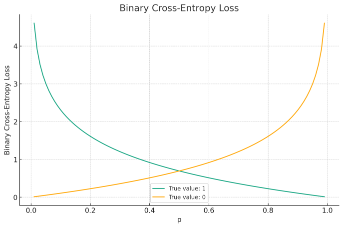
Introduction to Machine Learning

Information Theory
Cross-Entropy and KL



Learning goals

- Know the cross-entropy
- Understand the connection between entropy, cross-entropy, and KL divergence

CROSS-ENTROPY - DISCRETE CASE

Cross-entropy measures the average amount of information required to represent an event from one distribution p using a predictive scheme based on another distribution q (assume they have the same domain $\mathcal{X}$ as in KL).

$$H(p\|q) = \sum_{x \in \mathcal{X}} p(x) \log\left(\frac{1}{q(x)}\right) = -\sum_{x \in \mathcal{X}} p(x) \log\left(q(x)\right) = -\mathbb{E}_{X \sim p}[\log(q(X))]$$

For now, we accept the formula as-is. More on the underlying intuition follows in the content on inf. theory for ML and sourcecoding.

- Entropy = Avg. amount of information if we optimally encode p
- Cross-Entropy = Avg. amount of information if we suboptimally encode p with q
- $DL_{KL}(p\|q)$: Difference between the two
- $H(p\|q)$ sometimes also denoted as $H_q(p)$ to set it apart from KL

CROSS-ENTROPY - CONTINUOUS CASE

For continuous density functions $p(x)$ and $q(x)$:

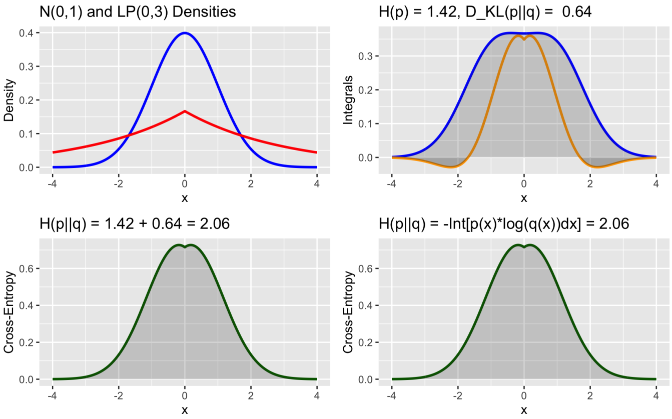
$$H(p\|q) = \int p(x) \log \left(\frac{1}{q(x)} \right) dx = -\int p(x) \log \left(q(x) \right) dx = -\mathbb{E}_{X \sim p}[\log(q(X))]$$

- It is not symmetric.
- As for the discrete case, $H(p\|q) = h(p) + D_{KL}(p\|q)$ holds.
- Can now become negative, as the $h(p)$ can be negative!

CROSS-ENTROPY EXAMPLE

Let $p(x) = N(0, 1)$ and $q(x) = LP(0, 3)$. We can visualize

$$H(p\|q) = H(p) + D_{KL}(p\|q)$$



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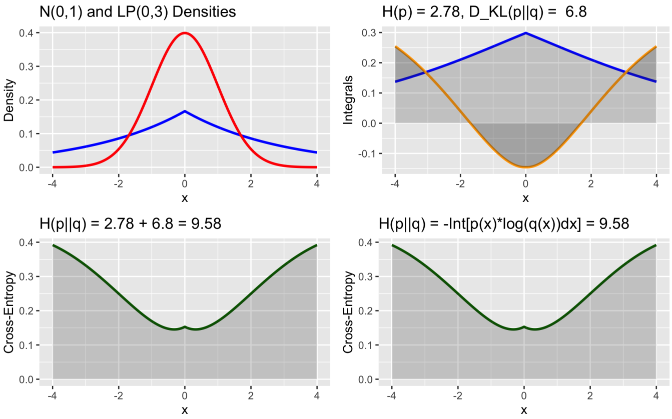

PROOF: MAXIMUM OF DIFFERENTIAL ENTROPY

Claim: For a given variance, the continuous distribution that maximizes differential entropy is the Gaussian.

Proof: Let $g(x)$ be a Gaussian with mean μ and variance σ^2 and $f(x)$ an arbitrary density function with the same variance. Since differential entropy is translation invariant, we can assume $f(x)$ and $g(x)$ have the same mean.

The KL divergence (which is non-negative) between $f(x)$ and $g(x)$ is:

$$
\begin{aligned}
0 \leq D_{KL}(f \| g) &= -h(f) + H(f \| g) \\
&= -h(f) - \int_{-\infty}^{\infty} f(x) \log(g(x)) dx
\end{aligned}
\tag{1}
$$