Examination
Mathematical Foundations for Software Engineering

Course codes: DIT022 / DIT023

Date:	2021-10-26
Time:	14:00-18:00
Place:	Lindholmen
Teacher:	Christian Berger Alexander Stotsky
Visit to e*xam hall:*	14:30 and 16:30
Questions:	6
Results:	Will be posted by 2021-11-16.
Grade Limits (DIT022):	Pass (3) 50% Pass with honors (VG) 90%
Grade Limits (DIT023):	Pass (3): 50% Pass with credit (4): 70% Pass with distinction (5): 90%
Allowed aids:	Calculators: Casio FX-82..., Texas TI-30... and Sharp EL-W531... Attached appendix with formulas and tables.

Please observe the following:

- DO NOT write your name on any answer sheet or exam sheet – write the anonymized code instead.
- Write in legible English (unreadable responses mean no points!).
- Motivate your answers and clearly state any assumptions made.
- Start each part of the exam on a new sheet!
- Write only on one side of the paper!
- Only answers written on the answer sheets will be graded, do not write on the exam sheets!
- Before handing in your exam, number and sort the sheets in task order!

NOTE:

Not following these instructions may result in the deduction of points!

Question 1 (1 + 1 + 1 + 2 = 5 pt)
"Languages and Grammars"

1.1 What language is generated by the given grammar G1? (1pt)
*Note that we are **not** asking for partially correct solution(s) but for the fully correct one(s). Also, "choosing multiple solutions when only one is correct" will not be assessed as correct. Mark your choice(s) on your answer sheets, not this exam sheet.*

G1 = (V, T, S, P) and V = {S, X}, where S is the start variable, T = {0,1} set of terminals and rules:

S → X0
X→ 1 | X1

a) $\mathcal{L}(G1)$ ={w∈ {0,1} * | w starts with a 1 or ends with a 0}
b) $\mathcal{L}(G1)$ ={w∈ {0,1} * | w contains at least one 1 and at least one 0}
c) $\mathcal{L}(G1)$ ={w∈ {0,1} * | w contains only 1s except for the last symbol which is a 0}
d) $\mathcal{L}(G1)$ ={w∈ {0,1} * | w contains 1s and 0s}
e) $\mathcal{L}(G1)$ ={w∈ {0,1} * | w contains firstly a random series of 1s and 0s, and a 0 as the last symbol}

1.2 What language is generated by the given grammar G2? (1pt)
*Note that we are **not** asking for partially correct solution(s) but for the fully correct one(s). Also, "choosing multiple solutions when only one is correct" will not be assessed as correct. Mark your choice(s) on your answer sheets, not this exam sheet.*

G2 = (V, T, S, P) and V = {S, A, B}, where S is the start variable, T = {0,1, λ} set of terminals and rules:

S → 0A1 | 1B0
B→ 0A1 | λ
A→ 1B0 | λ

a) $\mathcal{L}(G2)$ ={w∈ {0,1} * | the length of w is even and the middle symbol is 1}
b) $\mathcal{L}(G2)$ ={w∈ {0,1} * | w contains the same amount of 0s and 1s in alternating order} (Example: 010101, 101010, …)
c) $\mathcal{L}(G2)$ ={w∈ {0,1} * | w is a random series of 1s and 0s}
d) $\mathcal{L}(G2)$ ={w∈ {0,1} * | w contains 0s followed by the same amount of 1s, or vice-versa} (Example: 000111, 111000, …)
e) $\mathcal{L}(G2)$ ={w∈ {0,1} * | the length of w is odd and the middle symbol is 0}

1.3 What language is generated by the given grammar G3? (1pt)
*Note that we are **not** asking for partially correct solution(s) but for the fully correct one(s). Also, "choosing multiple solutions when only one is correct" will not be assessed as correct. Mark your choice(s) on your answer sheets, not this exam sheet.*

G3 = (V, T, S, P) and V = {S, W, 1, 2, 3}, where S is the start variable, T = {1, 2, 3, λ} set of terminals and rules P:

S → 1SW3 | λ
1W → 12
2W → 22
3W → W3

Please note that the empty word (λ) is allowed.

a) $\mathcal{L}(G3) = \{1^m 2^n 3^n \mid \text{where } m, n \geq 1\}$
b) $\mathcal{L}(G3) = \{1^n 2^n 3^n \mid \text{where } n \geq 0\}$
c) $\mathcal{L}(G3) = \{1^n 2^{2n} 3^n 3^m \mid \text{where } n \geq 0, m = 0\}$
d) $\mathcal{L}(G3) = \{1^n 2^n 3^n \mid \text{where } n \geq 1\}$
e) $\mathcal{L}(G3) = \{1^n 2^n 3^{2n} \mid \text{where } n \geq 1\}$

