

# Game-theoretic Foundations of Multi-agent Systems

Lecture 3: Games in Normal Form

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# Outline

1. Normal-form Games: Definition, Notations, and Examples
2. Dominant Strategy Equilibrium
3. Nash Equilibrium
4. Price of Anarchy
5. Minmax Theorem
6. Rationalizability
7. Correlated Equilibrium



## Normal-form Games

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- Agents choose their actions without knowledge of other agents' actions
- Such games are referred to as **strategic-form games** or **normal-form games**

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- Set of all action profiles is denoted by  $A = \prod A_i$
- Agent  $i$  has a utility function  $u_i$  that maps outcomes to real numbers

## Some Notations

- $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$  is an action profile of all agents except  $i$
- $A_{-i} = \prod_{j \neq i} A_j$  is set of action profiles of all agents except  $i$
- $a = (a_i, a_{-i}) \in A$  is another way of denoting an action profile (or an outcome)



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- For 2 agents and small action sets, game can be represented in **matrix form**

		Agent 2	
		x	y
		a, b	e, f
Agent 1	m	c, d	g, h
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- Each cell indexed by row  $r$  and column  $c$  contains a pair,  $(p, q)$ , where  $p = u_1(r, c)$  and  $q = u_2(r, c)$ .

## Example: Matching Pennies

- Each agent has a penny and independently chooses to display either heads or tails
- Agents compare their pennies
- If they are the same, agent 1 pockets both, otherwise agent 2 pockets them

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- **Zero-sum game:** Utility of one agent is negative of utility of other agent

## Example: Rock, Paper, Scissors Game

- Three-strategy generalization of the matching-pennies game

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0



## Example: Coordination Game

- Two drivers driving towards each other in a country with no traffic rules
- Drivers must independently decide whether to drive on the left or on the right
- If drivers choose the same side (left or right) they have some high utility, and otherwise they have a low utility

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- **Team game:** For all outcomes  $s$ , and any pair of agents  $i$  and  $j$ , it is the case that  $u_i(a) = u_j(a)$  (also known as **common-payoff game** or **pure-coordination game**)

## Example: Cournot Competition

- Two firms producing a homogeneous good for the same market
- Action of each firm is the amount of good it produces ( $a_i \in [0, \infty]$ )
- Utility of each firm is its total revenue minus its total cost

$$u_i(a_1, a_2) = a_i p(a_1 + a_2) - c a_i$$



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- $p(\cdot)$  is the price function that maps total production to a price
- $c$  is a unit cost
- E.g.,  $p(x) = \max(0, 2 - x)$  and  $c = 1$



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- Support of mixed strategy  $s_i$  is set of pure strategies,  $a_i$ , such that  $s_i(a_i) > 0$
- Expected utility of agent  $i$  for a (mixed) strategy profile  $s = (s_1, \dots, s_n)$  is

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j \in N} s_j(a_j)$$

## Example

		Agent 2		
		R ( $\frac{2}{3}$ )	P (0)	S ( $\frac{1}{3}$ )
Agent 1		R ( $\frac{1}{3}$ )	0, 0	-1, 1
		P ( $\frac{2}{3}$ )	1, -1	0, 0
		S (0)	-1, 1	1, -1

- $u_1 = 2/9 \times 0 + 1/9 \times 1 + 4/9 \times 1 - 2/9 \times 1 = 1/3$
- $u_2 = 2/9 \times 0 - 1/9 \times 1 - 4/9 \times 1 + 2/9 \times 1 = -1/3$



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- $s_i$  is strictly/weakly dominated if another strategy strictly/weakly dominates it
- $s = (s_1, \dots, s_n)$  is dominant strategy equilibrium if  $s_i$  is dominant strategy for all  $i$

## Example: Prisoner's Dilemma

- Two prisoners suspected of a crime are taken to separate interrogation rooms
- Each can either confess to the crime or deny it

	D	C
D	−2, −2	−4, −1
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- The dilemma: (D,D) is better for both prisoners, but they won't play it!

