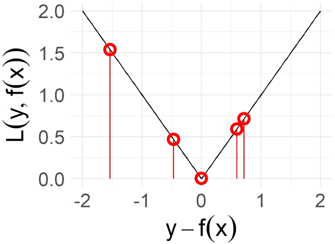
Introduction to Machine Learning

Advanced Risk Minimization
L1 Risk Minimizer (Deep-Dive)



Learning goals

- Derive the risk minimizer of the L1-loss
- Derive the optimal constant model for the L1-loss

L1-LOSS: RISK MINIMIZER

Optimal constant model under L1 loss is

$$f_c^* = \arg\min_c \mathbb{E}_y[|y - c|] = \text{med}[y]$$

Proof: Let $p(y)$ be the density function of y. Then:

$$\arg\min_c \mathbb{E}\left[|y - c|\right] = \arg\min_c \int_{-\infty}^{\infty} |y - c| \; p(y) \mathrm{d}y$$

$$= \arg\min_c \int_{-\infty}^{c} -(y - c) \; p(y) \; \mathrm{d}y + \int_{c}^{\infty} (y - c) \; p(y) \; \mathrm{d}y$$

We now compute the derivative of the above term and set it to 0

$$0 = \frac{\partial}{\partial c}\left(\int_{-\infty}^{c} -(y - c) \; p(y) \; \mathrm{d}y + \int_{c}^{\infty} (y - c) \; p(y) \; \mathrm{d}y\right)$$

$$\overset{*\text{Leibniz}}{=} \int_{-\infty}^{c} p(y) \; \mathrm{d}y - \int_{c}^{\infty} p(y) \; \mathrm{d}y = \mathbb{P}_y(y \leq c) - (1 - \mathbb{P}_y(y \leq c))$$

$$= 2 \cdot \mathbb{P}_y(y \leq c) - 1 \Leftrightarrow 0.5 = \mathbb{P}_y(y \leq c) \; \Rightarrow c = \text{med}[y]$$

NB: replacing $p(y)$ by $p(y|\mathbf{x})$, we obtain the point-wise conditional risk minimizer
$f^*(\tilde{\mathbf{x}}) = \arg\min_c \mathbb{E}_{y|x}\left[|y - c|\right] = \text{med}\left[y \mid \mathbf{x} = \tilde{\mathbf{x}}\right]$

L1-LOSS: RISK MINIMIZER

* **Note** that since we are computing the derivative w.r.t. the integration boundaries we need to use Leibniz integration rule

$$
\begin{aligned}
\frac{\partial}{\partial c}\left(\int_a^c g(c,y)\,\mathrm{d}y\right) &= g(c,c) + \int_a^c \frac{\partial}{\partial c} g(c,y)\,\mathrm{d}y \\
\frac{\partial}{\partial c}\left(\int_c^a g(c,y)\,\mathrm{d}y\right) &= -g(c,c) + \int_c^a \frac{\partial}{\partial c} g(c,y)\,\mathrm{d}y
\end{aligned}
$$

We get

$$
\begin{aligned}
&\frac{\partial}{\partial c}\left(\int_{-\infty}^c -(y-c)\,p(y)\,\mathrm{d}y + \int_c^\infty (y-c)\,p(y)\,\mathrm{d}y\right) \\
=&\frac{\partial}{\partial c}\left(\int_{-\infty}^c \underbrace{-(y-c)\,p(y)}_{g_1(c,y)}\,\mathrm{d}y\right) + \frac{\partial}{\partial c}\left(\int_c^\infty \underbrace{(y-c)\,p(y)}_{g_2(c,y)}\,\mathrm{d}y\right) \\
=&\underbrace{g_1(c,c)}_{=0} + \int_{-\infty}^c \frac{\partial}{\partial c}(-(y-c))\,p(y)\,\mathrm{d}y - \underbrace{g_2(c,c)}_{=0} + \int_c^\infty \frac{\partial}{\partial c}(y-c)\,p(y)\,\mathrm{d}y \\
=&\int_{-\infty}^c p(y)\,\mathrm{d}y + \int_c^\infty -\,p(y)\,\mathrm{d}y
\end{aligned}
$$

