Game-theoretic Foundations of Multi-agent Systems

Lecture 6: Repeated Games

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Outline

- 1. Finitely Repeated Games
- 2. Infinitely Repeated Games
- 3. Folk Theorem

4. Repeated Games with Imperfect Monitoring



Repeated Games

- In a (typical) repeated game:
 - Agents play a given game (aka. stage game)
 - Then, they get their utilities
 - And, they play again ...

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 - Agents play a given game (aka. stage game)
 - Then, they get their utilities
 - And, they play again ...
- Can be repeated finitely or infinitely many times
- Really, an extensive form game
 - Would like to find subgame-perfect equilibria

Repeated Games (cont.)

- One subgame-perfect equilibrium:
 - Keep repeating some Nash equilibrium of the stage game
 - Memoryless strategy, called a stationary strategy



Repeated Games (cont.)

- One subgame-perfect equilibrium:
 - Keep repeating some Nash equilibrium of the stage game
 - Memoryless strategy, called a stationary strategy
- But are there other equilibria?
 - Strategy space of repeated game is much richer than that of stage game

Key Questions

- Do agents see what the other agents played earlier?
- Do they remember what they knew?
- Given utility of each stage game, what is the utility of the entire repeated game?

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 - Discount factor is $0 \le \delta \le 1$
 - Game is denoted by $G^R(\delta)$
- Given sequence of utilities $u_i^{(1)},...,u_i^{(R)}$, $u_i=\sum_{r=1}^R \delta^{r-1}u_i^{(r)}$

D C D
$$-2, -2$$
 $-4, -1$ C $-1, -4$ $-3, -3$

ullet Two agents play Prisoner's Dilemma for R rounds $(\delta=1)$

D C
$$D = -2, -2 = -4, -1$$

$$C = -1, -4 = -3, -3$$

• Starting from last round, (C, C) is dominant strategy

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- So, (C, C) is dominant strategy at this round as well
- The unique SPE is (C, C) at each round

SPE in Finitely Repeated Games

[Theorem]

• If stage game G has unique strategy equilibrium s^* , then $G^R(\delta)$ has unique SPE in which $s^{(r)} = s^*$ for all r = 1, ..., R, regardless of history

[Proof]

- By backward induction, at round R, we have $s^{(R)} = s^*$
- Given this, then we have $s^{(R-1)}=s^*$, and continuing inductively, $s^{(r)}=s^*$ for all r=1,...,R, regardless of history



	D1	D2	С	
D1	4,4	1, 1	6,0	
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 - In round 1, cooperate;
 - In round 2, if someone defected in round 1, play D2; otherwise, play D1



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- If both agents play this, is that SPE?

	D	Crazy	C
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Crazy	0,1	0,0	0, 1
C	0,6	1,0	5, 5

ullet Two agents play the following game for 2 rounds $(\delta=1)$

	D	Crazy	С
D	4,4	1,0	6,0
Crazy	0,1	0,0	0, 1
C	0,6	1,0	5, 5

• What are the subgame perfect equilibria?

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- What are the subgame perfect equilibria?
- Consider the following strategy:
 - In round 1, cooperate;
 - In round 2, if someone played D or Crazy in round 1, play Crazy; otherwise, play D



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- What are the subgame perfect equilibria?
- Consider the following strategy:
 - In round 1, cooperate;
 - In round 2, if someone played D or Crazy in round 1, play Crazy; otherwise, play D
- If both agents play this, is that NE (not SPE)?

- If G has multiple equilibria, then $G^R(\delta)$ does not have unique SPE
- Consider following example

	X	у	Z
X	3,3	0,4	-2, 0
у	4,0	1, 1	-2, 0
Z	0, -2	0, -2	-1, -1

- Stage game has two pure NE: (y, y) and (z, z)
- Consider the following policy:
 - Play x in first round
 - Play y in second round if opponent played x; otherwise, play z
- Is both agents playing this SPE?



