

Game-theoretic Foundations of Multi-agent Systems

Lecture 8: Bayesian Games

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Outline

1. Introduction and Definitions
2. Strategies and Equilibria
3. Auctions
4. Extensive-form Games of Incomplete-Info



Bayesian Games: Games of Incomplete Information

- So far, we assumed **all agents know** what game they are playing
 - Number of agents
 - Actions available to each agent
 - Utilities associated with each outcome
- In extensive-form games, **taken actions** could be unknown, but **game itself** is
- **Bayesian games** allow us to represent uncertainties about game
 - **Commonly known probability distribution** over possible games



Assumptions

- All games have **same number of agents** and **same action sets** for each agents
- Possible games only differ in agents' utilities for each outcome
- Beliefs are **posteriors**, obtained by conditioning common prior on private signals



Bayesian Games: Formal Definition

- N is finite set of agents
- A_i is set of actions available to agent i
- Θ_i is type space of agent i
- $p : \Theta \mapsto [0, 1]$ is common prior over types
- $u_i : A \times \Theta \mapsto \mathbb{R}$ is utility function for agent i



Example I: Bayesian Entry-deterrence Game

- Firm 1 decides whether to fight, Firm 2 decides whether to enter
- Firm 1 knows its cost
- Firm 2 is uncertain if 1's cost is 4 w.p. p or 1 w.p. $1 - p$
- Game takes one of following two forms

	Enter	Stay out
Fight	0, -1	2, 0
Don't fight	2, 1	3, 0

θ_{11} : High Cost

	Enter	Stay out
Fight	3, -1	5, 0
Don't fight	2, 1	3, 0

θ_{12} : Low Cost

- $\Theta_1 = \{\theta_{11}, \theta_{12}\}$ and $\Theta_2 = \{\theta_{21}\}$



Example II

	θ_{21}	θ_{22}								
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Types: Discussion

- Types encapsulate information possessed by agents that is **not** common knowledge
 - E.g., agents' knowledge of their private utility function
- Type could also include
 - Agent's beliefs about other agents' utilities
 - Other agents' beliefs about the agent's own utility
 - And any other higher-order beliefs



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Strategies

- Before the game starts, agents only know the common prior
- Agent i 's strategy is $s_i : \Theta_i \mapsto \Delta(A_i)$ is contingency plan for all $\theta_i \in \Theta_i$
- $s_i(\theta_i)$ specifies agent i 's (mixed) strategy when i 's type is θ_i
- $s_i(a_i \mid \theta_i)$ specifies probability of agent i taking action a_i when i 's type is θ_i
- Type of agents is revealed to them once the game starts
- Once agents know their type, they follow their strategy for that particular type



Expected Utilities

- We can calculate expected utility depending on what agents know



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- **Ex ante**: Agents only know the common prior on types (before the game starts)
- **Interim**: Agents only knows about their own type (after types are reveals)
- **Ex post**: Agents know everyone's type (hypothetical – before they take actions)

Expected Utilities (cont.)

- **Ex-post** expected utility (a):

$$EU_i(s, \theta) = \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j \mid \theta_j) \right) u_i(a, \theta)$$



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- **Interim** expected utility:

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- **Ex-ante** expected utility:

$$EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s, \theta_i) = \sum_{\theta \in \Theta} p(\theta) EU_i(s, \theta)$$

Dominated Strategies

- **Ex-ante dominated strategy:** Alternative strategy provides greater ex ante utility regardless of all other agents' strategies
- **Interim dominated strategy:** For a given type, alternative strategy provides greater interim utility regardless of all other agents' strategies



Best Response in Bayesian Games

- Agent i 's **best response** to strategy s_{-i} is

$$BR_i(s_{-i}) = \operatorname{argmax}_{s_i} EU_i(s_i, s_{-i})$$

- To play best response, i must know strategy of **all agents** for **each of their types**
- Without this information, it is not possible to evaluate $EU_i(s_i, s_{-i})$



Best Response in Bayesian Games (cont.)

- Best response is defined based on agent i 's **ex ante** expected utility, $EU_i(s_i, s_{-i})$



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$$BR_i(s_{-i}) = \operatorname{argmax}_{s_i} \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s_i, s_{-i}, \theta_i)$$



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- Observe that $EU_i(s_i, s_{-i}, \theta_i)$ **does not depend on** $s_i(\theta'_i)$ for all $\theta'_i \neq \theta_i$



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- So, maximizing $EU_i(s_i, s_{-i})$ is equal to maximizing $EU_i(s_i, s_{-i}, \theta_i)$ for all $\theta_i \in \Theta_i$
- Intuitively, if certain action is best after a signal is revealed, it is also the best **conditional plan** devised **ahead of time** for what to do should that signal be received

Bayes-Nash Equilibrium

- Bayes-Nash equilibrium (BNE) is strategy profile s^* , such that

$$s_i^* \in BR_i(s_{-i}^*) \quad \forall i$$

- [Theorem] Any finite Bayesian game has BNE



Example

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$$EU_2(UD, LR) = \sum_{\theta \in \Theta} p(\theta) EU_2(UD, LR, \theta)$$

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$$\begin{aligned}
 EU_2(UD, LR) &= \sum_{\theta \in \Theta} p(\theta) EU_2(UD, LR, \theta) \\
 &= p(\theta_{11}, \theta_{2,1}) u_2(U, L, \theta_{11}, \theta_{2,1}) + p(\theta_{11}, \theta_{2,2}) u_2(U, R, \theta_{11}, \theta_{2,2}) + \\
 &\quad p(\theta_{12}, \theta_{2,1}) u_2(D, L, \theta_{12}, \theta_{2,1}) + p(\theta_{12}, \theta_{2,2}) u_2(D, R, \theta_{12}, \theta_{2,2}) \\
 &= 0.3 \times 0 + 0.1 \times 3 + 0.2 \times 0 + 0.4 \times 2 = 1.1
 \end{aligned}$$



Example (cont.)

- Continuing in this manner, complete payoff matrix can be constructed as

	LL	LR	RL	RR
UU	2, 1	1, 0.7	1, 1.2	0, 0.9
UD	0.8, 0.2	1, 1.1	0.4, 1	0.6, 1.9
DU	1.5, 1.4	0.5, 1.1	1.7, 0.4	0.7, 0.1
DD	0.3, 0.6	0.5, 1.5	1.1, 0.2	1.3, 1.1



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- Note that row agent's best response to RL is DU

Example (cont.)

- Once row agent receives the signal θ_{11} , we can calculate interim utilities

	LL	LR	RL	RR
UU	2, 0.5	1.5, 0.75	0.5, 2	0, 2.25
UD	2, 0.5	1.5, 0.75	0.5, 2	0, 2.25
DU	0.75, 1.5	0.25, 1.75	2.25, 0	1.75, 0.25
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- Row agent's payoffs are now **independent** of action taken upon observing θ_{12}
- Note that DU is **still best response** to RL
- What has changed is how much better it is compared to other strategies

Ex-post Equilibrium

- Strategy profile s^* is **ex-post equilibrium** if

$$s_i^* \in \operatorname{argmax}_{s_i} EU_i(s_i, s_{-i}^*, \theta) \quad \forall i, \theta \in \Theta$$



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 - Agents are not assumed to know θ
 - Even if they knew θ , agents would never want to deviate
 - Ex-post equilibrium is **not guaranteed** to exist

Example: Incomplete Information Cournot

- Two firms decide on their production level $q_i \in [0, \infty)$
- Price is given by $P(q)$ where $q = q_1 + q_2$
- Firm 1 has marginal cost equal to c which is common knowledge
- Firm 2's marginal cost is private information
 - c_L with probability x and c_H with probability $(1 - x)$, where $c_L < c_H$
- Utility of agents are ($t \in \{L, H\}$ type of firm 2)
 - $u_1((q_1, q_2), t) = q_1 P(q_1, q_2) - c$
 - $u_2((q_1, q_2), t) = q_2 P(q_1, q_2) - c_t$



Example: Incomplete Information Cournot (cont.)

