3. Hafta Cuma Dersi

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$\underbrace{\forall x \in D,\ P(x)}_{\text{varsay}} \Rightarrow \underbrace{Q(x)}_{\text{ulaşmaya çalış}}$ doğrudan ispat

değili : $\exists x \in D : P(x) \wedge \sim Q(x)$

ters örnek

Örn $\left(\forall a,b \in \mathbb{Z},\ \text{if } 3 \mid (a+b) \Rightarrow 3 \mid (a-b)\right)$

ispat (ters örnek): $a=5,\ b=1 \in \mathbb{Z}$, $3 \mid (5+1)$, $3 \nmid (5-1)$ (3 bölmez 4)

$6 = 3 \cdot \underbrace{2}_{\in \mathbb{Z}} \Rightarrow 3 \mid 6$ $\qquad 4 = 3 \cdot k \quad k = \frac{4}{3} \notin \mathbb{Z}$

Tek/Çift Tam Sayı:	Asal Sayı:	Rasyonel Sayı:	Bölünebilme:
n çifttir $\Leftrightarrow \exists k \in \mathbb{Z}: n = 2k$ n tektir $\Leftrightarrow \exists k \in \mathbb{Z}: n = 2k+1$	$1 < n$ asaldır $\Leftrightarrow (n = rs \Leftrightarrow (r=1 \wedge s=n) \vee (r=n \wedge s=1))$	$r \in \mathbb{R}$ rasyonel sayıdır $\Leftrightarrow (\exists a,b \in \mathbb{Z}: r = a/b \wedge b \neq 0)$	$d \mid n \Leftrightarrow (\exists k \in \mathbb{Z}: n = dk)$

24. For all integers a, b, and c, if $a \mid b$ and $a \mid c$ then $a \mid (2b - 3c)$.

$\forall a,b,c \in \mathbb{Z}\ (a \mid b \wedge a \mid c) \Rightarrow a \mid (2b-3c)$

ispat: $a,b,c \in \mathbb{Z}$ ve $a \mid b$ ve $a \mid c$ olsun.

$\Rightarrow b = a \cdot k,\ \exists k \in \mathbb{Z} \qquad c = ak',\ \exists k' \in \mathbb{Z}$

$\Rightarrow 2b - 3c = 2(ak) - 3(ak') = 2ak - 3ak' = a(\underbrace{2k - 3k'}_{\in \mathbb{Z}})$

$\Rightarrow a \mid (2b - 3c)$. ∎

29. For all integers a and b, if $a \mid b$ then $a^2 \mid b^2$.

$\forall a,b \in \mathbb{Z}\ a \mid b \Rightarrow a^2 \mid b^2$

$b^2 = a^2 \cdot \underbrace{?}_{\in \mathbb{Z}}$

ispat: $a,b \in \mathbb{Z}$ $a \mid b$ olsun.

$\Rightarrow b = a \cdot k,\ \exists k \in \mathbb{Z}$

$\Rightarrow b^2 = (ak) = a^2 \underbrace{k^2}_{\in \mathbb{Z}} \Rightarrow a^2 \mid b^2$ ∎

30. For all integers a and n, if $a \mid n^2$ and $a \leq n$ then $a \mid n$.

$\forall a,n \in \mathbb{Z}\ (a \mid n^2 \wedge a \leq n) \Rightarrow a \mid n$.

ispat. $a,n \in \mathbb{Z}$, $a \mid n^2$ ve $a \leq n$ olsun.

$\Rightarrow n^2 = a \cdot k,\ \exists k \in \mathbb{Z}$

Ters örnek: $a \mid n^2 \wedge a \leq n \wedge a \nmid n$

$a = 8\ n = 12 \in \mathbb{Z}$. $8 \mid 144$ $\quad 8 \leq 12$ $\quad 8 \nmid 12$

$\Rightarrow n^2 = a.k \,, \exists k \in \mathbb{Z}$

$\Rightarrow |n| = \sqrt{a.k}$

Ters örnek: $a \mid n^2 \wedge a \leq n \wedge a \nmid n$

$a=8, n=12 \in \mathbb{Z}, \quad 8 \mid 144 \quad 8 \leq 12 \quad 8 \nmid 12$

$\Rightarrow$ ifade yanlıştır. ∎

İspatı Durumlara Ayırma (Proof by cases) → doğrudan ispat tekniği

$\forall x \in D, \ p(x) \Rightarrow q(x)$

varsay

$\forall x \in D, \ p(x) \wedge q(x) \Rightarrow r(x)$

$\forall x \in D, \ (p(x) \vee q(x)) \Rightarrow r(x)$

ispat. 1. Durum: $x \in D$, $p(x)$ olsun.

