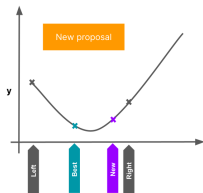


## Univariate optimization

### Golden ratio

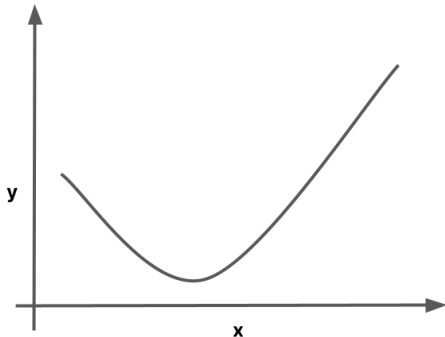


- Simple nesting procedure
- Golden ratio

# UNIVARIATE OPTIMIZATION

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

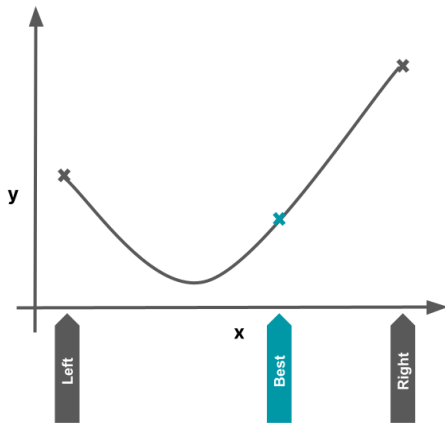
**Goal:** Iteratively improve eval points. Assume function is unimodal. Will not rely on gradients, so this also works for black-box problems.



## SIMPLE NESTING PROCEDURE

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

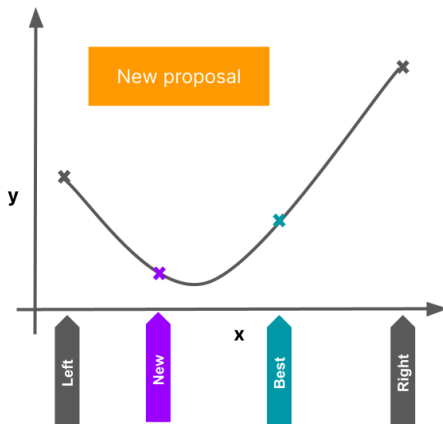
Always maintain three points: left, right, and current best.



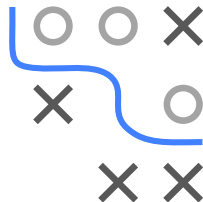
# SIMPLE NESTING PROCEDURE

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

Propose random point in interval.



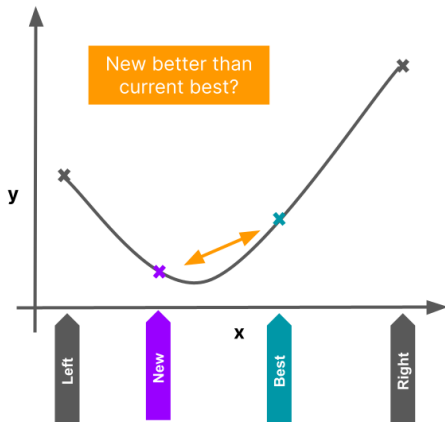
NB: Later we will define the optimal choice for a new proposal.



# SIMPLE NESTING PROCEDURE

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

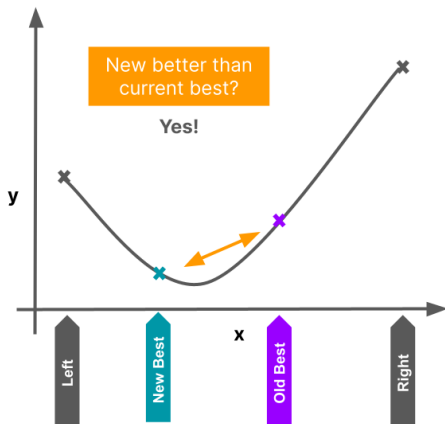
Compare proposal against current best.