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	L	C	R
U	3, 1	0, 2	0, 0
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- Order of elimination does not matter when removing strictly dominated strategies (**Church–Rosser property**)
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- If the procedure ends with a single strategy for each agent, then the game is said to be **dominance solvable**

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- Best response is not a solution concept
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  - Because agents do not know what strategies others will play
- However, we can leverage the idea of best response to define what is arguably the most central notion in game theory, the Nash equilibrium

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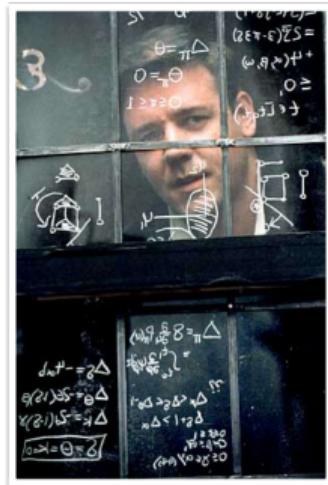


Fig. A Beautiful Mind, from s.hanux.com



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John Forbes Nash Jr.  
1928-2015



## Example: Battle of Sexes

- Husband and wife wish to meet this evening, but have a choice between two events to attend: football or opera
- Husband would prefer to go to football, wife would prefer opera
- Both would prefer to go to the same event rather than different ones

		Wife	
		Football	Opera
Husband	Football	2,1	0,0
	Opera	0,0	1,2



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- Husband would prefer to go to football, wife would prefer opera
- Both would prefer to go to the same event rather than different ones

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		Football	Opera
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- Are these the only Nash equilibria?



## Example: Battle of Sexes (cont.)

	$F (p)$	$O (1 - p)$
$F$	2, 1	0, 0
$O$	0, 0	1, 2

- In general, it is tricky to compute mixed-strategy equilibria (will discuss this later)



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- Husband must be indifferent between  $F$  and  $O$  (why?):

$$u_H(F) = u_H(O) \Rightarrow 2 \times p = (1 - p) \Rightarrow p = 1/3$$



## Example: Battle of Sexes (cont.)

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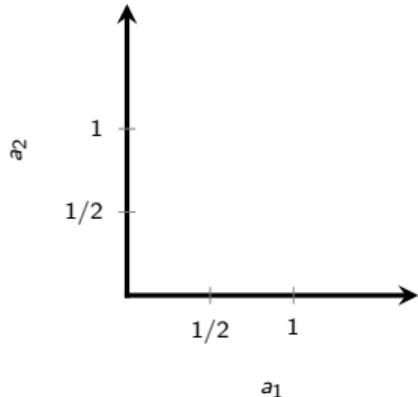
- You can show that the unique mixed-strategy NE is  $\{(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\}$

## Example: Cournot Competition

- $u_i(a_1, a_2) = a_i \max(0, 2 - a_1 - a_2) - a_i$
- Using first order optimality conditions, we have

$$BR_i(a_{-i}) = \underset{a_i \geq 0}{\operatorname{argmax}} a_i(2 - a_i - a_{-i}) - a_i$$

$$= \begin{cases} (1 - a_{-i})/2 & \text{if } a_{-i} < 1, \\ 0 & \text{Otherwise.} \end{cases}$$

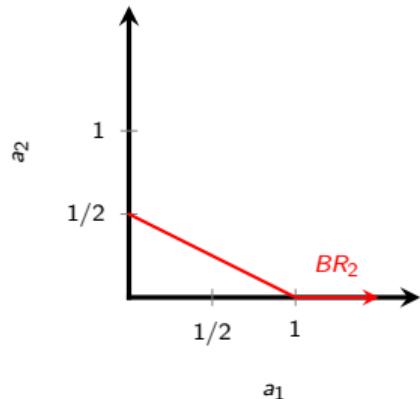


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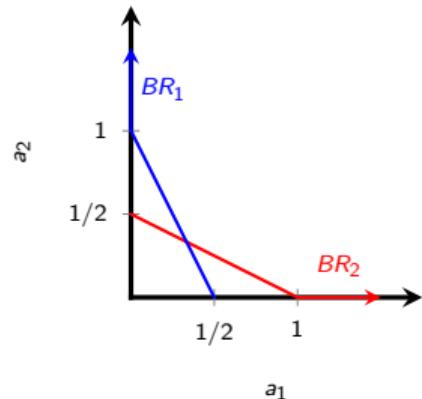
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- You are about to play a game that you have never played before with a person that you have never met



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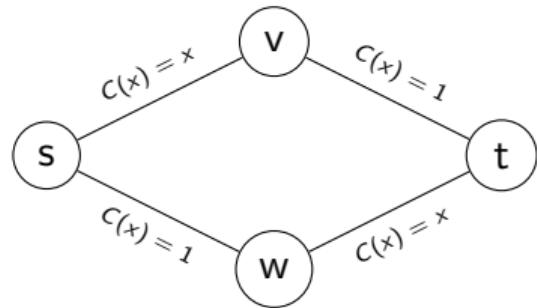
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  - Equilibrium that is the convergence point of some learning process
  - An equilibrium that is easy to compute
  - ...
- Equilibrium selection is a difficult problem

