Introduction to Machine Learning

Information Theory
Mutual Information under
Reparametrization (Deep-Dive)





Learning goals

Understand why MI is invariant under certain reparametrizations

MUTUAL INFORMATION PROPERTIES

ullet MI is invariant w.r.t. injective reparametrizations that are in \mathcal{C}^1 :

Let $f, g: \mathbb{R}^d \to \mathbb{R}^d \in \mathcal{C}^1$ be injective transformations and X, Y be continuous random variables in \mathbb{R}^d then by the change of variables the joint and marginal densities of $\tilde{X} = f(X), \tilde{Y} = g(Y)$

$$ilde{p}(ilde{x}, ilde{y}) = p(f^{-1}(ilde{x}),g^{-1}(ilde{y}))\cdot |J_{f^{-1}}(ilde{x})|\cdot |J_{g^{-1}}(ilde{y})|, \ ilde{p}(ilde{x}) = p(f^{-1}(ilde{x}))\cdot |J_{f^{-1}}(ilde{x})|, \ ilde{p}(ilde{y}) = p(g^{-1}(ilde{y}))\cdot |J_{g^{-1}}(ilde{y})|,$$

where p(x, y) is the joint density of X and Y and p(x), p(y) are the respective marginal densities. (J denotes the Jacobian)

With this, it follows that

$$I(\tilde{X}; \tilde{Y}) = \int \tilde{p}(\tilde{x}, \tilde{y}) \log \left(\frac{\tilde{p}(\tilde{x}, \tilde{y})}{\tilde{p}(\tilde{x}) \tilde{p}(\tilde{y})} \right) d\tilde{x} d\tilde{y} = *$$



MUTUAL INFORMATION PROPERTIES

$$\begin{split} * &= \int p(f^{-1}(\tilde{x}), g^{-1}(\tilde{y})) \cdot |J_{f^{-1}}(\tilde{x})| \cdot |J_{g^{-1}}(\tilde{y})| \\ &\cdot \log \left(\frac{p(f^{-1}(\tilde{x}), g^{-1}(\tilde{y})) \cdot |J_{f^{-1}}(\tilde{x})| \cdot |J_{g^{-1}}(\tilde{y})|}{p(f^{-1}(\tilde{x}))|J_{f^{-1}}(\tilde{x})| \cdot p(g^{-1}(\tilde{y}))|J_{g^{-1}}(\tilde{y})|} \right) d\tilde{x} d\tilde{y} \\ &= \int p(f^{-1}(f(x)), g^{-1}(g(y))) \cdot |J_{f^{-1}}(f(x))| \cdot |J_{g^{-1}}(g(y))| \\ &\cdot \log \left(\frac{p(f^{-1}(f(x)), g^{-1}(g(y)))}{p(f^{-1}(f(x)))p(g^{-1}(g(y)))} \right) |J_{f}(x)| \cdot |J_{g}(y)| dx dy \\ &= \int p(x, y) \cdot |J_{f^{-1}}(f(x))J_{f}(x)| \cdot |J_{g^{-1}}(g(y))J_{g}(y)| \log \left(\frac{p(x, y)}{p(x)p(y)} \right) dx dy \\ &= \int p(x, y) \cdot \log \left(\frac{p(x, y)}{p(x)p(y)} \right) dx dy = I(X; Y). \end{split}$$

(The fourth equality holds by the inverse function theorem)

