Univariate Optimization 1

Exercise 1: Golden Ratio, Brent's Method

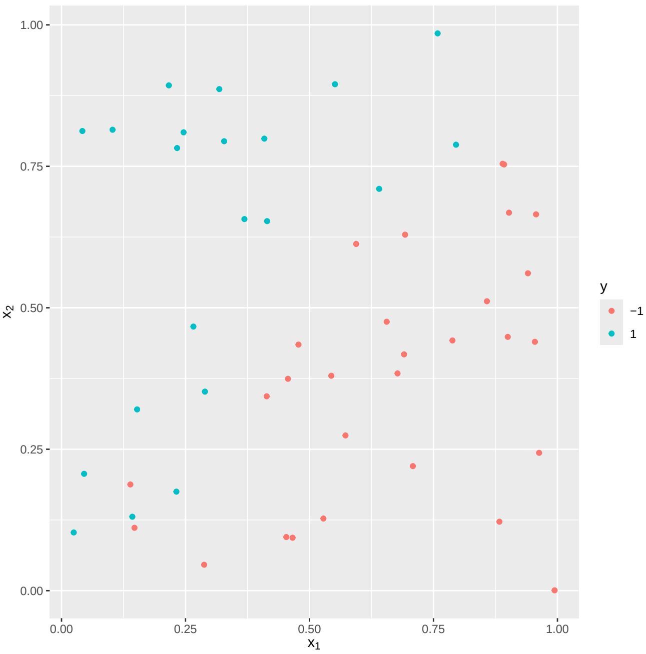
You are given the following data situation:

```
library(ggplot2)

set.seed(123)

X = matrix(runif(100), ncol = 2)
y = -((X %*% c(-1, 1) + rnorm(50, 0, 0.1) < 0) * 2 - 1)
df = as.data.frame(X)
df$type = as.character(y)

ggplot(df) +
  geom_point(aes(x = V1, y = V2, color=type)) +
  xlab(expression(x[1])) +
  ylab(expression(x[2])) +
  labs(color="y")
```



In the following we want to estimate a linear SVM without intercept and with $\lambda = 1$. We assume we know that $\boldsymbol{\theta}_2 = 2$.

(a) Show that if $f : \mathbb{R}^2 \to \mathbb{R}$ is convex then $g_c : \mathbb{R} \to \mathbb{R}, x \mapsto f(x, c) \quad \forall c \in \mathbb{R}$ is convex.

(b) Explain why the non-geometric primal linear SVM formulation should be used rather than the geometric one if we want to find $\boldsymbol{\theta}_1$ via the golden ratio algorithm[1].

(c) Find $\boldsymbol{\theta}_1$ via the golden ratio algorithm. Implement the algorithm in `R`. For the termination criterion, use an absolute error of 0.01. Use $[-3, 3]$ as the starting interval.

[1] We choose this algorithm for educational purposes; in practice, we typically use more advanced algorithms.

(d) Given the three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ show that the parameters $a, b, c \in \mathbb{R}$ of the interpolating parabola can be found via $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ when the parabola equation is given by $p(x) = ax^2 + bx + c$.

(e) Find $\boldsymbol{\theta}_1$ via Brent's method[2]. Implement a simplified version[3] of the algorithm in `R`. For the termination criterion, use an absolute error of 0.01. Use $[-3, 3]$ as the starting interval. For the first step, use a golden ratio step.

(f) Now, assume we do not know $\boldsymbol{\theta}_2$. Our initial guess is $\boldsymbol{\theta}_2 = 0$. We now alternately minimize w.r.t. either $\boldsymbol{\theta}_1$ or $\boldsymbol{\theta}_2$ via the golden ratio method (the starting interval is always reset to $[-3, 3]$) while the other parameter is held constant. We switch to minimizing the other parameter when the absolute error is smaller than 0.01. Repeat this procedure 10 times.

(g) How does the optimization trace of f) look in parameter space?

[2] We choose this algorithm for educational purposes; in practice, we typically use more advanced algorithms.
[3] Only check if the proposed point is in the current interval