Exam
DIT008: Discrete Mathematics

University of Gothenburg
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26 August 2025, 14:00 – 18:00

No calculators or other aids are allowed.

You must show your calculations or describe your argument. Unless otherwise stated, if you provide just the final answer, you will not be awarded any points.

The grade boundaries are

- pass (3): at least 20 points (50%)
- pass with credit (4): at least 28 points (70%)
- distinction (5): at least 36 points (90%)

In case of questions, call Jan Gerken at 031-772-14-37.

Good luck!

Problem:	1	2	3	4	5	6	7	8	9	10	11	12	Total
Points:	2	3	6	3	4	3	4	3	3	3	4	2	40

Problem 1 (2 points)

Use a truth table to establish the truth of the following statement: "The converse and inverse of a conditional statement are logically equivalent to each other."

Solution:

p	q	$\sim p$	$\sim q$	$q \to p$	$\sim p \to \sim q$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	T	T	T

The truth table shows that $q \to p$ and $\sim p \to \sim q$ always have the same truth values and thus are logically equivalent. It follows that the converse and inverse of a conditional statement are logically equivalent to each other.

Problem 2 (3 points)

For this question, you do not need to justify your answers.

(a) (2 points) Which of the following statements are true?

- $\{5\} \in \{1, 3, 5\}$
- $\{5\} \subseteq \{1, 3, 5\}$
- $\{5\} \in \{\{1\}, \{3\}, \{5\}\}$
- $\{5\} \subseteq \{\{1\}, \{3\}, \{5\}\}$

(b) (1 point) Let $A = \{a, b, c\}$ and $B = \{u, v\}$. Write $A \times B$ and $B \times A$.

Solution:

(a)
- $\{5\} \notin \{1, 3, 5\}$
- $\{5\} \subseteq \{1, 3, 5\}$
- $\{5\} \in \{\{1\}, \{3\}, \{5\}\}$
- $\{5\} \not\subseteq \{\{1\}, \{3\}, \{5\}\}$

(b) $A \times B = \{(a, u), (a, v), (b, u), (b, v), (c, u), (c, v)\}$
$B \times A = \{(u, a), (u, b), (u, c), (v, a), (v, b), (v, c)\}$

Problem 3 (6 points)

Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the *composition* of g and f, denoted by $g \circ f$, is a function $g \circ f : X \to Z$ defined by $(g \circ f)(x) = g(f(x))\ \forall x \in X$. Prove the following statements

(a) (3 points) If f and g are injective, then $g \circ f$ is also injective.

(b) (3 points) If f and g are surjective, then $g \circ f$ is also surjective.

Solution:

(a) *Proof*: Let $x_1, x_2 \in X$ such that $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then, $g(f(x_1)) = g(f(x_2))$ and by injectivity of g, $f(x_1) = f(x_2)$. Since f is also injective, $x_1 = x_2$. Hence, $g \circ f$ is injective.

(b) *Proof*: Let $z \in Z$. Since g is surjective, $\exists\, y \in Y$ such that $g(y) = z$. Since f is surjective, $\exists\, x \in X$ such that $f(x) = y$. Therefore, $(g \circ f)(x) = g(f(x)) = z$. Since z was generic and arbitrary, $g \circ f$ is surjective.

Problem 4 (3 points)
Prove that if one solution for a quadratic equation of the form $x^2 + bx + c = 0$ is rational (where b and c are rational), then the other solution is also rational. Use the fact that if the solutions of the equation are r and s, then $x^2 + bx + c = (x - r)(x - s)$, $\forall x \in \mathbb{R}$.

Solution: *Proof*: Given a quadratic equation

$$x^2 + bx + c = 0\,,$$

where b and c are rational numbers, suppose one solution, r, is rational. Call the other solution s. Then

$$\begin{aligned} x^2 + bx + c &= (x - r)(x - s) \\ &= x^2 - (r + s)x + rs. \end{aligned}$$

By equating the coefficients of x,

$$b = -(r + s).$$

Solving for s yields

$$\begin{aligned} s &= -r - b \\ &= -(r + b). \end{aligned}$$

Because s is the negative of a sum of two rational numbers, s also is rational.

