\exists x R(x,x)$	$\exists e$ 1, 10-12
14.	$\bot$	$\neg e$ 9, 13
15.	$\forall z \neg R(z,z)$	$\bot e$ 14

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d)

1.	$\exists x\,(P(x) \land \forall y(P(y) \to y=x))$	premise
2.	$P(x_0) \land \forall y\,(P(y) \to y = x_0)$	fresh x_0, assumption
3.	$P(x_0)$	$\land e_1$ 2
4.	$\forall y\, P(y) \to y = x_0$	$\land e_2$ 2
5.	$\exists x\, P(x)$	$\exists i$ 3, $x_0 \to x$
6.	y_0	fresh y_0
7.	z_0	fresh z_0
8.	$P(y_0) \land P(z_0)$	assumption
9.	$P(y_0)$	$\land e_1$ 8
10.	$P(y_0) \to y_0 = x_0$	$\forall_y e$ 4, y_0
11.	$y_0 = x_0$	$\to e$ 9, 10
12.	$P(z_0)$	$\land e_2$ 8
13.	$P(z_0) \to z_0 = x_0$	$\forall_y e$ 4, z_0
14.	$z_0 = x_0$	$\to e$ 12, 13
15.	$y_0 = z_0$	$= e$ 11, 14
16.	$P(y_0) \land P(z_0) \to y_0 = z_0$	$\to i$ 8-15
17.	$\forall z(P(y_0) \land P(z) \to y_0 = z)$	$\forall i$ 7-16
18.	$\forall y \forall z\,(P(y) \land P(z) \to y = z)$	$\forall i$ 6-17
19.	$\exists x P(x) \land \forall y \forall z\,(P(y) \land P(z) \to y = z)$	$\land i$ 5, 18
20.	$\exists x\, P(x) \land \forall y \forall z(P(y) \land P(z) \to y = z)$	$\exists e$ 1, 2-19

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a) A model $\mathcal{M}$ for the language has a non-empty universe $A \neq \emptyset$, and two unary predicates defined as subsets of the universe, $P^{\mathcal{M}} \subseteq A$ and $Q^{\mathcal{M}} \subseteq A$.

b) (i) Holds. Consider an arbitrary $\mathcal{M}$, its universe A and its interpretation of P, $P^{\mathcal{M}}$. Either $P^{\mathcal{M}} = A$ or $P^{\mathcal{M}} \subset A$. In the first case when $P^{\mathcal{M}} = A$ then for any $x_0 \in A$ we have $x_0 \in P^{\mathcal{M}}$, so $\mathcal{M} \models_{[x \to x_0]} P(x)$; since $P^{\mathcal{M}} = A$, $\mathcal{M} \models \forall x P(x)$, so $\mathcal{M} \models_{x \to x_0} P(x) \to \forall x P(x)$ so $\mathcal{M} \models \exists x (P(x) \to \forall x P(x))$.

Otherwise $P^{\mathcal{M}} \subset A$. So there's some $x_0 \in A$ such that $x_0 \notin P^{\mathcal{M}}$. For this x_0 then, $\mathcal{M} \not\models_{x \to x_0} P(x)$ so vacuously $\mathcal{M} \models_{[x \to x_0]} P(x) \to \forall x P(x)$ and therefore $\mathcal{M} \models \exists x (P(x) \to \forall x P(x))$.

(ii) Doesn't hold. Consider $\mathcal{M} :=$

$$A = \{0, 1\}, \quad P^{\mathcal{M}} = \{0\}, \quad Q^{\mathcal{M}} = \emptyset$$

$\mathcal{M} \models \exists x P(x)$ since $0 \in A$ and $0 \in P^{\mathcal{M}}$.
$\mathcal{M} \models \exists x (P(x) \to Q(x))$ vacuously since $1 \in A$ and $1 \notin P^{\mathcal{M}}$. But $\mathcal{M} \not\models \exists x Q(x)$ since $Q^{\mathcal{M}} = \emptyset$ so the entailment doesn't hold.

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Points for question (to be filled in by teacher) / Poäng på uppgiften (ifylles av lärare): –

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(iii) Holds. Consider arbitrary $\mathcal{M}$ for the language which satisfies the premises. Since $\mathcal{M} \models \exists x\, Q(x)$ there's an $x_0 \in A$ such that $x_0 \in Q^{\mathcal{M}}$. Since $\mathcal{M} \models \forall x\,(P(x) \to \neg Q(x))$ it must be the case that $x_0 \notin P^{\mathcal{M}}$ since otherwise, if it was, then $\mathcal{M} \models_{x \mapsto x_0} P(x)$ and by the premise $\mathcal{M} \models_{x \mapsto x_0} P(x) \to \neg Q(x)$ so $\mathcal{M} \models_{x \mapsto x_0} \neg Q(x)$ and therefore $x_0 \notin Q^{\mathcal{M}}$, a contradiction. But then since $x_0 \notin P^{\mathcal{M}}$ immediately $\mathcal{M} \models_{x \mapsto x_0} \neg P(x)$ therefore $\mathcal{M} \models \exists x \neg P(x)$.

