

Game-theoretic Foundations of Multi-agent Systems

Lecture 3: Games in Normal Form

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Outline

1. Normal-form Games: Definition, Notations, and Examples
2. Dominant Strategy Equilibrium
3. Nash Equilibrium
4. Price of Anarchy
5. Minmax Theorem
6. Rationalizability
7. Correlated Equilibrium



Normal-form Games

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- Agents choose their actions without knowledge of other agents' actions
- Such games are referred to as **strategic-form games** or **normal-form games**

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- Agent i has a utility function u_i that maps outcomes to real numbers

Some Notations

- $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ is an action profile of all agents except i
- $A_{-i} = \prod_{j \neq i} A_j$ is set of action profiles of all agents except i
- $a = (a_i, a_{-i}) \in A$ is another way of denoting an action profile (or an outcome)



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		Agent 2	
		x	y
		a, b	e, f
Agent 1	m	c, d	g, h
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- Each cell indexed by row r and column c contains a pair, (p, q) , where $p = u_1(r, c)$ and $q = u_2(r, c)$.

Example: Matching Pennies

- Each agent has a penny and independently chooses to display either heads or tails
- Agents compare their pennies
- If they are the same, agent 2 pockets both, otherwise agent 1 pockets them

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- **Zero-sum game:** Utility of one agent is negative of utility of other agent

Example: Rock, Paper, Scissors Game

- Three-strategy generalization of the matching-pennies game

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0



Example: Coordination Game

- Two drivers driving towards each other in a country with no traffic rules
- Drivers must independently decide whether to drive on the left or on the right
- If drivers choose the same side (left or right) they have some high utility, and otherwise they have a low utility

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- **Team game:** For all outcomes s , and any pair of agents i and j , it is the case that $u_i(a) = u_j(a)$ (also known as **common-payoff game** or **pure-coordination game**)

Example: Cournot Competition

- Two firms producing a homogeneous good for the same market
- Action of each firm is the amount of good it produces ($a_i \in [0, \infty]$)
- Utility of each firm is its total revenue minus its total cost

$$u_i(a_1, a_2) = a_i p(a_1 + a_2) - c a_i$$



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- $p(\cdot)$ is the price function that maps total production to a price
- c is a unit cost
- E.g., $p(x) = \max(0, 2 - x)$ and $c = 1$



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- Expected utility of agent i for a (mixed) strategy profile $s = (s_1, \dots, s_n)$ is

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j \in N} s_j(a_j)$$

Example

		Agent 2		
		R ($\frac{2}{3}$)	P (0)	S ($\frac{1}{3}$)
Agent 1		R ($\frac{1}{3}$)	0, 0	-1, 1
		P ($\frac{2}{3}$)	1, -1	0, 0
		S (0)	-1, 1	1, -1

- $u_1 = 2/9 \times 0 + 1/9 \times 1 + 4/9 \times 1 - 2/9 \times 1 = 1/3$
- $u_2 = 2/9 \times 0 - 1/9 \times 1 - 4/9 \times 1 + 2/9 \times 1 = -1/3$



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- s_i **weakly dominates** s'_i if
 - $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ for all $s_{-i} \in S_{-i}$, and
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- s_i is strictly/weakly dominated if another strategy strictly/weakly dominates it
- $s = (s_1, \dots, s_n)$ is dominant strategy equilibrium if s_i is dominant strategy for all i

Example: Prisoner's Dilemma

- Two prisoners suspected of a crime are taken to separate interrogation rooms
- Each can either confess to the crime or deny it

	D	C
D	−2, −2	−4, −1
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- Absolute value of utilities are the length of jail term each prisoner gets



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- (C,C) is a strict dominant strategy equilibrium
- The dilemma: (D,D) is better for both prisoners, but they won't play it!