1.4 Provide **two** different non-empty words of length 4 that are generated by the given grammar G4. (2pt)

G4 is given in the Backus-Naur form as (starting symbol is **⟨lastname⟩**):

```
⟨lastname⟩ ::= ⟨ucletter⟩|
               ⟨ucletter⟩⟨lcletter⟩|
               ⟨ucletter⟩⟨lcletter⟩⟨lcletter⟩|
               ⟨ucletter⟩⟨lcletter⟩⟨lcletter⟩⟨lcletter⟩|
               ⟨ucletter⟩⟨lcletter⟩⟨lcletter⟩⟨lcletter⟩⟨lcletter⟩|
               ⟨ucletter⟩⟨lcletter⟩⟨lcletter⟩⟨lcletter⟩⟨lcletter⟩⟨lcletter⟩

⟨ucletter⟩ ::= A|B|C| · · · |Z
⟨lcletter⟩ ::= a|b|c| · · · |z
```

Question 2 (1 + 1 + 7 + 14 = 23 pt)
"Automata"

2.1 What does automaton A1 do? (1pt)
*Note that we are **not** asking for partially correct solution(s) but for the fully correct one(s). Also, "choosing multiple solutions when only one is correct" will not be assessed as correct. Mark your choice(s) on your answer sheets, not this exam sheet.*

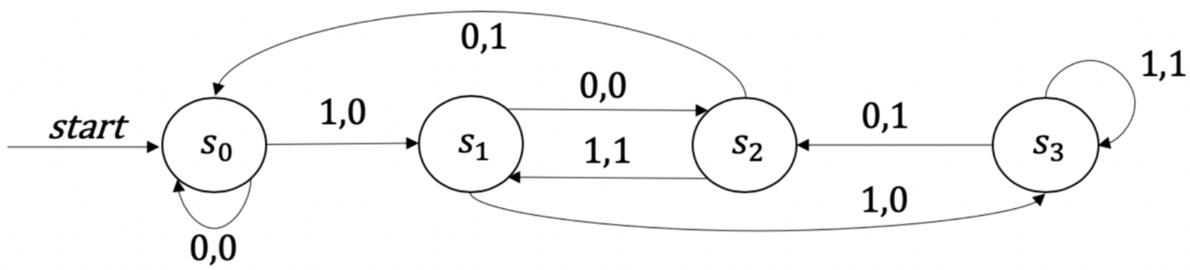


Figure 1: A1

a) It recognizes when the output equals the input delayed by two 0's.
b) It recognizes when the bit string "11000" is on the input.
c) It recognizes bit strings with an even number of '1's on the input.
d) It recognizes all possible input bit strings of length 2 (e.g., '00' is a bit string of length 2).
e) It recognizes all possible input bit strings of length 3 (e.g., '000' is a bit string of length 3).

2.2 What does automaton A2 do? (1pt)

*Note that we are **not** asking for partially correct solution(s) but for the fully correct one(s). Also, "choosing multiple solutions when only one is correct" will not be assessed as correct. Mark your choice(s) on your answer sheets, not this exam sheet.*

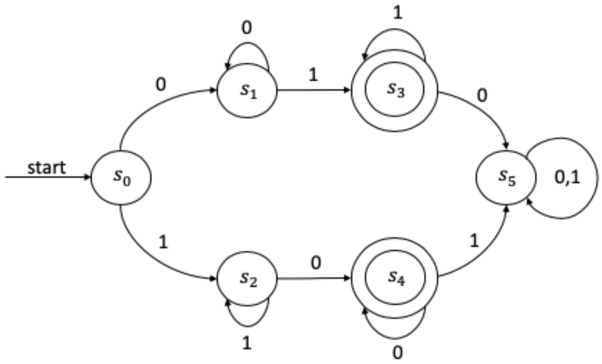


Figure 2: A2

a) It describes the language defined by grammar G = (V, S, T, P), where V = {0, 1, $s_0, s_1, s_2, s_3, s_4, s_5$, T}, s_0 is the starting state, T = {0, 1} and the production rules P define the acceptance of only the four bit strings '01', '001', '10', and '110'.
b) It describes the language defined by grammar G = (V, S, T, P), where V = {0, 1, $s_0, s_1, s_2, s_3, s_4, s_5$, T}, s_0 is the starting state, T = {0, 1} and the production rules P define the acceptance of bit strings with exactly one transition between 0 and 1 (transition either from 0 to 1 or from 1 to 0).
c) It describes the language defined by grammar G = (V, S, T, P), where V = {0, 1, $s_0, s_1, s_2, s_3, s_4, s_5$, T}, s_0 is the starting state, T = {0, 1} and the production rules P define the acceptance of all bit strings containing an odd number of '1's.
d) It describes the language defined by grammar G = (V, S, T, P), where V = {0, 1, $s_0, s_1, s_2, s_3, s_4, s_5$, T}, s_0 is the starting state, T = {0, 1} and the production rules P define the acceptance of all bit strings containing an even number of '0's.
e) It describes the language defined by grammar G = (V, S, T, P), where V = {0, 1, $s_0, s_1, s_2, s_3, s_4, s_5$, T}, s_0 is the starting state, T = {0, 1} and the production rules P define the acceptance only of bit strings starting with '01' or '10'.

