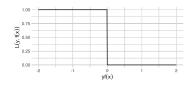
Introduction to Machine Learning

Advanced Risk Minimization Classification and 0-1-Loss





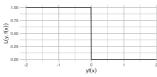
Learning goals

- 0-1 loss
- Risk minimizer(s) / Optim. predictions
- Bayes error rate
- Generative approach in classification

0-1-LOSS

0-1-loss

- Discrete classifier $h: \mathcal{X} \to \mathcal{V}$
- Maybe most "natural":



$$L(y, h(\mathbf{x})) = \mathbb{I}_{\{y \neq h(\mathbf{x})\}}$$

• For $\mathcal{Y} \in \{-1, +1\}$ and scoring classifier $f(\mathbf{x})$ can write it in terms of margin $\nu = yf(\mathbf{x})$

$$L(y, f(\mathbf{x})) = \mathbb{I}_{\{\nu < 0\}} = \mathbb{I}_{\{yf(\mathbf{x}) < 0\}}$$

• For $\mathcal{Y} \in \{0,1\}$ and prob. classifier $\pi(\mathbf{x})$

$$L(y,\pi(\mathbf{x})) = y \mathbb{I}_{\{\pi(\mathbf{x})<0.5\}} + (1-y)\mathbb{I}_{\{\pi(\mathbf{x})\geq0.5\}} = \mathbb{I}_{\{(2y-1)(\pi(\mathbf{x})-0.5)<0\}}$$

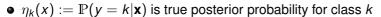
• Analytical properties: Not continuous, even for linear $f(\mathbf{x})$ optim. problem NP-hard = close to intractable Feldman et al. 2012



RISK MINIMIZER FOR DISCRETE CLASSIFIERS

Again, unravel with law of total expectation (here for multiclass)

$$\mathcal{R}(f) = \mathbb{E}_{xy} \left[L(y, f(\mathbf{x})) \right] = \mathbb{E}_{x} \left[\mathbb{E}_{y|x} [L(y, f(\mathbf{x})) \mid \mathbf{x}] \right]$$
$$= \mathbb{E}_{x} \left[\sum_{k \in \mathcal{Y}} L(k, f(\mathbf{x})) \mathbb{P}(y = k \mid \mathbf{x}) \right]$$



ullet For binary case, we denote $\eta(\mathbf{x}) := \mathbb{P}(y = 1 \mid \mathbf{x})$ and get:

$$\mathcal{R}(f) = \mathbb{E}_{\mathbf{x}} \left[L(1, \pi(\mathbf{x})) \cdot \eta(\mathbf{x}) + L(0, \pi(\mathbf{x})) \cdot (1 - \eta(\mathbf{x})) \right]$$

- Above formulas work for any loss, not only 0-1; and hard labelers, scorers or prob. classifiers
- Especially for hard labelers and arbitrary misclassif. costs, we see: produces cost-optimal decision, weighted by posterior probs; (we see this again in cost-senslearning)



0-1-LOSS: OPTIMAL PREDICTIONS

- For multiclass and hard labeler $h(\mathbf{x})$
- Optimal constant

$$\begin{split} h_c^* &= \arg\min_{k \in \mathcal{Y}} \sum_{I \in \mathcal{Y}} L(I, k) \cdot \mathbb{P}(y = I) \\ &= \arg\min_{k \in \mathcal{Y}} \sum_{k \neq I} \mathbb{P}(y = I) \\ &= \arg\min_{k \in \mathcal{Y}} 1 - \mathbb{P}(y = k) \\ &= \arg\max_{k \in \mathcal{Y}} \mathbb{P}(y = k) \end{split}$$

- Translation: Predict most probable class
- Empirical version: $\hat{h}_c = \underset{k \in \mathcal{Y}}{\operatorname{arg max}} \, \hat{\pi}_k$
- Risk minimizer / optim. cond. prediction / Bayes optim. classifier: $h^*(\tilde{\mathbf{x}}) = \arg\max_{k \in \mathcal{Y}} \mathbb{P}(y = k \mid \mathbf{x} = \tilde{\mathbf{x}})$



BAYES RISK / BAYES ERROR RATE

$$\mathcal{R}^* = 1 - \mathbb{E}_{x} \left[\max_{k \in \mathcal{Y}} \mathbb{P}(y = k \mid \mathbf{x}) \right]$$



For binary case, can write risk minimizer and Bayes risk as:

$$h^*(\mathbf{x}) = egin{cases} 1 & \eta(\mathbf{x}) \geq rac{1}{2} \ 0 & \eta(\mathbf{x}) < rac{1}{2} \end{cases}$$

$$\mathcal{R}^* = \mathbb{E}_{\mathbf{x}}\left[\min(\eta(\mathbf{x}), 1 - \eta(\mathbf{x}))\right] = 1 - \mathbb{E}_{\mathbf{x}}\left[\max(\eta(\mathbf{x}), 1 - \eta(\mathbf{x}))\right]$$

GENERATIVE CLASSIFIERS

- So, $rg \max_{k \in \mathcal{Y}} \mathbb{P}(y = k \mid \mathbf{x} = \tilde{\mathbf{x}})$ is what we want to do
- Assume we can model densities given classes and use Bayes:

$$\mathbb{P}(y = k \mid \mathbf{x} = \tilde{\mathbf{x}}) = \frac{\rho(\tilde{\mathbf{x}}|y = k)\mathbb{P}(y = k)}{\rho(\tilde{\mathbf{x}})}$$

Then

$$\arg\max_{k\in\mathcal{Y}} \mathbb{P}(y=k\mid \mathbf{x}=\tilde{\mathbf{x}}) = \arg\max_{k\in\mathcal{Y}} \rho(\tilde{\mathbf{x}}|y=k) \mathbb{P}(y=k)$$

- Then we can estimate these conditional densities and the prior probs, and classify via them
- This idea we will see in so-called "generative approaches" for classification, so in LDA, QDA, etc.



EXAMPLE

- Assume $\mathbb{P}(y=1)=\frac{1}{2}$
- And conditional densities of x per class as normal

$$\begin{cases} \phi_{\mu_1,\sigma^2}(x) & \text{for } y = 0 \\ \phi_{\mu_2,\sigma^2}(x) & \text{for } y = 1 \end{cases}$$

• Bayes optimal classifier = orange; Bayes error = red

