University of Toronto

Faculty of Arts and Science
Term Test 1
MAT237Y1Y - Advanced Calculus
Duration - 2 hours
No Aids Permitted

Surname: ______________________________

First Name: ______________________________

Student Number: ______________________________

Tutorial:

T0101	T0102	T0201	T0202	T0301	T5101
MW2 BA2135	MW2 BAB024	MW3 BA2135	MW3 BAB024	TR4 BA12340	TR 5 BA1240

This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of illegal aids include, but are not limited to textbooks, notes, calculators, or any electronic device.

Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	8	
2	6	
3	6	
4	10	
5	10	
6	10	
Total:	50	

1. For each of the following questions, determine whether the statement is true or false.

No justification is necessary.

(a) (2 points) Suppose that $(x_n)_{n=1}^{\infty}$ is a divergent sequence in $\mathbb{R}$. If $f : \mathbb{R} \to \mathbb{R}$ is continuous then $(f(x_n))_{n=1}^{\infty}$ diverges.

True	**False**

Solution: False. For example, if f is the constant function then the image of every sequence converges.

(b) (2 points) If $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ is bounded and increasing then $(x_n)_{n=1}^{\infty}$ converges.

True	**False**

Solution: True. This follows from the Monotone Convergence Theorem.

(c) (2 points) If $(\mathbf{x}_n)_{n=1}^{\infty}$ is bounded in $\mathbb{R}^m$ then $(\mathbf{x}_n)_{n=1}^{\infty}$ has a convergent subsequence.

True	**False**

Solution: True. This is a Theorem 1.44 of the notes.

(d) (2 points) If $f : \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^m \to \mathbb{R}^n$ are continuous functions such that $f(\mathbf{p}) = g(\mathbf{p})$ for infinitely many points $\mathbf{p} \in \mathbb{R}^n$ then $f(\mathbf{x}) = g(\mathbf{x})$ for all points $\mathbf{x} \in \mathbb{R}^m$.

True	**False**

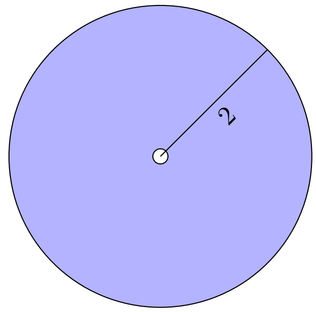
Solution: False. For example, the function $f(x) = \sin(2\pi x)$ and $g(x) = x\sin(2\pi x)$ both agree on the integers $\mathbb{Z}$; however, they are clearly not equal. The statement is true if they agree on a *dense set*.

2. Consider the set

$$S = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 4\}.$$

(a) (2 points) Sketch this set in the space provided.

Solution:



(b) (4 points) Using any technique available to you, show that S is open.

Solution: The simplest solution is the realize this sets as the preimage of an open set under a continuous map. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2+y^2$, which is a polynomial and hence continuous. Then

$$S = f^{-1}\left((0, 4)\right).$$

The set $(0, 4)$ is open, and since preimages of open sets are open, we conclude that S is open as required.

3. (6 points) Let $A_i \subseteq \mathbb{R}^n$ be an open set, for $i \in \mathbb{N}$. Show that $\displaystyle\bigcup_{i=1}^{\infty} A_i$ is open.

Solution: Let $x \in \bigcup_{i=1}^{\infty} A_i$. By definition of the union, there must exists some $j \in \mathbb{N}$ such that $x \in A_j$. Since A_j is open, there exists $r > 0$ such that $B_r(x) \subseteq A_j$, but then we have

$$x \in B_r(x) \subseteq A_j \subseteq \bigcup_{i=1}^{\infty} A_j$$

showing that x is an interior point of the union, as required.

4. (a) (5 points) Determine the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$. Be sure to justify your answer.

Solution: We use the squeeze theorem. Recall that $|x| \leq \sqrt{x^2+y^2}$, and so

$$0 \leq \left|\frac{xy}{x^2+y^2}\right| = \frac{|x||y|}{\sqrt{x^2+y^2}} \leq \frac{|y|\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = |y|.$$

In the limit as $(x,y) \to (0,0)$ we have that $|y| \to 0$, and so by the Squeeze Theorem we have

$$\lim_{(x,y)\to(0,0)} \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = 0, \quad \text{which implies} \quad \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0.$$

(b) (5 points) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y}{\sqrt{x^2+y^2}}$ does not exist.