2.3 Add missing elements (states, transitions, or labels) or remove incorrect elements (states, transitions, or labels) in the deterministic finite-state automaton A3 with no output so that it accepts the set of all bit strings that contain EXACTLY two 0s, the position of which is irrelevant. (7pt)

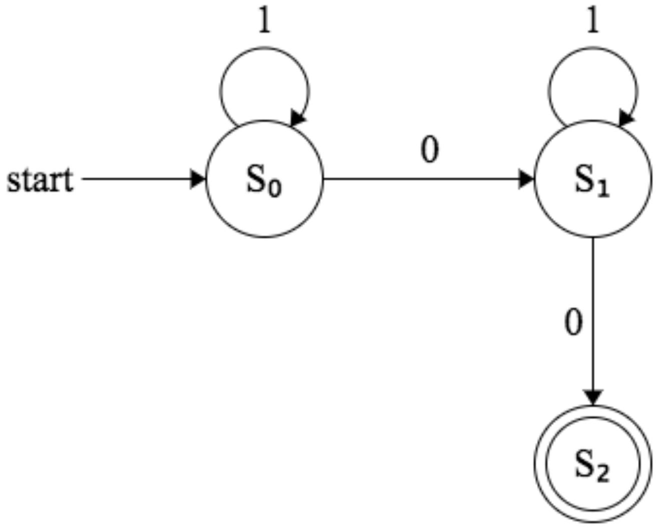


Figure 3: A3

2.4 Draw a deterministic finite-state machine with no output that accepts all possible bit strings of length 4 that have a ‘0’ in their last position. (14pt)
Note that unreadable drawings will be awarded with 0 points.

Question 3 (8 + 1 + 1 + 10 = 20 pt)
"Logic"

3.1 Provide the complete truth table for the given compound preposition. (8pt)

$$((p \land \neg q) \to ((r \lor p) \land (\neg r \leftrightarrow \neg p)))$$

3.2 Express this statement using predicates and quantifiers. (1pt)
*Note that we are **not** asking for partially correct solution(s) but for the fully correct one(s). Also, "choosing multiple solutions when only one is correct" will not be assessed as correct. Mark your choice(s) on your answer sheets, not this exam sheet.*

"There are exactly two vegan people in the group."

a) $\exists x \exists y((x = y) \land V(x) \land V(y) \land \forall z(V(z) \to (z = x \land z = y)))$ where *V(n)* denotes a statement as "n is vegan" and the domain for x, y and z are all people in the group.
b) $\forall x \forall y((x = y) \land V(x) \land V(y) \land \forall z(V(z) \to (z = x \land z = \neg y)))$ where *V(n)* denotes a statement as "n is vegan" and the domain for x, y and z are all people in the group.
c) $\exists x \exists y((x \neq y) \land V(x) \land V(y) \land \forall z(V(z) \to (z = x \lor z = y)))$ where *V(n)* denotes a statement as "n is vegan" and the domain for x, y and z are all people in the group.
d) $\forall x \forall y((x \neq y) \land V(x) \land V(y) \land \forall z(V(z) \to (z = x \lor z = y)))$ where *V(n)* denotes a statement as "n is vegan" and the domain for x, y and z are all people in the group.
e) $\forall x \forall y((x = y) \land V(x) \lor V(y) \land \forall z(V(z) \to (z = x \land z = y)))$ where *V(n)* denotes a statement as "n is vegan" and the domain for x, y and z are all people in the group.

3.3 Let D(x) be "x is a day", S(x) be "x is a student" and E(x,y) be "x needs to attend the examination of the course DIT-023 on day y". Use quantifiers to express the following statement. (1pt)

"There would be someday that no student needs to attend the examination of the course DIT-023."

*Note that we are **not** asking for partially correct solution(s) but for the fully correct one(s). Also, choosing multiple solutions, when only one is correct will not be assessed as correct. Mark clearly the correct solution. Mark your choice(s) on your answer sheets, not this exam sheet.*

a) $\exists x(D(x)) \rightarrow \forall y(\neg S(y) \rightarrow E(y,x))$
b) $\forall x(D(x)) \rightarrow \forall y(S(y) \rightarrow \neg E(y,x))$
c) $\exists x(D(x)) \leftrightarrow \forall y(S(y) \rightarrow \neg E(y,x))$
d) $\forall x(D(x)) \leftrightarrow \forall y(\neg S(y) \rightarrow E(y,x))$
e) $\exists x(D(x)) \rightarrow \forall y(S(y) \rightarrow \neg E(y,x))$