- What are firms best responses?

$$B_1(q_L, q_H) = \arg \max_{q \geq 0} \left((xP(q + q_L) + (1 - x)P(q + q_H) - c)q \right)$$

$$B_2^L(q_1) = \arg \max_{q \geq 0} \left((P(q_1 + q) - c_L)q \right)$$

$$B_2^H(q_1) = \arg \max_{q \geq 0} \left((P(q_1 + q) - c_H)q \right)$$

- BNE of this game is vector (q_1^*, q_L^*, q_H^*) such that

$$q_1^* \in B_1(q_L^*, q_H^*), q_L^* \in B_2^L(q_1^*), q_H^* \in B_2^H(q_1^*)$$



Example: Incomplete Information Cournot (cont.)

- For example, if $P(q) = \max(\alpha - q, 0)$, then we have:

$$q_1^* = \frac{1}{3}(\alpha - 2c + xc_L + (1-x)c_H)$$

$$q_L^* = \frac{1}{3}(\alpha - 2c_L + c) - \frac{1}{6}(1-x)(c_H - c_L)$$

$$q_H^* = \frac{1}{3}(\alpha - 2c_H + c) + \frac{1}{6}x(c_H - c_L)$$



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- Extracting private valuations could be challenging
- E.g., giving painting for free to bidder with highest valuation would create incentive for all bidders to overstate their valuations

Different Auctions

- **English auction:** bid must be higher than previous one, last bidder wins, pays last bid



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- **Japanese auction:** price rises, bidders drop out, last bidder wins at price of last dropout



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- **Japanese auction:** price rises, bidders drop out, last bidder wins at price of last dropout
- **First-price auction:** bidders bid simultaneously, highest bid wins, winner pays winning bid
- **Second-price action:** similar to first price, except that winner pays second highest bid

Valuations

- **Private valuations:** valuation of each bidder is independent of others' valuations
- **Common valuations:** bidders' valuations are correlated to common value



Sealed-bid Auctions (First- and Second-price Auctions)

- Suppose that there are N bidders and single object for sale



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- Bidders only know their own realized value (type)
- Bidders are risk neutral, maximizing their expected utility
- Pure strategy for bidder i is map $b_i : [0, \bar{v}] \rightarrow \mathbb{R}_+$

Second-price Auction

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- Agent i submit bid b_i simultaneously with other agents
- Agent with highest bid wins, and pays second highest bid
- Agent i 's profit is $v_i - \max_{j \neq i} b_j$ if i wins, and 0 otherwise
- [Proposition] Truthful bidding (i.e., $b_i = v_i$) is BNE in second price auction



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- Agent i submit bid b_i simultaneously with other agents
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 - If other bidders bid truthfully, does loser want to change their bid?

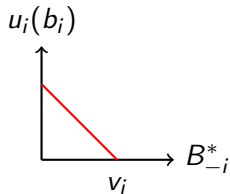
Truthful Bidding in Second-price Auction

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- **[Proof sketch]** Define maximum bid excluding i 's bid as $B_{-i}^* = \max_{j \neq i} b_j$



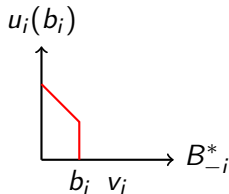
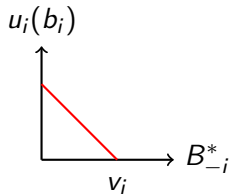
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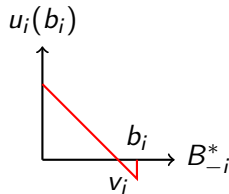
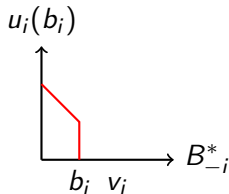
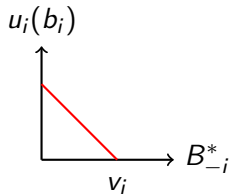
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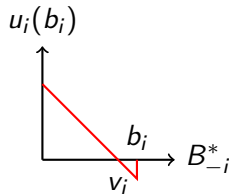
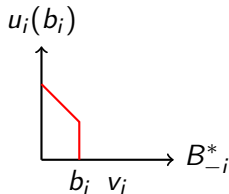
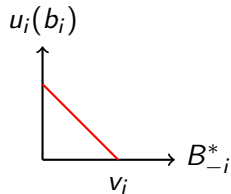
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- Truthful equilibrium is also the unique BNE



Expected Payment in Second-price Auctions

- Define random variable $y_i = \max_{j \neq i} v_j$
 - CDF of y_i is $G_{y_i}(v) = F(v)^{N-1}$
 - PDF of y_i is $g_{y_i}(v) = (N-1)f(v)F(v)^{N-2}$
- Expected payment of bidder i with value v_i is given by

$$\begin{aligned} p(v_i) &= P(v_i \text{ wins}) \times \mathbb{E}[y_i \mid y_i \leq v_i] \\ &= P(y_i \leq v_i) \times \mathbb{E}[y_i \mid y_i \leq v_i] \\ &= G_{y_i}(v_i) \times G_{y_i}(v_i)^{-1} \int_0^{v_i} y g_{y_i}(y) dy = \int_0^{v_i} y g_{y_i}(y) dy \end{aligned}$$



First-price Auctions

- Utility of agent i is $v_i - b_i$ if $b_i > \max_{j \neq i} b_j$ and zero otherwise



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- This implies that bidder i wins whenever $y_i < \beta^{-1}(b_i)$
- Optimal bid of bidder i is $b_i = \operatorname{argmax}_{b \geq 0} G_{y_i}(\beta^{-1}(b))(v_i - b)$

First-price Auctions (cont.)

- First-order (necessary) **optimality conditions** imply¹:

$$\frac{g_{y_i}(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}(v_i - b_i) - G_{y_i}(\beta^{-1}(b_i)) = 0$$

- In symmetric equilibrium, $b_i = \beta(v_i)$, therefore we have:

$$v_i g_{y_i}(v_i) = \beta'(v_i) G_{y_i}(v_i) + \beta(v_i) g_{y_i}(v_i) = \frac{d}{dv} (\beta(v_i) G_{y_i}(v_i))$$

- With **boundary condition** $\beta(0) = 0$, we have:

$$\beta(v_i) = G_{y_i}^{-1}(v_i) \int_0^{v_i} y g_{y_i}(y) dy = \mathbb{E}[y_i \mid y_i \leq v_i]$$

¹Derivative of $\beta^{-1}(b)$ is $1/\beta'(\beta^{-1}(b))$.