$\Rightarrow$

$\Rightarrow$

$+ \quad \Rightarrow \ldots \quad \Rightarrow r(x)$ ✓

2. Durum: $x \in D$, $q(x)$ olsun.

$\Rightarrow$

$\Rightarrow$

$\Rightarrow \ldots \quad \Rightarrow r(x)$ ✓

∎

$(p \vee q \vee r) \Rightarrow t(x)$

19. Prove that for all integers n, $n^2 - n + 3$ is odd.

$\forall n \in \mathbb{Z}, \ n^2 - n + 3$ tektir.

ispat 1. durum: $n \in \mathbb{Z}$ tek olsun.

$\Rightarrow n = 2k+1 \,, \exists k \in \mathbb{Z}$

$\Rightarrow n^2 - n + 3 = (2k+1)^2 - (2k+1) + 3 = 4k^2 + 4k + \cancel{1} - 2k - \cancel{1} + 3$

$= 4k^2 + 2k + \underset{2+1}{3} = 2\underbrace{(2k^2 + k + 1)}_{\in \mathbb{Z}} + 1$

$\Rightarrow n^2 - n + 3$ tektir. ✓

2. durum: $n \in \mathbb{Z}$ çift olsun.

$\Rightarrow n = 2k \,, \exists k \in \mathbb{Z}$

$\Rightarrow n^2 - n + 3 = (2k)^2 - (2k) + 3 = 4k^2 - 2k + 3$

$= 2\underbrace{(2k^2 - k + 1)}_{\in \mathbb{Z}} + 1$

$\Rightarrow n^2 - n + 3$ tektir. ✓

Sonuç olarak, ifade doğrudur. ∎

Bölüm-kalan Teoremi

- Bir n tam sayısı ve bir d pozitif tam sayısı için $n = dq + r$ ve $0 \le r < d$ olacak şekilde tek bir (q,r) tam sayı ikilisi vardır.

$n \,|\, d$, q, r

Örn

Örn $n \,|\, 3$

3'e bölünebilme 0, 1, 2

$\forall n \in \mathbb{Z}$, $n = 3k$, $n = 3k+1$, $n = 3k+2$

The square of any odd integer has the form $8m + 1$ for some integer m.

$\forall n \in \mathbb{Z}$ n tek ise $n^2 = 8m+1$, $\exists m \in \mathbb{Z}$ formunda yazılabilir.

$n = 2k+1$,

$\Rightarrow n^2 = 4k^2 + 4k + 1$

$8 \nmid 9$

$n = 3k$, $3k+1$, $3k+2$ $\Rightarrow n^2 = 9k^2$ - - -

$\cancel{n = 4k}$, $n = 4k+1$, $\cancel{n = 4k+2}$, $n = 4k+3$

İspat: $n \in \mathbb{Z}$ tek olsun.

1. Durum: $n = 4k+1$, $\exists k \in \mathbb{Z}$ olsun.

$\Rightarrow n^2 = (4k+1)^2 = 16k^2 + 8k + 1 = 8(\underbrace{2k^2 + k}_{\in \mathbb{Z}}) + 1$ ✓

2. Durum: $n = 4k+3$, $\exists k \in \mathbb{Z}$ olsun.

$\Rightarrow n^2 = (4k+3)^2 = 16k^2 + 24k + \underbrace{9}_{8+1} = 8(\underbrace{2k^2 + 3k + 1}_{\in \mathbb{Z}}) + 1$ ✓

$\therefore \Rightarrow n^2 = 8m+1$, $\exists m \in \mathbb{Z}$ formunda yazılır.