

## Solution 1: Kernels

- (a) The function is symmetric, as we can prove in the following way:

$$k(\mathbf{x}, \tilde{\mathbf{x}}) = \mathbf{x}^T \tilde{\mathbf{x}} = \tilde{\mathbf{x}}^T \mathbf{x} = k(\tilde{\mathbf{x}}, \mathbf{x}) \quad (1)$$

To check if the kernel gram matrix is positive definite, we will remember the definition of  $\mathbf{X}$

$$\mathbf{X} = \left\{ \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)} \right\} \in \mathbb{R}^{p \times n} \quad (2)$$

Taking that into account, the kernel gram matrix is defined as :

$$\begin{aligned} \mathbf{K} &= \begin{pmatrix} \langle \mathbf{x}^{(1)}, \mathbf{x}^{(1)} \rangle & \langle \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \rangle & \dots & \langle \mathbf{x}^{(1)}, \mathbf{x}^{(n)} \rangle \\ \langle \mathbf{x}^{(2)}, \mathbf{x}^{(1)} \rangle & \langle \mathbf{x}^{(2)}, \mathbf{x}^{(2)} \rangle & \dots & \langle \mathbf{x}^{(2)}, \mathbf{x}^{(n)} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{x}^{(n)}, \mathbf{x}^{(1)} \rangle & \langle \mathbf{x}^{(n)}, \mathbf{x}^{(2)} \rangle & \dots & \langle \mathbf{x}^{(n)}, \mathbf{x}^{(n)} \rangle \end{pmatrix} \\ &= \left\{ \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)} \right\}^T \left\{ \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)} \right\} \\ &= \mathbf{X}^T \mathbf{X} \in \mathbb{R}^{n \times n} \end{aligned} \quad (3)$$

$$\begin{aligned} \mathbf{a}^T \mathbf{K} \mathbf{a} &= \mathbf{a}^T \mathbf{X}^T \mathbf{X} \mathbf{a} \\ &= (\underbrace{\mathbf{a} \mathbf{X}}_{\mathbf{z}})^T (\mathbf{a} \mathbf{X}) \\ &= \mathbf{z}^T \mathbf{z} \\ &= \|\mathbf{z}\|_2^2 > 0 \end{aligned} \quad (4)$$

We can conclude that this function is a kernel.

- (b) The function is not a kernel, as we can prove that the kernel gram matrix is not positive definite. We will use  $x^{(1)} = \frac{\pi}{2}$  and  $a = 1$ . For this case, the kernel gram matrix is a scalar:

$$\begin{aligned} K &= k(x^{(1)}, x^{(1)}) = \cos\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = -1 \\ a \cdot K \cdot a &= 1 \cdot -1 \cdot 1 = -1 < 0 \end{aligned} \quad (5)$$

- (c) The function is not a kernel, we can prove that the kernel gram matrix is not positive definite by using a counter-example, we will use  $x^{(1)} = 1$  and  $x^{(2)} = 2$ , and  $\mathbf{a} = (1, -1)^T$  :

$$\mathbf{K} = \begin{pmatrix} \max(x^{(1)}, x^{(1)}) & \max(x^{(1)}, x^{(2)}) \\ \max(x^{(2)}, x^{(1)}) & \max(x^{(2)}, x^{(2)}) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \quad (6)$$

$$\begin{aligned} \mathbf{a}^T \mathbf{K} \mathbf{a} &= (1, -1) \cdot \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \cdot (1, -1)^T \\ &= (-1, 0) \cdot (1, -1)^T \\ &= -1 \end{aligned} \quad (7)$$

- (d) In this case, we can use properties of the kernels to prove that the function is indeed a kernel.

- Multiplying a kernel by a non-negative scalar always results in a kernel. Accordingly, we can say that  $\alpha k_1(\mathbf{x}, \tilde{\mathbf{x}})$  and  $\beta k_2(\mathbf{x}, \tilde{\mathbf{x}})$  are kernels.
- The sum of two kernels always results in a kernel. Accordingly, we can say that  $\alpha k_1(\mathbf{x}, \tilde{\mathbf{x}}) + \beta k_2(\mathbf{x}, \tilde{\mathbf{x}})$  is a kernel.