Game-theoretic Foundations of Multi-agent Systems

Lecture 4: Computing Solution Concepts of Normal-form Games

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Outline

1. Brief Overview of (Mixed Integer) Linear Programming

2. Dominated Strategies

3. Minmax and Maxmin Strategies

4. Nash Equilibrium

5. Correlated NE

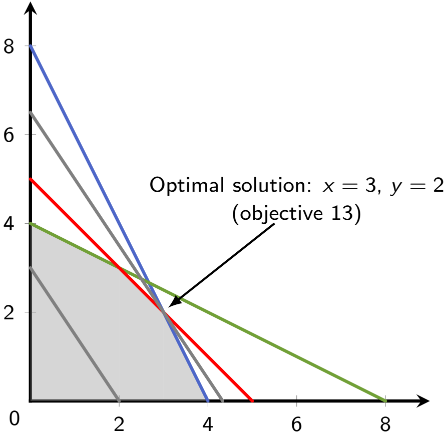
Example: Reproduction of Two Paintings

- Painting 1 sells for \$30
- Painting 2 sells for \$20
- We have 16 units of blue, 8 green, 5 red
- Painting 1 requires 4 blue, 1 green, 1 red
- Painting 2 requires 2 blue, 2 green, 1 red

$$\begin{aligned} \text{max.} \quad & 3x + 2y \\ \text{s.t.} \quad & 4x + 2y \leq 16 \\ & x + 2y \leq 8 \\ & x + y \leq 5 \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$

Solving Linear Program Graphically

$$\begin{aligned} \text{max.} \quad & 3x + 2y \\ \text{s.t.} \quad & 4x + 2y \leq 16 \\ & x + 2y \leq 8 \\ & x + y \leq 5 \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$



Optimal solution: $x = 3$, $y = 2$
(objective 13)

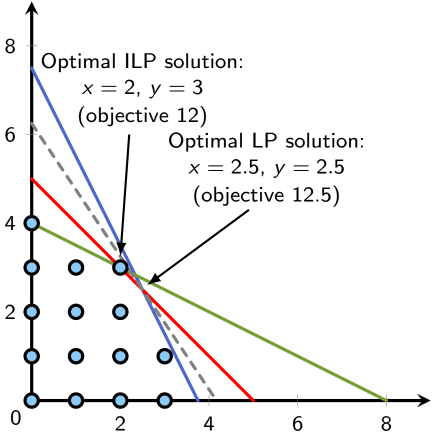
Modified LP

$$
\begin{array}{ll}
\text{max.} & 3x + 2y \\
\text{s.t.} & 4x + 2y \leq 16 \\
\text{s.t.} & 4x + 2y \leq \boxed{15} \\
 & x + 2y \leq 8 \\
 & x + y \leq 5 \\
 & x \geq 0 \\
 & y \geq 0
\end{array}
$$

- Optimal solution: x = 2.5, y = 2.5
- Objective = 7.5 + 5 = 12.5
- Can we sell half paintings?

Integer Linear Program

$$\begin{aligned} \text{max.} \quad & 3x + 2y \\ \text{s.t.} \quad & 4x + 2y \leq 15 \\ & x + 2y \leq 8 \\ & x + y \leq 5 \\ & x \in \mathbb{N}_0 \\ & y \in \mathbb{N}_0 \end{aligned}$$

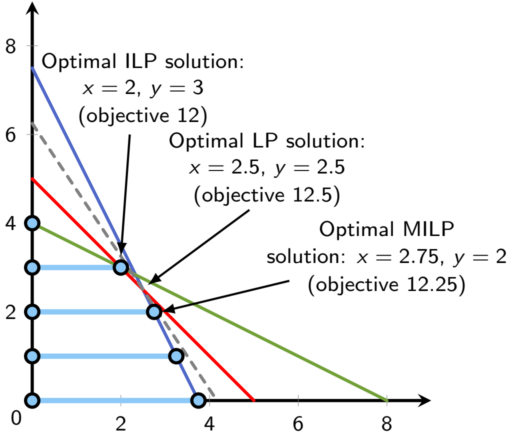


Optimal ILP solution:
$x = 2,\ y = 3$
(objective 12)

Optimal LP solution:
$x = 2.5,\ y = 2.5$
(objective 12.5)

Mixed Integer Linear Program

$$\begin{aligned} \text{max.} \quad & 3x + 2y \\ \text{s.t.} \quad & {\color{blue} 4x + 2y \le 15} \\ & {\color{olive} x + 2y \le 8} \\ & {\color{red} x + y \le 5} \\ & x \ge 0 \\ & y \in \mathbb{N}_0 \end{aligned}$$



Optimal ILP solution:
$x = 2, \ y = 3$
(objective 12)

Optimal LP solution:
$x = 2.5, \ y = 2.5$
(objective 12.5)

Optimal MILP
solution: $x = 2.75, \ y = 2$
(objective 12.25)

Solving Mixed Linear/Integer Programs

- Linear programs can be solved efficiently
  - Simplex, ellipsoid, interior point methods, etc.