Iterated Elimination of Strictly Dominated Strategies

- All strictly dominated pure strategies can be ignored



Iterated Elimination of Strictly Dominated Strategies

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	L	C	R
U	3, 1	0, 2	0, 0
M	1, 2	1, 1	5, 0
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- Order of elimination does not matter when removing strictly dominated strategies (**Church–Rosser property**)
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- If the procedure ends with a single strategy for each agent, then the game is said to be **dominance solvable**

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 - Because agents do not know what strategies others will play
- However, we can leverage the idea of best response to define what is arguably the most central notion in game theory, the Nash equilibrium

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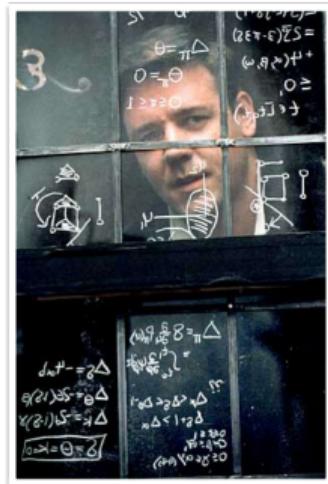


Fig. A Beautiful Mind, from s.hanux.com



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John Forbes Nash Jr.
1928-2015



Example: Battle of Sexes

- Husband and wife wish to meet this evening, but have a choice between two events to attend: football or opera
- Husband would prefer to go to football, wife would prefer opera
- Both would prefer to go to the same event rather than different ones

		Wife	
		Football	Opera
Husband		Football	2,1
		Opera	0,0

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- Are these the only Nash equilibria?



Example: Battle of Sexes (cont.)

	$F (p)$	$O (1 - p)$
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$$u_H(F) = u_H(O) \Rightarrow 2 \times p = (1 - p) \Rightarrow p = 1/3$$



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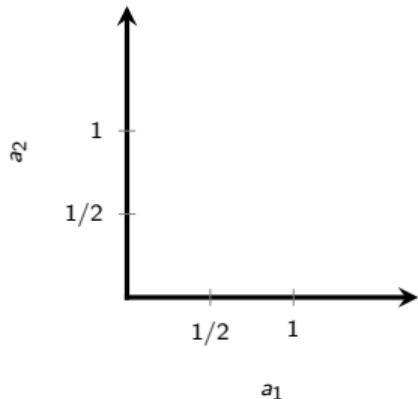
- You can show that the unique mixed-strategy NE is $\{(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\}$

Example: Cournot Competition

- $u_i(a_1, a_2) = a_i \max(0, 2 - a_1 - a_2) - a_i$
- Using first order optimality conditions, we have

$$BR_i(a_{-i}) = \underset{a_i \geq 0}{\operatorname{argmax}} a_i(2 - a_i - a_{-i}) - a_i$$

$$= \begin{cases} (1 - a_{-i})/2 & \text{if } a_{-i} < 1, \\ 0 & \text{Otherwise.} \end{cases}$$

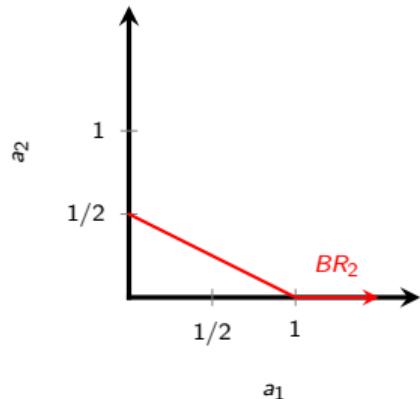


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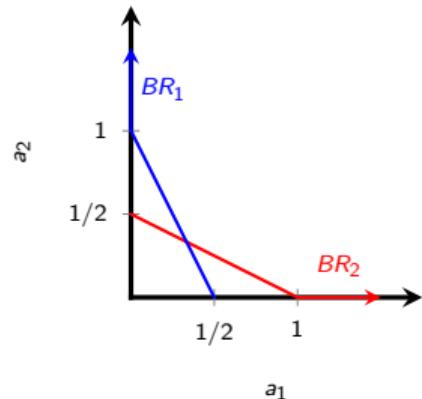
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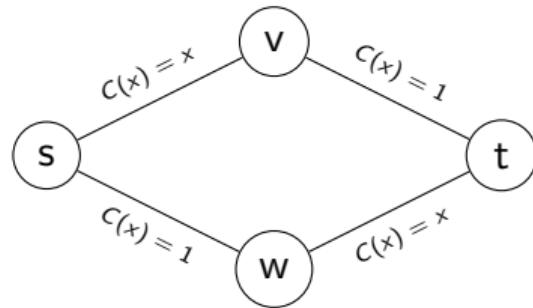
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 - ...
- Equilibrium selection is a difficult problem

Outline

1. Normal-form Games: Definition, Notations, and Examples
2. Dominant Strategy Equilibrium
3. Nash Equilibrium
4. Price of Anarchy
5. Minmax Theorem
6. Rationalizability
7. Correlated Equilibrium