Expected Payment in First-price Auctions

- Expected payment of bidder i with value v_i is:

$$\begin{aligned} p(v_i) &= P(v_i \text{ wins}) \times \beta(v_i) \\ &= P(y_i \leq v_i) \times \mathbb{E}[y_i \mid y_i \leq v_i] \\ &= G_{y_i}(v_i) \times G_{y_i}(v_i)^{-1} \int_0^{v_i} y g_{y_i}(y) dy = \int_0^{v_i} y g_{y_i}(y) dy \end{aligned}$$

- This establishes somewhat surprising results that both first and second price auction formats yield **same expected revenue** to auctioneer



Revenue Equivalence

- In **standard auctions**, item is sold to bidder with highest submitted bid



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- Suppose that values are *i.i.d* and all bidders are risk neutral



Revenue Equivalence

- In **standard auctions**, item is sold to bidder with highest submitted bid
- Suppose that values are *i.i.d* and all bidders are risk neutral
- **[Theorem]** Any symmetric and increasing equilibria of any standard auction (such that expected payment of bidder with value zero is zero) yields same expected revenue to auctioneer

Oil-field Example: Common Values with Correlated Recommendations

- Suppose that there are two bidders bidding to lease oil field
- Oil field could be worth \$0, \$25M, or \$50M w.p. 0.25, 0.5, and 0.25, respectively
- Bidders hire their own consultant to evaluate value of oil field
- Bidders get private recommendations, r_1 and r_2
- If field is worth \$0, then $r_1 = r_2 = L$
- If field is worth \$25M, then $r_1 = H, r_2 = L$ or $r_1 = L, r_2 = H$ (both equally likely)
- If field is worth \$50M, then $r_1 = r_2 = H$
- Given their private recommendation, how should bidders bid?



Oil-field Example: Expected Value

- What is expected value of oil field if one receives L recommendation?
- Given L , oil field is worth either \$0 or \$25



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- Bidding \$0 leads to utility \$0 and is **profitable deviation**
- **Truthful bidding is not BNE in second-price auction with common values and dependent recommendations**

Winner's Curse

- Winning means bidder received highest or **most optimistic** recommendation
- Condition on winning, value of item is lower than what recommendation says
- Ignoring this leads to paying, on average, **more than** true value of item
- To avoid this curse, bidders should assume their recommendation is optimistic
- In oil-field example, we can show that the following bidding strategy is BNE
 - Bid 0 upon receiving L
 - Bid \$50M upon receiving H



Oil-field Example II: Common Values and Independent Recommendations

- Consider two bidders interested in buying oil field that has part A and B
- Each bidder values A and B but is more interested in one of them
- Bidders hire their own consultant to evaluate value of their part
- Bidder 1 gets private recommendation r_1 about value of part A
- Bidder 2 gets private recommendation r_2 about value of part B
- Suppose that both recommendations are **uniformly distributed** over $[0, 1]$
- Suppose value of oil field to each bidder is as follows
 - $v_i = a.r_i + b.r_{-i}$ with $a \geq b \geq 0$
 - Private values are **special case** where $a = 1$ and $b = 0$



Oil-field Example II: Second-price Auction

- Similar to previous example, **truthful bidding is not BNE**



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- Instead, we show that both bidders following $\beta(r_i) = (a + b)r_i$ is BNE
- If $-i$ follows this, then probability that i wins by bidding b_i is:

$$P(\beta(r_{-i}) < b_i) = P((a + b)r_{-i} < b_i) = b_i / (a + b)$$



Oil-field Example II: Second-price Auction

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- Bidder i 's payment if i wins is $\beta(r_{-i}) = (a + b)r_{-i}$

Oil Field Example II: Second-price Auction (cont.)

- Expected payment of i condition on i winning is:

$$\mathbb{E}[(a+b)r_{-i} \mid r_{-i} < b_i/(a+b)] = b_i/2$$

- Expected value of $-i$'s signal condition on i winning is:

$$\mathbb{E}[r_{-i} \mid r_{-i} < b_i/(a+b)] = b_i/2(a+b)$$

- Expected utility of bidding b_i for recommendation r_i is

$$\begin{aligned} EU(b_i, r_i) &= P(b_i \text{ wins}) \times (a.r_i + b.\mathbb{E}[r_{-i} \mid b_i \text{ wins}] - \mathbb{E}[(a+b)r_{-i} \mid b_i \text{ wins}]) \\ &= b_i/(a+b) \times (a.r_i + b.b_i/2(a+b) - b_i/2) \end{aligned}$$

- Maximizing this with respect to b_i (for given r_i) leads to $b_i^* = (a+b)r_i$



Oil Field Example II: First-price Auction

- Analysis is similar to that of first-price auctions with private values
- It can be shown that unique symmetric BNE is for each bidder to bid $\beta(r_i) = (a + b)r_i/2$
- It can be shown that expected revenue is equal to first price auction
- Revenue equivalence principle **continues to hold** for common values



Outline

1. Introduction and Definitions
2. Strategies and Equilibria
3. Auctions
4. Extensive-form Games of Incomplete-Info



Incomplete Information in Extensive-form Games

- Incomplete-information games cannot always be represented as static games



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- Extensive-form games can capture explicit order of moves or dynamic games



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- Incomplete-information games cannot always be represented as **static games**
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- We can use **information sets** to represent what each agent knows



Incomplete Information in Extensive-form Games

- Incomplete-information games cannot always be represented as static games
- Extensive-form games can capture explicit order of moves or dynamic games
- We can use information sets to represent what each agent knows
- We need to modify BNE to include notion of perfection (as in subgame perfection)

Equilibrium Concepts

		Timing	
		Simultaneous	Sequential
Information	Complete	Nash	SPE
	Incomplete	Bayesian Nash	



Equilibrium Concepts

		Timing	
		Simultaneous	Sequential
Information	Complete	Nash	SPE
	Incomplete	Bayesian Nash	Perfect Bayesian



Extensive-form Games of Incomplete Information: Definition

- $N, A, H, Z, \alpha, \beta, \rho, u$, and I are the same as extensive-form games
- Θ_i is type space of agent i
- $p : \Theta \mapsto [0, 1]$ is common prior over types
- $u_i : Z \times \Theta \mapsto \mathbb{R}$ is utility function for agent i

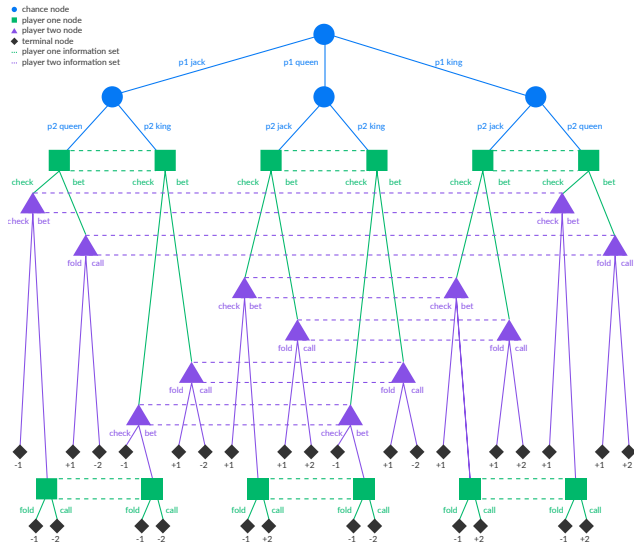


The “Nature” with Chance Moves

- To capture common prior, we can add special agent called **Nature**
- Nature makes **probabilistic choices**
- Nature **does not** have utility function (can be viewed as having constant utility)
- Nature has unique strategy of randomizing in **commonly known** way
- Agents receive individual signals about Nature's choice



Example: Kune Poker



Beliefs and Strategies

- Agents have **beliefs** about which node they are for each information set (**info**set)
- For each info



Requirements for Perfect Bayesian Equilibrium (PBE)

