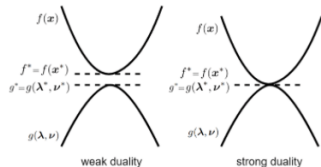


# Optimization in Machine Learning

## Nonlinear programs and Lagrangian



### Learning goals

- Lagrangian for general constrained optimization
- Geometric intuition for Lagrangian duality
- Properties and examples

# NONLINEAR CONSTRAINED OPTIMIZATION

Previous lecture: **Linear programs**

$$\begin{array}{ll} \min_{\mathbf{x} \in \mathbb{R}^d} & f(\mathbf{x}) := \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{Gx} = \mathbf{h} \end{array}$$



Related to its (Lagrange) dual formulation by the *Lagrangian*

$$\mathcal{L}(\mathbf{x}, \alpha, \beta) = \mathbf{c}^\top \mathbf{x} + \alpha^\top (\mathbf{Ax} - \mathbf{b}) + \beta^\top (\mathbf{Gx} - \mathbf{h}).$$

**Weak duality:** For  $\alpha \geq 0$  and  $\beta$ :

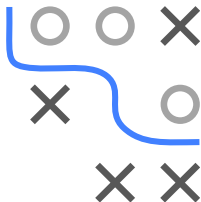
$$f(\mathbf{x}^*) \geq \min_{\mathbf{x} \in \mathcal{S}} \mathcal{L}(\mathbf{x}, \alpha, \beta) \geq \min_{\mathbf{x} \in \mathbb{R}^d} \mathcal{L}(\mathbf{x}, \alpha, \beta) =: g(\alpha, \beta)$$

**Recall:** Implicit domain constraint in *Lagrange dual function*  $g(\alpha, \beta)$ .

# NONLINEAR CONSTRAINED OPTIMIZATION / 2

General form of a constraint optimization problem

$$\begin{array}{ll}\min_{\mathbf{x} \in \mathbb{R}^d} & f(\mathbf{x}) \\ \text{s.t.} & g_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, k, \\ & h_j(\mathbf{x}) = 0, \quad j = 1, \dots, \ell.\end{array}$$



- Functions  $f$ ,  $g_i$ ,  $h_j$  generally nonlinear
- Transfer the Lagrangian from linear programs:

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}) := f(\mathbf{x}) + \sum_{i=1}^k \alpha_i g_i(\mathbf{x}) + \sum_{j=1}^{\ell} \beta_j h_j(\mathbf{x})$$

- Dual variables  $\alpha_i \geq 0$  and  $\beta_j$  are also called *Lagrange multipliers*.

# CONSTRAINED PROBLEMS: THE DIRECT WAY

Simple constrained problems can be solved in a direct way.

**Example 1:**

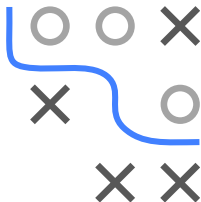
$$\begin{array}{ll}\min_{x \in \mathbb{R}} & 2 - x^2 \\ \text{s.t.} & x - 1 = 0\end{array}$$

**Solution:** Resolve the constraint by

$$\begin{aligned}x - 1 &= 0 \\ x &= 1\end{aligned}$$

and insert it into the objective:

$$x^* = 1, \quad f(x^*) = 1$$



# CONSTRAINED PROBLEMS: THE DIRECT WAY / 2

Example 2:

$$\begin{array}{ll}\min_{\mathbf{x} \in \mathbb{R}^2} & -2 + x_1^2 + 2x_2^2 \\ \text{s.t.} & x_1^2 + x_2^2 - 1 = 0\end{array}$$

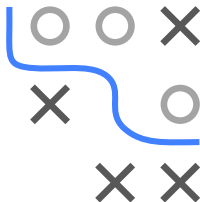
**Solution:** Resolve the constraint

$$x_1^2 = 1 - x_2^2$$

and insert it into the objective

$$\begin{aligned}f(x_1, x_2) &= -2 + (1 - x_2^2) + 2x_2^2 \\ &= -1 + x_2^2.\end{aligned}$$

$\Rightarrow$  Minimum at  $\mathbf{x}^* = (\pm 1, 0)^\top$ . However, direct way mostly not possible.



