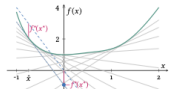


# Optimization in Machine Learning

## Other forms of duality



(a) The function  $f$  (solid) and the linear function of  $x^* = -4$  (dashed) and its shifted tangent (dotted).



(b) The Fenchel conjugate  $f^*$  of  $f$ .

### Learning goals

- Dual norms
- Conjugate functions
- Fenchel duality
- Examples in statistics

# CONSTRAINED MINIMIZATION AND DUAL NORMS

Consider the problem of norm minimization under linear constraints in its primal form:

$$\begin{array}{ll} \min_{\mathbf{x} \in \mathbb{R}^d} & \|\mathbf{x}\| \\ \text{s.t.} & \mathbf{G}\mathbf{x} = \mathbf{h}, \end{array}$$

where  $\|\cdot\|$  is some norm function. For instance, if the norm is the  $L1$  norm, this problem is the famous [basis pursuit](#) problem.

**Question:** Is there a more straightforward way to solve constrained optimization problems involving norms?



# CONSTRAINED MINIMIZATION AND DUAL NORMS

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Here, the concept of the dual norm from functional analysis can be helpful.

**Definition:** Let  $\|\mathbf{x}\|$  be the norm of  $\mathbf{x}$ . Then the dual norm  $\|\mathbf{x}\|_*$  is defined as

$$\|\mathbf{x}\|_* = \max_{\|\mathbf{z}\| \leq 1} \mathbf{z}^T \mathbf{x}$$

Using this definition, one can show that if  $\|\mathbf{x}\|$  is a norm and  $\|\mathbf{x}\|_*$  is the dual norm of it,  $\|\mathbf{z}^T \mathbf{x}\| \leq \|\mathbf{z}\| \|\mathbf{x}\|_*$  holds.

**Examples:** The dual norm of the  $L_p$  norm  $\|\cdot\|_p$  is the  $L_q$  norm  $\|\cdot\|_q$  where it holds that  $1/p + 1/q = 1$ .



# CONSTRAINED PROBLEMS AND CONJUGATE FUNCTIONS