Problem 5 (4 points)
Use mathematical induction to prove that $\forall n \in \mathbb{Z}$ with $n \geq 2$, $2^n < (n + 1)!$.

Solution: Let $P(n)$ be the property

$$2^n < (n+1)!\,.$$

Base case ($n = 2$): $P(2)$ is true because the left-hand side is

$$2^2 = 4$$

and the right-hand side is

$$(2+1)! = 6$$

and $4 < 6$.

Inductive step: Let k be any integer with $k \geq 2$, and suppose $P(k)$, i.e. suppose

$$2^k < (k+1)!\,.$$

Then, by the inductive hypothesis

$$2^{k+1} = 2 \cdot 2^k < 2 \cdot (k+1)!\,.$$

Together with

$$2 \cdot (k+1)! < 2 \cdot (k+1)! + \underbrace{k \cdot (k+1)!}_{>0} = (k+2) \cdot (k+1)! = (k+2)!\,,$$

we arrive at

$$2^{k+1} < (k+2)!\,.$$

This is just $P(k+1)$ and hence the inductive step is proven.

Problem 6 (3 points)

In a *Triple Tower of Hanoi*, there are three poles in a row and $3n$ disks, three of each of n different sizes, where n is any positive integer. Initially, one of the poles contains all the disks placed on top of each other in triples of decreasing size. Disks are transferred one by one from one pole to another, but at no time may a larger disk be placed on top of a smaller disk. However, a disk may be placed on top of one of the same size. Let t_n be the minimum number of moves needed to transfer a tower of $3n$ disks from one pole to another.

(a) (1 point) Find t_1 and t_2. Justify your answer.

(b) (2 points) Find a recursion relation for t_n, $n \leq 2$. Justify your answer.

Solution:

(a) $t_1 = 3$, since 3 disks can be moved directly from A to C.
$t_2 = 3 + 3 + 3 = 9$ since the following moves are required to move 2 blocks of 3 disks:

- Move 3 disks from A to B
- Move 3 disks from A to C
- Move 3 disks from B to C

(b) To move $3n$ disks from A to C, first move $3(n-1)$ disks from A to B in t_{n-1} moves, then 3 disks from A to C in 3 moves, finally $3(n-1)$ disks from B to C in t_{n-1} moves. In total, $t_n = 2t_{n-1} + 3$.

Problem 7 (4 points)
A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions: (a) Rabbit pairs are not fertile during their first two months of life, but thereafter they give birth to four new male/female pairs at the end of every month; (b) No deaths occur. Let s_n be the number of pairs of rabbits alive at the end of month n, for each integer $n \leq 1$, and let $s_0 = 1$. Find a recursion relation for s_n with suitable initial conditions. Justify your answer.

Solution: Rabbit pairs are not fertile in the first two months of their life. Hence, $s_1 = s_2 = 1$. For $n \geq 3$, we can use the following recursion relation: s_n is the sum of the rabbits alive at the end of the previous month, s_{n-1}, plus the number of newborn rabbit pairs in month n. All rabbit pairs which are fertile in month n were alive at the end of month $n-3$ (all other rabbit pairs entering month n are younger than two months). Since each fertile rabbit pair produces four rabbit pairs of offspring, we have in total $s_n = 4s_{n-3} + s_{n-1}$.

Problem 8 (3 points)
Given any set of 30 integers, must there be two that have the same remainder when they are divided by 25? Write an answer that would convince a good but skeptical fellow student who has learned the statement of the pigeonhole principle but not seen an application like this one. Either describe the pigeons, the pigeonholes, and how the pigeons get to the pigeonholes, or describe a function by giving its domain, co-domain, and how elements of the domain are related to elements of the co-domain.

Solution: Yes.

