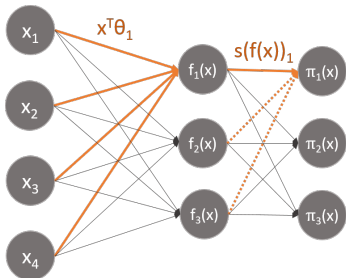


# Introduction to Machine Learning

## Softmax Regression



### Learning goals

- Know softmax regression
- Understand that softmax regression is a generalization of logistic regression

# FROM LOGISTIC REGRESSION ...

Remember **logistic regression** ( $\mathcal{Y} = \{0, 1\}$ ): We combined the hypothesis space of linear functions, transformed by the logistic function  $s(z) = \frac{1}{1+\exp(-z)}$ , i.e.

$$\mathcal{H} = \left\{ \pi : \mathcal{X} \rightarrow \mathbb{R} \mid \pi(\mathbf{x}) = s(\boldsymbol{\theta}^\top \mathbf{x}) \right\},$$

with the Bernoulli (logarithmic) loss:

$$L(y, \pi(\mathbf{x})) = -y \log(\pi(\mathbf{x})) - (1 - y) \log(1 - \pi(\mathbf{x})).$$

**Remark:** We suppress the intercept term for better readability. The intercept term can be easily included via  $\boldsymbol{\theta}^\top \tilde{\mathbf{x}}$ ,  $\boldsymbol{\theta} \in \mathbb{R}^{p+1}$ ,  $\tilde{\mathbf{x}} = (1, \mathbf{x})$ .

## ... TO SOFTMAX REGRESSION

There is a straightforward generalization to the multiclass case:

- Instead of a single linear discriminant function we have  $g$  linear discriminant functions

$$f_k(\mathbf{x}) = \boldsymbol{\theta}_k^\top \mathbf{x}, \quad k = 1, 2, \dots, g,$$

each indicating the confidence in class  $k$ .

- The  $g$  score functions are transformed into  $g$  probability functions by the **softmax** function  $s : \mathbb{R}^g \rightarrow \mathbb{R}^g$

$$\pi_k(\mathbf{x}) = s(f(\mathbf{x}))_k = \frac{\exp(\boldsymbol{\theta}_k^\top \mathbf{x})}{\sum_{j=1}^g \exp(\boldsymbol{\theta}_j^\top \mathbf{x})},$$

instead of the **logistic** function for  $g = 2$ . The probabilities are well-defined:  $\sum \pi_k(\mathbf{x}) = 1$  and  $\pi_k(\mathbf{x}) \in [0, 1]$  for all  $k$ .

## ... TO SOFTMAX REGRESSION

- The softmax function is a generalization of the logistic function. For  $g = 2$ , the logistic function and the softmax function are equivalent.
- Instead of the **Bernoulli** loss, we use the multiclass **logarithmic loss**

$$L(y, \pi(\mathbf{x})) = - \sum_{k=1}^g \mathbb{1}_{\{y=k\}} \log(\pi_k(\mathbf{x})).$$

- Note that the softmax function is a “smooth” approximation of the arg max operation, so  $s((1, 1000, 2)^T) \approx (0, 1, 0)^T$  (picks out 2nd element!).
- Furthermore, it is invariant to constant offsets in the input:

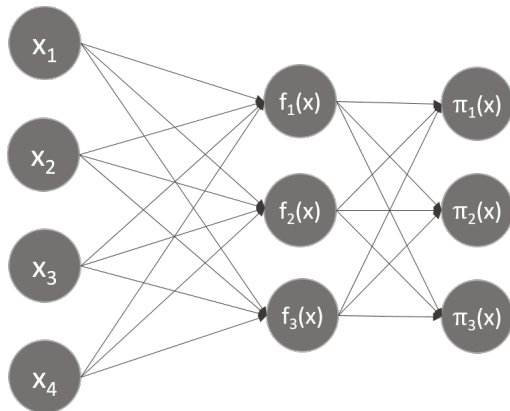
$$s(f(\mathbf{x}) + \mathbf{c}) = \frac{\exp(\theta_k^\top \mathbf{x} + c)}{\sum_{j=1}^g \exp(\theta_j^\top \mathbf{x} + c)} = \frac{\exp(\theta_k^\top \mathbf{x}) \cdot \exp(c)}{\sum_{j=1}^g \exp(\theta_j^\top \mathbf{x}) \cdot \exp(c)} = s(f(\mathbf{x}))$$