- (Mixed) integer programs are NP-hard to solve
  - Many standard NP-complete problems can be modeled as MILP
  - Search type algorithms such as branch and bound

- Standard packages for solving these
  - Gurobi, MOSEK, GNU Linear Programming Kit, CPLEX, CVXOPT, etc.

- LP relaxation of (M)ILP: remove integrality constraints
  - Gives upper bound on MILP ($\sim$ admissible heuristic)

Exercise I: Knapsack-type Problem

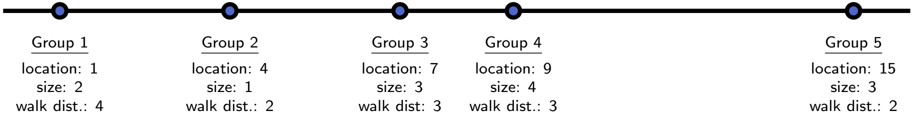
- We arrive in room full of precious objects
- Can carry only 30kg out of the room
- Can carry only 20 liters out of the room
- Want to maximize our total value
- Unit of object A: 16kg, 3 liters, sells for $11 (3 units available)
- Unit of object B: 4kg, 4 liters, sells for $4 (4 units available)
- Unit of object C: 6kg, 3 liters, sells for $9 (1 unit available)
- What should we take?

Exercise II: Cellphones (Set Cover)

- We want to have a working phone in every continent (besides Antarctica)
- But we want to have as few phones as possible
- Phone A works in NA, SA, Af
- Phone B works in E, Af, As
- Phone C works in NA, Au, E
- Phone D works in SA, As, E
- Phone E works in Af, As, Au
- Phone F works in NA, E

Exercise III: Hot-dog Stands

- We have two hot-dog stands to be placed in somewhere along beach
- We know where groups of people who like hot dogs are
- We also know how far each group is willing to walk
- Where do we put our stands to maximize # hot dogs sold? (price is fixed)



Group 1
location: 1
size: 2
walk dist.: 4

Group 2
location: 4
size: 1
walk dist.: 2

Group 3
location: 7
size: 3
walk dist: 3

Group 4
location: 9
size: 4
walk dist.: 3

Group 5
location: 15
size: 3
walk dist.: 2

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Recall: Strict Dominance

a_i strictly dominates s_i if $u_i(a_i, s_{-i}) > u_i(s_i, s_{-i}) \; \forall s_{-i} \in S_{-i}$

Dominance by Pure Strategy

Algorithm 1: Determine whether s_i is strictly dominated by any pure strategy

```
for all $a_i \in A_i$ where $a_i \neq s_i$ do
    $dom \leftarrow true$;
    forall $a_{-i} \in A_{-i}$ do
        if $u_i(s_i, a_{-i}) \geq u_i(a_i, a_{-i})$ then
            $dom \leftarrow false$;
            break;
    if $dom = true$ then
        return $true$;
return $false$;
```

Dominance by Pure Strategy: Discussion

- Complexity of the Algorithm is $O(|A|)$, linear in the size of normal-form game
- Recall: a_i strictly dominates s_i if $u_i(a_i, s_{-i}) > u_i(s_i, s_{-i})\ \forall s_{-i} \in S_{-i}$
- This definition refers to mixed-strategy profile of other agents
- In Alg. (1), we do not check every mixed-strategy profile of others, why?
  - Suppose a_i strictly dominates s_i for all a_{-i}
  - Then, there is no s_{-i} for which $u_i(a_i, s_{-i}) \geq u_i(s_i, s_{-i})$
  - This holds because of the linearity of expectation

Weak Dominance by Mixed Strategy

- Checking if strategy s_i is weakly dominated by any mixed strategy

$$\begin{aligned}
\text{max.} \quad & \sum_{a_{-i} \in A_{-i}} \left[\left(\sum_{a_i \in A_i} p_{a_i} u_i(a_i, a_{-i}) \right) - u_i(s_i, a_{-i}) \right] & \\
\text{s.t.} \quad & \sum_{a_i \in A_i} p_{a_i} u_i(a_i, a_{-i}) \geq u_i(s_i, a_{-i}) & \forall a_{-i} \in A_{-i} \\
& \sum_{a_i \in A_i} p_{a_i} = 1 & \\
& p_{a_i} \geq 0, & \forall a_i \in A_i
\end{aligned}$$