3.4 Look at the following compound proposition and answer the questions below. (10pt in total)

$$((p \wedge \neg q) \rightarrow \Big((r \vee p) \wedge \big((\neg r \vee \neg p) \wedge p\big)\Big)$$

i) What is a "tautology"? (1pt)

ii) When is a compound proposition "not satisfiable"? (1pt)

iii) When is a compound proposition "not a contingency"? (1pt)

iv) Why is the given compound proposition not a tautology? (6pt)

v) Turn the given compound proposition into a tautology by making changes to the compound proposition *(in other words, changing ∧, ∨, → and/or ↔, but not removing individual propositions p, q or r)*. (1pt)

Question 4 (4 + 3 = 7 pt)
"Proofs"

4.1 Prove that $\sum_{i=1}^{n} i = 1 + \cdots + n = \frac{3n^2+3n}{6}$ for n > 0. (4pt)

4.2 Prove by contradiction that there is no "largest even number". (3pt)

Question 5 (1 + 1 + 1 + 8 + 2 + 2 + 5 = 20 pt)
"Complexity and Graphs"

5.1 What is the complexity of the following code snippet (in terms of Big-O notation)? (1pt)

```
function(int n){
   int sum = 0;
   for (int i=n; i<=n; i++){
      for (int j=1; j<=n; j++){
         if (j % 2 == 0){
            for (int k=1; k<=j; k++) {
               sum++;
            }
         }
      }
   }
}
```

5.2 What is the complexity of the following code snippet (in terms of Big-O notation)? (1pt)

```
function(int n){
   for (int i=0; i<n; i= i*2){
      for (int j=0; j<n; j++){
         System.out.println("Hej!"); // This is O(1)
      }
   }
}
```

5.3 What is the complexity of the following code snippet (in terms of Big-O notation)? (1pt)

```
function(int n){
   // a[..] is an array defined outside this function with at least n elements
   int i = 0;
   while (i < n) {
      bool found = false;
      int j = 0;
      while (j < n) {
          if (a[i] == a[j]) {
               found = true;
               break; // break means leaving the current loop
          }
          j++;
      }
      if (found == false) {
          System.out.println("not found");
      }
      i++;
   }
   return true;
```

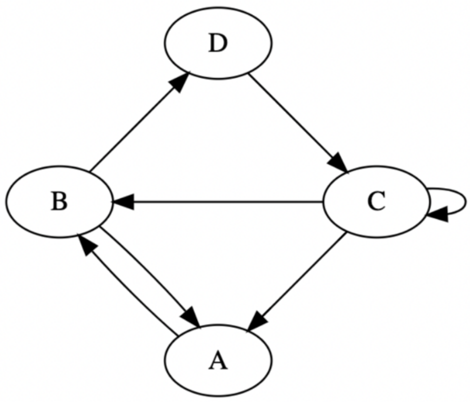
5.4 Trace the execution of the **selection sort** algorithm over the array below. Show each pass of the algorithm until the array is sorted. (8pt)

Array: 3 15 -8 5 13 -1 9 19 -4

5.5 What is the best-case and the average-case complexity (in terms of Big-O) of the Selection sort algorithm? (2pt)

5.6 What is the best-case and the average-case complexity (in terms of Big-O) of the Bubble sort algorithm? (2pt)

5.7 Look at the directed graph G5 = (V, E) below, where V is the set of vertices V={A,B,C,D} and E is the set of edges, and answer the questions below.



5.7.1) Determine $in = \sum_{v \in V} deg^{-}(v)$ and $out = \sum_{v \in V} deg^{+}(v)$ (2pt)

5.7.2) How many different Euler paths do exist in G5? (1pt)

5.7.3) Provide one possible Euler path from G5 if there is one. (1pt)

5.7.4) Is the following statement "true" or "false": $\forall v \in V$: there is a Hamilton circuit in G5. (1pt)

Question 6 (4 +3.5 + 1.5 + 4.5 + 6 + 5.5 = 25 pt)
"Statistics"

6.1 The sales results of the Company in millions of U.S. dollars (MM USD) (y variable) in the years 2017 – 2020 (x variable) are given in the following table (please, read the whole problem before starting calculations):

x	y
2017	1
2018	2
2019	3
2020	4

1) Please find the parameters of the linear regression $\hat{y} = \alpha + \beta x$, 2 pt.
Please change the variable x in order to simplify calculations, i.e. please use $x = 0, 1, 2, 3$ instead of $x = 2017, 2018, 2019, 2020$. Please report the regression parameters α, β in the following (x, y) coordinates:

x	y
0	1
1	2
2	3
3	4

2) Please report the following sum of squares $\sum_{i=1}(y_i - \hat{y}_i)^2$, where $y_i - \hat{y}_i$ is the differences between observed values y_i and fitted values $\hat{y}_i$, $i = 1, .., 4$, 0.5 pt.

