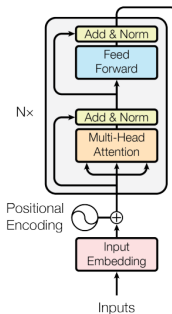


Transformer

The Encoder

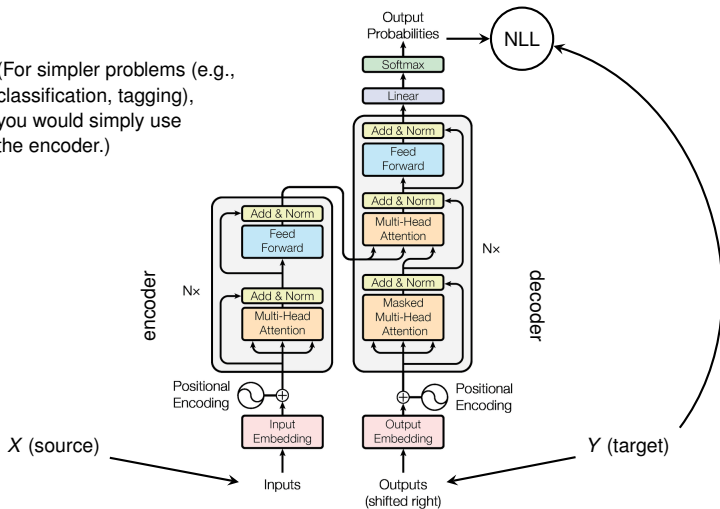


Learning goals

- Understand Self-Attention and the role of position embeddings
- Understand subtleties of parallelized multi-head attention

THE TRANSFORMER ARCHITECTURE

(For simpler problems (e.g., classification, tagging), you would simply use the encoder.)



► Source: Vaswani et al., 2017

ATTENTION IN THE TRANSFORMER

- We can use attention on many different “things”, including:
 - The pixels of images
 - The nodes of knowledge graphs
 - The words of a vocabulary
- Here, we focus on scenarios where the query, key and value vectors represent tokens (e.g., words, characters, etc.) in sequences (e.g., sentences, paragraphs, etc.).

ATTENTION IN THE TRANSFORMER

Cross-Attention:

- Let $X = (x_1 \dots x_{J_x})$, $Y = (y_1 \dots y_{J_y})$ be two sequences (e.g., source and target in a sequence-to-sequence problem)
- The query vectors represent tokens in Y and the key/value vectors represent tokens in X (“ Y attends to X ”)

Self-Attention:

- There is only one sequence $X = (x_1 \dots x_J)$
- The query, key and value vectors represent tokens in X (“ X attends to itself”)

CROSS-ATTENTION (1)

- We start by describing cross-attention, self-attention can easily be derived by assuming $\vec{X} = \vec{Y}$.
- Let $\vec{X} \in \mathbb{R}^{J_x \times d_x}$, $\vec{Y} \in \mathbb{R}^{J_y \times d_y}$ be representations of X , Y (e.g., stacked word embeddings, or the outputs of a previous layer)
- Let $\theta = \{\vec{W}^{(q)} \in \mathbb{R}^{d_y \times d_q}, \vec{W}^{(k)} \in \mathbb{R}^{d_x \times d_k}, \vec{W}^{(v)} \in \mathbb{R}^{d_x \times d_v}\}$ be trainable weight matrices
- We transform \vec{Y} into a matrix of query vectors:

$$\vec{Q} = \vec{Y}\vec{W}^{(q)}$$

- We transform \vec{X} into matrices of key and value vectors:

$$\vec{K} = \vec{X}\vec{W}^{(k)}; \quad \vec{V} = \vec{X}\vec{W}^{(v)}$$

CROSS-ATTENTION (2)

- To calculate the e scores, Vaswani et al. use a parameter-less scaled dot product instead of Bahdanau's complicated FFN
- Step 1 of the basic recipe: *Scores*

$$e_{j,j'} = a(\vec{q}_j, \vec{k}_{j'}) = \frac{\vec{q}_j^T \vec{k}_{j'}}{\sqrt{d_k}}$$

- Note: This requires that $d_q = d_k$

CROSS-ATTENTION (3)