- I. Beliefs: In addition to strategy profile s , beliefs μ must be specified



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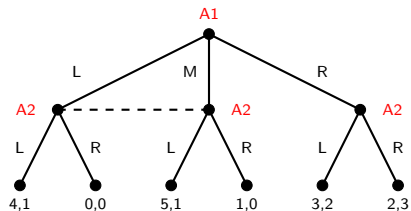
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- III. **On-the-path consistency**: For any on-the-equilibrium-path info set, μ must be derived from s according to **Bayes' rule**
- IV. **Off-the-path consistency**: For any off-the-equilibrium-path info set, μ must be derived from s according to Bayes' rule **whenever possible**

Weak and Strong PBE

- I-III define **weak PBE**, and I-IV define strong PBE
- PBE is defined for all extensive-form games with imperfect information

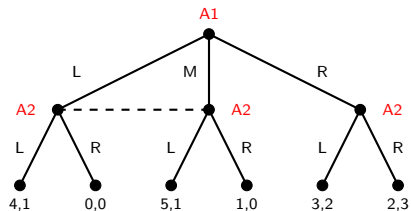


Example I (from Lecture 5)



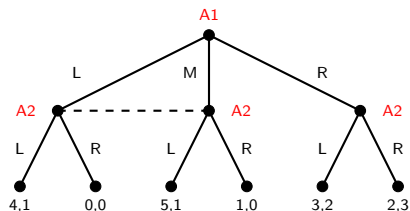
- $(R, (R, R))$ is NE and SPE, but it is not PBE, why?

Example I (from Lecture 5)



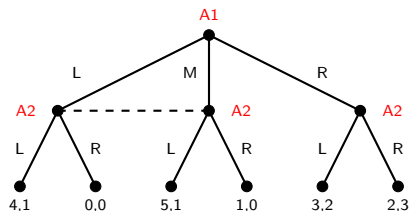
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 - R in A2's left-side info set is not optimal for any belief of A2

Example I (from Lecture 5)



- $(R, (R, R))$ is NE and SPE, but it is not PBE, why?
 - R in A2's left-side info set is not optimal for any belief of A2
- $(M, (L, R))$ + believing that A1 takes M with probability 1 is weak PBE

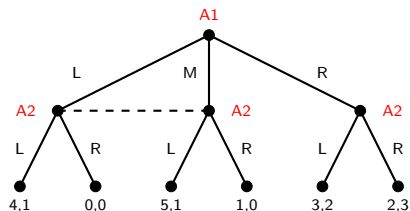
Example I (from Lecture 5)



- $(R, (R, R))$ is NE and SPE, but it is not PBE, why?
 - R in A2's left-side info set is not optimal for any belief of A2
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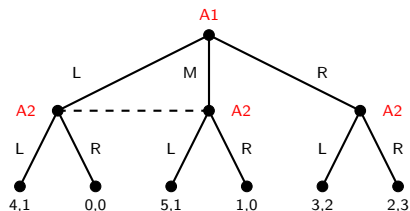
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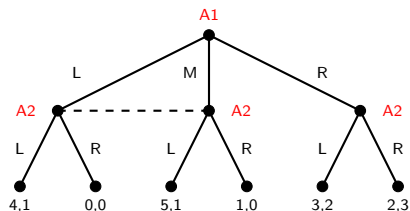
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- $(M, (L, R))$ + believing that A1 takes M with probability 1 is also strong PBE



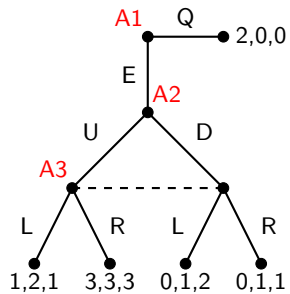
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 - On-the-path beliefs are **consistent** with the equilibrium strategy
- $(M, (L, R))$ + believing that A1 takes M with probability 1 is also strong PBE
 - Off-the-path beliefs are also consistent (right-side info set has single node)

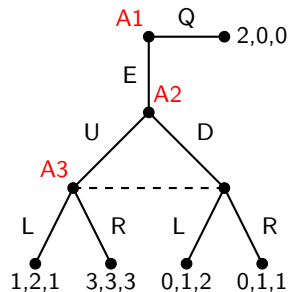
Example II: Strong vs. Weak PBE

- U is A2's dominant strategy



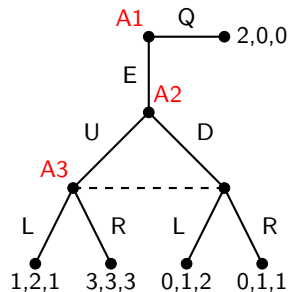
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- U is A2's dominant strategy
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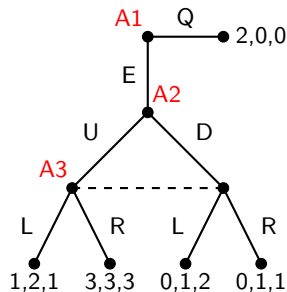
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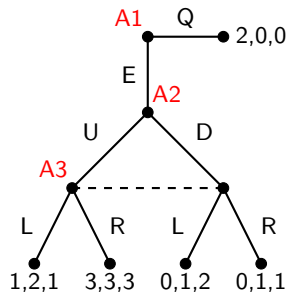
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- U is A2's dominant strategy
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- (E, U, R) + A3 believing that A2 takes U w.p 1 is PBE (S&W)



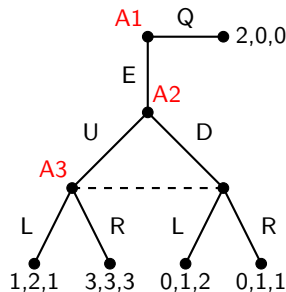
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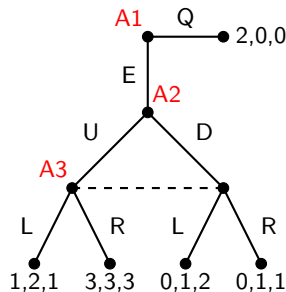
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- What about (Q, U, L) + A3 believing that A2 takes D w.p. 1?
- Q is best respond to (U, L) and U is dominant strategy



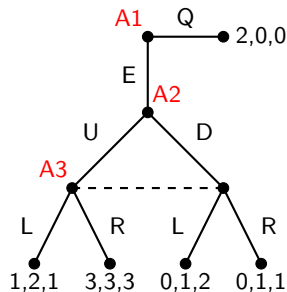
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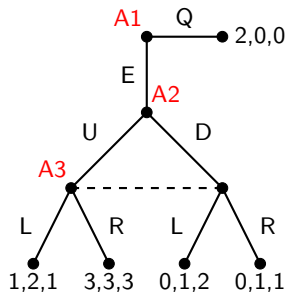
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- So, it is weak PBE, but is it also strong PBE?

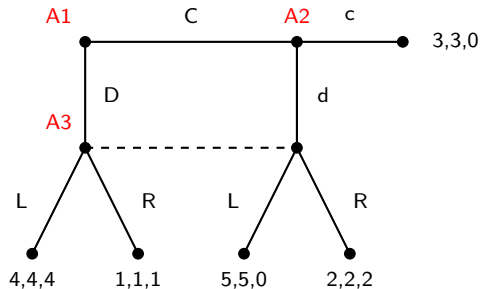


Example II: Strong vs. Weak PBE

- U is A2's dominant strategy
- NE of A2's subgame is (U, R)
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- (E, U, R) + A3 believing that A2 takes U w.p 1 is PBE (S&W)
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- Q is best respond to (U, L) and U is dominant strategy
- L is best respond to believing that A2 takes D w.p. 1
- So, it is weak PBE, but is it also strong PBE?
- No! IV does not hold; A3's belief is **inconsistent** with A2's strategy



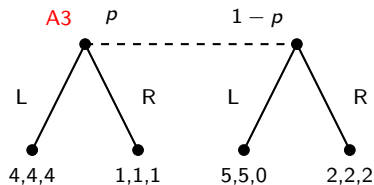
Example II: Selten's Horse



Reinhard Selten²
(1930-2016)

¹Photograph by Stefan Schickler

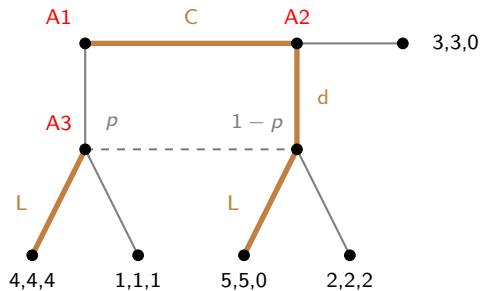
Example II: Selten's Horse (cont.)



