4. Hafta Pazartesi Dersi

15 Mart 2021 Pazartesi 11:39

Singüler Matris: Tersi olmayan matrislere singüler matris diyoruz.

Elementer Matrisler:

$$I_n = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}_{n\times n} \xrightarrow[\text{satır operasyonu}]{\text{Sadece 1 tane}} \text{Elementer Matris}$$

Birim Matris

Satır Operasyonları

1) $r_i \leftrightarrow r_j \longrightarrow$ 1. Tip Elementer Matris

2) $cr_i \rightarrow r_i \longrightarrow$ 2. Tip " "

3) $cr_j + r_i \rightarrow r_i \longrightarrow$ 3. Tip " "

Örn

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E \rightarrow \text{1. tip Elementer Matris}$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad I_5 \xrightarrow{3r_2 \rightarrow r_2} E \longrightarrow \text{2. tip Elementer Matris}$$

$$E = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{2r_2 + r_1 \rightarrow r_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E$$

↳ 3. tip elementer matris

* ! Bir matrisi soldan (sağdan) bir elementer matrisle çarpmak =

* Bu matrise o elementer matrisi üreten satır (sütun) operasyonunu uygulamak

EA

$$A \xrightarrow{r\ldots} EA$$

Örn

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times 3} \xrightarrow{-5r_1 + r_2 \to r_2} \begin{bmatrix} 1 & 2 & 3 \\ -1 & -5 & -9 \\ 7 & 8 & 9 \end{bmatrix} =$$

$-5 \quad -10 \quad -15$

$4 \quad 5 \quad 6$

$-1 \quad -5 \quad -9$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3x3

$$EA = \overset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -5 & -9 \\ 7 & 8 & 9 \end{bmatrix}$$

$-5.1 + 1.4 + 0.7$

$-5.2 + 1.5 + 0.8$

$-5.3 + 1.6 + 0.9$

$-5c_2 + c_1 \to c_1$

$$AE = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 2 & 3 \\ -21 & 5 & 6 \\ -33 & 8 & 9 \end{bmatrix}$$

$4.1 + 5.-5 + 0$

$7.1 + 8.-5 + 9.0$

Soldan E ile çarpma → satır operasyon

sağdan E ile çarpma → sütun operasyon

SECTION 1.5 EXERCISES

1. Which of the matrices that follow are elementary matrices? Classify each elementary matrix by type.

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. Find the inverse of each matrix in Exercise 1. For each elementary matrix, verify that its inverse is an elementary matrix of the same type.

3. For each of the following pairs of matrices, find an elementary matrix E such that $EA = B$.

(a) $A = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 2 \\ 5 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 3 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ -2 & 4 & 5 \end{bmatrix}$

(c) $A = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ -2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ 0 & 3 & 5 \end{bmatrix}$

4. For each of the following pairs of matrices, find an elementary matrix E such that $AE = B$.

(a) $A = \begin{bmatrix} 4 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & 4 \\ 4 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{bmatrix}$,

$C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{bmatrix}$

(a) Find an elementary matrix E such that $EA = B$.

(b) Find an elementary matrix F such that $FB = C$.

$$A_{(n\times n)} \longrightarrow \text{İSEF}$$

$$\underset{n\times n}{A} \longrightarrow \text{İSEF}$$

Örn/

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow[E_1]{-3r_1+r_2\to r_2} \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}}_{=E_1 A} \xrightarrow[E_2]{-\frac{1}{2}r_2\to r_2} \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{=E_2(E_1 A)} \to \text{SEF}$$

$$\xrightarrow[E_3]{-2r_2+r_1\to r_1} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=E_3(E_2(E_1 A))} \to \text{İSEF}$$

$$\to I_2 \to \text{Birim matrisi elde ettik}$$

$$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{\underbrace{E_3 E_2 E_1}_{=A^{-1}} A = I}$$

$$\Rightarrow E_3 E_2 E_1 I = A^{-1}$$

$$E_3 \to E_2 \to E_1 \to I \Rightarrow A^{-1}$$

$$A \xrightarrow{\text{işlemler}} \text{İSEF} = I$$

$$I \xrightarrow{\text{aynı işlemler}} A^{-1}$$

$$\left[A \;\vdots\; I \right] \xrightarrow{\text{işlemler}} \left[\begin{matrix}A'\text{nın} \\ \text{İSEF hali} \\ =I\end{matrix} \;\vdots\; A^{-1} \right]$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^{-1} = E_3 E_2 E_1 \Rightarrow \quad E_2 E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3/2 & -1/2 \end{bmatrix}$$

$$E_3 \begin{bmatrix} 1 & 0 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3/2 & -1/2 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}}_{=A^{-1}}$$

$$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{-3r_1+r_2\to r_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$A \to \text{İSEF} = I$ $\qquad I \Rightarrow A^{-1}$

$$\xrightarrow{-\frac{1}{2}r_2\to r_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

$$\overset{\shortparallel}{I} \qquad \overset{\shortparallel}{A^{-1}} \qquad \xrightarrow{} \begin{bmatrix} \cdot & \cdot & | & & \\ 0 & 1 & | & 3/2 & -1/2 \end{bmatrix}$$