# A CLASSIC EXAMPLE: "MILKMAID PROBLEM"

**Question 1:** Is there a general recipe for solving general constrained nonlinear optimization problems?

**Question 2:** Can we understand this recipe geometrically?

**Question 3:** How does this relate to the Lagrange function approach?

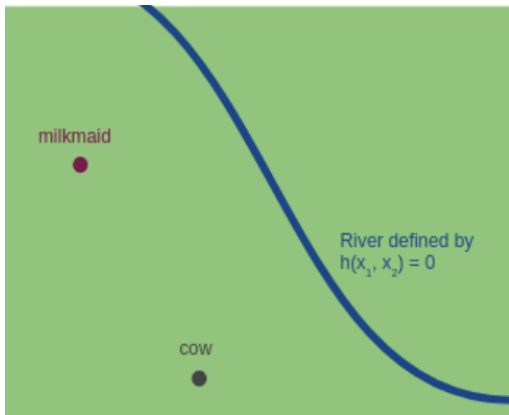
For this purpose, we consider the classical “milkmaid problem”; the following example is taken from *Steuard Jensen, An Introduction to Lagrange Multipliers* (but the example works of course equally well with a “milk man”).

- Assume a milk maid is sent to the field to get the day’s milk
- The milkmaid wants to finish her job as quickly as possible
- However, she has to clean her bucket first at the nearby river.



# A CLASSIC EXAMPLE: "MILKMAID PROBLEM" / 2

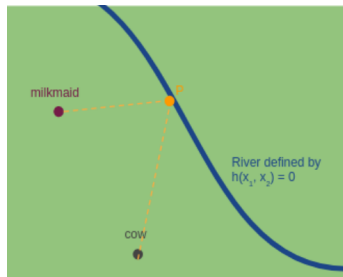
Where is the best point  $P$  to clean her bucket?



# A CLASSIC EXAMPLE: "MILKMAID PROBLEM" / 3

**Aim:** Find point  $P$  at the river for minimum total distance  $f(P)$

- $f(P) := d(M, P) + d(P, C)$  ( $d$  measures distance)
- $h(P) = 0$  describes the river

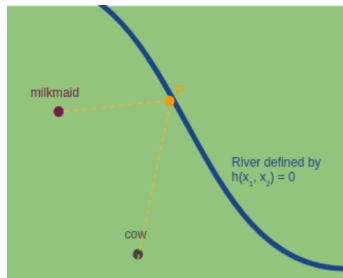




# A CLASSIC EXAMPLE: "MILKMAID PROBLEM" / 4

Corresponding optimization problem:

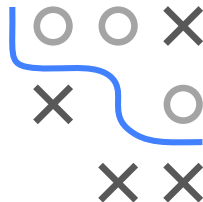
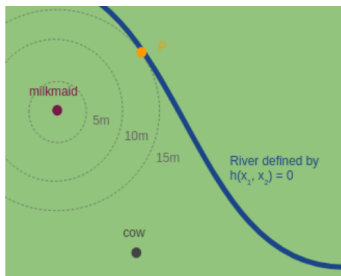
$$\begin{array}{ll}\min_{x_1, x_2} & f(x_1, x_2) \\ \text{s.t.} & h(x_1, x_2) = 0\end{array}$$



# A CLASSIC EXAMPLE: "MILKMAID PROBLEM" / 5

**Question:** How far can the milkmaid get for a fixed total distance  $f(P)$ ?

**Assume:** We only care about  $d(M, P)$ .

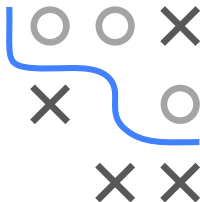
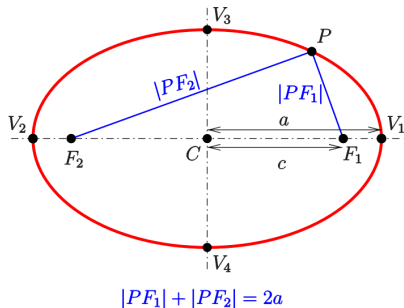


