Exam
DIT008: Discrete Mathematics

University of Gothenburg
Lecturer: Jan Gerken

27 October 2025, 8:30 – 12:30

No calculators or other aids are allowed.

You must show your calculations or describe your argument. Unless otherwise stated, if you provide just the final answer, you will not be awarded any points.

The grade boundaries are

- pass (3): at least 20 points (50%)
- pass with credit (4): at least 28 points (70%)
- distinction (5): at least 36 points (90%)

In case of questions, call Jan Gerken at 031-772-14-37.

Good luck!

Problem:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	3	3	4	3	7	3	3	3	3	4	4	40

Problem 1 (3 points)

Consider the argument form:

$$\begin{array}{rl} & p \to \sim q \\ & q \to \sim p \\ \therefore & p \vee q \end{array}$$

Use a truth table to determine whether this form of argument is valid or invalid. Annotate the table as appropriate and include a few words explaining how the truth table supports your answer.

Problem 2 (3 points)

The following is the definition for $\lim_{x\to a} f(x) = L$:

> For every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for every real number x, if $a - \delta < x < a + \delta$ and $x \neq a$ then
>
> $$L - \varepsilon < f(x) < L + \varepsilon\,.$$

Write what it means for $\lim_{x\to a} f(x) \neq L$. In other words, write the negation of the statement above. Do not use negations of elementary operators and quantifiers such as $\not\forall$, $\nexists$, $\not<$ etc. You can use $\neq$. No further justification necessary.

Problem 3 (4 points)

Prove by induction:

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2 \quad \forall n \in \mathbb{Z}^{\geq 0}\,.$$

Problem 4 (3 points)

Prove the following statement by contradiction:

> $\forall n \in \mathbb{Z}$, if n^3 is even, then n is even.

Problem 5 (7 points)

Define $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows: $F(x, y) = (3y - 1, 1 - x)\ \forall\,(x, y) \in \mathbb{R} \times \mathbb{R}$.

(a) (1 point) Compute $F(0, 0)$ and $F(1, 4)$.

(b) (3 points) Is F injective? Prove or give a counterexample.

(c) (2 points) Is F surjective? Prove or give a counterexample.

(d) (1 point) Is F bijective? If not, explain why not. If yes, find F^{-1}.

Problem 6 (3 points)

Consider the set S of all strings of a's and b's. For each integer $n \geq 0$, let

a_n = the number of strings of length n that do not contain the pattern bb.

Find a recurrence relation with appropriate initial conditions for $a_0, a_1, a_2, \ldots$ and explain your answer carefully.

Problem 7 (3 points)

The binomial theorem states that $\forall\, a, b \in \mathbb{R}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \quad \forall\, n \in \mathbb{Z}^{\geq 0}.$$

(a) (1 point) Use this theorem to compute $(2x+1)^4$. Write out the intermediate steps of your calculation.

(b) (2 points) Use this theorem to prove that,

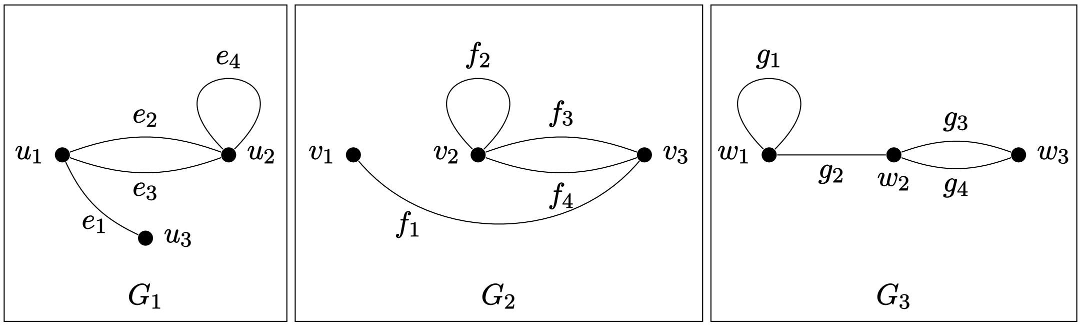
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k} 2^k = 1 \quad \forall\, n \in \mathbb{Z}^{\geq 0}.$$

Problem 8 (3 points)

If six integers are chosen from the set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, must there be at least two integers with the property that the sum of the smaller plus the larger is 11? Why or why not? Use the pigeonhole principle. Describe the function appearing in the pigeonhole principle by giving its domain, co-domain, and how elements of the domain are related to elements of the co-domain.

Problem 9 (3 points)

Determine for each of the pairs of graphs given below if they are isomorphic. If they are, give vertex and edge functions that define the isomorphism. If they are not, explain why not.



(a) (1 point) G_1 and G_2

(b) (1 point) G_1 and G_3

(c) (1 point) G_2 and G_3

Problem 10 (4 points)

Prove that $\lceil \log_2(n) \rceil$ is $\Theta(\log_2(n))$.

Problem 11 (4 points)

(a) (3 points) Find the total number of additions and multiplications that must be performed when the following algorithm is executed. Show your work carefully.

```
for i := 1 to n
    for j := i to n
        a := 2 · (5 · i + j + 1)
    next j
next i
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(b) (1 point) Find the order for the number of additions and multiplications in the algorithm segment of part (a). Justify your answer.