$-2 \cdot \frac{3}{2} + 1$

$-2 \cdot -\frac{1}{2}$

$$\xrightarrow{-2r_2 + r_1 \to r_1} \begin{bmatrix} 1 & 0 & | & -2 & 1 \\ 0 & 1 & | & 3/2 & -1/2 \end{bmatrix}$$

$\underbrace{\begin{matrix}1 & 0\\ 0 & 1\end{matrix}}_{\text{İSEF} = I_2}$ $\quad \underbrace{\begin{matrix}-2 & 1\\ 3/2 & -1/2\end{matrix}}_{= A^{-1}}$

$-2 + 3 \qquad 1 + 2 \cdot -\frac{1}{2}$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark \rightarrow \text{sağlama}$$

$3 \cdot -2 + 4 \cdot \frac{3}{2} \qquad 3 \cdot 1 + 4 \cdot -\frac{1}{2}$

$-6 + 6 \qquad 3 + -2$

Örn

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \qquad A^{-1} = ?$$

$$[\,A \mid I\,] \longrightarrow \quad A \to \text{İSEF} = I, \quad I \to A^{-1}$$

İSEF

$$\begin{bmatrix} 2 & 0 & 1 & | & 1 & 0 & 0 \\ 1 & -1 & 1/2 & | & 0 & 1 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}r_1 \to r_1} \begin{bmatrix} 1 & 0 & 1/2 & | & 1/2 & 0 & 0 \\ 1 & -1 & 1/2 & | & 0 & 1 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-r_1 + r_2 \to r_2} \begin{bmatrix} 1 & 0 & 1/2 & | & 1/2 & 0 & 0 \\ 0 & -1 & 0 & | & -1/2 & 1 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-1 r_2 \to r_2} \begin{bmatrix} 1 & 0 & 1/2 & | & 1/2 & 0 & 0 \\ 0 & 1 & 0 & | & 1/2 & -1 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-r_2 + r_3 \to r_3} \begin{bmatrix} \boxed{1} & 0 & 1/2 & | & 1/2 & 0 & 0 \\ 0 & \boxed{1} & 0 & | & 1/2 & -1 & 0 \\ 0 & 0 & 0 & | & -1/2 & 1 & 1 \end{bmatrix}$$

İSEF $\checkmark \neq I_3$

$\Rightarrow A^{-1}$ yoktur.

A terslenebilir değildir.

A singülerdir.

Örn

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$[\,A \,\vdots\, I\,] \quad \text{İSEF} = I \quad \rightarrow A^{-1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2r_1+r_3 \to r_3]{-2r_1+r_2 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -2 & 1 & 0 \\ 0 & 0 & -3 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}r_3 \to r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2/3 & 0 & -1/3 \end{array}\right]$$

$$\xrightarrow[5r_3+r_2 \to r_2]{-2r_3+r_1 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 0 & 2/3 \\ 0 & 1 & 0 & 4/3 & 1 & -5/3 \\ 0 & 0 & 1 & 2/3 & 0 & -1/3 \end{array}\right]$$

$= I_3$ $\qquad$ $= A^{-1}$

$$A^{-1} = \begin{bmatrix} -1/3 & 0 & 2/3 \\ 4/3 & 1 & -5/3 \\ 2/3 & 0 & -1/3 \end{bmatrix} \longrightarrow$$

A terslenebilirdir.

A singüler değildir.

$$A A^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -1/3 & 0 & 2/3 \\ 4/3 & 1 & -5/3 \\ 2/3 & 0 & -1/3 \end{bmatrix}}_{A^{-1}}$$

$-\frac{1}{3} + 0 + 2 \cdot \frac{2}{3}$

$+\frac{3}{3} = 1$

$\frac{4}{3} + 0 + \frac{-1}{3}$

$\frac{3}{3} = 1$

$1 \cdot \frac{2}{3} + 0 + 2 \cdot -\frac{1}{3} = 0$

$2 \cdot -\frac{1}{3} + 1 \cdot \frac{4}{3} + -1 \cdot \frac{2}{3} = 0$

$\cancel{2 \cdot 0} + 1 \cdot 1 + \cancel{1 \cdot 0} = 1$

$2 \cdot \frac{2}{3} + 1 \cdot -\frac{5}{3} + -1 \cdot -\frac{1}{3} = 0$

$-\frac{2}{3} + \frac{2}{3}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark\checkmark$$

$\left(2 \cdot \frac{2}{3}\right) + \left(1 \cdot \frac{\ }{3}\right)$

var

$\to A x = b$

$A^{-1} A x = A^{-1} b$

$x = A^{-1} b$

$[A \,\vdots\, b] \longrightarrow$

$A \to I$