Multi-Criteria Optimization

Exercise 1: Concepts in Multi-Criteria Optimization

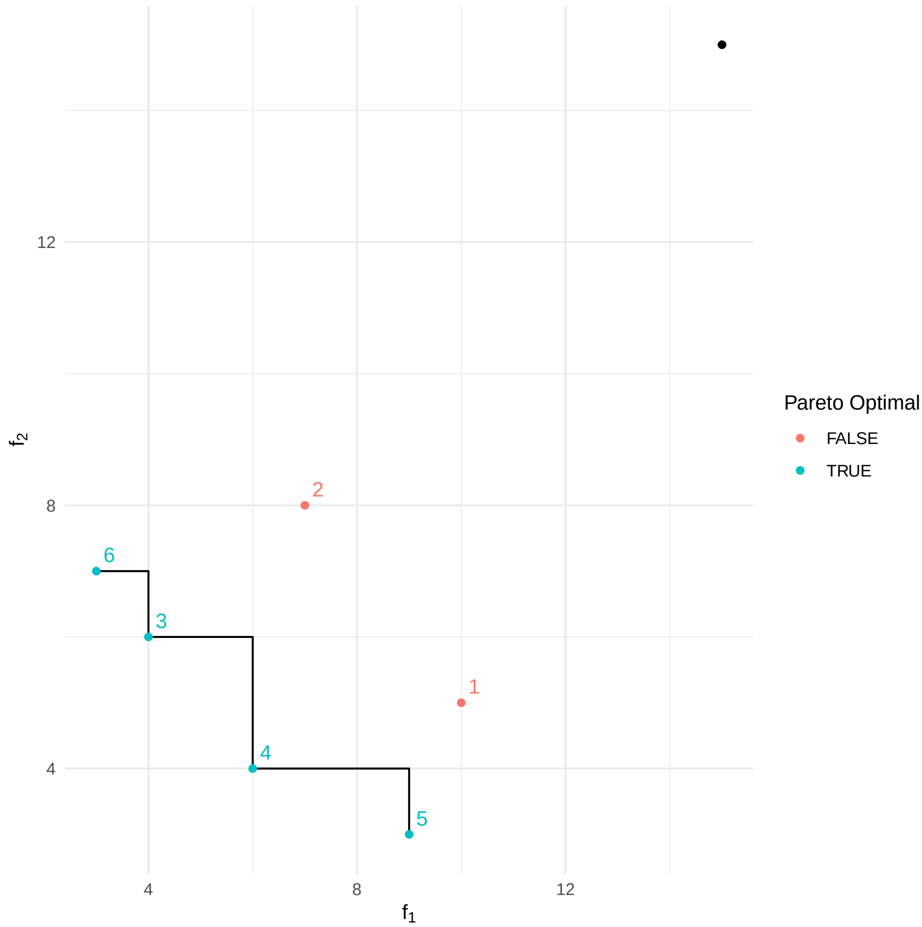
(a)
- $\mathbf{x}^{(1)}$ with $\mathbf{f}^{(1)} = (10, 5)$ e.g., dominated by $\mathbf{x}^{(4)}$ with $\mathbf{f}^{(4)} = (6, 4)$.
- $\mathbf{x}^{(2)}$ with $\mathbf{f}^{(2)} = (7, 8)$ e.g., dominated by $\mathbf{x}^{(3)}$ with $\mathbf{f}^{(3)} = (4, 6)$.
- $\mathbf{x}^{(3)}$ with $\mathbf{f}^{(3)}$ not dominated.
- $\mathbf{x}^{(4)}$ with $\mathbf{f}^{(4)}$ not dominated.
- $\mathbf{x}^{(5)}$ with $\mathbf{f}^{(5)}$ not dominated.
- $\mathbf{x}^{(6)}$ with $\mathbf{f}^{(6)}$ not dominated.