# LOGISTIC VS. SOFTMAX REGRESSION

	Logistic Regression	Softmax Regression
$\mathcal{Y}$	$\{0, 1\}$	$\{1, 2, \dots, g\}$
Discriminant fun.	$f(\mathbf{x}) = \boldsymbol{\theta}^\top \mathbf{x}$	$f_k(\mathbf{x}) = \boldsymbol{\theta}_k^\top \mathbf{x}, k = 1, 2, \dots, g$
Probabilities	$\pi(\mathbf{x}) = \frac{1}{1 + \exp(-\boldsymbol{\theta}^\top \mathbf{x})}$	$\pi_k(\mathbf{x}) = \frac{\exp(\boldsymbol{\theta}_k^\top \mathbf{x})}{\sum_{j=1}^g \exp(\boldsymbol{\theta}_j^\top \mathbf{x})}$
$L(y, \pi(\mathbf{x}))$	Bernoulli / logarithmic loss $-y \log(\pi(\mathbf{x})) - (1 - y) \log(1 - \pi(\mathbf{x}))$	Multiclass logarithmic loss $-\sum_{k=1}^g [y = k] \log(\pi_k(\mathbf{x}))$

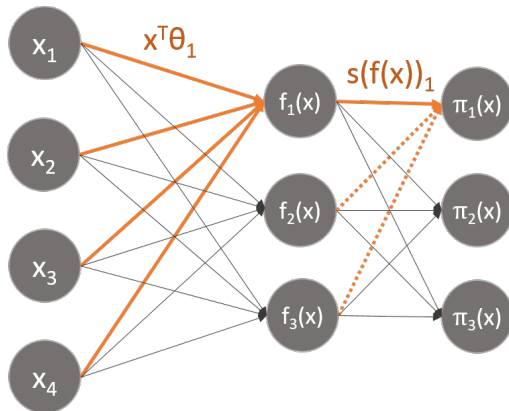
# LOGISTIC VS. SOFTMAX REGRESSION

We can schematically depict softmax regression as follows:



# LOGISTIC VS. SOFTMAX REGRESSION

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# LOGISTIC VS. SOFTMAX REGRESSION

Further comments:

- We can now, for instance, calculate gradients and optimize this with standard numerical optimization software.
- Softmax regression has an unusual property in that it has a “redundant” set of parameters. If we subtract a fixed vector from all  $\theta_k$ , the predictions do not change at all. Hence, our model is “over-parameterized”. For any hypothesis we might fit, there are multiple parameter vectors that give rise to exactly the same hypothesis function. This also implies that the minimizer of  $\mathcal{R}_{\text{emp}}(\theta)$  above is not unique! Hence, a numerical trick is to set  $\theta_g = 0$  and only optimize the other  $\theta_k$ . This does not restrict our hypothesis space, but the constrained problem is now convex, i.e., there exists exactly one parameter vector for every hypothesis.
- A similar approach is used in many ML models: multiclass LDA, naive Bayes, neural networks and boosting.



# SOFTMAX: LINEAR DISCRIMINANT FUNCTIONS

Softmax regression gives us a **linear classifier**.

- The softmax function  $s(\mathbf{z})_k = \frac{\exp(\mathbf{z}_k)}{\sum_{j=1}^g \exp(\mathbf{z}_j)}$  is
  - a rank-preserving function, i.e. the ranks among the elements of the vector  $\mathbf{z}$  are the same as among the elements of  $s(\mathbf{z})$ . This is because softmax transforms all scores by taking the  $\exp(\cdot)$  (rank-preserving) and divides each element by **the same** normalizing constant.

Thus, the softmax function has a unique inverse function  $s^{-1} : \mathbb{R}^g \rightarrow \mathbb{R}^g$  that is also monotonic and rank-preserving.

Applying  $s_k^{-1}$  to  $\pi_k(\mathbf{x}) = \frac{\exp(\theta_k^\top \mathbf{x})}{\sum_{j=1}^g \exp(\theta_j^\top \mathbf{x})}$  gives us  $f_k(\mathbf{x}) = \theta_k^\top \mathbf{x}$ .

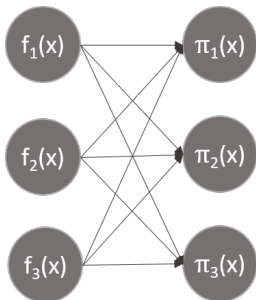
Thus, softmax regression is a linear classifier.

# GENERALIZING SOFTMAX REGRESSION

Instead of simple linear discriminant functions we could use **any** model that outputs  $g$  scores

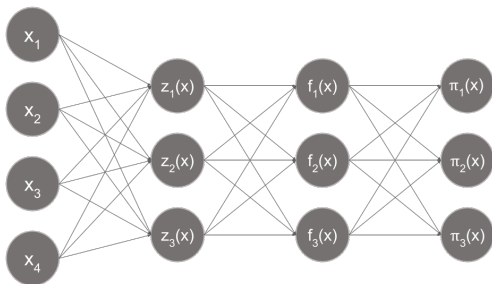
$$f_k(\mathbf{x}) \in \mathbb{R}, k = 1, 2, \dots, g$$

We can choose a multiclass loss and optimize the score functions  $f_k, k \in \{1, \dots, g\}$  by multivariate minimization. The scores can be transformed to probabilities by the **softmax** function.



# GENERALIZING SOFTMAX REGRESSION

For example for a **neural network** (note that softmax regression is also a neural network with no hidden layers):



**Remark:** For more details about neural networks please refer to the lecture **Deep Learning**.