# University of Toronto

**Faculty of Arts and Science**
**Term Test 1**
**MAT237Y1Y - Advanced Calculus**
**Duration - 2 hours**
**No Aids Permitted**

**Surname:** ______________________________

**First Name:** ______________________________

**Student Number:** ______________________________

**Tutorial:**

| T0101 | T0102 | T0201 | T0202 | T0301 | T5101 |
|---|---|---|---|---|---|
| MW2 BA2135 | MW2 BAB024 | MW3 BA2135 | MW3 BAB024 | TR4 BA12340 | TR 5 BA1240 |
| | | | | | |

This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of illegal aids include, but are not limited to textbooks, notes, calculators, or any electronic device.

Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

| Problem | Points | Score |
|---|---|---|
| 1 | 8 | |
| 2 | 6 | |
| 3 | 6 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| Total: | 50 | |

1. For each of the following questions, determine whether the statement is true or false.

**No justification is necessary.**

(a) (2 points) Suppose that $(x_n)_{n=1}^{\infty}$ is a divergent sequence in $\mathbb{R}$. If $f : \mathbb{R} \to \mathbb{R}$ is continuous then $(f(x_n))_{n=1}^{\infty}$ diverges.

| **True** | **False** |
|---|---|
| | |

(b) (2 points) If $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ is bounded and increasing then $(x_n)_{n=1}^{\infty}$ converges.

| **True** | **False** |
|---|---|
| | |

(c) (2 points) If $(\mathbf{x}_n)_{n=1}^{\infty}$ is bounded in $\mathbb{R}^m$ then $(\mathbf{x}_n)_{n=1}^{\infty}$ has a convergent subsequence.

| **True** | **False** |
|---|---|
| | |

(d) (2 points) If $f : \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^m \to \mathbb{R}^n$ are continuous functions such that $f(\mathbf{p}) = g(\mathbf{p})$ for infinitely many points $\mathbf{p} \in \mathbb{R}^n$ then $f(\mathbf{x}) = g(\mathbf{x})$ for all points $\mathbf{x} \in \mathbb{R}^m$.

| **True** | **False** |
|---|---|
| | |

2. Consider the set

$$S = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 4\}.$$

(a) (2 points) Sketch this set in the space provided.

(b) (4 points) Using any technique available to you, show that $S$ is open.

3. (6 points) Let $A_i \subseteq \mathbb{R}^n$ be an open set, for $i \in \mathbb{N}$. Show that $\bigcup_{i=1}^{\infty} A_i$ is open.

4. (a) (5 points) Determine the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$. Be sure to justify your answer.

(b) (5 points) Show that $\lim_{(x,y)\to(0,0)} \frac{x^2+y}{\sqrt{x^2+y^2}}$ does not exist.

5. (a) (2 points) Define what it means for a sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in $\mathbb{R}^m$ to be Cauchy.

(b) (2 points) Give any example of a Cauchy sequence. You do not need to give justification.

(c) (6 points) Give an example of a sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in $\mathbb{R}^m$ such that $|\mathbf{x}_{n+1} - \mathbf{x}_n| \xrightarrow{n\to\infty} 0$ but $(\mathbf{x}_n)_{n=1}^{\infty}$ is not a Cauchy sequence. Justify that the sequence is not Cauchy, and that the above limit indeed tends to 0.

6. (a) (5 points) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $S \subseteq \mathbb{R}^m$. Show that $f^{-1}(S^c) = [f^{-1}(S)]^c$.

(b) (5 points) Let $f$ and $S$ be as above. Assume that $f$ is a continuous function. Show that if $S$ is closed then $f^{-1}(S)$ is closed.

**This page is for additional work and will not be marked.**