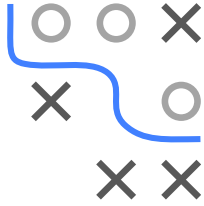
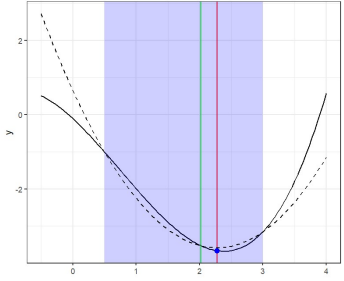
# Optimization in Machine Learning



## Univariate optimization
## Brent's method




**Learning goals**

- Quadratic interpolation
- Brent's procedure

# QUADRATIC INTERPOLATION

Similar to golden ratio procedure but select $x^{\text{new}}$ differently: $x^{\text{new}}$ as minimum of a parabola fitted through

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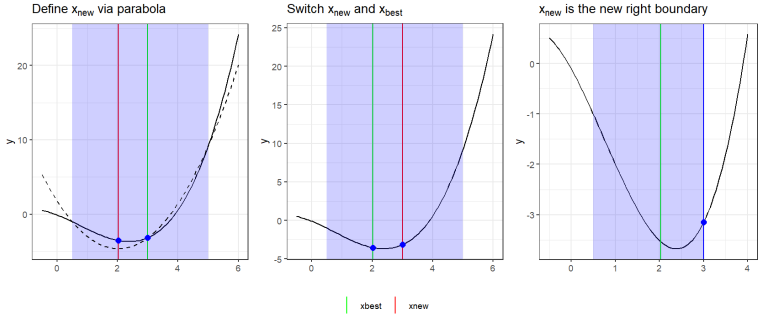


Left: Fit parabola (dashed) and propose minimum (red) as new point. Middle: Switch / not switch with $x^{\text{best}}$. Right: New interval.

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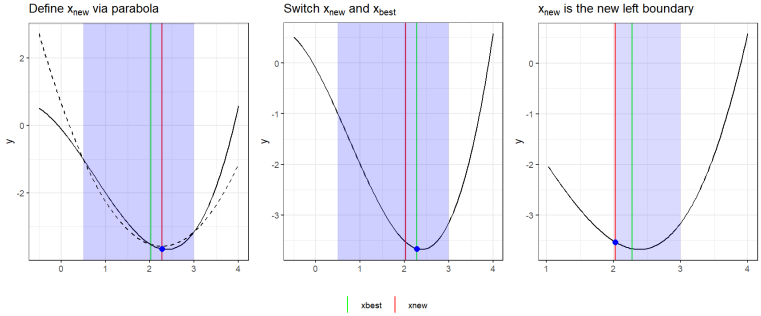


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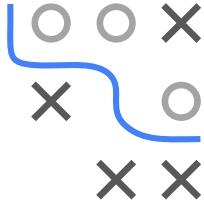

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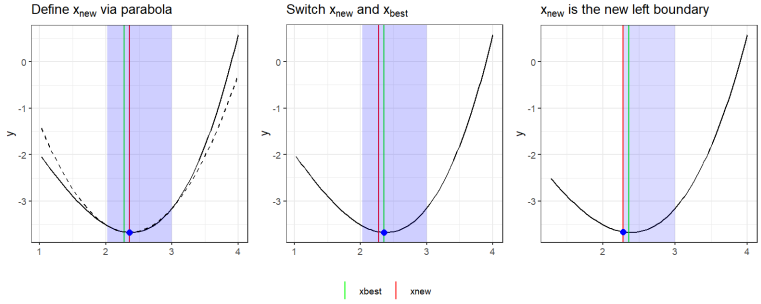


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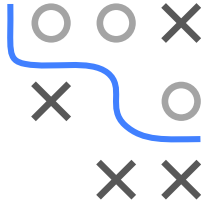

