Mathematical Concepts 1

Exercise 1: Gradient

Consider the bivariate function $f : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto x_1^2 + 0.5x_2^2 + x_1x_2$.

(a) Show that f is smooth (as defined in the lecture).

(b) Find the direction of greatest increase of f at $\mathbf{x} = (1, 1)$.

(c) Find the direction of greatest decrease of f at $\mathbf{x} = (1, 1)$.

(d) Find a direction in which f does not instantly change at $\mathbf{x} = (1, 1)$.

(e) Assume there exists a differentiable parametrization of a curve $\tilde{\mathbf{x}} : \mathbb{R} \to \mathbb{R}^2, t \mapsto \tilde{\mathbf{x}}(t)$ such that $\forall t \in \mathbb{R}$: $f(\tilde{\mathbf{x}}(t)) = f(1, 1)$. Show that at each point of the curve $\tilde{\mathbf{x}}$ the tangent line $\frac{\partial \tilde{\mathbf{x}}}{\partial t}$ is perpendicular to the gradient $\nabla f(\tilde{\mathbf{x}})$.

(f) Interpret (d), (e) geometrically

Exercise 2: Convexity

Consider two convex functions $f, g : \mathbb{R} \to \mathbb{R}$.

(a) Show that $f + g : \mathbb{R} \to \mathbb{R}, x \mapsto f(x) + g(x)$ is convex.

(b) Now, assume that g is additionally non-decreasing, i.e., $g(y) \geq g(x)\ \forall x \in \mathbb{R}, \forall y \in \mathbb{R}$ with $y > x$. Show that $g \circ f$ is convex.

Exercise 3: Taylor polynomials

Consider the bivariate function $f : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto \exp(\pi \cdot x_1) - \sin(\pi \cdot x_2) + \pi \cdot x_1 \cdot x_2$

(a) Compute the gradient of f for an arbitrary $\mathbf{x}$.

(b) Compute the Hessian of f for an arbitrary $\mathbf{x}$.

(c) State the first order taylor polynomial $T_{1,\mathbf{a}}(\mathbf{x})$ expanded around the point $\mathbf{a} = (0, 1)$.

(d) State the second order taylor polynomial $T_{2,\mathbf{a}}(\mathbf{x})$ expanded around the point $\mathbf{a} = (0, 1)$.

(e) Determine if $T_{2,\mathbf{a}}$ is a convex function.