- If optimal solution is strictly positive, then s_i is weakly dominated by $\{p_{a_i}\}$

Strict Dominance by Mixed Strategies

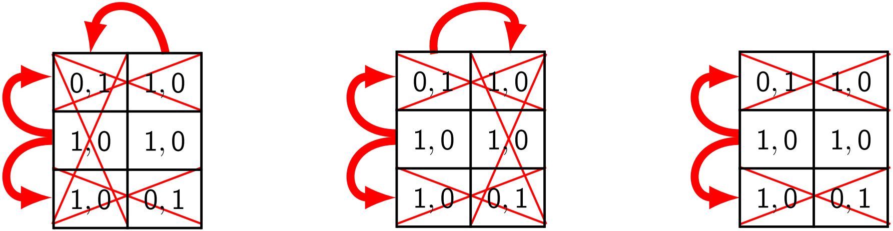
- Checking if strategy s_i is strictly dominated by any mixed strategy

$$\begin{aligned}
\text{max.} \quad & \epsilon \\
\text{s.t.} \quad & \sum_{a_i \in A_i} p_{a_i} u_i(a_i, a_{-i}) \geq u_i(s_i, a_{-i}) + \epsilon \quad \forall a_{-i} \in A_{-i} \\
& \sum_{a_i \in A_i} p_{a_i} = 1 \\
& p_{a_i} \geq 0, \qquad \forall a_i \in A_i
\end{aligned}$$

- If optimal solution is strictly positive, then s_i is strictly dominated by $\{p_{a_i}\}$

Path Dependency of Iterated Dominance

- Iterated weak dominance is path-dependent:
  - Sequence of eliminations may determine which solution we get (if any)



~~0, 1~~	~~1, 0~~
~~1, 0~~	1, 0
~~1, 0~~	~~0, 1~~

~~0, 1~~	~~1, 0~~
1, 0	~~1, 0~~
~~1, 0~~	~~0, 1~~

~~0, 1~~	~~1, 0~~
1, 0	1, 0
~~1, 0~~	~~0, 1~~

- Iterated strict dominance is path-independent:
  - Elimination process will always terminate at the same point

Computational Questions for Iterated Dominance

- Is there some elimination path under which s_i is eliminated?
- Is there maximally reduced game where each agent has exactly 1 action?
- For strict dominance, both can be solved in polynomial time
  - Due to path-independence
  - Check if any strategy is dominated, remove it, repeat
  - With or without dominance by mixed strategies
- For weak dominance, both questions are NP-hard[1]
  - Even when all utilities are 0 or 1
  - With or without dominance by mixed strategies

[1] [Conitzer, Sandholm 05] and weaker version proved by [Gilboa, Kalai, Zemel 93]

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Recall: Minmax and Maxmin

- Maxmin strategy for agent i (maxmin value for agent i)

$$\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

- Minmax strategy against agent i (minmax value for agent i)

$$\arg\min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

Maxmin Strategy and Value

- Finding maxmin strategy of agent i

$$\begin{aligned}
\text{max.} \quad & U_i \\
\text{s.t.} \quad & \sum_{a_i \in A_i} p_{a_i} u_i(a_i, a_{-i}) \geq U_i, \quad \forall a_{-i} \in A_{-i} \\
& \sum_{a_i \in A_i} p_{a_i} = 1 \\
& p_{a_i} \geq 0, \qquad\qquad\qquad\qquad \forall a_i \in A_i
\end{aligned}$$

- Given p_{a_i}, first constraint ensures that U_i is less than any achievable expected utility for any pure strategies of opponents
- Objective of this LP, U_i, is maxmin value of agent i

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NE of Two-player, Zero-sum Games

- Maxmin value for agent 1

$$\begin{aligned} \text{max.} \quad & U_1 \\ \text{s.t.} \quad & \sum_{a_1 \in A_1} p_{a_1} u_1(a_1, a_2) \geq U_1 && \forall a_2 \in A_2 \\ & \sum_{a_1 \in A_1} p_{a_1} = 1 \\ & p_{a_1} \geq 0, && \forall a_1 \in A_1 \end{aligned}$$