3) Is it possible to evaluate the regression error using information about the sum of squares without any calculations ? Please report the regression error, if possible, 0.5 pt.

4) Is it possible to evaluate the correlation coefficient using information about the sum of squares without calculations ? Please report the correlation coefficient, if possible, 0.5 pt.

5) Please make prediction of the sales of the Company for the year 2021 using the regression line, 0.5 pt.

The total number of points for this problem is 4 pt.

6.2 Suppose that the test detects a number of virus particles in the body with the probability which follows Poisson distribution with the parameter $\lambda = 4$. Suppose that a new test (detection method) has been introduced. It is stated by the manufacturer that the detection probability follows Poisson distribution with the parameter $\lambda = 2$ for 95% of the test cases. For other 5% a new test is not effective, but the number of virus particles in the body can still be detected with the probability which follows Poisson distribution with the parameter $\lambda = 4$.
How likely that a new method is beneficial for detection of 0 (zero) number of virus particles ? (3.5 pt)
The probabilities are defined as follows:

$P(0\ number|beneficial)$ is Poisson random variable ($X = 0$, where X is the number of particles) with the parameter $\lambda = 2$

$P(0\ number|not\ beneficial)$ is Poisson random variable ($X = 0$) with the parameter $\lambda = 4$

$P(beneficial) = 0.95$
$P(not\ beneficial) = 0.05$

Use Bayes theorem for probability inversion $P(A|B) = P(B|A)P(A)/P(B)$, which has the following form in this problem formulation:
$P(benefical|0\ number) = P(0\ number|beneficial)P(beneficial)/P(0\ number)$

Hint: Please use the formula for total probability for evaluation of $P(0\ number)$.

6.3 In the fast food restaurant 1 out of 5 customers (in average) purchases tea. A random sample of 10 customers is selected.
Please find the probability that exactly 5 customers purchase tea, 1.5 pt.
Please, apply the formula for the binomial distribution.

6.4 Suppose that the student gets precalculated table for ANOVA with three groups of data with the sample size n, mean values $\overline{X}_i$ and standard deviations S_i for each group respectively, $i = 1, 2, 3$, please see the table with precalculated data. In addition, the student gets precalculated values of SSB[1] and SSW[2] for the data, SSB = 390.6 and SSW = 3462.89

n	$\overline{X}$	S
30	10.3	5.4
30	15.4	7.6
30	12.7	5.7

1. Test whether the mean values are equal for the groups, please use $\alpha = 1\%$ (totally 4.5 points, see the distribution below:
2. State the null and alternative hypothesis. (2 pt)
3. Calculate the appropriate test statistic. (1 pt)
   Please report the test statistic with two decimal place accuracy.[3]
4. Find the critical value. (1 pt) (The table for critical values is enclosed).
5. What is the decision rule? (0.5 pt)

[1]SSB = Sum of Squares between groups: a measure of between groups variability.
[2]SSW = Sum of Squares within groups: a measure of within groups variability
[3]The decimal place accuracy of a number is the number of digits to the right of the decimal point

6.5 Food delivery service knows that the standard deviation for normally distributed waiting times for customers on Friday afternoon is 7.25 minutes with individual lines at its various windows. The food delivery service makes the following measurements of the waiting times for a single main waiting line:

Waiting times for a single line, minutes	1.5	10.1	8.3	2.4	11.2	8	2.2	13.1	6.3

With a significance level of $\alpha = 5\%$, test the claim that a single line of the food delivery service causes lower variation among waiting times (shorter waiting times) for customers.

1) State the null and alternative hypothesis. (2 pt)
2) Calculate the appropriate test statistic. (2 pt)
Please report the test statistic with three decimal place accuracy.[4]

3) Find the critical value. (1 pt)
4) What is your decision and interpretation of the findings ? (1 pt).

The table for critical values is enclosed.

The total number of points for this problem is 6 pt.

Hint: For calculation of the sample variance of a finite sample of n observations the following relation can be used:

$$(n-1)s^2 = \sum_{i=1}^{n} x_i^2 - \frac{(\sum_{i=1}^{n} x_i)^2}{n}$$

6.6 A group of 7 randomly selected school kids who were diagnosed with asthma was tested to see whether a new educational video is effective in increasing the children's knowledge about asthma. The nurse gave the children the test containing questions about asthma before and after seeing the video. The test scores before and after watching the video are the following :

Child	1	2	3	4	5	6	7
After	6	7	7	4	4	7	3
Before	5	6	7	4	3	8	2
After - Before	1	1	0	0	1	-1	1

Please find, at the $\alpha = 5\%$ level of significance, whether the data given in the Table provide sufficient evidence to conclude that there is improvement in the test scores.
1) State the null and alternative hypothesis. (2 pt)
2) Calculate the appropriate test statistic. (2 pt)
3) Find the critical value. (1 pt)
4) What is your decision and interpretation of the findings ? (0.5 pt).

