#6 Set Theory and Proofs

- In this section we will go over **Sets** by means of logical definitions and mathematical proofs.

Definitions

<u>Set:</u> $A = \{x \in E : P(x)\}$ or $A = \{x \in E \mid P(x)\}$

<u>Subset:</u> $A \subseteq B \iff \{\forall x,\ x \in A \Rightarrow x \in B\}$

$$A \nsubseteq B \iff \{\exists x:\ x \in A \land x \notin B\}$$

<u>Proper Subset:</u> $A \subset B \iff A \subseteq B \land \{\exists x \in B:\ x \notin A\}$

Proving $A \subseteq B$ (Element Method)

1. Let $x \in A$
2. $\Rightarrow$ Use properties of being an element of A, try to reach properties of being an element of B.
3. $\Rightarrow x \in B$ concluded.

Direct proof…

To Prove that a set is empty

1. Assume that the set is not empty ($A \neq \emptyset$)
2. Try to reach to a contradiction

Proof by contradiction...

Example

- $\mathrm{A} = \{m \in \mathbb{Z} \mid m = 6r + 12, \exists r \in \mathbb{Z}\}$ and $B = \{n \in \mathbb{Z} \mid n = 3s, \exists s \in \mathbb{Z}\}$ given.

Show that $A \subseteq B$ and $B \nsubseteq A$.

Equality of Sets

$$A = B \iff A \subseteq B \land B \subseteq A$$

To prove that $A = B$:

1. $(\subseteq)$: Prove $A \subseteq B$.
2. $(\supseteq)$: Prove $B \subseteq A$.

Biconditional proof…

Example

- $\mathrm{A} = \{x \in \mathbb{Z} \mid m = 6k + 4, \exists k \in \mathbb{Z}\}$

  $B = \{y \in \mathbb{Z} \mid y = 18m - 2, \exists m \in \mathbb{Z}\}$

  $\mathrm{C} = \{z \in \mathbb{Z} \mid z = 18n + 16, \exists n \in \mathbb{Z}\}$ given.
- $A \subseteq B$? $B \subseteq A$? $B = C$?

Definitions Related to Sets

1. Venn Schemas
2. Sets of Numbers
3. Reel Number Intervals

Operations on Sets

<u>Union:</u> $A \cup B = \{x \in E \mid x \in A \lor x \in B\}$

<u>Intersection :</u> $A \cap B = \{x \in E \mid x \in A \land x \in B\}$

<u>Difference:</u> $A - B = \{x \in E \mid x \in A \land x \notin B\}$

<u>Complement:</u> $A^C = \{x \in E \mid x \notin A\}$

Operations on Sets

Intersection and Union of More Than One Sets:

Union: $\bigcup_{i=1}^{n} A_i = \{x \in E \mid x \in A_i, \exists i \in \{1, .., n\}\}$

Intersection: $\bigcap_{i=1}^{n} A_i = \{x \in E \mid x \in A_i, \forall i \in \{1, .., n\}\}$

Example

- $A_i = \{x \in R \mid -\frac{1}{i} < x < \frac{1}{i}\}$ defined.
- $\bigcup_{i=1}^{3} A_i = ?$ $\bigcap_{i=1}^{3} A_i = ?$

Definitions on Sets

<u>Disjoint Sets:</u> A and B are disjoint $\Leftrightarrow$ $A \cap B = \emptyset$

<u>Mutually Disjoint Sets:</u>

$A_1, A_2, \ldots, A_n$ are mutually disjoint $\Leftrightarrow A_i \cap A_j = \emptyset, \forall i \neq j$.

<u>Partition:</u>

$[A_1, A_2, \ldots, A_n]$ is a partition of A $\Leftrightarrow$

$(A_1, A_2, \ldots, A_n$ are mutually disjoint $\wedge \bigcup_{i=1}^{n} A_i = A\ )$

Definitions on Sets

<u>Power Set:</u> $\wp(A)$ = Set of all subsets of A

$$|\wp(A)| = 2^{n}$$

<u>Cartesian Product:</u>

$$A \times B = \{(x, y) |\, x \in A \wedge y \in B, \forall x \in A, y \in B\}$$

Set Identities

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets, A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets A,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

Set Identities

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

8. *Universal Bound Laws:* For all sets A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$