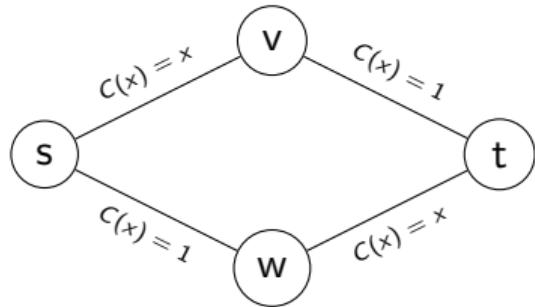
Braess's Paradox



- Suppose there are $2k$ drivers commuting from s to t



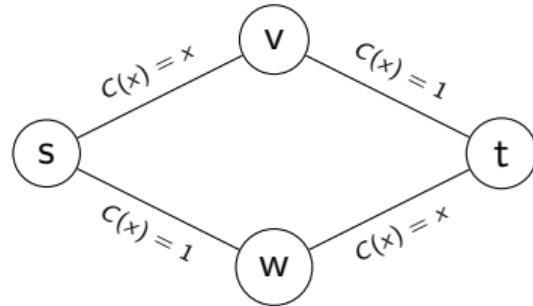
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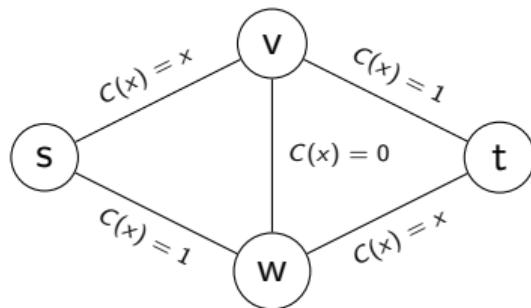


Braess's Paradox



- Suppose there are $2k$ drivers commuting from s to t
- $C(x)$ indicates travel time in hours for x fraction of drivers
- k drivers going through v , and k going through w is NE (why?)

Braess's Paradox (cont.)



- Suppose we install a teleportation device allowing instant travel from v to w
- What is new NE?
- What is optimal commute time?
- **Price of anarchy:** ratio between (worst) NE performance and optimal performance
 - Ratio between 2 and 3/2 in Braess's Paradox

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Maxmin Strategy

- Maxmin strategy for agent i is

$$\operatorname{argmax}_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

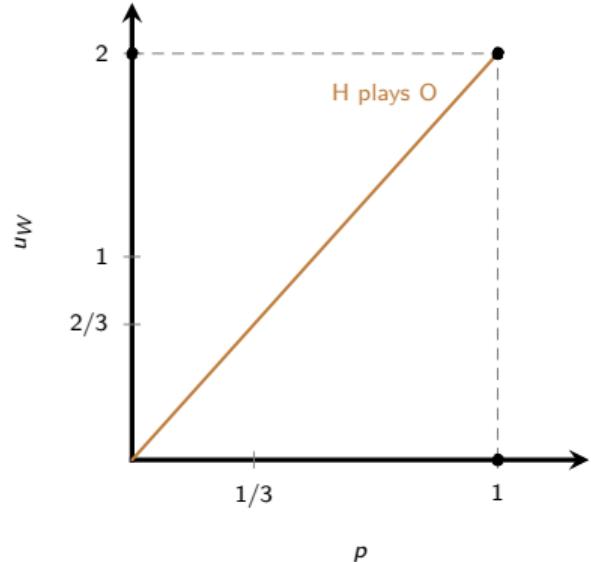
- Maxmin value for agent i is

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- If i plays maxmin strategy and others play arbitrarily, i still receives expected payoff of at least their maxmin value