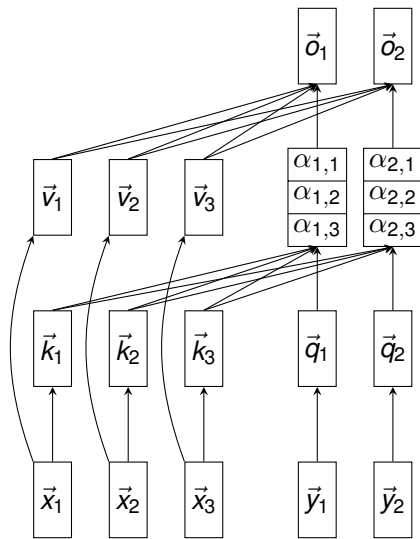
- Attention weights and outputs are defined like before
- Step 2 of the basic recipe: *Weights*

$$\alpha_{j,j'} = \frac{\exp(\mathbf{e}_{j,j'})}{\sum_{j''=1}^{J_x} \exp(\mathbf{e}_{j,j''})}$$

- Step 3 of the basic recipe: *Outputs*

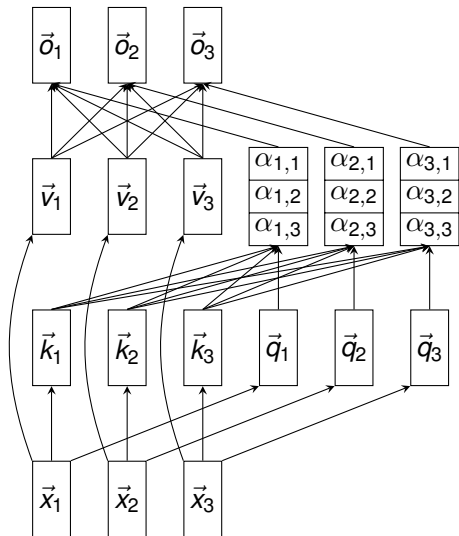
$$\vec{o}_j = \sum_{j'=1}^{J_x} \alpha_{j,j'} \vec{v}_{j'}$$

CROSS-ATTENTION (4)



Cross-attention

SELF-ATTENTION



Self-attention

SELF-ATTENTION FORMALIZED

- Let $\vec{X} \in \mathbb{R}^{J_x \times d_x}$ be a representation of X (e.g., stacked word embeddings, or the outputs of a previous layer)
- Let $\theta = \{\vec{W}^{(q)} \in \mathbb{R}^{d_x \times d_q}, \vec{W}^{(k)} \in \mathbb{R}^{d_x \times d_k}, \vec{W}^{(v)} \in \mathbb{R}^{d_x \times d_v}\}$ be trainable weight matrices
- We transform \vec{X} into a matrix of query vectors:

$$\vec{Q} = \vec{X} \vec{W}^{(q)}$$

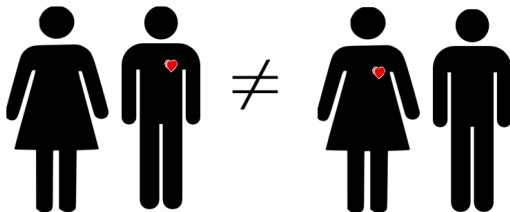
- And we transform \vec{X} into matrices of key and value vectors:

$$\vec{K} = \vec{X} \vec{W}^{(k)}; \quad \vec{V} = \vec{X} \vec{W}^{(v)}$$

- Execute Steps 1 – 3 of the basic recipe

CAN SELF-ATTENTION MODEL WORD ORDER?

- Hypothetical model:
 - simple word embedding lookup layer
 - self-attention layer
- Let $X^{(1)}$, $X^{(2)}$ be two sentences of the same length J , which contain the same words, but in a different order
- Example: “john loves mary” vs. “mary loves john”



CAN SELF-ATTENTION MODEL WORD ORDER?

- Definition of \vec{o}_j :

$$\vec{o}_j = \sum_{j'=1}^J \alpha_{j,j'} \vec{v}_{j'}$$

- Since addition is commutative, and the permutation is bijective, it is sufficient to show that:

$$\forall_{j \in \{1, \dots, J\}, j' \in \{1, \dots, J\}} \alpha_{j,j'}^{(1)} \vec{v}_{j'}^{(1)} = \alpha_{g_j, g_{j'}}^{(2)} \vec{v}_{g_{j'}}^{(2)}$$

CAN SELF-ATTENTION MODEL WORD ORDER?

- Step 1: Let's show that $\forall_j \vec{v}_j^{(1)} = \vec{v}_{g_j}^{(2)}$
- Definition of \vec{v}_j :

$$\vec{v}_j = \vec{W}^{(v)T} \vec{x}_j$$

- Then:

$$\vec{x}_j^{(1)} = \vec{x}_{g_j}^{(2)} \implies \vec{W}^{(v)T} \vec{x}_j^{(1)} = \vec{W}^{(v)T} \vec{x}_{g_j}^{(2)} \implies \vec{v}_j^{(1)} = \vec{v}_{g_j}^{(2)}$$

CAN SELF-ATTENTION MODEL WORD ORDER?