The table for critical values is enclosed.

The total number of points for this problem is 5.5 pt.

Formulas for Examen: Statistical Part

1 Linear Regression

1.1 Pearson Correlation Coefficient and Variances:

$$
\begin{aligned}
r &= \frac{1}{s_x s_y}\frac{1}{(n-1)}\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y}) \\
&= \frac{\displaystyle\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})}{\sqrt{\displaystyle\sum_{i=1}^{n}(x_i-\overline{x})^2\sum_{i=1}^{n}(y_i-\overline{y})^2}} \\
s_x^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\overline{x})^2 \\
s_y^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(y_i-\overline{y})^2
\end{aligned}
$$

1.2 Equation for Linear Regression:

$$
\begin{aligned}
y-\overline{y} &= r\frac{s_y}{s_x}(x-\overline{x}) \\
y-\overline{y} &= b(x-\overline{x}) \\
b &= \frac{\displaystyle\sum_{i=1}^{n}x_i y_i - n\,\overline{x}\,\overline{y}}{\displaystyle\sum_{i=1}^{n}x_i^2 - n(\overline{x})^2}
\end{aligned}
$$

1.3 Regression Error Estimation:

$$
\epsilon = \sqrt{\frac{\displaystyle\sum_{i=1}^{n}(y_i-\hat{y}_i)^2}{n-2}}
$$

2 Probability and Distributions

2.1 Bayes Theorem and Total Probablity

$$P(B|A) = \frac{P(B)P(A|B)}{P(A)}$$
$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

2.2 Poisson Distribution

$$P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!}$$
$$E(x) = \lambda, \quad Var(X) = \lambda$$

2.3 Binomial Distribution

$$P_{i,n} = \binom{n}{i} p^i q^{n-i}, \ \binom{n}{i} = \frac{n!}{i!(n-i)!}, \ q = 1 - p, \ i = 0, 1, ..., n$$

3 ANOVA

$$MSTR = \frac{n \sum_{i=1}^{k} (\bar{x}_i - \bar{X})^2}{(k-1)}$$
$$MSE = \frac{(n-1) \sum_{i=1}^{k} s_i^2}{(n_t - k)}, \quad n_t = k \ n$$
$$F = \frac{MSTR}{MSE} = \frac{\frac{SSB}{df_{SSB}}}{\frac{SSW}{df_{SSW}}}$$
$$SSB = n \sum_{i=1}^{k} (\overline{x}_i - \overline{X})^2$$
$$SSW = \sum \sum x_{ij}^2 - SSB - n \ k \ \overline{X}^2$$

$$\text{critical value}, \quad F_{k-1, \ n_t - k, \ \alpha}$$

4 Hypothesis Testing

4.1 One Sample χ^2 test (Test on Variance)

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$
$$\text{critical value,} \quad \chi^2_{\alpha,n-1}$$

4.2 F-Test

$$\begin{aligned} F &= \frac{s_1^2}{s_2^2} \\ df_1 &= n_1 - 1 \\ df_2 &= n_2 - 1 \end{aligned}$$

4.3 Two Sample t-Test

$$\begin{aligned} t &= \frac{\overline{x} - \overline{y}}{S\sqrt{1/n + 1/m}} \\ S &= \sqrt{\frac{(n-1)S_x^2 + (m-1)S_y^2}{(n-1)+(m-1)}} \\ df &= n + m - 2 \end{aligned}$$

4.4 Paired t-Test

$$t = \frac{\overline{d} - \mu_0}{S/\sqrt{n}}$$
$$\text{critical value,} \quad t_{\alpha,n-1}$$

Table of critical values for the F distribution (for use with ANOVA):

How to use this table:

There are two tables here. The first one gives critical values of F at the p = 0.05 level of significance.
The second table gives critical values of F at the p = 0.01 level of significance.

1. Obtain your F-ratio. This has (x,y) degrees of freedom associated with it.

2. Go along x columns, and down y rows. The point of intersection is your critical F-ratio.

3. If your obtained value of F is equal to or larger than this critical F-value, then your result is significant at that level of probability.

An example: I obtain an F ratio of 3.96 with (2, 24) degrees of freedom.
I go along 2 columns and down 24 rows. The critical value of F is 3.40. My obtained F-ratio is larger than this, and so I conclude that my obtained F-ratio is likely to occur by chance with a p<.05.