# SIMPLE NESTING PROCEDURE

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

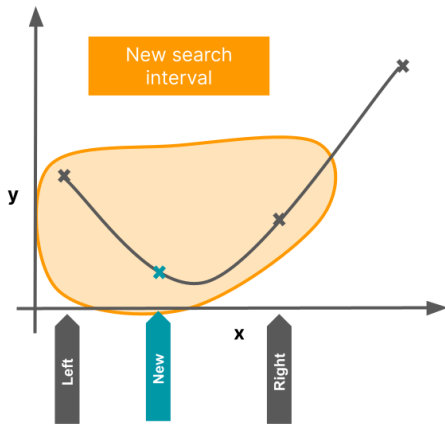
If it is better: proposal becomes current best.



# SIMPLE NESTING PROCEDURE

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

New search interval: around current best.



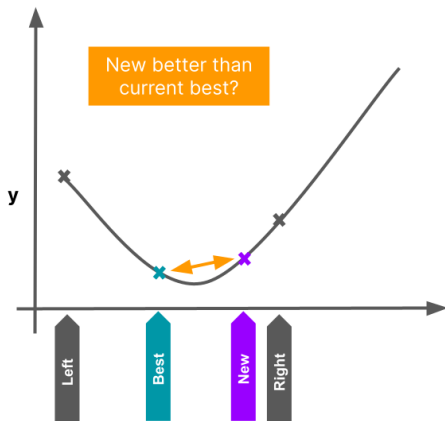




# SIMPLE NESTING PROCEDURE

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

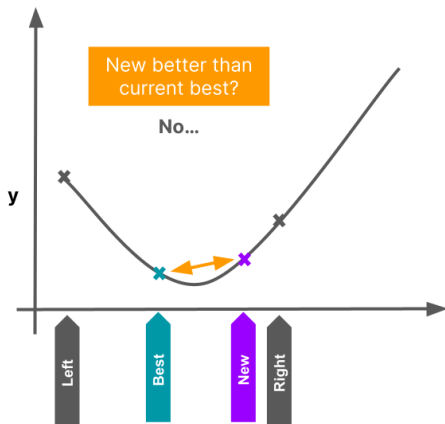
Compare proposal against current best.



# SIMPLE NESTING PROCEDURE

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

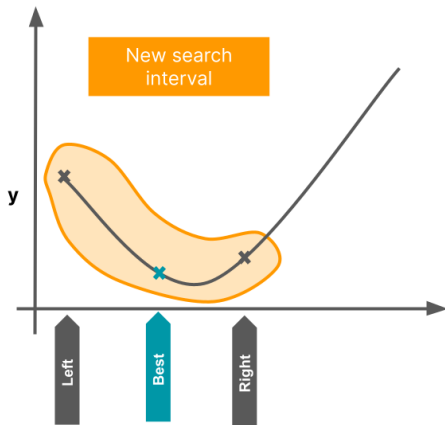
If it is better: proposal becomes current best.



# SIMPLE NESTING PROCEDURE

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

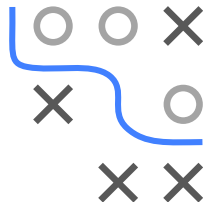
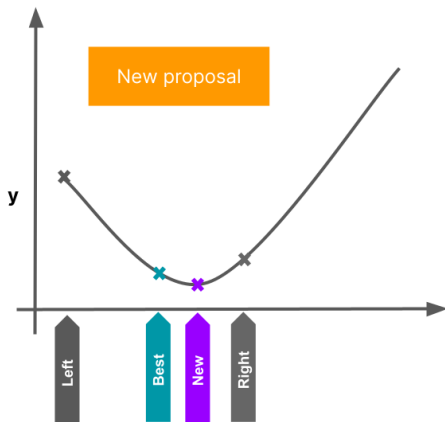
New search interval: around current best.



# SIMPLE NESTING PROCEDURE

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

Propose a random point.



## SIMPLE NESTING PROCEDURE

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

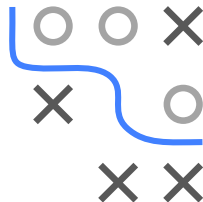
Compare proposal against current best.



