Diagonalization $\left(A_{n\times n} \to X_{n\times n} D_{n\times n} X^{-1}_{n\times n}\right)$

D: a diagonal matrix

✓ $\Rightarrow$ A is diagonalizable.

* If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of A,
$\Rightarrow$ the corresponding eigenvectors $\vec{x}_1, \vec{x}_2, \ldots$ are linearly independent.

* ! $A_{n\times n}$ is diagonalizable $\Leftrightarrow$ $A_{n\times n}$ has n linearly independent eigenvectors.

* If A is diagonalizable $\Rightarrow$ Eigenvectors of A = Columns of X
$\Rightarrow$ Eigenvalues of A = diagonal elements of D

$A = XDX^{-1}$!

not unique

$$X = \begin{bmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$$

if columns are lin. dep $\Rightarrow$ $\det(X) = 0$
$\Rightarrow$ X is not invertible.

* If A has n distinct eigenvalues, $\Rightarrow$ ($\Rightarrow$ n linearly indep. eigenvectors) A is diagonalizable.

$A_{n\times n} \to$ [Look at the number of distinct eigenvalues $= k$] $\xrightarrow{k=n}$ A is diagonalizable $\Rightarrow A = XDX^{-1}$

$\downarrow k<n$

[Look at the number of linearly independent eigenvectors $= \ell$] $\xrightarrow{\ell = n}$ A is diagonalizable

$\downarrow \ell<n$

A is NOT diagonalizable.

Ex) $A = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}_{2\times 2}$ $\qquad A - \lambda I = \begin{bmatrix} 2-\lambda & -3 \\ 2 & -5-\lambda \end{bmatrix}$

$$\det(A-\lambda I) = (2-\lambda)(-5-\lambda)+6 = 0$$
$$\Rightarrow p(\lambda) = \lambda^2 + 3\lambda - 4 = 0 \to \text{characteristic poly.}$$
$$= (\lambda+4)(\lambda-1) = 0 \Rightarrow \boxed{\lambda_1 = -4, \ \lambda_2 = 1}$$

$\Rightarrow A$ is diagonalizable. $\Rightarrow A = XDX^{-1} \to D = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$.

$\lambda_1 = -4$: $(A-\lambda I)\vec{x} = 0$ $\quad \begin{bmatrix} 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} \ 0 \ \ldots \ 2x_1 - x_2 = 0 \ \begin{bmatrix} 1 \end{bmatrix}$

$\Rightarrow A$ is diagonalizable. $\Rightarrow A = XDX^{-1}$

$\underline{\lambda_1 = -4}$:
$$(A-\lambda I)\vec{x} = 0$$
$$(A+4I)\vec{x} = 0 \rightarrow \begin{bmatrix} 6 & -3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow 2x_1 - x_2 = 0 \rightarrow r\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$x_1 = r \in \mathbb{R}$

$\Rightarrow x_2 = 2r$

eigenvector

$\underline{\lambda_2 = 1}$:
$$(A-1I)\vec{x} = 0 \rightarrow \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 - 3x_2 = 0 \rightarrow r\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$x_2 = r \in \mathbb{R}$

$x_1 = 3r$

eigenvector

$$D = \begin{bmatrix} \overset{\lambda_1}{-4} & 0 \\ 0 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \qquad X^{-1} = \begin{bmatrix} -1/5 & 3/5 \\ 2/5 & -1/5 \end{bmatrix} \Rightarrow A = XDX^{-1}$$

you may also write $D = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$, $X = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow A = XDX^{-1}$

Ex

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}_{3\times3} \qquad A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (2-\lambda)(4-\lambda)(2-\lambda) = 0$$

$$\lambda_1 = 4 \qquad \lambda_2 = 2 \longrightarrow \text{repeated root}$$

$\Rightarrow$ We can not say anything about diagonalizability of A for now. We should check lin. indep. eigenvectors.

$\underline{\lambda_1 = 4}$:
$$(A-4I)\vec{x} = 0 \rightarrow \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right] \Rightarrow \begin{array}{l} -2x_1 = 0 \Rightarrow x_1 = 0 \\ x_1 - 2x_3 = 0 \Rightarrow x_3 = 0 \\ x_2 = r \in \mathbb{R} \end{array} \rightarrow r\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

eigenvector

$\underline{\lambda_2 = 2}$:
$$(A-2I)\vec{x} = 0 \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} 2x_2 = 0 \\ x_1 = 0 \\ x_3 = r \in \mathbb{R} \end{array} \rightarrow r\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

eigenvector

$\Rightarrow$ $\underline{2}$ linearly independent eigenvectors $\Rightarrow$ A is $\underline{\underline{\text{NOT}}}$ diagonalizable!

Ex

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \end{bmatrix} \qquad \lambda_1 = 0 \qquad \underline{\lambda_2 = 1} \rightarrow \text{repeated root}$$

$$\begin{bmatrix} 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

→ repeated root

$\lambda_1 = 0$: Eigenvector → $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$: Eigenvectors → $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

3 linearly independent eigenvectors ⇒ A is diagonalizable.

$A = XDX^{-1}$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow X^{-1} = \dots$$

$\lambda_2 = 1$ → repeated root

1. ~~Find the eigenvalues and the corresponding eigenspaces for each of the following matrices:~~

(a) $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & -8 \\ 2 & 3 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (h) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix}$

(i) $\begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ (j) $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

(k) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ (l) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Check diagonalizability of A. (Why?)

Find D, X, X^{-1} $\quad A = XDX^{-1}$ if A is diagonalizable