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a) No, consider a model $\mathcal{M} ::$

$A = \{0\}$ $\quad s^{\mathcal{M}} : s^{\mathcal{M}}(0) = 0 \quad zero^{\mathcal{M}} = 0$

$\mathcal{M} \models \forall x \forall y (s(x) = s(y) \to x = y)$. There's only one element in A, 0, and $s^{\mathcal{M}}(0) = s^{\mathcal{M}}(0) = 0$ and $0 = 0$ so the implication is satisfied. But $\mathcal{M} \not\models \forall x (s(x) \neq zero)$. There's $x_0 \in A$, 0, for which $s(0) = 0 = zero^{\mathcal{M}}$. Since the semantic entailment doesn't hold, by contraposition of soundness we have $\Psi_0 \nvdash \Psi_1$.

b) Consider arbitrary $\mathcal{M} \mid \mathcal{M} \models \Psi_0, \Psi_1$. Assume for contradiction that $\mathcal{M} \models s^2(zero) = s(zero)$.

Then by Ψ_0 we know $\mathcal{M} \models s(zero) = zero$.

But this contradicts the fact that $\mathcal{M} \models \Psi_1$ i.e. $\mathcal{M} \models \forall x (s(x) \neq zero)$ and in particular $\mathcal{M} \models_{x \to zero^{\mathcal{M}}} s(x) \neq zero$. So $\mathcal{M} \not\models s^2(zero) = s(zero)$.

We can formalize by induction that $\mathcal{M} \models s^{n+1}(zero) \neq s^n(zero)$ for all $n \in \mathbb{N}$. The base case is immediate by Ψ_1, in particular we have $\mathcal{M} \models_{[x \to zero^{\mathcal{M}}]} s(x) \neq zero$.

For the inductive hypothesis assume $\mathcal{M} \models s^{n+1}(zero) \neq s^n(zero)$. We want to show $\mathcal{M} \models s^{n+2}(zero) \neq s^{n+1}(zero)$. Assume for contradiction that $\mathcal{M} \models s^{n+2}(zero) = s^{n+1}(zero)$. By Ψ_0 then $\mathcal{M} \models s^{n+1}(zero) = s^n(zero)$, contradicting the inductive hypothesis. So $\mathcal{M} \models s^{n+2}(zero) \neq s^{n+1}(zero)$ completing the induction.

c) Any finite subset of the given theory T, T', has a model. In particular, any model $\mathcal{M}$ with a universe of $A = \mathbb{N}$ (natural numbers); and $s^{\mathcal{M}}(x) = x+1$; and $c^{\mathcal{M}} = n+1 \in A$ where n is the greatest n that appears in a formula $c = s^n(\text{zero})$ in T'; and $\text{zero}^{\mathcal{M}} = 0 \in A$. This satisfies T' because, if ψ_0 is in T', we know it's satisfied since $s^{\mathcal{M}}$ as defined is injective; if ψ_1 is in T' we know it's satisfied because, for all $x \in \mathbb{N}$, $x+1 \neq 0$; and for any formula $c \neq s^n(\text{zero})$ in T' we know it's satisfied since $c^{\mathcal{M}} = n+1$ where n is the greatest natural number that appears in such a formula in T'. (We know a greatest such n exists because T' is finite). So in particular $c \neq s^n(\text{zero})$ is satisfied because we know $c^{\mathcal{M}} > s^{\mathcal{M}(n)}(\text{zero}^{\mathcal{M}})$ by definition. Since $\mathcal{M} \models T'$ and T' is an arbitrary finite subset of T, by compactness we know T is satisfiable.