Critical values of F for the 0.01 significance level:

	1	2	3	4	5	6	7	8	9	10
1	4052.19	4999.52	5403.34	5624.62	5763.65	5858.97	5928.33	5981.10	6022.50	6055.85
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05
6	13.75	10.93	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10
14	8.86	6.52	5.56	5.04	4.70	4.46	4.28	4.14	4.03	3.94
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.90	3.81
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69
17	8.40	6.11	5.19	4.67	4.34	4.10	3.93	3.79	3.68	3.59
18	8.29	6.01	5.09	4.58	4.25	4.02	3.84	3.71	3.60	3.51
19	8.19	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26
23	7.88	5.66	4.77	4.26	3.94	3.71	3.54	3.41	3.30	3.21
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17
25	7.77	5.57	4.68	4.18	3.86	3.63	3.46	3.32	3.22	3.13
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09
27	7.68	5.49	4.60	4.11	3.79	3.56	3.39	3.26	3.15	3.06
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03
29	7.60	5.42	4.54	4.05	3.73	3.50	3.33	3.20	3.09	3.01
30	7.56	5.39	4.51	4.02	3.70	3.47	3.31	3.17	3.07	2.98
31	7.53	5.36	4.48	3.99	3.68	3.45	3.28	3.15	3.04	2.96
32	7.50	5.34	4.46	3.97	3.65	3.43	3.26	3.13	3.02	2.93

33	7.47	5.31	4.44	3.95	3.63	3.41	3.24	3.11	3.00	2.91
34	7.44	5.29	4.42	3.93	3.61	3.39	3.22	3.09	2.98	2.89
35	7.42	5.27	4.40	3.91	3.59	3.37	3.20	3.07	2.96	2.88
36	7.40	5.25	4.38	3.89	3.57	3.35	3.18	3.05	2.95	2.86
37	7.37	5.23	4.36	3.87	3.56	3.33	3.17	3.04	2.93	2.84
38	7.35	5.21	4.34	3.86	3.54	3.32	3.15	3.02	2.92	2.83
39	7.33	5.19	4.33	3.84	3.53	3.31	3.14	3.01	2.90	2.81
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80
41	7.30	5.16	4.30	3.82	3.50	3.28	3.11	2.98	2.88	2.79
42	7.28	5.15	4.29	3.80	3.49	3.27	3.10	2.97	2.86	2.78
43	7.26	5.14	4.27	3.79	3.48	3.25	3.09	2.96	2.85	2.76
44	7.25	5.12	4.26	3.78	3.47	3.24	3.08	2.95	2.84	2.75
45	7.23	5.11	4.25	3.77	3.45	3.23	3.07	2.94	2.83	2.74
46	7.22	5.10	4.24	3.76	3.44	3.22	3.06	2.93	2.82	2.73
47	7.21	5.09	4.23	3.75	3.43	3.21	3.05	2.92	2.81	2.72
48	7.19	5.08	4.22	3.74	3.43	3.20	3.04	2.91	2.80	2.72
49	7.18	5.07	4.21	3.73	3.42	3.20	3.03	2.90	2.79	2.71
50	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.79	2.70
51	7.16	5.05	4.19	3.71	3.40	3.18	3.01	2.88	2.78	2.69
52	7.15	5.04	4.18	3.70	3.39	3.17	3.01	2.87	2.77	2.68
53	7.14	5.03	4.17	3.70	3.38	3.16	3.00	2.87	2.76	2.68
54	7.13	5.02	4.17	3.69	3.38	3.16	2.99	2.86	2.76	2.67
55	7.12	5.01	4.16	3.68	3.37	3.15	2.98	2.85	2.75	2.66
56	7.11	5.01	4.15	3.67	3.36	3.14	2.98	2.85	2.74	2.66
57	7.10	5.00	4.15	3.67	3.36	3.14	2.97	2.84	2.74	2.65
58	7.09	4.99	4.14	3.66	3.35	3.13	2.97	2.84	2.73	2.64
59	7.09	4.98	4.13	3.66	3.35	3.12	2.96	2.83	2.72	2.64
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63
61	7.07	4.97	4.12	3.64	3.33	3.11	2.95	2.82	2.71	2.63
62	7.06	4.97	4.11	3.64	3.33	3.11	2.94	2.81	2.71	2.62
63	7.06	4.96	4.11	3.63	3.32	3.10	2.94	2.81	2.70	2.62
64	7.05	4.95	4.10	3.63	3.32	3.10	2.93	2.80	2.70	2.61
65	7.04	4.95	4.10	3.62	3.31	3.09	2.93	2.80	2.69	2.61
66	7.04	4.94	4.09	3.62	3.31	3.09	2.92	2.79	2.69	2.60
67	7.03	4.94	4.09	3.61	3.30	3.08	2.92	2.79	2.68	2.60
68	7.02	4.93	4.08	3.61	3.30	3.08	2.91	2.79	2.68	2.59
69	7.02	4.93	4.08	3.60	3.30	3.08	2.91	2.78	2.68	2.59
70	7.01	4.92	4.07	3.60	3.29	3.07	2.91	2.78	2.67	2.59
71	7.01	4.92	4.07	3.60	3.29	3.07	2.90	2.77	2.67	2.58
72	7.00	4.91	4.07	3.59	3.28	3.06	2.90	2.77	2.66	2.58
73	7.00	4.91	4.06	3.59	3.28	3.06	2.90	2.77	2.66	2.57
74	6.99	4.90	4.06	3.58	3.28	3.06	2.89	2.76	2.66	2.57
75	6.99	4.90	4.05	3.58	3.27	3.05	2.89	2.76	2.65	2.57
76	6.98	4.90	4.05	3.58	3.27	3.05	2.88	2.76	2.65	2.56
77	6.98	4.89	4.05	3.57	3.27	3.05	2.88	2.75	2.65	2.56
78	6.97	4.89	4.04	3.57	3.26	3.04	2.88	2.75	2.64	2.56
79	6.97	4.88	4.04	3.57	3.26	3.04	2.87	2.75	2.64	2.55
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.64	2.55
81	6.96	4.88	4.03	3.56	3.25	3.03	2.87	2.74	2.63	2.55
82	6.95	4.87	4.03	3.56	3.25	3.03	2.87	2.74	2.63	2.55