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Utilities in Infinitely Repeated Games

Limit-average utility:

$$u_i = \lim_{R \to \infty} \frac{\sum_{r=1}^R u_i^{(r)}}{R}$$

Future-discounted utility:

$$u_i = (1 - \delta) \sum_{r=1}^{\infty} \delta^{r-1} u_i^{(r)},$$

for some $0 \ge \delta < 1$



Subgame Perfection in Infinitely Repeated Games

- One-shot deviation from strategy s means deviating from s in single stage and conforming to it thereafter
- Strategy profile s* is SPE if and only if there are no profitable one-shot deviation for each subgame and every agent
- This follows from principle of optimality of dynamic programming
- This applies to finitely repeated games as well

Trigger Strategies (TS)

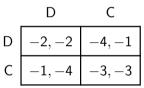
- Agents get punished if they deviate from agreed profile
- In non-forgiving TS (or grim TS), punishment continues forever

$$s_i^{(t)} = egin{cases} s_i^* & ext{if } s^{(r)} = s^* & orall r < t, \ \underline{s}_i^j & ext{otherwise} \end{cases}$$

- Here, s^* is agreed profile, and \underline{s}_i^j is punishment strategy of i against agent j
- Single deviation by j triggers agent i to switch to \underline{s}_{i}^{j} forever

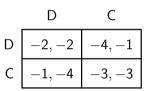


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 - Deny as long as everyone denies
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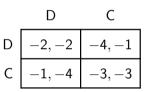


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- Is both agents playing this SPE?
- Does it depend on δ ?



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 - Deviation is not beneficial if $\delta \geq 1/2$
- Type-2 subgames: (C is best response to C)
 - Other agents will always play C, thus C is best response

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- Is both agents playing this SPE?

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- Is both agents playing this NE?
- Is both agents playing this SPE?
- What about one playing TFT and other trigger?

Remarks

- If s^* is NE of G, then "each agent plays s_i^* " is SPE of $G^R(\delta)$
 - Future play of other agents is independent of how each agent plays
 - Optimal play is to maximize current utility, i.e., play static best response
- Sets of equilibria for finite and infinite horizon versions can be different
 - ullet Multiplicity of equilibria in repeated prisoner's dilemma only occurs at $R=\infty$
 - For any finite R (thus for $R \to \infty$), repeated prisoners' dilemma has unique SPE

Repetition Could Lead to Bad Outcomes

Consider the following game

	×	У	Z
х	2, 2	2, 1	0,0
у	1,2	1, 1	-1, 0
z	0,0	0, -1	-1, -1

- Strategy x strictly dominates y and z for both agents
- Unique NE of stage game is (x, x)
- If $\delta \geq 1/2$, this game has SPE in which (y, y) is played in every round
- It is supported by slightly more complicated strategy than grim trigger
 - I. Play y in every round unless someone deviates, then go to II
 - II. Play z. If no one deviates go to I. If someone deviates stay in II



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- Folk theorem states that utility vector can be realized by some NE iff it is both feasible and enforceable

Feasible Utilities: Formal Definition

• Utility profile $u=(u_1,u_2,\ldots,u_n)$ is feasible if there exist rational, non-negative values $\{\alpha_a\}$ such that for all i, $u_i=\sum_{a\in A}\alpha_au_i(a)$, with $\sum_{a\in A}\alpha_a=1$



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- You could think of feasible utilities as convex hull of possible outcomes:

$$U = \mathsf{Conv}\{u \in \mathbb{R}^{|N|} \mid \mathsf{there} \; \mathsf{exists} \; a \in A \; \mathsf{such} \; \mathsf{that} \; u(a) = u\}$$



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• Note that $U \neq \{u \in \mathbb{R}^{|N|} \mid \text{ there exists } s \in S \text{ such that } u(s) = u\}$

	Left	Right
Left	2, 2	0,3
Right	3,0	1, 1

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- What about (3, 0.1)?

