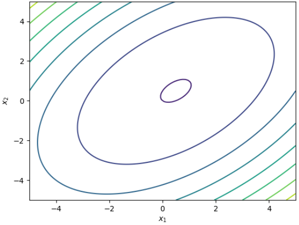
Optimization in Machine Learning

First order methods
GD on quadratic forms



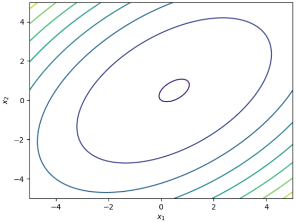
Learning goals

- Eigendecomposition of quadratic forms
- GD steps in eigenspace

QUADRATIC FORMS & GD

- We consider the quadratic function $q(\mathbf{x}) = \mathbf{x}^\top \mathbf{A}\mathbf{x} - \mathbf{b}^\top \mathbf{x}$.
- We assume that Hessian $\mathbf{H} = 2\mathbf{A}$ has full rank
- Optimal solution is $\mathbf{x}^* = \frac{1}{2}\mathbf{A}^{-1}\mathbf{b}$
- As $\nabla q(\mathbf{x}) = 2\mathbf{A}\mathbf{x} - \mathbf{b}$, iterations of gradient descent are

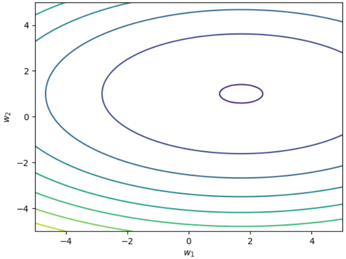
$$\mathbf{x}^{[t+1]} = \mathbf{x}^{[t]} - \alpha(2\mathbf{A}\mathbf{x}^{[t]} - \mathbf{b})$$



The following slides follow the blog post "Why Momentum Really Works", Distill, 2017.
`http://doi.org/10.23915/distill.00006`

EIGENDECOMPOSITION OF QUADRATIC FORMS

- We want to work in the coordinate system given by q
- **Recall:** Coordinate system is given by the eigenvectors of $\mathbf{H} = 2\mathbf{A}$
- Eigendecomposition of $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top$
- $\mathbf{V}$ contains eigenvectors $\mathbf{v}_i$ and $\mathbf{\Lambda} = \text{diag}(\lambda_1, ..., \lambda_n)$ eigenvalues
- Change of basis: $\mathbf{w}^{[t]} = \mathbf{V}^\top(\mathbf{x}^{[t]} - \mathbf{x}^*)$



GD STEPS IN EIGENSPACE

With $\mathbf{w}^{[t]} = \mathbf{V}^\top(\mathbf{x}^{[t]} - \mathbf{x}^*)$, a single GD step

$$\mathbf{x}^{[t+1]} = \mathbf{x}^{[t]} - \alpha(2\mathbf{A}\mathbf{x}^{[t]} - \mathbf{b})$$

becomes

$$\mathbf{w}^{[t+1]} = \mathbf{w}^{[t]} - 2\alpha\boldsymbol{\Lambda}\mathbf{w}^{[t]}.$$

Therefore:

$$\begin{aligned} w_i^{[t+1]} &= w_i^{[t]} - 2\alpha\lambda_i w_i^{[t]} \\ &= (1 - 2\alpha\lambda_i)w_i^{[t]} \\ &= \cdots \\ &= (1 - 2\alpha\lambda_i)^{t+1} w_i^{[0]} \end{aligned}$$

GD STEPS IN EIGENSPACE / 2

Proof (for $\mathbf{w}^{[t+1]} = \mathbf{w}^{[t]} - 2\alpha\boldsymbol{\Lambda}\mathbf{w}^{[t]}$):

- A single GD step means

$$\mathbf{x}^{[t+1]} = \mathbf{x}^{[t]} - \alpha(2\mathbf{A}\mathbf{x}^{[t]} - \mathbf{b})$$

- Then:

$$\begin{aligned}
\mathbf{V}^\top(\mathbf{x}^{[t+1]} - \mathbf{x}^*) &= \mathbf{V}^\top(\mathbf{x}^{[t]} - \mathbf{x}^*) - \alpha\mathbf{V}^\top(2\mathbf{A}\mathbf{x}^{[t]} - \mathbf{b}) \\
\mathbf{w}^{[t+1]} &= \mathbf{w}^{[t]} - \alpha\mathbf{V}^\top(2\mathbf{A}\mathbf{x}^{[t]} - \mathbf{b}) \\
\mathbf{w}^{[t+1]} &= \mathbf{w}^{[t]} - \alpha\mathbf{V}^\top(2\mathbf{A}(\mathbf{x}^{[t]} - \mathbf{x}^*) + \underbrace{2\mathbf{A}\mathbf{x}^* - \mathbf{b}}_{=0}) \\
&= \mathbf{w}^{[t]} - 2\alpha\boldsymbol{\Lambda}\mathbf{V}^\top(\mathbf{x}^{[t]} - \mathbf{x}^*) \\
&= \mathbf{w}^{[t]} - 2\alpha\boldsymbol{\Lambda}\mathbf{w}^{[t]}
\end{aligned}$$

GD ERROR IN ORIGINAL SPACE

- Move back to **original space**:

$$\mathbf{x}^{[t]} - \mathbf{x}^* = \mathbf{V}\mathbf{w}^{[t]} = \sum_{i=1}^{d} (1 - 2\alpha\lambda_i)^t w_i^{[0]} \mathbf{v}_i$$

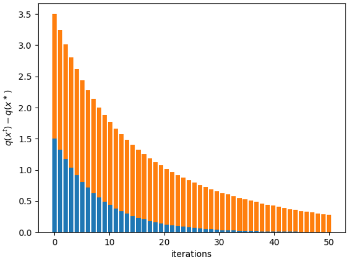
- **Intuition:** Initial error components $w_i^{[0]}$ (in the eigenbasis) decay with rate $1 - 2\alpha\lambda_i$

- **Therefore:** For sufficiently small step sizes α, error components along eigenvectors with large eigenvalues decay quickly

GD ERROR IN ORIGINAL SPACE / 2

We now consider the contribution of each eigenvector to the total loss

$$q(\mathbf{x}^{[t]}) - q(\mathbf{x}^*) = \frac{1}{2}\sum_i^d (1-2\alpha\lambda_i)^{2t}\lambda_i(w_i^{[0]})^2$$


$q(x^t) - q(x^*)$
iterations