# Outline

1. Normal-form Games: Definition, Notations, and Examples
2. Dominant Strategy Equilibrium
3. Nash Equilibrium
4. Price of Anarchy
5. Minmax Theorem
6. Rationalizability
7. Correlated Equilibrium



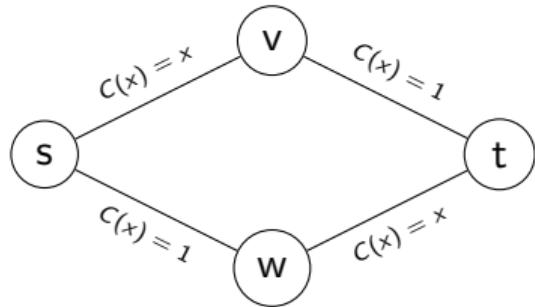
## Braess's Paradox



- Suppose there are  $2k$  drivers commuting from  $s$  to  $t$



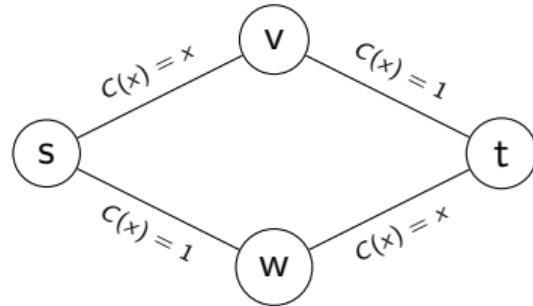
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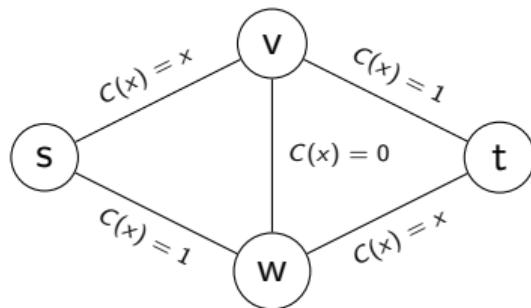


## Braess's Paradox



- Suppose there are  $2k$  drivers commuting from  $s$  to  $t$
- $C(x)$  indicates travel time in hours for  $x$  fraction of drivers
- $k$  drivers going through  $v$ , and  $k$  going through  $w$  is NE (why?)

## Braess's Paradox (cont.)



- Suppose we install a teleportation device allowing instant travel from  $v$  to  $w$
- What is new NE?
- What is optimal commute time?
- **Price of anarchy:** ratio between (worst) NE performance and optimal performance
  - Ratio between 2 and 3/2 in Braess's Paradox

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## Maxmin Strategy

- Maxmin strategy for agent  $i$  is

$$\operatorname{argmax}_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

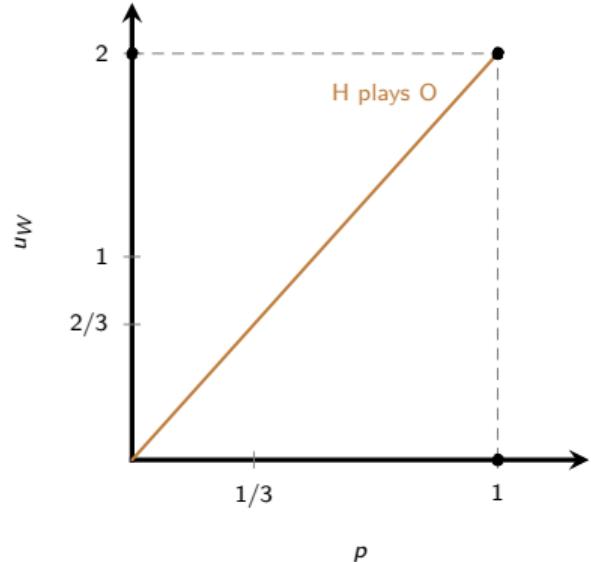
- Maxmin value for agent  $i$  is

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

- If  $i$  plays maxmin strategy and others play arbitrarily,  $i$  still receives expected payoff of at least their maxmin value



