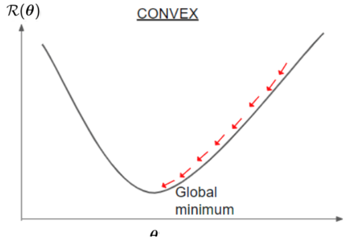
# Optimization in Machine Learning

## Deep dive
## Gradient descent and optimality

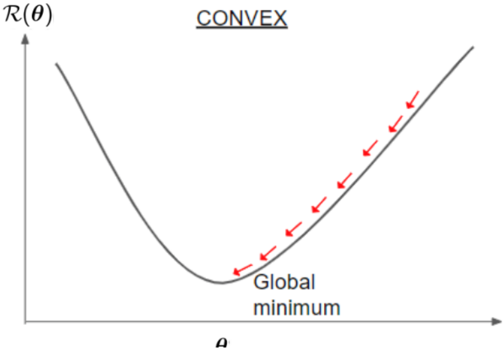



**Learning goals**

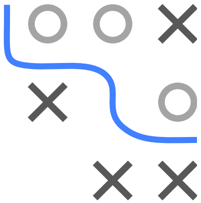
- Convergence of GD
- Proof strategy and tools
- Descent lemma

# SETTING

- GD is **greedy**: **locally optimal** moves in each iteration
- If $f$ is **convex**, **differentiable** and has a **Lipschitz gradient**, GD converges to global minimum for sufficiently small step sizes.



# SETTING



- **Assumptions:**
  - $f$ convex and differentiable
  - Global minimum $\mathbf{x}^*$ exists
  - $f$ has Lipschitz gradient ($\nabla f$ does not change too fast)

$$\|\nabla f(\mathbf{x}) - \nabla f(\tilde{\mathbf{x}})\| \leq L\|\mathbf{x} - \tilde{\mathbf{x}}\| \quad \text{for all } \mathbf{x}, \tilde{\mathbf{x}}$$

**Theorem** (Convergence of GD)**.** GD with step size $\alpha \leq 1/L$ yields

$$f(\mathbf{x}^{[k]}) - f(\mathbf{x}^*) \leq \frac{\|\mathbf{x}^{[0]} - \mathbf{x}^*\|^2}{2\alpha k}.$$

In other words: GD converges with rate $\mathcal{O}(1/k)$.

# PROOF STRATEGY

1. Show that $f(\mathbf{x}^{[t]})$ **strictly decreases** with each iteration $t$

**Descent lemma:**

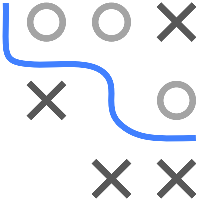
$$f(\mathbf{x}^{[t+1]}) \leq f(\mathbf{x}^{[t]}) - \frac{\alpha}{2}\|\nabla f(\mathbf{x}^{[t]})\|^2$$

2. Bound **error of one step**

$$f(\mathbf{x}^{[t+1]}) - f(\mathbf{x}^*) \leq \frac{1}{2\alpha}\left(\|\mathbf{x}^{[t]} - \mathbf{x}^*\|^2 - \|\mathbf{x}^{[t+1]} - \mathbf{x}^*\|^2\right)$$

3. Finalize by **telescoping** argument

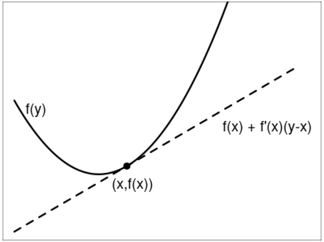
# MAIN TOOL



- **Recall:** First order condition of convexity

Every tangent line of $f$ is always below $f$.

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x})$$



# DESCENT LEMMA

- **Recall:** $\nabla f$ Lipschitz $\implies \nabla^2 f(\mathbf{x}) \preccurlyeq L \cdot \mathbf{I}$ for all $\mathbf{x}$
- This gives convexity of $g(\mathbf{x}) := \frac{L}{2}\|\mathbf{x}\|^2 - f(\mathbf{x})$ since

$$\nabla^2 g(\mathbf{x}) = L \cdot I - \nabla^2 f(\mathbf{x}) \succcurlyeq 0.$$

- First order condition of convexity of $g$ yields

$$\begin{aligned}
& & g(\mathbf{x}) &\geq g(\mathbf{x}^{[t]}) + \nabla g(\mathbf{x}^{[t]})^\top (\mathbf{x} - \mathbf{x}^{[t]}) \\
&\Leftrightarrow & \frac{L}{2}\|\mathbf{x}\|^2 - f(\mathbf{x}) &\geq \frac{L}{2}\|\mathbf{x}^{[t]}\|^2 - f(\mathbf{x}^{[t]}) + (L\mathbf{x}^{[t]} - \nabla f(\mathbf{x}^{[t]}))^\top (\mathbf{x} - \mathbf{x}^{[t]}) \\
&\Leftrightarrow & &\vdots \\
&\Leftrightarrow & f(\mathbf{x}) &\leq f(\mathbf{x}^{[t]}) + \nabla f(\mathbf{x}^{[t]})^\top (\mathbf{x} - \mathbf{x}^{[t]}) + \frac{L}{2}\|\mathbf{x} - \mathbf{x}^{[t]}\|^2
\end{aligned}$$