Example: Battle of Sexes

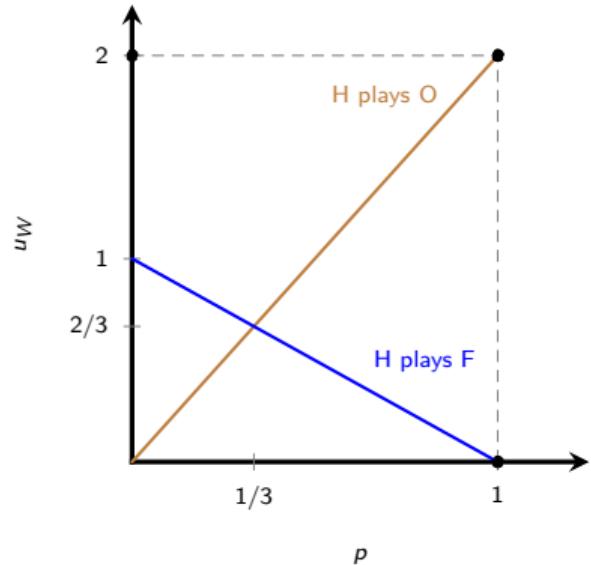


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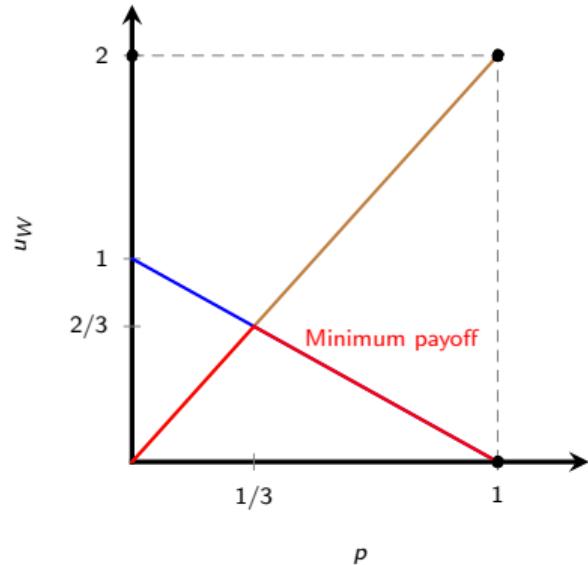


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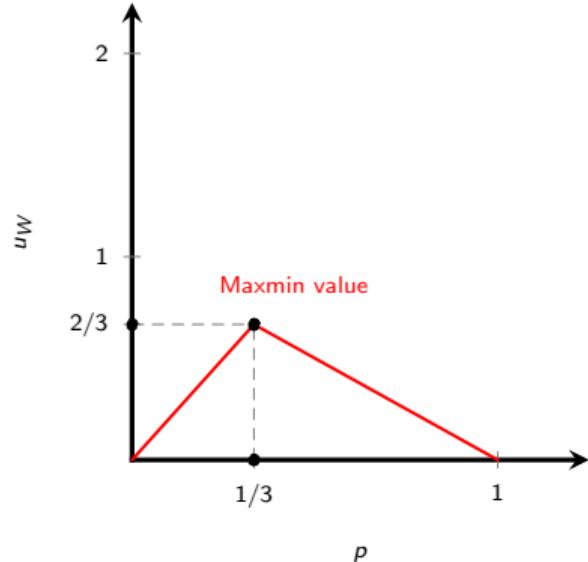


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- Minmax strategy against i keeps maximum payoff of agent i at minimum
- Agents' maxmin value is always less than or equal to their minmax value (try to show this!)



Minimax Theorem (John von Neumann, 1928)

In any finite, two-player, zero-sum game, in any Nash equilibrium¹, each agent receives a payoff that is equal to both their maxmin value and their minmax value

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¹You might wonder how a theorem from 1928 can use the term "Nash equilibrium," when Nash's work was published in 1950. John von Neumann used different terminology and proved the theorem in a different way; however, the given presentation is probably clearer in the context of modern game theory



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- Minimax theorem does not hold with pure strategies only (example?)



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Example

		Agent 2	
		Left	Right
		Up	20, -20
Agent 1	Up	20, -20	0, 0
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- What is maximin value of agent 1 with and without mixed strategies?



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- What is minimax value of agent 1 with and without mixed strategies?
- What is NE of this game?

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- Rationalizable strategy: Perfectly rational agent could justifiably play it



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 - ... (infinite regress)

Example: Matching Pennies

	H	T
H	-1, 1	1, -1
T	1, -1	-1, 1

- Col playing H is rationalizable



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- ...
- In this game, all pure strategies are rationalizable

Rationalizability: Properties

- Nash equilibrium strategies are always rationalizable
- Some rationalizable strategies are not Nash
 - Set of rationalizable strategies in finite games is nonempty



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- To find rationalizable strategies:
 - In **2-player** games, use iterated elimination of strictly dominated strategies
 - In ***n*-player** games, iterated elimination of **never-best response** strategies
 - Eliminate strategies that are not optimal against any belief about others' strategies

Example: 2/3-Beauty Contest Game

- No agent plays more than 100



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- ...
- Only rationalizable action is playing 1

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- Can they both do better by coordinating?
- Agents can observe random coin flip and condition their strategies on its outcome

Example: Battle of Sexes (cont.)