**Observe:** Radius of circle touching river first is the minimal distance.

# A CLASSIC EXAMPLE: "MILKMAID PROBLEM" / 6

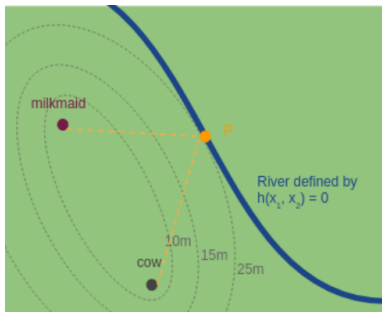
- For  $f(P) = d(M, P) + d(P, C)$ : Use an **ellipse**.
- **Def.:** Given two focal points  $F_1, F_2$  and distance  $2a$ :

$$E = \{P \in \mathbb{R}^2 \mid d(F_1, P) + d(P, F_2) = 2a\}$$



# A CLASSIC EXAMPLE: "MILKMAID PROBLEM" / 7

- Let  $M$  and  $C$  be focal points of a collection of ellipses
- Find **smallest** ellipse touching the river  $h(x_1, x_2)$
- Define  $P$  as the touching point



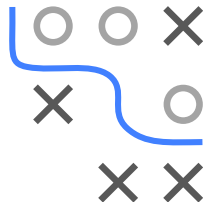
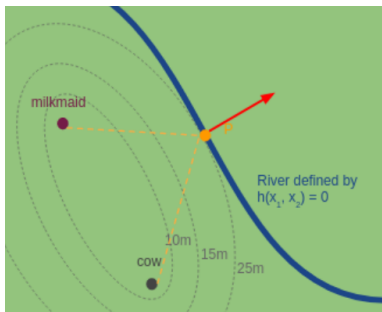
**Question:** How can we make sense of this mathematically?

# A CLASSIC EXAMPLE: "MILKMAID PROBLEM" / 8

- **Recall:** Gradient is normal (perpendicular) to contour lines
- Since contour lines of  $f$  and  $h$  touch, gradients are parallel:

$$\nabla f(P) = \beta \nabla h(P)$$

- Multiplier  $\beta$  is called **Lagrange multiplier**.



# LAGRANGE FUNCTION

**General:** Solve problem with single equality constraint by:

$$\nabla f(\mathbf{x}) = \beta \nabla h(\mathbf{x})$$

$$h(\mathbf{x}) = 0$$

- **First line:** Parallel gradients | **Second line:** Constraint

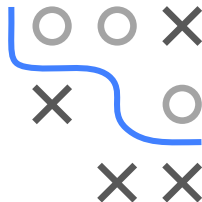
**Observe:** Above system is equivalent to

$$\nabla \mathcal{L}(\mathbf{x}, \beta) = \mathbf{0}$$

for **Lagrange function** (or **Lagrangian**)  $\mathcal{L}(\mathbf{x}, \beta) := f(\mathbf{x}) + \beta h(\mathbf{x})$

Indeed:

$$\begin{pmatrix} \nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \beta) \\ \nabla_{\beta} \mathcal{L}(\mathbf{x}, \beta) \end{pmatrix} = \begin{pmatrix} \nabla f(\mathbf{x}) + \beta \nabla h(\mathbf{x}) \\ h(\mathbf{x}) \end{pmatrix}$$

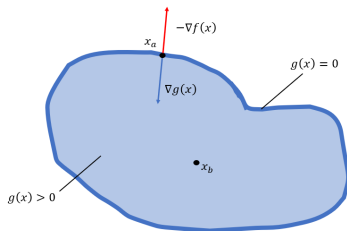


# LAGRANGE FUNCTION / 2

Idea can be extended to **inequality** constraints  $g(\mathbf{x}) \leq 0$ .

There are two possible cases for a solution:

- Solution  $\mathbf{x}_b$  inside feasible set: constraint is inactive ( $\alpha_b = 0$ )
- Solution  $\mathbf{x}_a$  on boundary  $g(\mathbf{x}) = 0$ :  $\nabla f(\mathbf{x}_a) = \alpha_a \nabla g(\mathbf{x}_a)$  ( $\alpha_a > 0$ )



# LAGRANGE FUNCTION AND PRIMAL PROBLEM

General constrained optimization problems:

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, k \\ & h_j(\mathbf{x}) = 0, \quad j = 1, \dots, \ell \end{aligned}$$



Extend Lagrangian ( $\alpha_i \geq 0$ ,  $\beta_i$  Lagrange multipliers):

$$\mathcal{L}(\mathbf{x}, \alpha, \beta) := f(\mathbf{x}) + \sum_{i=1}^k \alpha_i g_i(\mathbf{x}) + \sum_{j=1}^{\ell} \beta_j h_j(\mathbf{x})$$