## Example: Battle of Sexes

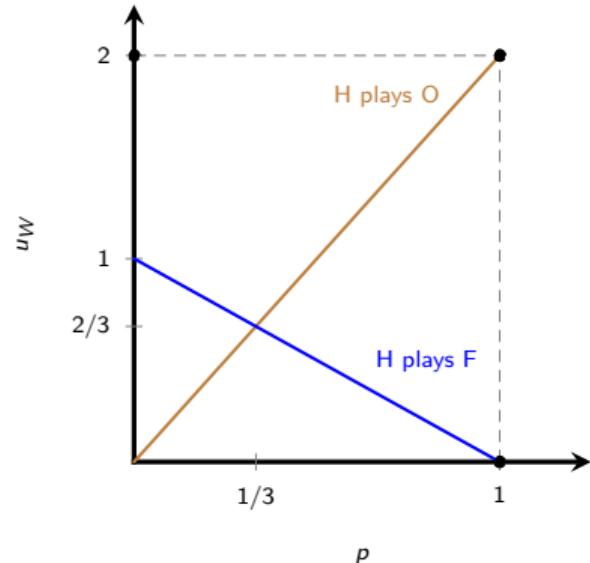


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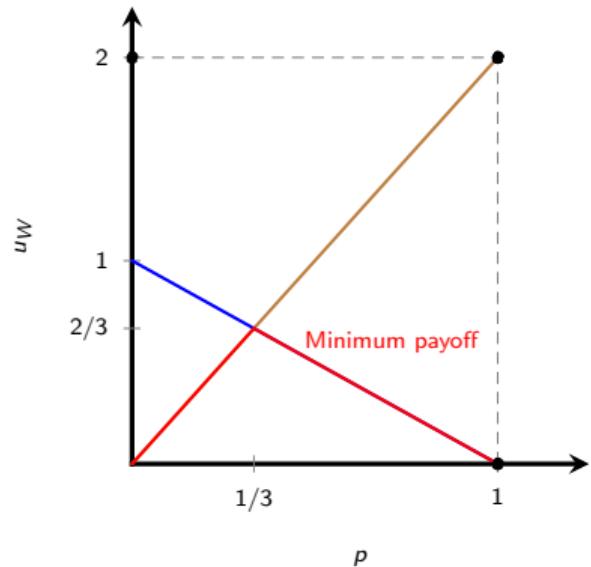


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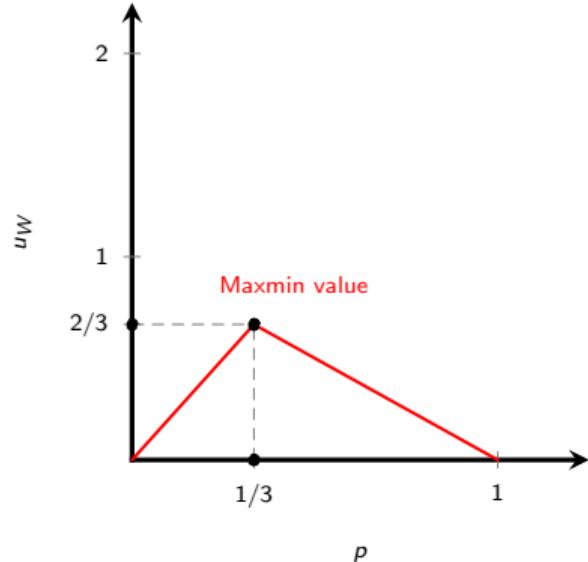


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F (1 -  $p$ )      O ( $p$ )

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$$\min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

- Minmax strategy against  $i$  keeps maximum payoff of agent  $i$  at minimum
- Agents' maxmin value is always less than or equal to their minmax value (try to show this!)



# Minimax Theorem (John von Neumann, 1928)

In any finite, two-player, zero-sum game, in any Nash equilibrium<sup>1</sup>, each agent receives a payoff that is equal to both their maxmin value and their minmax value

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<sup>1</sup>You might wonder how a theorem from 1928 can use the term "Nash equilibrium," when Nash's work was published in 1950. John von Neumann used different terminology and proved the theorem in a different way; however, the given presentation is probably clearer in the context of modern game theory



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- Minimax theorem does not hold with pure strategies only (example?)



---

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## Example

		Agent 2	
		Left	Right
		Up	20, -20
Agent 1	Up	20, -20	0, 0
	Down	0, 0	10, -10

- What is maximin value of agent 1 with and without mixed strategies?



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- What is minimax value of agent 1 with and without mixed strategies?
- What is NE of this game?