Enforceable and Individually Rational Utilities

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- Utility profile $u=(u_1,u_2,\ldots,u_n)$ is enforceable if it is individually rational

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- If u is utility profile for any NE of repeated G, then u_i is enforceable for all i
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- Folk theorem can be stated for agents with discounted utilities as well

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- NE involves non-forgiving TS which may be costly for punishers
- NE may include non-credible threats
- NE may not be subgame perfect

Example

	L	R
U	6,6	0, -100
D	7, 1	0, -100

- Unique NE in this game is (D, L)
- \bullet Minmax values are given by $\underline{v}_1=0$ and $\underline{v}_2=1$
- Minmax strategy against agent 1 requires agent 2 to play R
- R is strictly dominated by L for agent 2



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Motivation

- So far, we assumed that agents observe actions of others at each round of game
- Next, we consider games where agents' actions may not be directly observable
- We assume that agents observe only an imperfect signal of stage game actions



Example: Cournot Competition with Noisy Demand

[Green and Porter, Non-cooperative Collusion under Imperfect Price Information, 1984]

- Firms set production levels $q_1^{(r)}, \ldots, q_n^{(r)}$ privately at round r
- Firms do not observe each others' output levels
- Market demand is stochastic
- Market price depends on total production and market demand
- Low price could be due to high production or low demand
- Firms utility depends on their own production and market price



Model

- We focus on game with public information
- At each round, all agents observe some public outcome
- Let $y^{(r)} \in Y$ denote publicly observed outcome at round r
- Each action profile a induces probability distribution over y
- Let $\pi(y, a)$ denote probability distribution of y under action profile a
- Public information at round r is $h^{(r)} = (y^{(1)}, \dots, y^{(r-1)})$
- Strategy of agent i is sequence of maps $s_i^{(r)}:h^{(r)} o S_i$



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- Agent i's realized utility at round r is $u_i(a_i^{(r)}, y^{(r)})$
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$$u_i(a) = \sum_{y \in Y} \pi(y, a) u_i(a_i, y)$$

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• Agent i's average discounted utility when sequence $\{a^{(t)}\}$ is played is

$$(1-\delta)\sum_{r=1}^{\infty}\delta^{r-1}u_i(a^{(r)})$$

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 - If a = (D, C) or (C, D), then y = X 2

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 - If a = (D, D), then y = X
 - If a = (D, C) or (C, D), then y = X 2
 - If a = (C, C), then y = X 4
- Normal-form stage game is

	D	С
D	1 + X, 1 + X	-1 + X, 2 + X
С	2 + X, -1 + X	X, X

Trigger-price Strategy

- Consider following trigger strategy
 - (I) Play (D, D) until $y \le y^*$, then go to (II)
 - (II) Play(C, C) for R rounds, then go back to (I)
- Notice that strategy is stationary and symmetric
- Also notice that punishment uses NE of stage game
- We can choose y^* and R such that this strategy profile is SPE



Trigger-price Strategy (cont.)

- We use one-shot deviation principle
- Deviation in (II) is obviously not beneficial
- In (I), if agents do not deviate, their expected utility is

$$v = (1 - \delta) \left((1 + 0) + \delta \left(F(y^*) \delta^R v + (1 - F(y^*)) v \right) \right)$$

• From this, we obtain

$$v = rac{1-\delta}{1-\delta(1-\delta)ig(1-F(y^*)(1-\delta^R)ig)}$$



Trigger-price Strategy (cont.)

• If some agent deviates in (1), then her expected utility is

$$v_d = (1 - \delta) \left((2 + 0) + \delta \left(F(y^* + 2) \delta^R v + (1 - F(y^* + 2)) v \right) \right)$$

- Deviation provides immediate utility, but increases probability of entering (II)
- To have SPE, we mush have $v > v_d$ which means

$$v \geq rac{2(1-\delta)}{1-\delta(1-\delta)ig(1-F(y^*+2)(1-\delta^R)ig)}$$

$$\Rightarrow F(y^*+2)-2F(y^*) \geq \frac{1-\delta(1-\delta)}{\delta(1-\delta)(1-\delta^R)}$$

- Any R and y* that satisfy this constraint construct SPE
- Best trigger-price strategy can be found by maximizing v s.t. this constraint



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