- Step 2: Let's show that $\forall_{j \in \{1, \dots, J\}, j' \in \{1, \dots, J\}} \alpha_{j,j'}^{(1)} = \alpha_{g_j, g_{j'}}^{(2)}$
- Definition of $\alpha_{j,j'}$:

$$\alpha_{j,j'} = \frac{\exp(\mathbf{e}_{j,j'})}{\sum_{j''=1}^J \exp(\mathbf{e}_{j,j''})}$$

- Since the sum in the denominator is commutative, and the permutation is bijective, it is sufficient to show that

$$\forall_{j \in \{1, \dots, J\}, j' \in \{1, \dots, J\}} \mathbf{e}_{j,j'}^{(1)} = \mathbf{e}_{g_j, g_{j'}}^{(2)}$$

CAN SELF-ATTENTION MODEL WORD ORDER?

- Definition of $e_{j,j'}$:

$$e_{j,j'} = \frac{1}{\sqrt{d_k}} \vec{q}_j^T \vec{k}_{j'} = \frac{1}{\sqrt{d_k}} (\vec{W}^{(q)T} \vec{x}_j)^T (\vec{W}^{(k)T} \vec{x}_{j'})$$

- Then:

$$\begin{aligned} \vec{x}_j^{(1)} &= \vec{x}_{g_j}^{(2)} \wedge \vec{x}_{j'}^{(1)} = \vec{x}_{g_{j'}}^{(2)} \\ \implies \vec{W}^{(q)T} \vec{x}_j^{(1)} &= \vec{W}^{(q)T} \vec{x}_{g_j}^{(2)} \wedge \vec{W}^{(k)T} \vec{x}_{j'}^{(1)} = \vec{W}^{(k)T} \vec{x}_{g_{j'}}^{(2)} \\ \implies \vec{q}_j^{(1)} &= \vec{q}_{g_j}^{(2)} \wedge \vec{k}_{j'}^{(1)} = \vec{k}_{g_{j'}}^{(2)} \\ \implies \vec{q}_j^{(1)T} \vec{k}_{j'}^{(1)} &= \vec{q}_{g_j}^{(2)T} \vec{k}_{g_{j'}}^{(2)} \\ \implies \frac{1}{\sqrt{d_k}} \vec{q}_j^{(1)T} \vec{k}_{j'}^{(1)} &= \frac{1}{\sqrt{d_k}} \vec{q}_{g_j}^{(2)T} \vec{k}_{g_{j'}}^{(2)} \\ \implies e_{j,j'}^{(1)} &= e_{g_j,g_{j'}}^{(2)} \end{aligned}$$

CAN SELF-ATTENTION MODEL WORD ORDER?

- So, $\forall_j \vec{o}_j^{(1)} = \vec{o}_{g_j}^{(2)}$
- In other words: The representation of **mary** is identical to that of **mary**, and the representation of **john** is identical to that of **john**

CAN SELF-ATTENTION MODEL WORD ORDER?

- **Question:** Can the other layers in the Transformer architecture (feed-forward net, layer normalization) help with the problem?
 - No, because they are apply the same function to all positions.
- **Question:** Would it help to apply more self-attention layers?
 - No. Since the representations of identical words are still identical in \vec{O} , the next self-attention layer will have the same problem.
- So... does that mean the Transformer is unusable?
- Luckily not. We just need to ensure that input embeddings of identical words at different positions are not identical.

POSITION EMBEDDINGS

- Add to every input word embedding a position embedding $\vec{p} \in \mathbb{R}^d$:
- Input embedding of word “mary” in position j : $\vec{x}_j = \vec{w}_{\mathcal{I}(\text{mary})} + \mathbf{p}_j$

$$\vec{w}_{\mathcal{I}(\text{mary})} + \vec{p}_j \neq \vec{w}_{\mathcal{I}(\text{mary})} + \vec{p}_{j'} \text{ if } j \neq j'$$

- Option 1 ► Source: Vaswani et al., 2017

(Deterministic) Sinusoidal position embeddings:

$$p_{j,i} = \begin{cases} \sin\left(\frac{j}{10000^{\frac{i}{d}}}\right) & \text{if } i \text{ is even} \\ \cos\left(\frac{j}{10000^{\frac{i-1}{d}}}\right) & \text{if } i \text{ is odd} \end{cases}$$