**Equivalent** primal problem:

$$\min_{\mathbf{x}} \max_{\alpha \geq 0, \beta} \mathcal{L}(\mathbf{x}, \alpha, \beta)$$

**Question:** Why?



# LAGRANGE FUNCTION AND PRIMAL PROBLEM / 2

For simplicity: Consider only single inequality constraint  $g(\mathbf{x}) \leq 0$

If  $\mathbf{x}$  **breaks** inequality constraint ( $g(\mathbf{x}) > 0$ ):

$$\max_{\alpha \geq 0} \mathcal{L}(\mathbf{x}, \alpha) = \max_{\alpha \geq 0} f(\mathbf{x}) + \alpha g(\mathbf{x}) = \infty$$

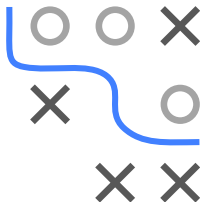
If  $\mathbf{x}$  **satisfies** inequality constraint ( $g(\mathbf{x}) \leq 0$ ):

$$\max_{\alpha \geq 0} \mathcal{L}(\mathbf{x}, \alpha) = \max_{\alpha \geq 0} f(\mathbf{x}) + \alpha g(\mathbf{x}) = f(\mathbf{x})$$

Combining yields **original formulation**:

$$\min_{\mathbf{x}} \max_{\alpha \geq 0} \mathcal{L}(\mathbf{x}, \alpha) = \begin{cases} \infty & \text{if } g(\mathbf{x}) > 0 \\ \min_{\mathbf{x}} f(\mathbf{x}) & \text{if } g(\mathbf{x}) \leq 0 \end{cases}$$

Similar argument holds for equality constraints  $h_j(\mathbf{x})$



## EXAMPLE: LAGRANGE FUNCTION FOR QP'S

We consider quadratic programming

$$\begin{array}{ll}\min_{\mathbf{x}} & f(\mathbf{x}) := \frac{1}{2} \mathbf{x}^\top \mathbf{Q} \mathbf{x} \\ \text{s.t.} & h(\mathbf{x}) := \mathbf{C} \mathbf{x} - \mathbf{d} = \mathbf{0}\end{array}$$

with  $\mathbf{Q} \in \mathbb{R}^{d \times d}$  symmetric,  $\mathbf{C} \in \mathbb{R}^{\ell \times d}$ , and  $\mathbf{d} \in \mathbb{R}^\ell$ .

Lagrange function:  $\mathcal{L}(\mathbf{x}, \boldsymbol{\beta}) = \frac{1}{2} \mathbf{x}^\top \mathbf{Q} \mathbf{x} + \boldsymbol{\beta}^\top (\mathbf{C} \mathbf{x} - \mathbf{d})$

Solve

$$\begin{aligned}\nabla \mathcal{L}(\mathbf{x}, \boldsymbol{\beta}) &= \begin{pmatrix} \partial \mathcal{L} / \partial \mathbf{x} \\ \partial \mathcal{L} / \partial \boldsymbol{\beta} \end{pmatrix} = \begin{pmatrix} \mathbf{Q} \mathbf{x} + \mathbf{C}^\top \boldsymbol{\beta} \\ \mathbf{C} \mathbf{x} - \mathbf{d} \end{pmatrix} = \mathbf{0} \\ \Leftrightarrow &\quad \begin{pmatrix} \mathbf{Q} & \mathbf{C}^\top \\ \mathbf{C} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\beta} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{d} \end{pmatrix}\end{aligned}$$

**Observe:** Solve QP by solving a linear system



# LAGRANGE DUALITY

Dual problem:

$$\max_{\alpha \geq 0, \beta} \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \alpha, \beta)$$

Define **Lagrange dual function**  $g(\alpha, \beta) := \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \alpha, \beta)$

Important characteristics of the dual problem:

- **Convexity** (pointwise minimum of *affine* functions)
  - Gives methods based on dual solutions
  - Might be computationally inefficient (expensive minimizations)

- **Weak duality:**

$$f(\mathbf{x}^*) \geq g(\alpha^*, \beta^*)$$

- **Strong duality** if primal problem satisfies *Slater's condition*<sup>(1)</sup>:

$$f(\mathbf{x}^*) = g(\alpha^*, \beta^*)$$

<sup>(1)</sup> **Slater's condition:** Primal problem convex and “strictly feasible” ( $\exists \mathbf{x} \forall i : g_i(\mathbf{x}) < 0$ ).

