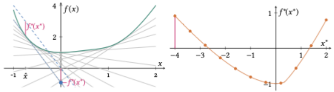
Optimization in Machine Learning

Other forms of duality



(a) The function f (solid) and the linear function of $x^* = -4$ (dashed) and its shifted tangent (dotted).

(b) The Fenchel conjugate f^* of f.

Learning goals

- Dual norms
- Conjugate functions
- Fenchel duality
- Examples in statistics

CONSTRAINED MINIMIZATION AND DUAL NORMS

Consider the problem of norm minimization under linear constraints in its primal form:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^d} \quad & \|\boldsymbol{x}\| \\ \text{s.t.} \quad & \boldsymbol{G}\boldsymbol{x} = \boldsymbol{h}\,, \end{aligned}$$

where $\|\cdot\|$ is some norm function. For instance, if the norm is the $L1$ norm, this problem is the famous basis pursuit problem.

Question: Is there a more straightforward way to solve constrained optimization problems involving norms?

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Here, the concept of the dual norm from functional analysis can be helpful.
Definition: Let $\|\boldsymbol{x}\|$ be the norm of $\boldsymbol{x}$. Then the dual norm $\|\boldsymbol{x}\|_*$ is defined as

$$\|\boldsymbol{x}\|_* = \max_{\|\boldsymbol{z}\| \leq 1} \boldsymbol{z}^T \boldsymbol{x}$$

Using this definition, one can show that if $\|\boldsymbol{x}\|$ is a norm and $\|\boldsymbol{x}\|_*$ is the dual norm of it, $\|\boldsymbol{z}^T \boldsymbol{x}\| \leq \|\boldsymbol{z}\| \|\boldsymbol{x}\|_*$ holds.

Examples: The dual norm of the Lp norm $\| \cdot \|_p$ is the Lq norm $\| \cdot \|_q$ where it holds that $1/p + 1/q = 1$.

CONSTRAINED PROBLEMS AND CONJUGATE FUNCTIONS