Solution: If we approach along any line of the form $y = mx$, or equivalently along the path $\gamma(t) = (t, mt)$ we have

$$\begin{aligned}\lim_{t\to 0} \frac{t^2+tm}{\sqrt{t^2+m^2t^2}} &= \lim_{t\to 0} \frac{t(t+m)}{|t|\sqrt{1+m^2}} \\ &= \pm\frac{m}{\sqrt{1+m^2}},\end{aligned}$$

where the sign depends on whether $t \to 0^+$ or $t \to 0^-$. In any case, the limit clearly depends upon the chosen path, and hence the limit does not exist.

5. (a) (2 points) Define what it means for a sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in $\mathbb{R}^m$ to be Cauchy.

Solution: A sequence $(\mathbf{x}_m)$ is Cauchy if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ then $\|\mathbf{x}_n - \mathbf{x}_m\| < \epsilon$.

(b) (2 points) Give any example of a Cauchy sequence. You do not need to give justification.

Solution: Every convergent sequence is Cauchy, so simply writing down such a sequence is sufficient.

(c) (6 points) Give an example of a sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in $\mathbb{R}^m$ such that $|\mathbf{x}_{n+1} - \mathbf{x}_n| \xrightarrow{n\to\infty} 0$ but $(\mathbf{x}_n)_{n=1}^{\infty}$ is not a Cauchy sequence. Justify that the sequence is not Cauchy, and that the above limit indeed tends to 0.

Solution: There are many possible solutions, one need only provide a sequence which grows slower as $n \to \infty$ but fails to converge. For example,

1. $x_n = \sum_{k=1}^{n} \frac{1}{n}$
2. $x_n = \sqrt{n}$
3. $x_n = \log(n)$,

all work. They cannot possibly be Cauchy as each sequence diverges. Showing that each sequence satisfies $|\mathbf{x}_{n+1} - \mathbf{x}_n| \xrightarrow{n\to\infty} 0$ depends on the sequence, but typically reduces to a simple limit computation.

6. (a) (5 points) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $S \subseteq \mathbb{R}^m$. Show that $f^{-1}(S^c) = [f^{-1}(S)]^c$.

Solution: We must show double subset inclusion.

($\subseteq$) Let $x \in f^{-1}(S^c)$. By definition of the preimage we know that $f(x) \in S^c$; equivalently, $f(x) \notin S$. This is precisely the condition that $x \in [f^{-1}(S)]^c$.

($\supseteq$) Let $x \in [f^{-1}(S)]^c$. By definition, $x \notin f^{-1}(S)$ implying that $f(x) \notin S$. But this is precisely the condition that $x \in f^{-1}(S^c)$.

Alternatively, one can play around with the set builder notation to see that these are equivalent.

$$\begin{aligned} f^{-1}(S^c) &= \{x \in \mathbb{R}^n : f(x) \in S^c\} \\ &= \{x \in \mathbb{R}^n : f(X) \notin S\} \end{aligned} \qquad \begin{aligned} f^{-1}(S)^c &= \{x \in \mathbb{R}^n : f(x) \in S\}^c \\ &= \{x \in \mathbb{R}^n : f(x) \notin S\}. \end{aligned}$$

Both statements have equivalent set-builder notation, and hence are equal.

(b) (5 points) Let f and S be as above. Assume that f is a continuous function. Show that if S is closed then $f^{-1}(S)$ is closed.

Solution: By definition of a closed set, it suffices to show that $[f^{-1}(S)]^c$ is open. By part (a), we have

$$[f^{-1}(S)]^c = f^{-1}(S^c)$$

and as S is closed, S^c is open. Since f is continuous, the preimage of open sets is open, so we conclude that $[f^{-1}(S)]^c$ is open, indicating that $f^{-1}(S)$ is closed, as required.

This page is for additional work and will not be marked.