# SIMPLE NESTING PROCEDURE

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

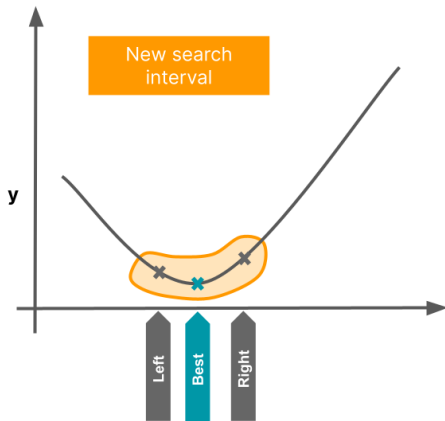
If it is better: proposal becomes current best.



# SIMPLE NESTING PROCEDURE

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

New search interval: around current best.



# SIMPLE NESTING PROCEDURE

- **Initialization:** Search interval  $(x^{\text{left}}, x^{\text{right}})$ ,  $x^{\text{left}} < x^{\text{right}}$
- Choose  $x^{\text{best}}$  randomly.
- For  $t = 0, 1, 2, \dots$ 
  - Choose  $x^{\text{new}}$  randomly in  $[x^{\text{left}}, x^{\text{right}}]$
  - If  $f(x^{\text{new}}) < f(x^{\text{best}})$ :
    - $x^{\text{best}} \leftarrow x^{\text{new}}$
  - New interval: Points around  $x^{\text{best}}$

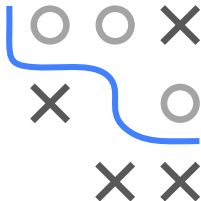
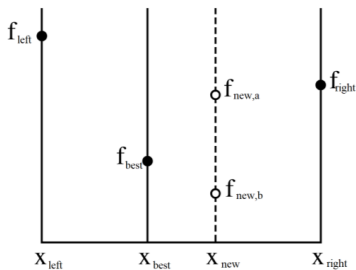




# GOLDEN RATIO

**Key question:** How can  $x^{\text{new}}$  be chosen better than randomly?

- **Insight 1:** Always in bigger subinterval to maximize reduction.
- **Insight 2:**  $x^{\text{new}}$  symmetrically to  $x^{\text{best}}$  for uniform reduction.

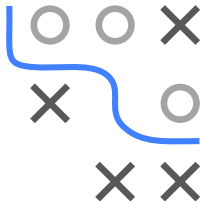
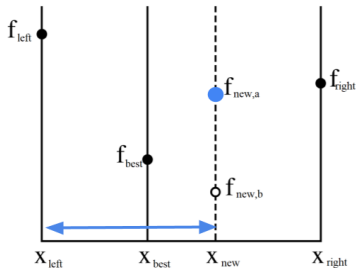


Consider two hypothetical outcomes  $x^{\text{new}}$ :  $f_{\text{new},a}$  and  $f_{\text{new},b}$ .

## GOLDEN RATIO / 2

If  $f_{new,a}$  is the outcome,  $x_{best}$  stays best and we search around  $x_{best}$  :

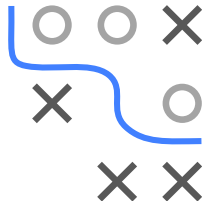
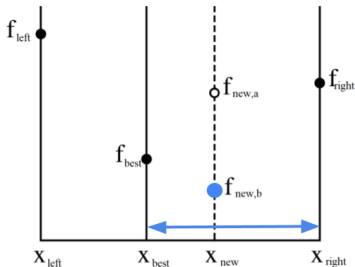
$[X_{left}, X_{new}]$



## GOLDEN RATIO / 3

If  $f_{new,b}$  is outcome,  $x_{new}$  becomes best point and search around  $x_{new}$  :