L1-LOSS: OPTIMAL CONSTANT MODEL

Optimal constant model for empirical risk under L1 loss is:

$$\hat{f}_c = \arg\min_c \frac{1}{n}\sum_{i=1}^{n} |y^{(i)} - c| = \hat{\theta} = \text{med}(y^{(1)}, \dots, y^{(n)})$$

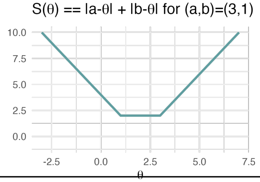
Proof:

- Firstly note that for $n = 1$ the median $\hat{\theta} = \text{med}(y^{(i)}) = y^{(1)}$ obviously minimizes the emp. risk $\mathcal{R}_{\text{emp}}$ using the $L1$ loss
- Hence let $n > 1$ in the following For $a, b \in \mathbb{R}$, define

$$S_{a,b} : \mathbb{R} \to \mathbb{R}_0^+, \theta \mapsto |a - \theta| + |b - \theta|$$

Any $\hat{\theta} \in [a, b]$ minimizes $S_{a,b}(\theta)$, because it holds that

$$S_{a,b}(\theta) = \begin{cases} |a - b|, & \text{for } \theta \in [a, b] \\ |a - b| + 2 \cdot \min\{|a - \theta|, |b - \theta|\}, & \text{otherwise} \end{cases}$$



L1-LOSS: OPTIMAL CONSTANT MODEL

W.l.o.g. assume now that all $y^{(i)}$ are sorted in increasing order.
Let us define $i_{\max} = n/2$ for n even and $i_{\max} = (n-1)/2$ for n odd and consider the intervals

$$\mathcal{I}_i := [y^{(i)}, y^{(n+1-i)}], i \in \{1, ..., i_{\max}\}$$

By construction $\mathcal{I}_{j+1} \subseteq \mathcal{I}_j$ for $j \in \{1, \dots, i_{\max} - 1\}$ and $\mathcal{I}_{i_{\max}} \subseteq \mathcal{I}_i$. With this $\mathcal{R}_{\text{emp}}$ can be expressed as

$$\begin{aligned}
\mathcal{R}_{\text{emp}}(\theta) &= \sum_{i=1}^{n} L(y^{(i)}, \theta) = \sum_{i=1}^{n} |y^{(i)} - \theta| \\
&= \underbrace{|y^{(1)} - \theta| + |y^{(n)} - \theta|}_{=S_{y^{(1)},y^{(n)}}(\theta)} + \underbrace{|y^{(2)} - \theta| + |y^{(n-1)} - \theta|}_{=S_{y^{(2)},y^{(n-1)}}(\theta)} + \dots \\
&= \begin{cases} \sum_{i=1}^{i_{\max}} S_{y^{(i)},y^{(n+1-i)}}(\theta) & \text{for } n \text{ is even} \\ \sum_{i=1}^{i_{\max}} (S_{y^{(i)},y^{(n+1-i)}}(\theta)) + |y^{((n+1)/2)} - \theta| & \text{for } n \text{ is odd} \end{cases}
\end{aligned}$$

L1-LOSS: OPTIMAL CONSTANT MODEL

From this follows that

- for "n is even": $\hat{\theta} \in \mathcal{I}_{i_{\text{max}}} = [y^{(n/2)}, y^{(n/2+1)}]$ minimizes S_i for all $i \in \{1, \dots, i_{\text{max}}\}$ $\Rightarrow$ it minimizes $\mathcal{R}_{\text{emp}}$

- for "n is odd": $\hat{\theta} = y^{(n+1)/2} \in \mathcal{I}_{i_{\text{max}}}$ minimizes S_i for all $i \in \{1, \dots, i_{\text{max}}\}$ and it's minimal for $|y^{((n+1)/2)} - \theta|$ $\Rightarrow$ it minimizes $\mathcal{R}_{\text{emp}}$

Since the median fulfills these conditions, we can conclude that it minimizes the $L1$ loss