- **Now:** One GD step with step size $\alpha \leq 1/L$:

$$\mathbf{x} \leftarrow \mathbf{x}^{[t+1]} = \mathbf{x}^{[t]} - \alpha \nabla f\left(\mathbf{x}^{[t]}\right)$$

# DESCENT LEMMA

$$
\begin{aligned}
f(\mathbf{x}^{[t+1]}) &\leq f(\mathbf{x}^{[t]}) + \nabla f(\mathbf{x}^{[t]})^\top (\mathbf{x}^{[t+1]} - \mathbf{x}^{[t]}) + \frac{L}{2}\|\mathbf{x}^{[t+1]} - \mathbf{x}^{[t]}\|^2 \\
&= f(\mathbf{x}^{[t]}) + \nabla f(\mathbf{x}^{[t]})^\top (\mathbf{x}^{[t]} - \alpha \nabla f(\mathbf{x}^{[t]}) - \mathbf{x}^{[t]}) \\
&\quad + \frac{L}{2}\|\mathbf{x}^{[t]} - \alpha \nabla f(\mathbf{x}^{[t]}) - \mathbf{x}^{[t]}\|^2 \\
&= f(\mathbf{x}^{[t]}) - \nabla f(\mathbf{x}^{[t]})^\top \alpha \nabla f(\mathbf{x}^{[t]}) + \frac{L}{2}\|\alpha \nabla f(\mathbf{x}^{[t]})\|^2 \\
&= f(\mathbf{x}^{[t]}) - \alpha \|\nabla f(\mathbf{x}^{[t]})\|^2 + \frac{L\alpha^2}{2}\|\nabla f(\mathbf{x}^{[t]})\|^2 \\
&\leq f(\mathbf{x}^{[t]}) - \frac{\alpha}{2}\|\nabla f(\mathbf{x}^{[t]})\|^2
\end{aligned}
$$

- **Note:** $\alpha \leq 1/L$ yields $L\alpha^2 \leq \alpha$
  - $\|\nabla f(\mathbf{x}^{[t]})\|^2 > 0$ unless $\nabla f(\mathbf{x}) = \mathbf{0}$
  - $f$ **strictly decreases** with each GD iteration until optimum reached
  - Descent lemma yields bound on **guaranteed progress** if $\alpha \leq 1/L$ (explains why GD may diverge if step sizes too large)

# ONE STEP ERROR BOUND

- Again, first order condition of convexity gives

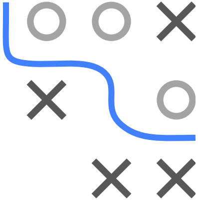
$$f(\mathbf{x}^{[t]}) - f(\mathbf{x}^*) \leq \nabla f(\mathbf{x}^{[t]})^\top (\mathbf{x}^{[t]} - \mathbf{x}^*).$$

- This and the descent lemma yields

$$\begin{aligned}
f(\mathbf{x}^{[t+1]}) - f(\mathbf{x}^*) &\leq f(\mathbf{x}^{[t]}) - \frac{\alpha}{2}\|\nabla f(\mathbf{x}^{[t]})\|^2 - f(\mathbf{x}^*) \\
&= f(\mathbf{x}^{[t]}) - f(\mathbf{x}^*) - \frac{\alpha}{2}\|\nabla f(\mathbf{x}^{[t]})\|^2 \\
&\leq \nabla f(\mathbf{x}^{[t]})^\top (\mathbf{x}^{[t]} - \mathbf{x}^*) - \frac{\alpha}{2}\|\nabla f(\mathbf{x}^{[t]})\|^2 \\
&= \frac{1}{2\alpha}\left(\|\mathbf{x}^{[t]} - \mathbf{x}^*\|^2 - \|\mathbf{x}^{[t]} - \mathbf{x}^* - \alpha\nabla f(\mathbf{x}^{[t]})\|^2\right) \\
&= \frac{1}{2\alpha}\left(\|\mathbf{x}^{[t]} - \mathbf{x}^*\|^2 - \|\mathbf{x}^{[t+1]} - \mathbf{x}^*\|^2\right)
\end{aligned}$$

- **Note:** Line 3 $\rightarrow$ 4 is hard to see (just expand line 4).

# FINALIZATION



- Summing over iterations yields

$$
\begin{aligned}
k(f(\mathbf{x}^{[k]}) - f(\mathbf{x}^*)) &\leq \sum_{t=1}^{k} [f(\mathbf{x}^{[t]}) - f(\mathbf{x}^*)] \\
&\leq \sum_{t=1}^{k} \frac{1}{2\alpha} \left[ \|\mathbf{x}^{[t-1]} - \mathbf{x}^*\|^2 - \|\mathbf{x}^{[t]} - \mathbf{x}^*\|^2 \right] \\
&= \frac{1}{2\alpha} \left( \|\mathbf{x}^{[0]} - \mathbf{x}^*\|^2 - \|\mathbf{x}^{[k]} - \mathbf{x}^*\|^2 \right) \\
&\leq \frac{1}{2\alpha} \left( \|\mathbf{x}^{[0]} - \mathbf{x}^*\|^2 \right).
\end{aligned}
$$

- **Arguments:** Descent lemma (line 1).
  Telescoping sum (line 2 $\rightarrow$ 3).

$$
\boxed{f(\mathbf{x}^{[t+1]}) - f(\mathbf{x}^*) \leq \frac{\|\mathbf{x}^{[0]} - \mathbf{x}^*\|^2}{2\alpha k}}
$$