$\rightarrow$ the set of Pareto optimal points is $\mathcal{P} = \{\mathbf{x}^{(3)}, \mathbf{x}^{(4)}, \mathbf{x}^{(5)}, \mathbf{x}^{(6)}\}$.

(b)
```r
library(ggplot2)

solutions = data.frame(f1 = c(10, 7, 4, 6, 9, 3), f2 = c(5, 8, 6, 4, 3, 7), id = 1:6)
solutions$pareto_optimal = c(FALSE, FALSE, TRUE, TRUE, TRUE, TRUE)

ggplot(aes(x = f1, y = f2, colour = pareto_optimal), data = solutions) +
  geom_step(data = solutions[solutions$pareto_optimal == TRUE, ],
            direction = "hv", colour = "black") +
  geom_point() +
  geom_text(aes(x = f1, y = f2, label = id),
            nudge_x = 0.25, nudge_y = 0.25, show.legend = FALSE) +
  geom_point(aes(x = f1, y = f1), colour = "black", data = data.frame(f1 = 15, f2 = 15)) +
  labs(x = expression(f[1]), y = expression(f[2]), colour = "Pareto Optimal") +
  theme_minimal()
```



(c) We can simply compute the area slices under each segment and sum them up. For the four rectangles from left to right:

- $(4 - 3) \cdot (15 - 7) = 8$
- $(6 - 4) \cdot (15 - 6) = 18$
- $(9 - 6) \cdot (15 - 4) = 33$
- $(15 - 9) \cdot (15 - 3) = 72$

$\rightarrow S(\mathcal{P}, R) = 8 + 18 + 33 + 72 = 131.$

(d) We start with the first front of non-dominated solutions $\mathcal{F}_1 = \{\mathbf{x}^{(3)}, \mathbf{x}^{(4)}, \mathbf{x}^{(5)}, \mathbf{x}^{(6)}\}$. After dropping these solutions, $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ remain. Neither of these solutions dominates the other solution. Therefore $\mathcal{F}_2 = \{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\}$.

(e) Crowding distance is always computed within a front. From (d) we have that $\mathbf{x}^{(3)} \in \mathcal{F}_1$. We start with the first dimension, f_1.
First, sort the values by f_1: $(3, 7), (4, 6), (6, 4), (9, 3)$. $(3, 7)$ and $(9, 3)$ are outermost and get an infinite partial distance for the f_1 dimension. Normalize the values by the minimum of 3 and maximum of 9 among the four points. For the point $\mathbf{x}^{(3)}$ (new index of $i = 2$) we compute:

$$\mathrm{CD}_1(\mathbf{x}^{(3)}) = \frac{(f_1^{(i+1)} - f_1^{(i-1)})}{(f_1^{(\max)} - f_1^{(\min)})} = \frac{(6 - 3)}{(9 - 3)} = 0.5.$$

For the second dimension, f_2, we analogously obtain:

$$\mathrm{CD}_2(\mathbf{x}^{(3)}) = \frac{(f_2^{(i+1)} - f_2^{(i-1)})}{(f_2^{(\max)} - f_2^{(\min)})} = \frac{(7 - 4)}{(7 - 3)} = 0.75.$$

$\rightarrow$ the total crowding distance is (when taking the sum) $0.5 + 0.75 = 1.25$.

(f) We know from (c) that the total dominated hypervolume is $S(\mathcal{P}, R) = 131$. To compute the hypervolume contribution of $\mathbf{x}^{(5)}$, we compute the hypervolume of $\mathcal{P} \setminus \{\mathbf{x}^{(5)}\}$ and substract it. Similar computations as in (c) but now for $\mathcal{P} \setminus \{\mathbf{x}^{(5)}\}$ yield $S(\mathcal{P} \setminus \{\mathbf{x}^{(5)}\}, R) = 125$. Therefore $\mathbf{x}^{(5)}$ has a hypervolume contribution of $131 - 125 = 6$.