Exercise 1: Entropy

A fair dice is rolled at the same time as a fair coin is tossed. Let A be the number on the upper surface of the dice and let B describe the outcome of the coin toss, where

$$B = \begin{cases} 1, & \text{head}\,, \\ 0, & \text{tail}\,. \end{cases}$$

Two random variables X and Y are given by $X = A + B$ and $Y = A - B$, respectively.

(a) Calculate the entropies $H(X)$ and $H(Y)$, the conditional entropies $H(Y|X)$ and $H(X|Y)$, the joint entropy $H(X, Y)$ and the mutual information $I(X; Y)$.

(b) Show that, for independent discrete random variables X and Y,

$$I(X; X + Y) - I(Y; X + Y) = H(X) - H(Y)$$

Exercise 2: Kullback-Leibler Divergence

(a) You want to approximate the binomial distribution with n number of trials and probability p with a Gaussian distribution with mean μ and variance σ^2. To find a suitable distribution you investigate the Kullback-Leibler divergence (KLD) in terms of the parameters $\boldsymbol{\theta} = (\mu, \sigma^2)^\top$.

  (i) Write down the KLD for the given setup.

  (ii) Derive the gradients with respect to $\boldsymbol{\theta}$.

  (iii) Is there an analytic solution for the optimal parameter setting? If yes, derive the corresponding solution. If no, give a short reasoning.

  (iv) Independent of the previous exercise, state a numerical procedure to minimize the KLD.

(b) Sample points according to the true distribution and visualize the KLD for different parameter settings of the Gaussian distribution (including the optimal one if available).

(c) Create a surface plot with axes n and p and colour value equal to the KLD for the optimal normal distribution.

(d) Based on the previous result,

  (i) how can the behaviour for varying p be explained?

  (ii) how can the behaviour for varying n be explained?