- Minmax value for agent 1

$$\begin{aligned} \text{min.} \quad & U_1 \\ \text{s.t.} \quad & \sum_{a_2 \in A_2} p_{a_2} u_1(a_1, a_2) \leq U_1 && \forall a_1 \in A_1 \\ & \sum_{a_2 \in A_2} p_{a_2} = 1 \\ & p_{a_2} \geq 0, && \forall a_2 \in A_2 \end{aligned}$$

- NE is expressed as LP $\Rightarrow$ NE can be computed in polynomial time

Maxmin Strategy for General-sum Games

- Agents could still play minmax strategy in general-sum games
  - I.e., pretend that the opponent is only trying to hurt you
- But this might not be rational:

		Agent 2	
		Left	Right
Agent 1	Up	0, 0	3, 1
	Down	1, 0	2, 1

  - If A2 was trying to hurt A1, she would play Left, so A1 should play Down
  - In reality, A2 will play Right (strictly dominant), so A1 should play Up

Hardness of Computing NE for General-sum Games

- Complexity was open for long time
  - "together with factoring [...] the most important concrete open question on the boundary of P today" [Papadimitriou STOC'01]
- Sequence of papers showed that computing any NE is PPAD-complete (even in 2-player games) [Daskalakis, Goldberg, Papadimitriou 2006; Chen, Deng 2006]
- All known algorithms require exponential time (in worst case)

Hardness of Computing NE for General-Sum Games (cont.)

- What about computing NE with specific property?
  - NE that is not Pareto-dominated
  - NE that maximizes expected social welfare (i.e., sum of all agents' utilities)
  - NE that maximizes expected utility of given agent
  - NE that maximizes expected utility of worst-off player
  - NE in which given pure strategy is played with positive probability
  - NE in which given pure strategy is played with zero probability
  - ...
- All of these are NP-hard (and the optimization questions are inapproximable assuming P != NP), even in 2-player games

  [Gilboa, Zemel 89; Conitzer & Sandholm IJCAI-03/GEB-08]

Search-based Approaches (for Two-player Games)

- We can use LP, if we know support X_i of each player i's mixed strategy

$$\begin{array}{lll}
\text{find} & (U_1, U_2) & \\
\text{s.t.} & p_{a_i} \geq 0, & \forall i, a_i \in A_i \\
& \displaystyle\sum_{a_i \in A_i} p_{a_i} = 1, & \forall i \\
& p_{a_i} = 0, & \forall i, a_i \in A_i / X_i \\
& \displaystyle\sum_{a_{-i} \in A_{-i}} p_{a_{-i}} u_i(a_i, a_{-i}) = U_i, & \forall i, a_i \in X_i \\
& \displaystyle\sum_{a_{-i} \in A_{-i}} p_{a_{-i}} u_i(a_i, a_{-i}) \leq U_i, & \forall i, a_i \in A_i / X_i
\end{array}$$

- Thus, we can search over possible supports, which is basic idea underlying methods in [Dickhaut & Kaplan 91; Porter, Nudelman, Shoham AAAI04/GEB08]

NE using MILP (for Two-player Games)

[Sandholm, Gilpin, Conitzer AAAI05]

$$
\begin{array}{lll}
\text{max.} & \text{whatever you like (e.g., social welfare)} & \\
\text{s.t.} & p_{a_i} \geq 0, & \forall i, a_i \in A_i \\
& \displaystyle\sum_{a_i \in A_i} p_{a_i} = 1, & \forall i \\
& \displaystyle\sum_{a_{-i} \in A_{-i}} p_{a_{-i}} u_i(a_i, a_{-i}) = u_{a_i}, & \forall i, a_i \in A_i \\
& u_{a_i} \leq u_i, & \forall i, a_i \in A_i \\
& p_{a_i} \leq b_{a_i}, & \forall i, a_i \in A_i \\
& u_i - u_{a_i} \leq M(1 - b_{a_i}), & \forall i, a_i \in A_i \\
& b_{a_i} \in \{0, 1\}, & \forall i, a_i \in A_i
\end{array}
$$

- b_{a_i} indicates whether a_i is in support of i's mixed strategy, and M is large number

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Correlated Equilibrium (N-player Games!)

- Variables are now p_a for all action profiles a (i.e., outcome)

$$\begin{aligned}
\text{max.}\quad & \text{whatever you like (e.g., social welfare)} \\
\text{s.t.}\quad & \sum_{a_{-i} \in A_{-i}} p_a u_i(a) \geq \sum_{a_{-i} \in A_{-i}} p_a u_i(t_i, a_{-i}) \qquad \forall i, a_i, t_i \in A_i \\
& \sum_{a \in A} p_a = 1 \\
& p_a \geq 0, \qquad \forall a \in A
\end{aligned}$$

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