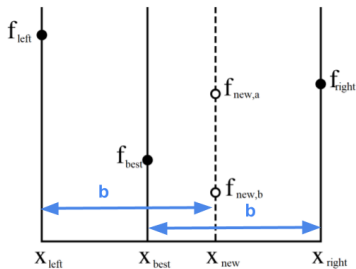
$[X_{best}, X_{right}]$



# GOLDEN RATIO / 4

For uniform reduction, require the two potential intervals equal sized:

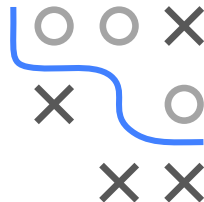
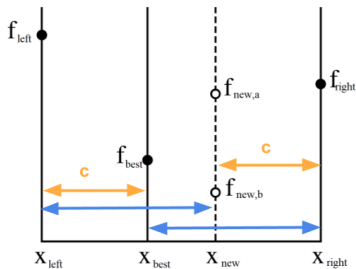
$$b := x_{\text{right}} - x_{\text{best}} = x_{\text{new}} - x_{\text{left}}$$



# GOLDEN RATIO / 5

One iteration ahead: require again the intervals to be of same size.

$$C := x_{\text{best}} - x_{\text{left}} = x_{\text{right}} - x_{\text{new}}$$



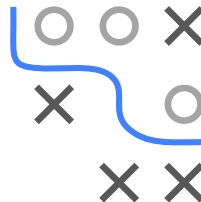
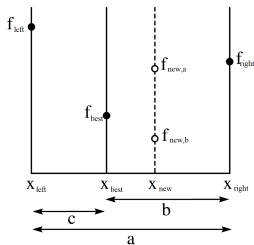
# GOLDEN RATIO / 6

To summarize, we require:

$$a = x^{right} - x^{left},$$

$$b = x_{right} - x_{best} = x_{new} - x_{left}$$

$$c = x_{best} - x_{left} = x_{right} - x_{new}$$



# GOLDEN RATIO / 7

- We require the same percentage improvement in each iteration
- For  $\varphi$  reduction factor of interval sizes ( $a$  to  $b$ , and  $b$  to  $c$ )

$$\varphi := \frac{b}{a} = \frac{c}{b}$$

$$\varphi^2 = \frac{b}{a} \cdot \frac{c}{b} = \frac{c}{a}$$

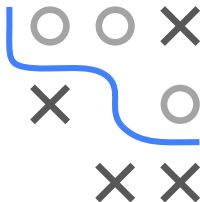
- Divide  $a = b + c$  by  $a$ :

$$\frac{a}{a} = \frac{b}{a} + \frac{c}{a}$$

$$1 = \varphi + \varphi^2$$

$$0 = \varphi^2 + \varphi - 1$$

- Unique positive solution is  $\varphi = \frac{\sqrt{5}-1}{2} \approx 0.618$ .



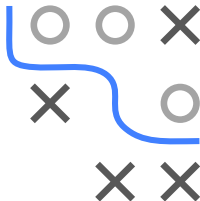
# GOLDEN RATIO / 8

- With  $x^{\text{new}}$  we always go  $\varphi$  percentage points into the interval.
- Given  $x^{\text{left}}$  and  $x^{\text{right}}$  it follows

$$\begin{aligned}x^{\text{best}} &= x^{\text{right}} - \varphi(x^{\text{right}} - x^{\text{left}}) \\ &= x^{\text{left}} + (1 - \varphi)(x^{\text{right}} - x^{\text{left}})\end{aligned}$$

and due to symmetry

$$\begin{aligned}x^{\text{new}} &= x^{\text{left}} + \varphi(x^{\text{right}} - x^{\text{left}}) \\ &= x^{\text{right}} - (1 - \varphi)(x^{\text{right}} - x^{\text{left}}).\end{aligned}$$





# GOLDEN RATIO / 9

Termination criterion:

- A reasonable choice is the absolute error, i.e. the width of the last interval:

$$|x^{best} - x^{new}| < \tau$$

- In practice, more complicated termination criteria are usually applied, for example in *Numerical Recipes in C, 2017*

$$|x^{right} - x^{left}| \leq \tau(|x^{best}| + |x^{new}|)$$

is proposed as a termination criterion.