- A3 believes that left and right nodes are reached w.p. p and $1 - p$, respectively
- Utility for playing L is $4p$ and $p + 2(1 - p)$ for playing R
- A3 must play R if $p < 2/5$, R or L if $p = 2/5$, and L if $p > 2/5$

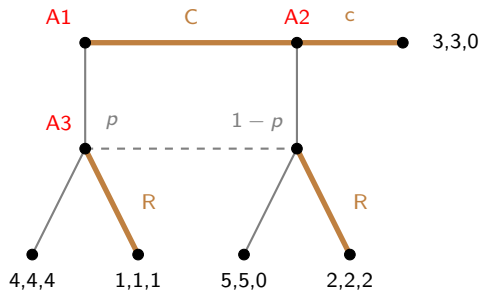


Example II: Selten's Horse (cont.)



- Is there any p with which (C, d, L) is weak PBE?
- Given (C, d), on-the-path belief for A3 must set $p = 0$
- For $p = 0$, A3 must take R, so the answer is **NO**

Example II: Selten's Horse (cont.)



- Is there any p with which (C, c, R) is weak PBE?
- Given (C, c), A3's info set is off the equilibrium path
- **Consistency** does not put any constraint on p ; **optimality** of R requires $p \leq 2/5$
- Is (C, c, R) + $p \leq 2/5$ strong PBE? Why?



Example III: Signaling Games

- **Informed** agent moves first to **signal** some information to uninformed agent
- Sending signal is more costly if it conveys false information
- E.g., producer provides warranty to signal that its products are unlikely to break
- E.g., employees acquire college degree to signal their ability to employers
- This is different from sending costless **messages** in **cheap talk** games
- Cheap talk is communication between agents that does not directly affect payoffs
- E.g., agents message each other on where they want to go in Battle of the Sexes



PBE Types in Signaling Games

- **Separating**: Informed agent sends distinct signal for each type



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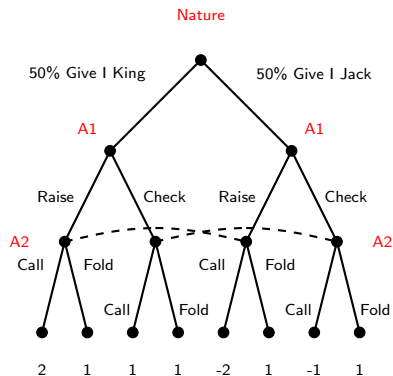


PBE Types in Signaling Games

- **Separating**: Informed agent sends distinct signal for each type
 - Signal always reveals sender's type
 - Receiver's beliefs become deterministic after seeing the signal
- **Pooling**: Informed agent sends the same signal for all types
 - Signal does not give any information to receiver
 - Receiver's beliefs are not updated after seeing the signal
- **Semi-separating** (a.k.a. partially pooling): Informed agent sends same signal for some types distinct signal for some other types

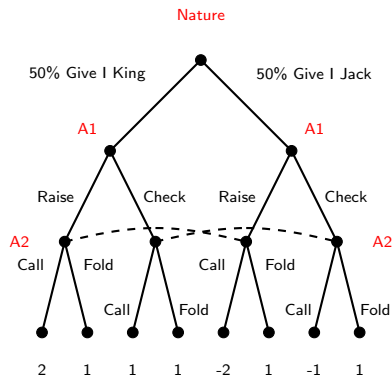
Simple Poker-like Game: Separating PBE

- Consider Raising for King and Checking for Jack



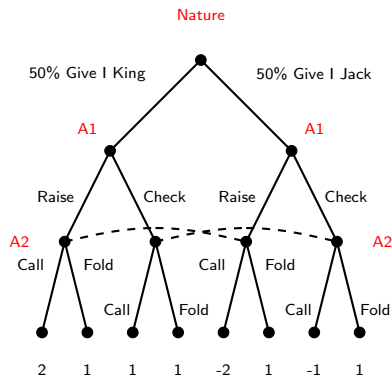
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- Consider Raising for King and Checking for Jack
- What is A2's posterior belief?



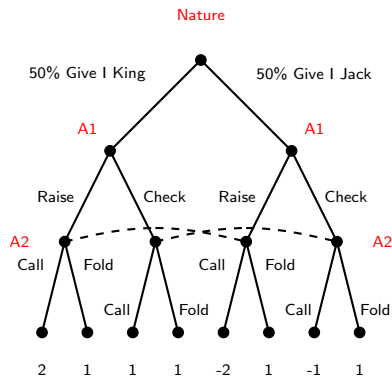
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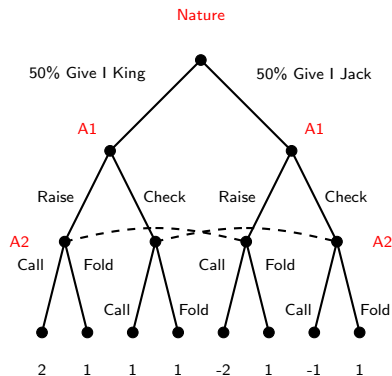
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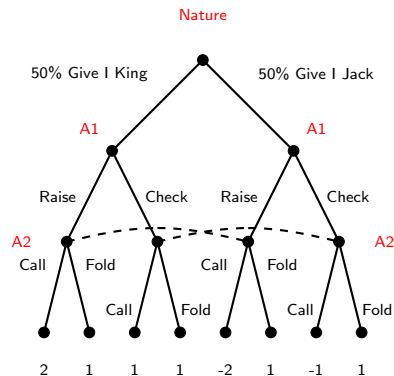
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- What is A2's optimal strategy?