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  - ... (infinite regress)

## Example: Matching Pennies

	H	T
H	-1, 1	1, -1
T	1, -1	-1, 1

- Col playing H is rationalizable



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- In this game, all pure strategies are rationalizable

## Rationalizability: Properties

- Nash equilibrium strategies are always rationalizable
- Some rationalizable strategies are not Nash
  - Set of rationalizable strategies in finite games is nonempty



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- To find rationalizable strategies:
  - In **2-player** games, use iterated elimination of strictly dominated strategies
  - In ***n*-player** games, iterated elimination of **never-best response** strategies
    - Eliminate strategies that are not optimal against any belief about others' strategies

## Example: 2/3-Beauty Contest Game

- No agent plays more than 100



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- $2/3$  of average is less than 45 ( $67 \times 2/3$ )



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- ...
- Only rationalizable action is playing 1

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		W				
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H	F	<table border="1" style="margin-left: auto; margin-right: auto;"><tr><td style="text-align: center;">2, 1</td><td style="text-align: center;">0, 0</td></tr><tr><td style="text-align: center;">0, 0</td><td style="text-align: center;">1, 2</td></tr></table>	2, 1	0, 0	0, 0	1, 2
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- In NE, agents randomize over strategies **independently**
- Can they both do better by coordinating?



## Example: Battle of Sexes

		W	
	Football	Opera	
H	F	2, 1	0, 0
	O	0, 0	1, 2

- Unique mixed strategy NE yields each agent expected payoff of  $2/3$
- In NE, agents randomize over strategies **independently**
- Can they both do better by coordinating?
- Agents can observe random coin flip and condition their strategies on its outcome

## Example: Battle of Sexes (cont.)

- Suppose there is **publicly observable** fair coin



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- Similar argument can be made when they see tails
- Expected utilities for this play of game **increases** to  $(1.5, 1.5)$

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- $\pi(r_i)$  represents marginal probability for  $R_i = r_i$
- Given  $r_i$ , agent  $i$  can use conditional probability to form beliefs others' signals

$$\pi(r_{-i}|r_i) = \frac{\pi(r_i, r_{-i})}{\sum_{r'_{-i} \in A_{-i}} \pi(r_i, r'_{-i})}$$

## Correlated Equilibrium: Formal Definition

- **Correlated equilibrium** of finite game is joint probability distribution  $\pi \in \Delta(A)$  such that if  $R$  is random variable distributed according to  $\pi$ , then for all  $i$ ,  $r_i \in A_i$  with  $\pi(r_i) > 0$ , and  $r'_i \in A_i$

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- No agent can benefit by deviating from their recommendation, assuming that other agents follow their recommendations

## Example: Game of Chicken

		Driver 2	
		Dare	Yield
		D	-5, -5   1, -1
Driver 1	D	-1, 1	0, 0
	Y		

- (D,Y) and (Y,D) are **strict** pure-strategy NE



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$$p - 5 \times (1 - p) = -(1 - p) \implies p = 4/5$$

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- Mixed-strategy NE utilities are  $(-0.2, -0.2)$ , people **die** with probability 0.04

## Example: Game of Chicken (cont.)

- Is this correlated equilibrium?

	D2					
	D	Y				
D1	<table border="1"><tr><td>-5, -5 0%</td><td>1, -1 40%</td></tr><tr><td>-1, 1 40%</td><td>0, 0 20%</td></tr></table>	-5, -5 0%	1, -1 40%	-1, 1 40%	0, 0 20%	
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- Is this correlated equilibrium?
- Suppose D1 gets Y recommendation

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## Example: Game of Chicken (cont.)

- Is this correlated equilibrium?
- Suppose D1 gets Y recommendation
- Conditional probability that D2 yields is 1/3

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## Example: Game of Chicken (cont.)

- Is this correlated equilibrium?
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- Expected utility of Y is  $1 \times -2/3$

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- Following recommendation is again better
- Similar analysis works for D2
- Expected utilites are  $(0, 0)$ , so nobody dies!

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- Joint distribution  $\pi \in \Delta(S)$  is correlated equilibrium of finite game iff

$$\sum_{r_{-i} \in A_{-i}} \pi(r) [u_i(r) - u_i(r'_i, r_{-i})] \geq 0, \quad \forall i, r_i, r'_i \in A_i \quad (1)$$



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- Denominator does not depend on variable of sum
- So it can be factored and canceled
- If  $\pi(r_i) = 0$ , LHS of (1) is zero regardless of  $i$  and  $r'_i$ , so equation always holds

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