4th Week

30 Ekim 2021 Cumartesi 12:31

Elementary Matrices

Elementary Row Operations

1.) $r_i \leftrightarrow r_j$

2.) $c \cdot r_i \to r_i$

3.) $c \cdot r_j + r_i \to r_i$

$$I_n = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{bmatrix}_{n \times n} \xrightarrow{\text{Only 1 row operation}} \text{An elementary matrix}$$

$$I_n \xrightarrow{\text{1st type of row op.}} \text{Type-I elementary matrix}$$

$$I_n \xrightarrow{\text{2nd of row op.}} \text{Type-II elementary matrix}$$

$$I_n \xrightarrow{\text{3rd of row op}} \text{Type-III elementary matrix}$$

Ex/

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{1st type}]{r_2 \leftrightarrow r_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \to \text{A type-I elementary matrix}$$

Ex/

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{2nd type}]{5r_3 \to r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \to \text{A type-II elementary matrix.}$$

Ex/

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{3rd type}]{3r_1 + r_3 \to r_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \text{A type-III elementary matrix.}$$

! Each elementary matrix corresponds to its creator row operation.

$$A \xrightarrow{\text{1 row op.}^{*}} A' \quad \langle \approx \rangle \quad A' = EA$$

E: the elementary matrix created by * row operation.

EA: left multiplication of A with E

Ex/

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & -1 & 0 \end{bmatrix} \xrightarrow[\text{3rd type row}]{2r_1 + r_2 \to r_2} \overset{A'}{\begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & 2 & 3 & 6 \end{bmatrix}}$$

$$EA \stackrel{?}{=} A'$$

Ex

$$A=\begin{bmatrix}1&0&2&3\\0&2&-1&0\\-2&0&3&1\end{bmatrix}_{3\times4}\xrightarrow{2r_1+r_2\to r_2}\begin{bmatrix}1&0&2&3\\2&2&3&6\\-2&0&3&1\end{bmatrix}$$

$EA \doteq A'$

corresponding elementary matrix

$$I_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\xrightarrow{2r_1+r_2\to r_2}\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\qquad E_1=\begin{bmatrix}1&0&0\\2&1&0\\0&0&0\end{bmatrix}$$

$= \quad E_1A = A'$

$$E_1A=\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}_{3\times3}\begin{bmatrix}1&0&2&3\\0&2&-1&0\\-2&0&3&1\end{bmatrix}_{3\times4}=\begin{bmatrix}1&0&2&3\\2&2&3&6\\-2&0&3&1\end{bmatrix}_{3\times4}=A'$$

$2+0+0 \quad 0+2+0 \quad 4-1+0 \quad 6+0+0$

$$\underbrace{\begin{bmatrix}1&0&2&3\\2&2&3&6\\-2&0&3&1\end{bmatrix}}_{A'}\xrightarrow{3r_1\to r_1}\begin{bmatrix}3&0&6&9\\2&2&3&6\\-2&0&3&1\end{bmatrix}\to A''$$

$$E_2=\begin{bmatrix}3&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$= \checkmark$

$$A''=E_2A'=\begin{bmatrix}3&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&2&3\\2&2&3&6\\-2&0&3&1\end{bmatrix}=\begin{bmatrix}3&0&6&9\\2&2&3&6\\-2&0&3&1\end{bmatrix}\to A''$$

$$A\xrightarrow{1^{st}\text{ row op}}A'\xrightarrow{2^{nd}\text{ row op}}A''$$

$E_1A = A'$ $\qquad$ $E_2A' = A''$

$$\underline{E_2(E_1A) = A''}$$

Ex

$$A\longrightarrow RREF$$

$1^{st}, 2^{nd}, \ldots, m$: $E_1, \ldots, E_m$

$\underline{E_m \cdots E_2E_1A} =$ RREF

! If $A_{n\times n}$ is a square matrix, and If the RREF of A becomes the identity matrix

$\Rightarrow$ A is invertible. (A^{-1} exists)

Why?

$$A\longrightarrow RREF = I_n$$

$$E_m \cdots E_3E_2E_1A = I_n \to \text{identity matrix}$$

$= A^{-1}$

$$\underbrace{E_m \cdots E_3 E_2 E_1}_{A^{-1}} A = I_n \rightarrow \text{identity matrix}$$

!

Ex

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow[E_1]{-\frac{1}{2} r_2 \to r_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow[E_2]{\frac{1}{4} r_3 \to r_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{REF}$$

$$E_3 \leftarrow \frac{3}{2} r_3 + r_2 \to r_2$$

$$E_4 \leftarrow -2 r_3 + r_1 \to r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \rightarrow \text{identity matrix}$$

RREF ✓

What is $A^{-1} = ?$

$$\underbrace{E_4 E_3 E_2 E_1}_{A^{-1}} A = I_3$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

$$E_3 (E_2 E_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 3/8 \\ 0 & 0 & 1/4 \end{bmatrix}$$

$$E_4 (E_3 E_2 E_1) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 3/8 \\ 0 & 0 & 1/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & -1/2 & 3/8 \\ 0 & 0 & 1/4 \end{bmatrix} \longrightarrow A^{-1}$$

$-1/2 \qquad 0 + \frac{1}{2}$

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 3 \\ 0 & 0 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & -1/2 & 3/8 \\ 0 & 0 & 1/4 \end{bmatrix}}_{A^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow I_3 \checkmark$$

$0 + -\frac{3}{4} + \frac{3}{4}$

$$[\, A \;\vdots\; I \,] \longrightarrow [\, \underset{= I_n}{\text{RREF}} \;\vdots\; A^{-1} \,]$$