- Option 2 ► Source: Devlin et al., 2019

Trainable position embeddings: $\vec{P} \in \mathbb{R}^{J^{\max} \times d}$

(Disadvantage: Cannot deal with sentences longer than J^{\max})

PARALLELIZED ATTENTION

- We want to apply our attention recipe to every query vector \vec{q}_j
- We could simply loop over all time steps $1 \leq j \leq J_x$ (or J_y) and calculate each \vec{o}_j independently.
- Then stack all \vec{o}_j into an output matrix $\vec{O} \in \mathbb{R}^{J_x \times d_v}$ (or $\mathbb{R}^{J_y \times d_v}$)
- But a loop does not use the GPU's capacity for parallelization, so it might be unnecessarily slow

PARALLELIZED SELF-ATTENTION

- Do some inputs (e.g., \vec{q}_j) depend on previous outputs (e.g., \vec{o}_{j-1})? If not, we can parallelize the loop into a single function:

$$\vec{O} = \mathcal{F}^{\text{attn}}(\vec{X}, \vec{X}; \theta)$$

- Attention in Transformers is usually parallelizable, unless we are doing autoregressive inference (more on that later).

PARALLELIZED SELF-ATTENTION

- By the way: The Bahdanau model is not parallelizable in this way, because s_i (a.k.a. the query of the $i + 1$ 'st step) depends on c_i (a.k.a. the attention output of the i 'th step), see last lecture:

The hidden state s_i of the decoder given the annotations from the encoder is computed by

$$s_i = (1 - z_i) \circ s_{i-1} + z_i \circ \tilde{s}_i,$$

where

$$\tilde{s}_i = \tanh(W E y_{i-1} + U[r_i \circ s_{i-1}] + C c_i)$$

$$z_i = \sigma(W_z E y_{i-1} + U_z s_{i-1} + C_z c_i)$$

$$r_i = \sigma(W_r E y_{i-1} + U_r s_{i-1} + C_r c_i)$$

PARALLELIZED SELF-ATTENTION

- **Step 1:** The parallel application of the scaled dot product to all query-key pairs can be written as:

$$\vec{E} = \frac{\vec{Q}\vec{K}^T}{\sqrt{d_k}}; \quad \vec{E} \in \mathbb{R}^{J_x \times J_x}$$

$$\frac{1}{\sqrt{d_k}} \begin{bmatrix} - & \vec{q}_1 & - \\ & \vdots & \\ - & \vec{q}_{J_x} & - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{k}_1 & \dots & \vec{k}_{J_x} \\ | & & | \end{bmatrix} = \begin{matrix} \downarrow \\ \text{queries} \\ \downarrow \end{matrix} \begin{matrix} \rightarrow \text{keys} \rightarrow \\ \begin{bmatrix} e_{1,1} & \dots & e_{1,J_x} \\ \vdots & \ddots & \vdots \\ e_{J_x,1} & \dots & e_{J_x,J_x} \end{bmatrix} \end{matrix}$$

PARALLELIZED SCALED DOT PRODUCT SELF-ATTENTION

- **Step 2:** Softmax with normalization over the second axis (key axis):

$$\alpha_{j,j'} = \frac{\exp(\mathbf{e}_{j,j'})}{\sum_{j''=1}^{J_x} \exp(\mathbf{e}_{j,j''})}$$

- Let's call this new normalized matrix $\vec{A} \in (0, 1)^{J_x \times J_x}$
- The rows of \vec{A} , denoted $\vec{\alpha}_j$, are probability distributions (one $\vec{\alpha}_j$ per \vec{q}_j)

PARALLELIZED SCALED DOT PRODUCT SELF-ATTENTION

- **Step 3:** Weighted sum

$$\vec{O} = \vec{A}\vec{V}; \vec{O} \in \mathbb{R}^{J_x \times d_v}$$

$$\begin{bmatrix} - & \alpha_1 & - \\ & \vdots & \\ - & \alpha_{J_x} & - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{v}_{:,1} & \dots & \vec{v}_{:,d_v} \\ | & & | \end{bmatrix} = \begin{matrix} \downarrow \\ \text{queries} \\ \downarrow \end{matrix} \begin{matrix} \rightarrow d_v(\text{value dims}) \rightarrow \\ \begin{bmatrix} o_{1,1} & \dots & o_{1,d_v} \\ \vdots & \ddots & \vdots \\ o_{J_x,1} & \dots & o_{J_x,d_v} \end{bmatrix} \end{matrix}$$

... AS A ONE-LINER

$$\vec{O} = \mathcal{F}^{\text{attn}}(\vec{X}, \vec{X}; \theta) = \text{softmax}\left(\frac{(\vec{X}\vec{W}^{(q)})(\vec{X}\vec{W}^{(k)})^T}{\sqrt{d_k}}\right)(\vec{X}\vec{W}^{(v)})$$

- GPUs like matrix multiplications
→ usually a lot faster than RNN!
- But: The memory requirements of \vec{E} and \vec{A} are $\mathcal{O}(J_x^2)$
- Sequence lengths are an important issue.

