Mathematical Concepts 3

Exercise 1: Optimality in 2 dimensions

Let $f : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto -\cos(x_1^2 + x_2^2 + x_1x_2)$

(a) Create a contour plot of f in the range $[-2, 2] \times [-2, 2]$ with `R`.

(b) Compute ∇f

(c) Compute $\nabla^2 f$

Now, we define the restriction of f to $S_r = \{(x_1, x_2) \in \mathbb{R}^2 | \ x_1^2 + x_2^2 + x_1x_2 < r\}$ with $r \in \mathbb{R}, r > 0$, i.e., $f_{|S_r} : S_r \to \mathbb{R}, (x_1, x_2) \mapsto f(x_1, x_2)$.

(d) Show that $f_{|S_{\overline{r}}}$ with $\overline{r} = \pi/4$ is convex.

(e) Find the local minimum $\mathbf{x}^*$ of $f_{|S_{\overline{r}}}$

(f) Is $\mathbf{x}^*$ a global minimum of f?

Exercise 2: Optimality in d dimensions

Let $\mathbf{X}$ be a d-dimensional random vector and let $\mathbf{Y}$ be a one-dimensional random vector with $\mathsf{Var}(\mathbf{X}) = \mathbf{\Sigma_X}$ and $\mathsf{Cov}(\mathbf{X}, \mathbf{Y}) = \mathbf{\Sigma_{X,Y}} \in \mathbb{R}^{d\times 1}$.
Further, let $f : \mathbb{R}^d \to \mathbb{R}, \mathbf{w} \mapsto \mathsf{Var}(\mathbf{w}^\top \mathbf{X} - \mathbf{Y})$.

(a) Show that f is convex.

(b) Compute ∇f and $\nabla^2 f$

(c) Under which condition exists a unique minimizer $\mathbf{w}^*$ of f. Is this a global minimum? (if it exists)

(d) Given the samples $(\mathbf{x}_i, y_i) \sim \mathbb{P}_{\mathbf{X},\mathbf{Y}}$, under which condition is the least squares estimator a consistent estimator of $\mathbf{w}^*$ in general?[1]

[1] This question is out of the scope of this lecture; however, it gives interesting insights into the entities we have computed.