83	6.95	4.87	4.03	3.55	3.25	3.03	2.86	2.73	2.63	2.54
84	6.95	4.87	4.02	3.55	3.24	3.03	2.86	2.73	2.63	2.54
85	6.94	4.86	4.02	3.55	3.24	3.02	2.86	2.73	2.62	2.54
86	6.94	4.86	4.02	3.55	3.24	3.02	2.85	2.73	2.62	2.53
87	6.94	4.86	4.02	3.54	3.24	3.02	2.85	2.72	2.62	2.53
88	6.93	4.86	4.01	3.54	3.23	3.01	2.85	2.72	2.62	2.53
89	6.93	4.85	4.01	3.54	3.23	3.01	2.85	2.72	2.61	2.53
90	6.93	4.85	4.01	3.54	3.23	3.01	2.85	2.72	2.61	2.52
91	6.92	4.85	4.00	3.53	3.23	3.01	2.84	2.71	2.61	2.52
92	6.92	4.84	4.00	3.53	3.22	3.00	2.84	2.71	2.61	2.52
93	6.92	4.84	4.00	3.53	3.22	3.00	2.84	2.71	2.60	2.52
94	6.91	4.84	4.00	3.53	3.22	3.00	2.84	2.71	2.60	2.52
95	6.91	4.84	4.00	3.52	3.22	3.00	2.83	2.70	2.60	2.51
96	6.91	4.83	3.99	3.52	3.21	3.00	2.83	2.70	2.60	2.51
97	6.90	4.83	3.99	3.52	3.21	2.99	2.83	2.70	2.60	2.51
98	6.90	4.83	3.99	3.52	3.21	2.99	2.83	2.70	2.59	2.51
99	6.90	4.83	3.99	3.52	3.21	2.99	2.83	2.70	2.59	2.51
100	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59	2.50

TABLE A2 *Values of* $x^2_{\alpha,n}$

n	$\alpha = .995$	$\alpha = .99$	$\alpha = .975$	$\alpha = .95$	$\alpha = .05$	$\alpha = .025$	$\alpha = .01$	$\alpha = .005$
1	.0000393	.000157	.000982	.00393	3.841	5.024	6.635	7.879
2	.0100	.0201	.0506	.103	5.991	7.378	9.210	10.597
3	.0717	.115	.216	.352	7.815	9.348	11.345	12.838
4	.207	.297	.484	.711	9.488	11.143	13.277	14.860
5	.412	.554	.831	1.145	11.070	12.832	13.086	16.750
6	.676	.872	1.237	1.635	12.592	14.449	16.812	18.548
7	.989	1.239	1.690	2.167	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.484	36.415	39.364	42.980	45.558
25	10.520	11.524	13.120	14.611	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	40.113	43.194	46.963	49.645
28	12.461	13.565	15.308	16.928	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	42.557	45.772	49.588	52.336
30	13.787	14.953	16.791	18.493	43.773	46.979	50.892	53.672

Other Chi-Square Probabilities:

$x^2_{.9,9} = 4.2$ $\quad P\{x^2_{16} < 14.3\} = .425$ $\quad P\{x^2_{11} < 17.1875\} = .8976.$

TABLE A3 *Values of $t_{\alpha,n}$*

n	$\alpha = .10$	$\alpha = .05$	$\alpha = .025$	$\alpha = .01$	$\alpha = .005$
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.474	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
∞	1.282	1.645	1.960	2.326	2.576

Other t Probabilities:
$P\{T_8 < 2.541\} = .9825$ $P\{T_8 < 2.7\} = .9864$ $P\{T_{11} < .7635\} = .77$ $P\{T_{11} < .934\} = .81$ $P\{T_{11} < 1.66\} = .94$ $P\{T_{12} < 2.8\} = .984$.