ADD-ON: CHANGES FOR CROSS ATTENTION

- **Step 1:** Scaled dot-product

$$\vec{E} = \frac{\vec{Q}\vec{K}^T}{\sqrt{d_k}}; \quad \vec{E} \in \mathbb{R}^{J_y \times J_x}$$

- **Step 2:** Softmax

$$\alpha_{j,j'} = \frac{\exp(\mathbf{e}_{j,j'})}{\sum_{j''=1}^{J_x} \exp(\mathbf{e}_{j,j''})}$$

- **Step 3:** Output

$$\vec{O} = \vec{A}\vec{V}; \quad \vec{O} \in \mathbb{R}^{J_y \times d_v}$$

- As one-liner:

$$\vec{O} = \mathcal{F}^{\text{attn}}(\vec{X}, \vec{Y}; \theta) = \text{softmax}\left(\frac{(\vec{Y}\vec{W}^{(q)})(\vec{X}\vec{W}^{(k)})^T}{\sqrt{d_k}}\right)(\vec{X}\vec{W}^{(v)})$$

MULTI-LAYER ATTENTION

- Sequential application of several attention layers, with separate parameters $\{\theta^{(1)} \dots \theta^{(N)}\}$
- In Transformer: sequential application of Transformer blocks
- There are some additional position-wise layers inside the Transformer block, i.e., $\vec{O}^{(n)}$ undergoes some additional transformations before becoming the input to the next Transformer block $n + 1$

MULTI-HEAD ATTENTION

- Application of several attention layers (“heads”) in parallel
- M sets of parameters $\{\theta^{(1)}, \dots, \theta^{(M)}\}$, with
 $\theta^{(m)} = \{\vec{W}^{(m,q)}, \vec{W}^{(m,k)}, \vec{W}^{(m,v)}\}$
- For every head, compute in parallel:

$$\vec{O}^{(m)} = \mathcal{F}^{\text{attn}}(\vec{X}, \vec{Y}; \theta^{(m)})$$

- Choice of dimensionality: $d_k = d_v = d_x/M$
- Concatenate all $\vec{O}^{(m)}$ along their last axis; then down-project the concatenation with an additional parameter matrix $\vec{W}^{(o)} \in \mathbb{R}^{Md_v \times d_v}$:

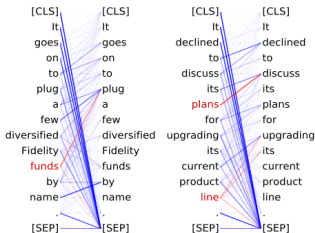
$$\vec{O} = [\vec{O}^{(1)}; \dots; \vec{O}^{(M)}] \vec{W}^{(o)}$$

MULTI-HEAD ATTENTION

- Conceptually, multi-head attention is to single-head attention like a filter bank is to a single filter (CNNs)
- Division of labor: different heads model different kinds of inter-word relationships

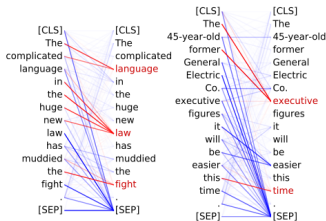
Head 8-10

- **Direct objects** attend to their verbs
- 86.8% accuracy at the dobj relation



Head 8-11

- **Noun modifiers** (e.g., determiners) attend to their noun
- 94.3% accuracy at the det relation



► Source: Clark et al., 2019

MODEL SIZE

- Embedding size: $d_x = 512$
- Number of heads: $M = 8$
- Embedding size *per head*: $d_k = d_v = 512/8 = 64$
 $\rightarrow \vec{W}^{(q)} \in \mathbb{R}^{512 \times 64}, \vec{W}^{(k)} \in \mathbb{R}^{512 \times 64}, \vec{W}^{(v)} \in \mathbb{R}^{512 \times 64}$
- Projection matrix after multi-head attention: $\vec{W}^{(o)} \in \mathbb{R}^{512 \times 512}$