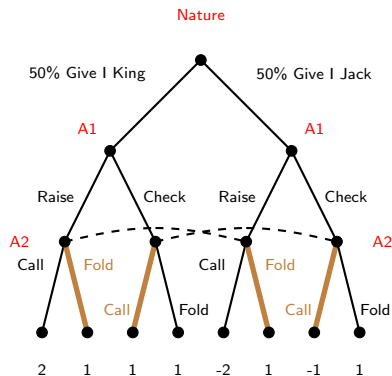
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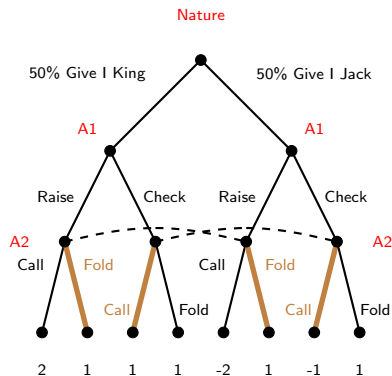
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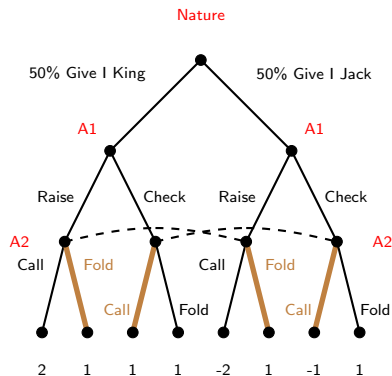
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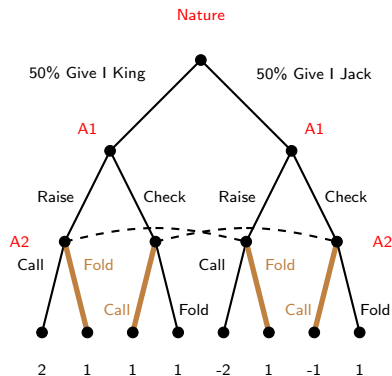
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 - Indifferent between Raise and Check if King ($1 = 1$)
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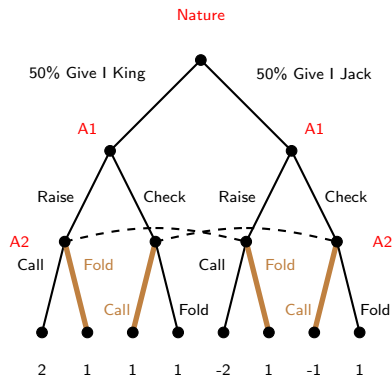
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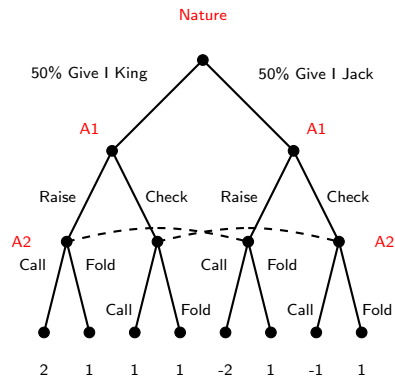
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- How about Checking for King and Raising for Jack?



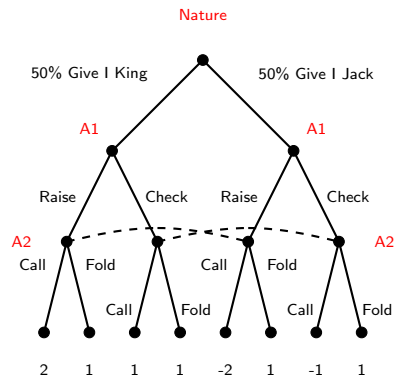
Simple Poker-like Game: Pooling PBE

- Consider Raising for both King and Jack



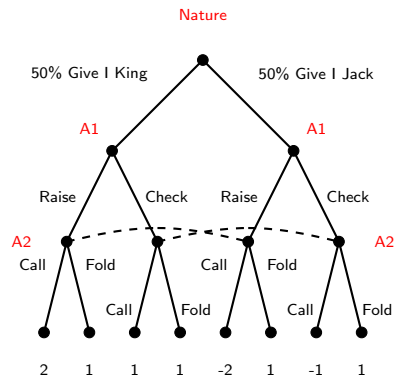
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- Consider Raising for both King and Jack
- A2's posterior beliefs are the same as prior beliefs



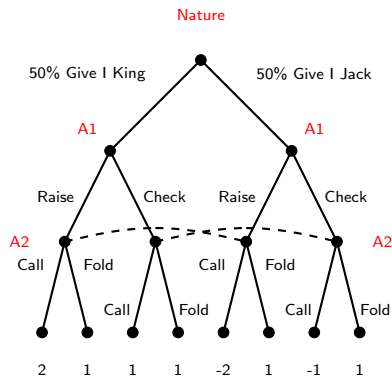
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- Consider Raising for both King and Jack
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 - King w.p. 0.5 and Jack w.p. 0.5



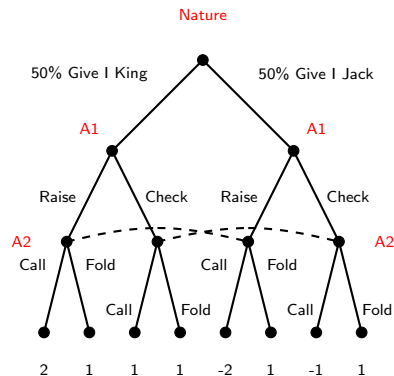
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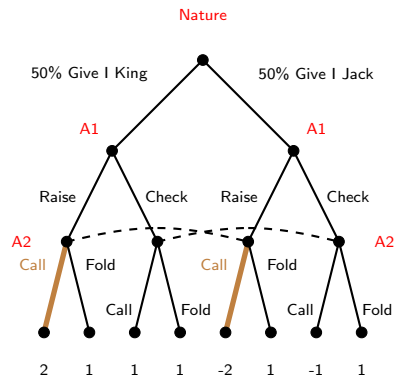
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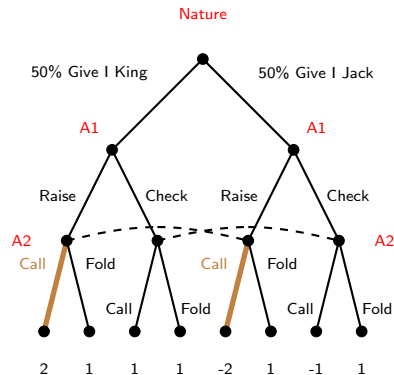
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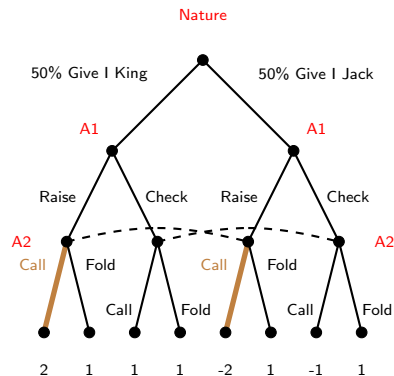
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 - Call give 0 ($-2 \times 0.5 + 2 \times 0.5$), Fold gives -1
 - A2 prefers Call on the equilibrium path
- What is A2's optimal strategy off equilibrium path (Check)?



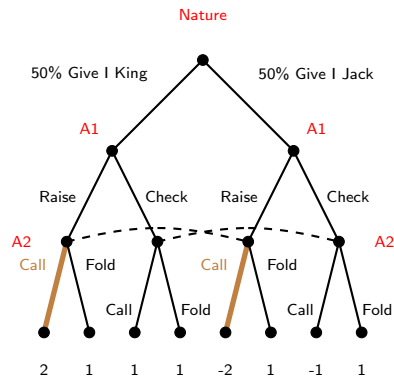
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 - A2 prefers Call on the equilibrium path
- What is A2's optimal strategy off equilibrium path (Check)?
 - Consistency does not put any restriction on beliefs



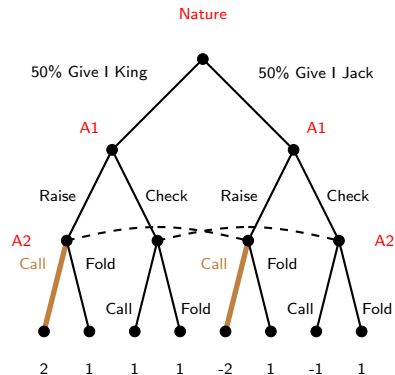
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 - Consider p for King and $1 - p$ for Jack