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(6) a) To compute the least fixpoint, set $X = \emptyset$ and start computing $F^n(X)$, where F^n is F applied to X n times, halting when $F^n(X) = F^{n+1}(X)$ for some n. $F^n(X)$ is the least fixpoint; this will halt since the existence of a least fixpoint for a monotone function over powersets is guaranteed, as is the existence of a greatest fixpoint. Similarly to compute the greatest fixpoint set $X = S$ and compute $F^n(X)$ stopping when $F^n(X) = F^{n+1}(X)$; $F^n(X)$ is the greatest fixpoint of F.

b) $F(\emptyset) = A - B$

$F(A-B) = ((A-B) \cup A) - B \quad = \boxed{A - B}$

least fixpoint

$F^1(X) = F^2(X)$

$F(S) = (S \cup A) - B = S - B$

$F(S-B) = ((S-B) \cup A) - B \quad = \boxed{S - B}$

$= (S \cup (B \cap A)) - B$

$= S - B$

greatest fixpoint

$F^1(X) = F^2(X)$

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① a) An LTL model is a tuple $(S^{\mathcal{M}}, \rightarrow^{\mathcal{M}}, L^{\mathcal{M}})$ where

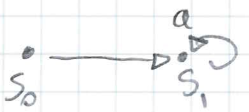
$S^{\mathcal{M}}$: some finite nonempty set of states

$\rightarrow^{\mathcal{M}}$: a binary relation over S where, for every $s_i \in S$, some $(s_i, s_j) \in \rightarrow^{\mathcal{M}}$ 1.5

$L^{\mathcal{M}}$: a labeling function from $S \rightarrow Pow(Atom)$ which labels each state in S with some subset of propositions from an 'Atom' set

b) (i) Valid. Consider an arbitrary path π in arbitrary $\mathcal{M}$ for which $\pi \vDash G(a \rightarrow b)$. Then $\pi^i \vDash a \rightarrow b \mid \forall i \geq 1$. Now if $\pi \nvDash Ga$ then vacuously $\pi \vDash Ga \rightarrow Gb$. Otherwise if $\pi \vDash Ga$ then $\pi^i \vDash a \mid \forall i \geq 1$. We already stated $\pi^i \vDash a \rightarrow b$ so $\pi^i \vDash b$. Since this holds for all $i \geq 1$ then $\pi \vDash Gb$, so $\pi \vDash Ga \rightarrow Gb$. 2

(ii) No, consider



$\mathcal{M} := S^{\mathcal{M}} = \{s_0, s_1\}$ 2

$\rightarrow^{\mathcal{M}} = \{(s_0, s_1), (s_1, s_1)\}$

$L^{\mathcal{M}}(s) = \begin{cases} s_0 : \emptyset \\ s_1 : a \end{cases}$

At $\pi = \{s_0, s_1, s_1 \ldots\}$ in this model, $\pi \vDash F(a \rightarrow b)$ since $\pi^1 \vDash a \rightarrow b$ since $\pi^1 \nvDash a$, and $\pi \vDash Fa$ since $\pi^2 \vDash a$, but $\pi \nvDash Fb$ since for no state s_i in π is $b \in L^{\mathcal{M}}(s_i)$.

NOTE: My convention is 1-indexing suffixes, i.e., $\pi^1 = \pi$

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(iii) Valid. Consider arbitrary $\pi \models Ga \wedge Fb$.
Then $\pi^i \models a \mid \forall i \geq 1$ and there's some $j \geq 1$ s.t.
$\pi^j \models b$. Since for all $i \geq 1$ $\pi^i \models a$ we trivially
have for all $i \geq j$, $\pi^i \models a$, so $\pi^j \models Ga$.
So $\pi^j \models Ga \wedge b$. So $\pi \models F(Ga \wedge b)$.

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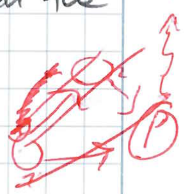
(8) a) Label all states s for which $s \models p$ with AFp. Then consider all predecessors of AFp-labeled states (where a predecessor is a state s_i where $(s_i, s_j) \in \xrightarrow{m}$ and $s_j \neq s_i$ is AFp-labelled). If for **all** successors $s_j \neq s_i$ of this predecessor s_i, where $(s_i, s_j) \in \xrightarrow{m}$, s_j is labeled AFp, label s_i with AFp. Repeat this step on the predecessors of the states that were just labeled, until there are no more states to consider. The labeled states satisfy AFp.



This considers states multiple times, essentially implementing the algorithm from the book using FP_s

On this model, we label s_4 since $s_4 \models p$. Then we look at its sole predecessor s_3. s_3 doesn't have any unlabelled successors so we label it AFp. We look at s_3's sole predecessor s_2. s_2 has an unlabelled successor, s_1. So we don't label it. There's no more states to consider. So $s_4, s_3 \models AFp$.

~~yields infinity these still considered~~

Repeat on state until no labelling changes!

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b) $s_2, s_1, s_0 \models EFq$.

Immediately $s_2 \models EFq$ since $s_2 \models q$.

$s_1 \models EFq$ since there's a path $\pi = \{s_1, s_0, s_0 \dots\}$ starting at s_1 which satisfies Fq since $\pi^2 \models q$.

And again trivially $s_0 \models EFq$ since $s_0 \models q$.