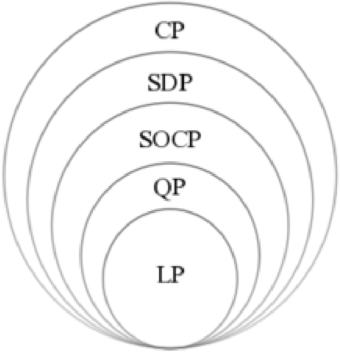
Optimization in Machine Learning

Constrained Optimization
Algorithms for linear programs



Learning goals

- Understand the Simplex algorithm for solving LPs
- Know the two-phase approach for finding a starting point
- Understand how Simplex traverses polytope corners

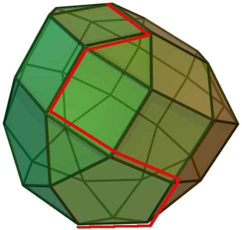
SIMPLEX ALGORITHM

- Most important method for solving linear programs
- Published in 1947 by Georg Dantzig

  Basic idea: Start from an arbitrary corner of the polytope.
  Run along edges as long as the solution improves. Find a new edge, ...

  Output: A path along corners of the polytope ending at the optimum

- Since LP is a **convex** optimization problem, the optimal corner found is also a global optimum



SIMPLEX ALGORITHM

- The Simplex algorithm consists of two phases:
  - **Phase I:** Determination of a **starting point**
  - **Phase II:** Determination of the **optimal solution**
- In **Phase I**, a feasible corner $\boldsymbol{x}_0$ must be found first
- In **Phase II**, this solution is iteratively improved by searching for an edge that improves the objective and running along it to the next corner

SIMPLEX – PHASE I

- Find starting point $\boldsymbol{x}_0$ by solving auxiliary LP with artificial variables $\boldsymbol{\epsilon}$:

$$\min_{\epsilon_1,\ldots,\epsilon_m} \sum_{i=1}^{m} \epsilon_i \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\epsilon} \geq \boldsymbol{b},\ \epsilon_1, \ldots, \epsilon_m \geq 0,\ \boldsymbol{x} \geq 0$$

- A feasible starting point for the auxiliary problem is $\boldsymbol{x} = \mathbf{0}$ and
$$\epsilon_i = \begin{cases} 0 & \text{if } b_i < 0 \\ b_i & \text{if } b_i \geq 0 \end{cases}$$
- We then apply Phase II of Simplex to the auxiliary problem
- If the original problem is feasible, the optimal solution **must** be $\boldsymbol{\epsilon} = (0, \ldots, 0)$ (all artificial variables disappear)
- If we find $\boldsymbol{\epsilon} = \mathbf{0}$, we have a valid starting point
- Otherwise, the original problem has no feasible solution

SIMPLEX – EXAMPLE

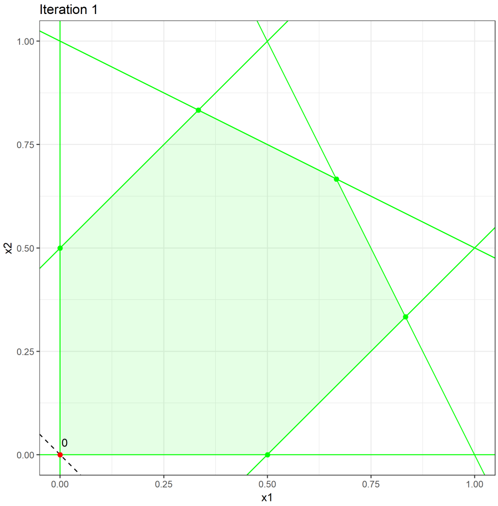
- Consider the following LP:

$$\min_{\boldsymbol{x}\in\mathbb{R}^2} -x_1 - x_2$$

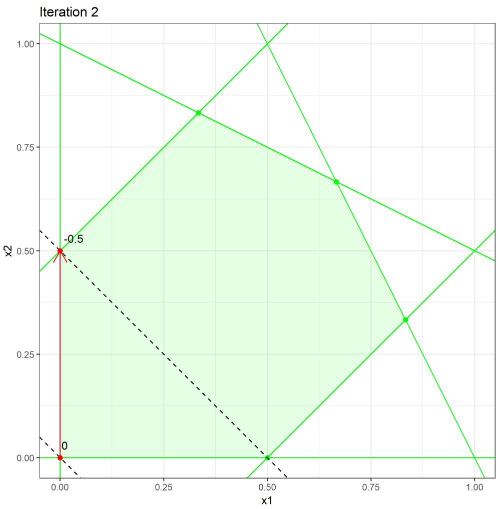
s.t. $x_1 - x_2 \geq -0.5,\ -x_1 - 2x_2 \geq -2,\ -2x_1 - x_2 \geq -2,\ -x_1 + x_2 \geq -0.5,\ \boldsymbol{x} \geq 0$

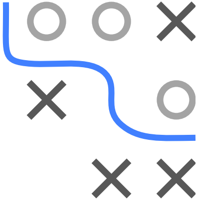
- Starting point is the corner $(\mathbf{0}, \mathbf{0})$

SIMPLEX – EXAMPLE

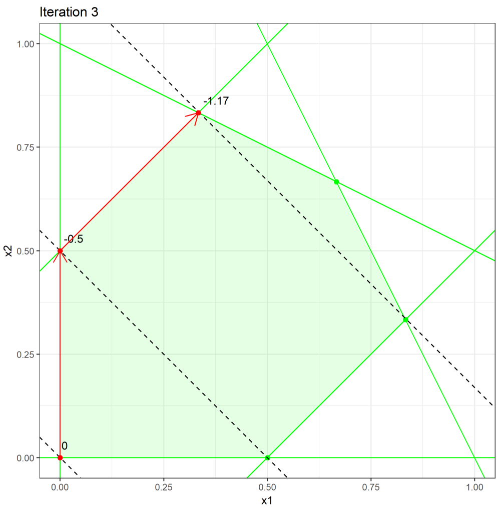


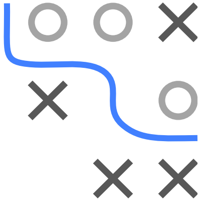
SIMPLEX – EXAMPLE





SIMPLEX – EXAMPLE





SIMPLEX – EXAMPLE