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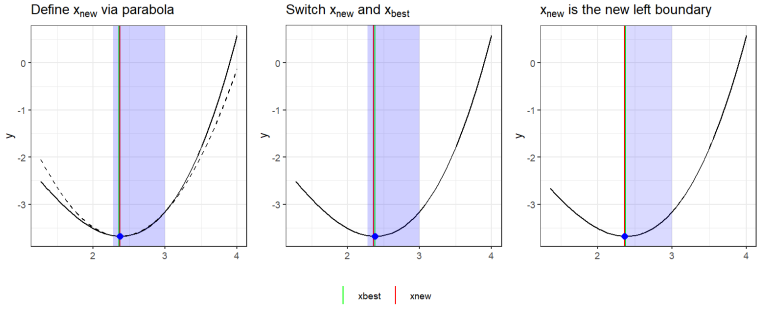


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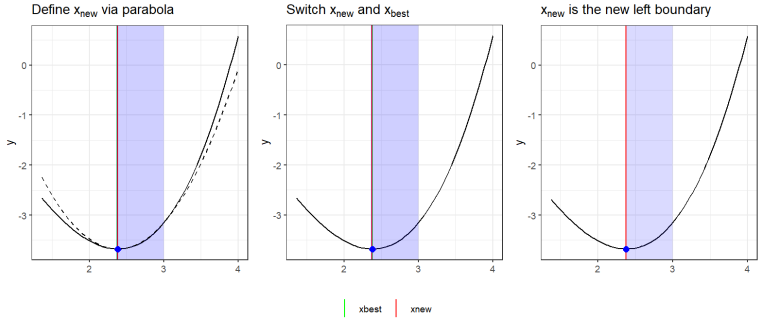


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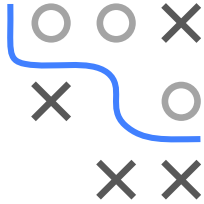
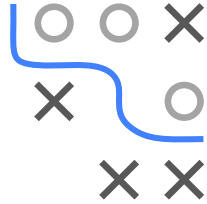
# QUADRATIC INTERPOLATION COMMENTS

- Quadratic interpolation **not robust**. The following may happen:
  - Algorithm suggests the same $x^{\text{new}}$ in each step,
  - $x^{\text{new}}$ outside of search interval,
  - Parabola degenerates to line and no real minimum exists
- Algorithm must then abort, finding a global minimum is not guaranteed.



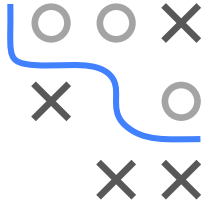
# BRENT'S METHOD



- Brent proposed an algorithm (1973) that alternates between golden ratio search and quadratic interpolation as follows:
  - Quadratic interpolation step acceptable if: (i) $x^{\text{new}}$ falls within $[x^{\text{left}}, x^{\text{right}}]$ (ii) $x^{\text{new}}$ sufficiently far away from $x^{\text{best}}$

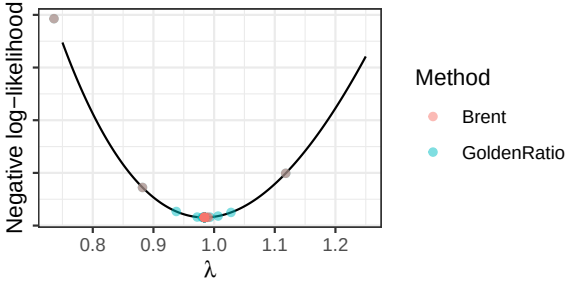
    (Heuristic: Less than half of movement of step before last)
  - Otherwise: Proposal via golden ratio
- Benefit: Fast convergence (quadratic interpolation), unstable steps (e.g. parabola degenerated) stabilized by golden ratio search
- Convergence guaranteed if the function $f$ has a local minimum
- Used in R-function `optimize()`

# EXAMPLE: MLE POISSON



- Poisson density: $f(k \mid \lambda) := \mathbb{P}(x = k) = \frac{\lambda^k \cdot \exp(-\lambda)}{k!}$
- Negative log-likelihood for $n$ observations:

$$-\ell(\lambda, \mathcal{D}) = -\log \prod_{i=1}^{n} f\left(x^{(i)} \mid \lambda\right) = -\sum_{i=1}^{n} \log f\left(x^{(i)} \mid \lambda\right)$$




GR and Brent converge to minimum at $x^* \approx 1$.
**But:** GR needs $\approx 45$ it., Brent only needs $\approx 15$ it. for same tolerance.