- Suppose there is **publicly observable** fair coin



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- Similar argument can be made when they see tails
- Expected utilities for this play of game **increases** to $(1.5, 1.5)$

Correlated Recommendations

- Let $R = (R_1, \dots, R_n)$ be random variable taking values in $A = \prod_i A_i$



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- $\pi(r_i)$ represents marginal probability for $R_i = r_i$



Correlated Recommendations

- Let $R = (R_1, \dots, R_n)$ be random variable taking values in $A = \prod_i A_i$
- Let R be distributed according to $\pi \in \Delta(A)$
- $r = (r_1, \dots, r_n)$ is an instantiation of R and a pure strategy profile
- $r_i \in A_i$ is called **recommendation to agent i**
- $\pi(r_i)$ represents marginal probability for $R_i = r_i$
- Given r_i , agent i can use conditional probability to form beliefs others' signals

$$\pi(r_{-i}|r_i) = \frac{\pi(r_i, r_{-i})}{\sum_{r'_{-i} \in A_{-i}} \pi(r_i, r'_{-i})}$$

Correlated Equilibrium: Formal Definition

- **Correlated equilibrium** of finite game is joint probability distribution $\pi \in \Delta(A)$ such that if R is random variable distributed according to π , then for all i , $r_i \in A_i$ with $\pi(r_i) > 0$, and $r'_i \in A_i$

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- No agent can benefit by deviating from their recommendation, assuming that other agents follow their recommendations

Example: Game of Chicken

		Driver 2	
		Dare	Yield
		D	-5, -5 1, -1
Driver 1	D	-1, 1	0, 0
	Y		

- (D,Y) and (Y,D) are **strict** pure-strategy NE



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$$p - 5 \times (1 - p) = -(1 - p) \implies p = 4/5$$

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- Mixed-strategy NE utilities are $(-0.2, -0.2)$, people **die** with probability 0.04

Example: Game of Chicken (cont.)

- Is this correlated equilibrium?

	D2	
	D	Y
D1	-5, -5 0%	1, -1 40%
Y	-1, 1 40%	0, 0 20%



Example: Game of Chicken (cont.)

- Is this correlated equilibrium?
- Suppose D1 gets Y recommendation

	D2					
	D	Y				
D1	<table border="1"><tr><td>-5, -5 0%</td><td>1, -1 40%</td></tr><tr><td>-1, 1 40%</td><td>0, 0 20%</td></tr></table>	-5, -5 0%	1, -1 40%	-1, 1 40%	0, 0 20%	
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Example: Game of Chicken (cont.)

- Is this correlated equilibrium?
- Suppose D1 gets Y recommendation
- Conditional probability that D2 yields is 1/3

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	D	Y				
D1	<table border="1"><tr><td>-5, -5 0%</td><td>1, -1 40%</td></tr><tr><td>-1, 1 40%</td><td>0, 0 20%</td></tr></table>	-5, -5 0%	1, -1 40%	-1, 1 40%	0, 0 20%	
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D						
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Example: Game of Chicken (cont.)

- Is this correlated equilibrium?
- Suppose D1 gets Y recommendation
- Conditional probability that D2 yields is $1/3$
- Expected utility of Y is $1 \times -2/3$

	D2					
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D1	<table border="1"><tr><td>-5, -5 0%</td><td>1, -1 40%</td></tr><tr><td>-1, 1 40%</td><td>0, 0 20%</td></tr></table>	-5, -5 0%	1, -1 40%	-1, 1 40%	0, 0 20%	
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	D	Y				
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- Following recommendation is again better
- Similar analysis works for D2
- Expected utilites are $(0, 0)$, so nobody dies!

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- Joint distribution $\pi \in \Delta(A)$ is correlated equilibrium of finite game iff

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- Denominator does not depend on variable of sum
- So it can be factored and canceled
- If $\pi(r_i) = 0$, LHS of (1) is zero regardless of i and r'_i , so equation always holds

Acknowledgment

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