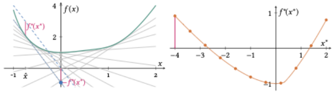
Optimization in Machine Learning

Constrained Optimization
Other forms of duality



(a) The function f (solid) and the linear function of $x^* = -4$ (dashed) and its shifted tangent (dotted).

(b) The Fenchel conjugate f^* of f.

Learning goals

- Dual norms
- Conjugate functions
- Fenchel duality
- Examples in statistics

CONSTRAINED MINIMIZATION AND DUAL NORMS

- Consider norm minimization under linear constraints in its primal form:

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} \|\boldsymbol{x}\| \quad \text{s.t. } \boldsymbol{G}\boldsymbol{x} = \boldsymbol{h}$$

- $\| \cdot \|$ is some norm function
- If the norm is the L_1 norm, this is the famous basis pursuit problem
  ▸ Click for source

- **Question:** Is there a more straightforward way to solve constrained optimization problems involving norms?

CONSTRAINED MINIMIZATION AND DUAL NORMS

- The concept of the **dual norm** from functional analysis can help

  Definition Let $\|\boldsymbol{x}\|$ be the norm of $\boldsymbol{x}$. The dual norm $\|\boldsymbol{x}\|_*$ is defined as

$$\|\boldsymbol{x}\|_* = \max_{\|\mathbf{z}\| \leq 1} \mathbf{z}^T \boldsymbol{x}$$

- If $\|\boldsymbol{x}\|$ is a norm and $\|\boldsymbol{x}\|_*$ its dual norm, then

$$|\mathbf{z}^T \boldsymbol{x}| \leq \|\mathbf{z}\| \|\boldsymbol{x}\|_*$$

Example The dual norm of the L_p norm $\| \cdot \|_p$ is the L_q norm $\| \cdot \|_q$ where $1/p + 1/q = 1$