Solution 1: There are 25 possible remainders that can be obtained when an integer is divided by 12, namely all the integers from 0 through 24. Apply the pigeonhole principle, thinking of the elements of the set of 30 integers as the pigeons and the possible remainders as the pigeonholes. Each pigeon flies into the pigeonhole that is the remainder obtained when it is divided by 25. Since $30 > 25$, the pigeonhole principle says that at least two pigeons must fly into the same pigeonhole. So at least two of the numbers must have the same remainder when divided by 25.

Solution 2: Let X be the set of 30 integers and Y the set of all possible remainders obtained through division by 25, and consider the function R from X (the pigeons) to Y (the pigeonholes) defined by the rule: $R(n) = n \bmod 25$. Now X has 30 elements and Y has 25 elements (the integers from 0 through 24). Hence by the pigeonhole principle, R is not one-to-one: $R(n_1) = R(n_2)$ for some integers n_1 and n_2 with $n_1 \neq n_2$. But this means that n_1 and n_2 have the same remainder when divided by 25.

Problem 9 (3 points)

Two new drugs are to be tested using a group of 9 laboratory mice, each tagged with a number for identification purposes. Drug A is to be given to 3 mice, drug B is to be given to another 3 mice, and the remaining 3 mice are to be used as controls. How many ways can the assignment of treatments to mice be made? (A single assignment involves specifying the treatment for each mouse – whether drug A, B, or no drug.)

Solution: First select 3 out of 9 mice to be given drug A. There are $\binom{9}{3}$ possibilities of doing so. Then select 3 out of the remaining 6 mice to be given drug B. There are $\binom{6}{3}$ possibilities of doing so. All the remaining mice are controls. In total, there are

$$\binom{9}{3} \cdot \binom{6}{3} = 84 \cdot 20 = 1680$$

possible assignments.

Problem 10 (3 points)

A password consists of 4 characters, where each character is either a letter in $\{A, B\}$ or a digit in $\{1, 2\}$.

(a) (1 point) How many different passwords are possible?

(b) (1 point) How many different passwords with exactly 2 letters and 2 digits are possible?

(c) (1 point) How many different passwords with at least one letter and at least one digit are possible?

Solution:

(a) For each character, there are 4 choices. Hence, there are $4^4 = 256$ possibilities.

(b) There are $\binom{4}{2}$ possible positions for the digits and 2^2 possible digit-combinations.

For the remaining 2 characters, there are 2^2 letter-combinations. In total, there are $\binom{4}{2} \cdot 2^2 \cdot 2^2 = 96$ possibilities.

(c) Passwords with only digits or only letters are forbidden. Hence, there are $2^4 + 2^4 = 32$ forbidden passwords. Subtracting these from the total from (a) yields $256 - 32 = 224$ possible passwords.

Problem 11 (4 points)

(a) (2 points) A certain connected graph has 68 vertices and 72 edges. Does it have a circuit? Explain.

(b) (2 points) A certain graph has 19 vertices, 16 edges, and no circuits. Is it connected? Explain.

Solution:

(a) Yes. The graph does has a circuit because if it did not have a circuit, then, since it is connected, it would be a tree and would have 67 edges instead of 72 edges.

(b) No. The graph is not connected because if it were connected, then, since it is circuit-free, it would be a tree and would have 18 edges instead of 19 edges.

Problem 12 (2 points)

Using the definition of the O-notation, prove that $15x^3 + 8x + 4$ is $O(x^3)$. Do not use the theorem on polynomial orders.

Solution: $\forall x > 1$,

$$0 \leq 15x^3 + 8x + 4\,,$$

because $15x^3$, $8x$ and 4 are all positive for $x > 1$. Furthermore,

$$15x^3 + 8x + 4 \leq 15x^3 + 8x^3 + 4x^3 = 27x^3 \quad \forall x > 1\,,$$

since $x < x^3$ and $1 < x^3$ for $x > 1$. Therefore,

$$0 \leq 15x^3 + 8x + 4 \leq 27x^3 \quad \forall x > 1\,.$$

Hence, by the definition of the O-notation, $15x^3 + 8x + 4$ is $O(x^3)$.