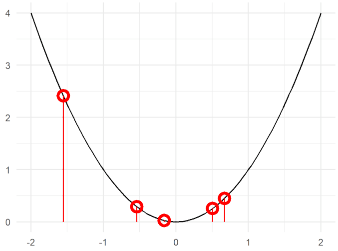
Introduction to Machine Learning

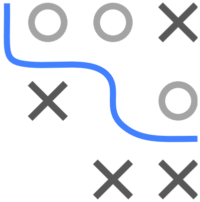
Advanced Risk Minimization
Regression Losses: L2 and L1 loss



Learning goals

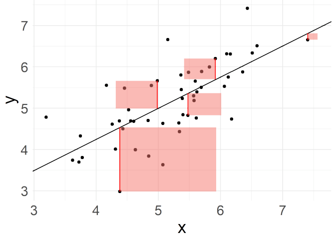
- L2 loss and risk minimizers
- L1 loss and risk minimizers

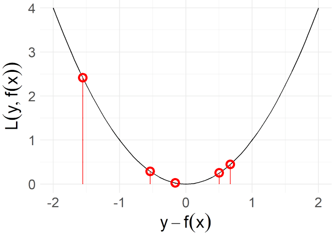
L2-LOSS



$$L(y, f(\mathbf{x})) = (y - f(\mathbf{x}))^2 \quad \text{or} \quad L(y, f(\mathbf{x})) = 0.5(y - f(\mathbf{x}))^2$$

- Tries to reduce large residuals
  If residual is twice as large, loss is 4 times as large
  Hence, sensitive to outliers in y
- Analytic properties: convex, differentiable





L2: OPTIMAL VALUE IS EXPECTATION

- Can derive a general result now for any $z \sim Q$

- Consider

$$\arg\min_{c \in \mathbb{R}} \mathbb{E}_z[L(z, c)] = \arg\min_{c \in \mathbb{R}} \mathbb{E}[(z - c)^2]$$

$$\mathbb{E}[(z - c)^2] = \mathbb{E}[z^2 - 2zc + c^2] = \mathbb{E}[z^2] - 2c\mathbb{E}[z] + c^2$$

- The RHS is minimized by $c = \mathbb{E}[z]$
  (simple quadratic, or take derivative and set to 0)

L2: OPTIMAL CONSTANT MODEL

- From the previous we immediately get for $Q = P_y$

$$f_c^* = \arg\min_{c \in \mathbb{R}} \mathbb{E}_y[(y - c)^2] = \mathbb{E}[y]$$

- For the best empirical constant we could minimize

$$\hat{f}_c = \arg\min_{c \in \mathbb{R}} \sum_{i=1}^{n} L(y^{(i)}, c)$$

  And later we will proceed like that
- But we can get the result for free from our previous consideration
- For data $y^{(1)}, \ldots, y^{(n)}$, empirical distribution is $P_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{y^{(i)}}$
- Hence: Optimal constant is sample mean

$$\hat{f}_c = \arg\min_{c \in \mathbb{R}} \sum_{i=1}^{n} L(y^{(i)}, c) = \mathbb{E}_{z \sim P_n}(z - c)^2 = \mathbb{E}[z] = \frac{1}{n} \sum_{i=1}^{n} y^{(i)} = \bar{y}$$

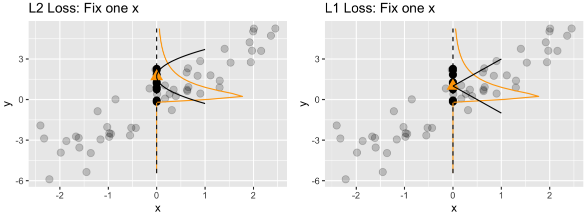
L2-LOSS: RISK MINIMIZER

- Let's minimize true risk for unrestricted hypothesis space and $L2$
- We know: At any point $\mathbf{x} = \tilde{\mathbf{x}}$, our loss-optimal prediction is

$$f^*(\tilde{\mathbf{x}}) = \arg\min_{c \in \mathbb{R}} \mathbb{E}_{y|x}\left[L(y, c) \mid \mathbf{x} = \tilde{\mathbf{x}}\right]$$

- We know from previously that $L2$ RM is the cond. exp.

$$f^*(\tilde{\mathbf{x}}) = \mathbb{E}_{y|x}\left[y \mid \mathbf{x} = \tilde{\mathbf{x}}\right].$$



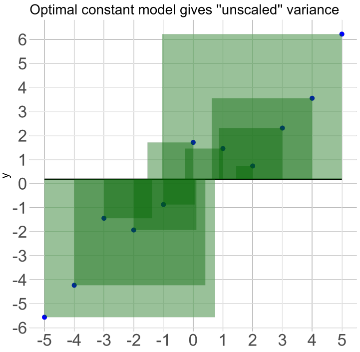
- For $L2$ loss, the pointwise loss is the (conditional) variance

$$\mathbb{E}_{y|x}[(L(y, f^*(\mathbf{x}))|\mathbf{x} = \tilde{\mathbf{x}}] = \mathbb{E}_{y|x}[(y - f^*(\mathbf{x}))^2|\mathbf{x} = \tilde{\mathbf{x}}] = \mathrm{Var}(y|\mathbf{x} = \tilde{\mathbf{x}})$$

This is trivially true, as we know $f^*(\tilde{\mathbf{x}}) = \mathbb{E}_{y|x}\left[y \mid \mathbf{x} = \tilde{\mathbf{x}}\right]$.

L2 LOSS MEANS MINIMIZING VARIANCE

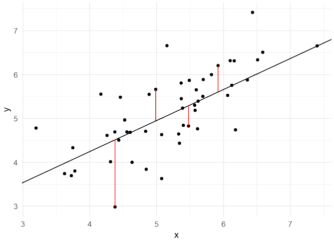
- Let's reconsider the previous
- Optimized for const whose squared dist to points is minimal (on avg)
- Result: $\hat{\theta} = \bar{y}$
- What is the associated risk? $\mathcal{R}(\hat{\theta}) = \sum_{i=1}^{n} (y_i - \bar{y})^2$
- Average this by $\frac{1}{n}$ or $\frac{1}{n-1}$ to obtain variance
- Same holds for the pointwise construction / conditional distribution considered before

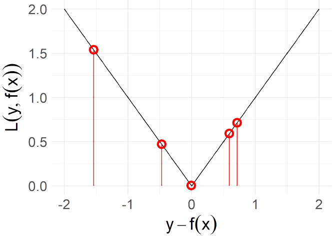


L1-LOSS

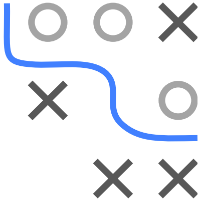
$$L\left(y, f(\mathbf{x})\right) = |y - f(\mathbf{x})|$$

- More robust than $L2$, outliers in y are less problematic
- Analytical properties: convex, not differentiable for $y = f(\boldsymbol{x})$ (optimization becomes harder)





L1-LOSS: OPTIMAL PREDICTIONS



- Optimal constant model is median: $f_c^* = \text{med}[y]$
- Empirical version: $\hat{f}_c = \text{med}(y^{(1)}, \ldots, y^{(n)})$
- Derivations slightly harder and in deep-dive
- Risk minimizer / optimal conditional prediction:

$$f^*(\tilde{\mathbf{x}}) = \arg\min_c \mathbb{E}_{y|x}\left[|y - c| \mid \mathbf{x} = \tilde{\mathbf{x}}\right] = \text{med}\left[y \mid \mathbf{x} = \tilde{\mathbf{x}}\right]$$