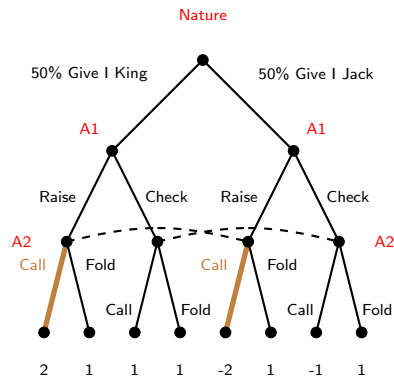
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 - Call give $-p + 1 - p$, Fold gives -1 and



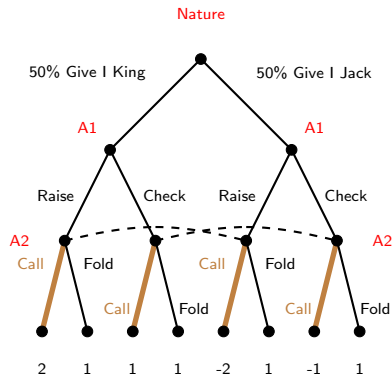
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 - Consider p for King and $1 - p$ for Jack
 - Call give $-p + 1 - p$, Fold gives -1 and
 - For $p < 1$, A2 prefers Call, for $p = 1$, A2 is indifferent



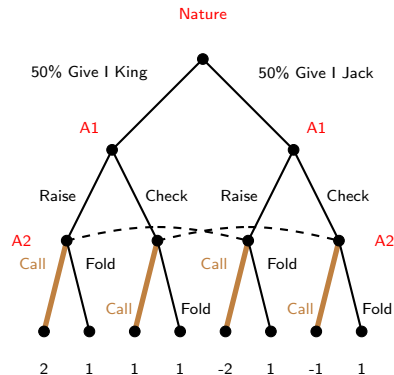
Simple Poker-like Game: Pooling PBE (cont.)

- If A2 Calls ($p \leq 1$), what is A1's best response?



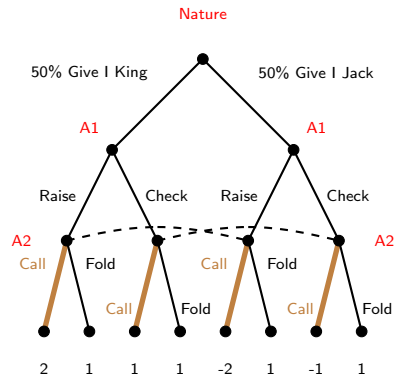
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- If A2 Calls ($p \leq 1$), what is A1's best response?
 - If King, A1 prefers Raise



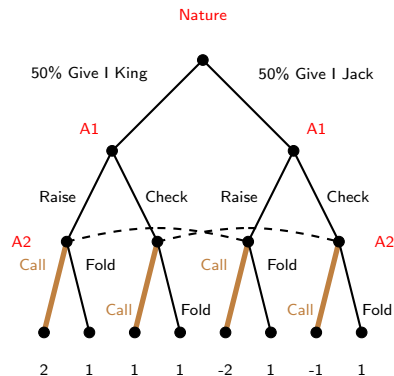
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- If A2 Calls ($p \leq 1$), what is A1's best response?
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 - If Jack, A1 prefers Check



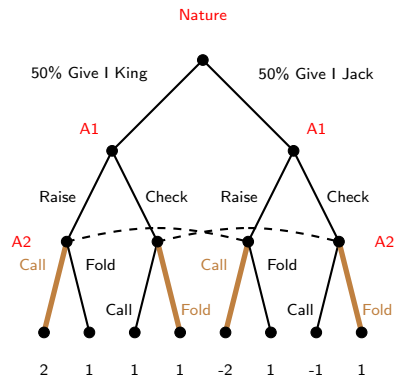
Simple Poker-like Game: Pooling PBE (cont.)

- If A2 Calls ($p \leq 1$), what is A1's best response?
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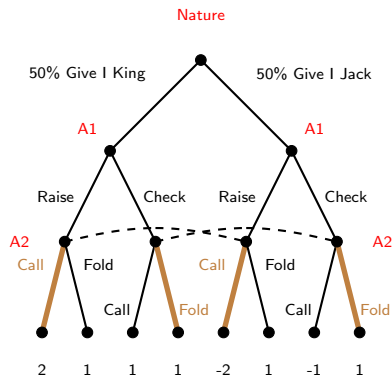
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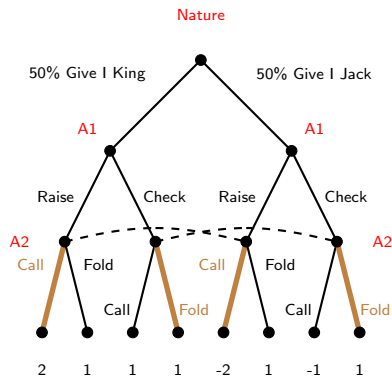
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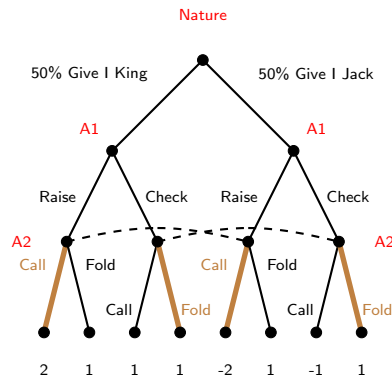
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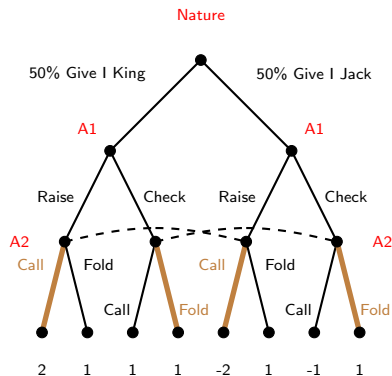
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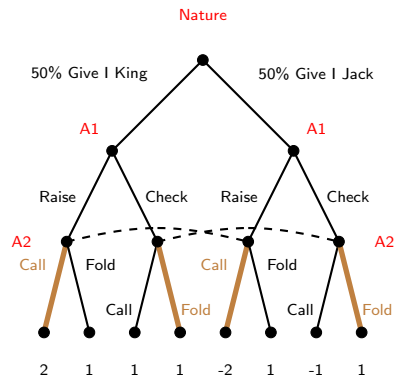
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- What if A2 Calls on and Folds off the path (for $p = 1$)?
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 - A1 wants to **deviate** from pooling strategy
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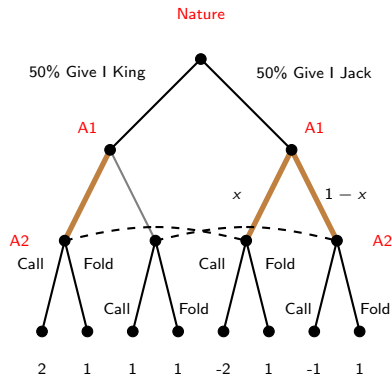
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- There is no p for which A1 wants to follow pooling
- What about Checking for both King and Jack?



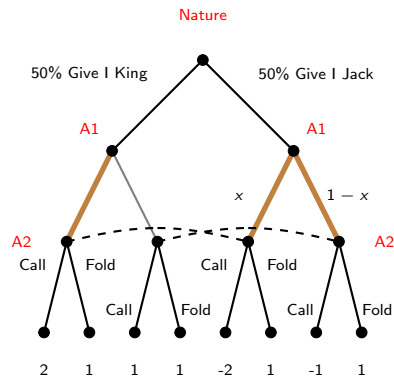
Simple Poker-like Game: Semi-separating PBE

- If King, A1 Raises - If Jack, A1 Raises w.p x



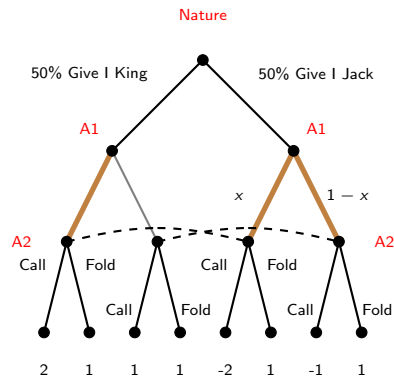
Simple Poker-like Game: Semi-separating PBE

- If King, A1 Raises - If Jack, A1 Raises w.p x
- What is A2's posterior belief?



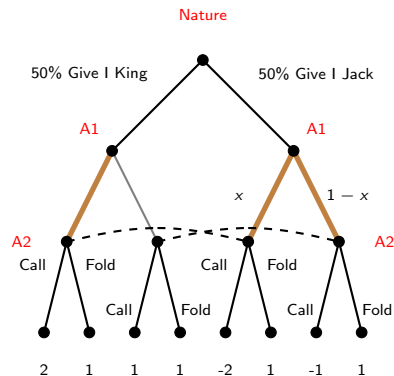
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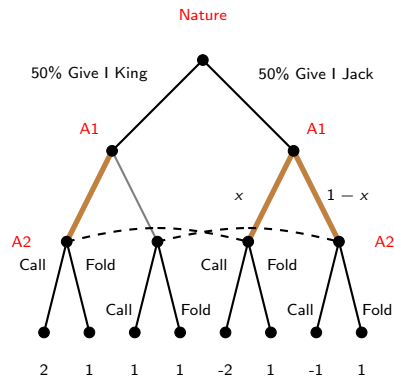
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- If King, A1 Raises - If Jack, A1 Raises w.p x
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 - If Raise, King w.p. $1/(1+x)$ and Jack w.p. $x/(1+x)$



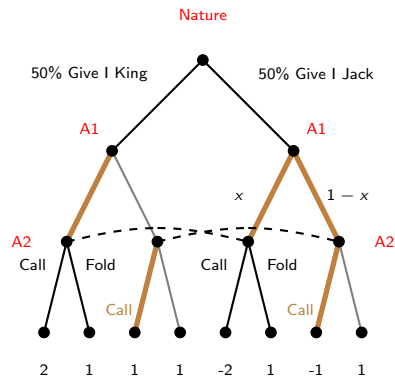
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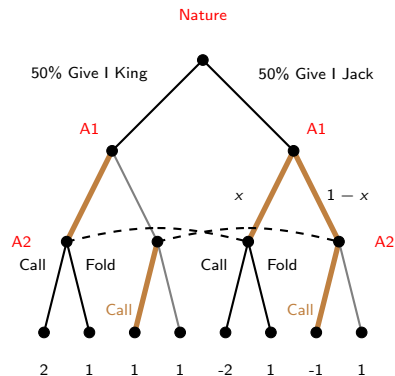
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 - A2 must Call (A2 believes Jack w.p. 1)



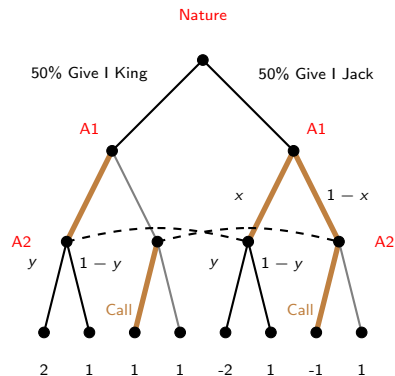
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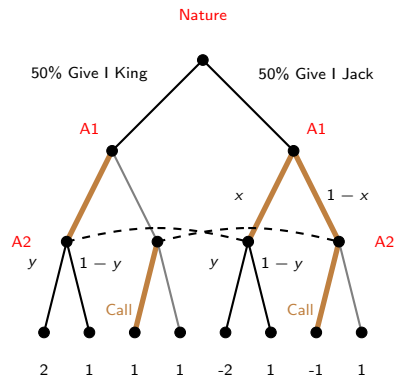
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 - Suppose A2 Calls w.p. y if A1 Raises



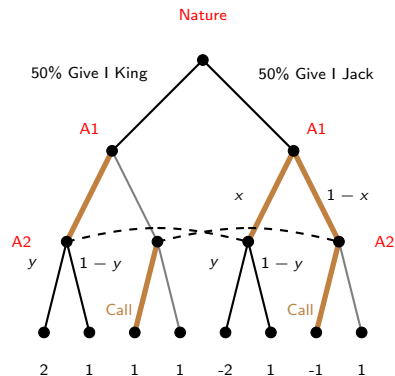
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 - Suppose A2 Calls w.p. y if A1 Raises
 - A1's utility for Raise is $-2y + 1 - y$



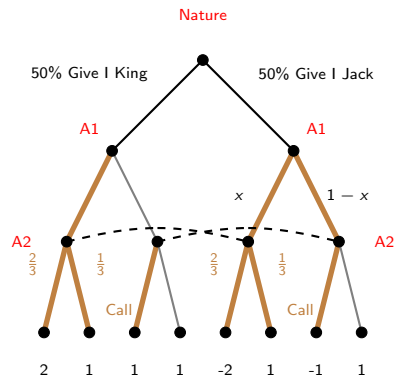
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 - Suppose A2 Calls w.p. y if A1 Raises
 - A1's utility for Raise is $-2y + 1 - y$
 - A1's utility for Check is -1

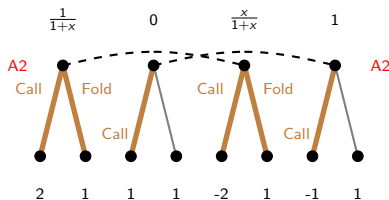


Simple Poker-like Game: Semi-separating PBE

- If King, A1 Raises - If Jack, A1 Raises w.p. x
- What is A2's posterior belief?
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- What is A2's best response if A1 Checks?
 - A2 must Call (A2 believes Jack w.p. 1)
- A2's strategy should make A1 indifferent if Jack
 - Suppose A2 Calls w.p. y if A1 Raises
 - A1's utility for Raise is $-2y + 1 - y$
 - A1's utility for Check is -1
 - $y = 2/3$ makes A1 indifferent



Simple Poker-like Game: Semi-separating PBE (cont.)



- x should be set s.t. A2 is indifferent between Call and Fold
- If A1 Raises, A2's utility for Call is $(2x - 2)/(1 + x)$
- If A1 Raises, A2's utility for Fold is -1
- $x = 1/3$ makes A2 indifferent between Call and Fold



Simple Poker-like Game: Final Semi-separating PBE

- A1 Raises w.p. 1 if King and w.p. $1/3$ if Jack
- A1 Checks w.p. 0 if King and w.p. $2/3$ if Jack
- A2 Calls w.p. 1 if A1 Checks and w.p. $2/3$ if A1 Raises
- A2 Folds w.p. 0 if A1 Checks and w.p. $1/3$ if A1 Raises
- If A1 Raises, A2 believes King w.p. $3/4$ and Jack w.p. $1/4$
- If A1 Checks, A2 believes Jack w.p. 1



Acknowledgment

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 - William Spaniel [Game Theory 101](#)
- Yiqin Huang helped with importing slides from PowerPoint to \LaTeX

