11th Week Monday Course

03 Mayıs 2021 Pazartesi 09:37

$B_1 \; V \to V \; B_2$

$[v]_{B_1} \mapsto [v]_{B_2}$

$\xrightarrow{}$ functions

$f: D \to R$

$x \to y$

$x \in D \quad f(x)=y$

$\exists y \in R$

Linear Transformations

mapping L between vector spaces,

$$L : \underset{\text{vector space}}{V} \longrightarrow \underset{\text{vector space}}{W}$$

$$\vec{v} \mapsto \vec{w}$$

$$(V, \oplus, \odot) \qquad (W, \boxplus, \boxdot)$$

satisfying ;

1) $\forall \vec{v}_1, \vec{v}_2 \in V \qquad L(\underbrace{\vec{v}_1 \oplus \vec{v}_2}_{\text{an operation in } V}) = \underbrace{L(\vec{v}_1) \boxplus L(\vec{v}_2)}_{\text{an operation in } W}$

LHS → result → check the equality ✓ ← result ← RHS.

2) $\forall \alpha \in \mathbb{R}, \forall \vec{v} \in V \qquad L(\alpha \odot \vec{v}) = \alpha \boxdot L(\vec{v})$

$\alpha \odot \vec{v}$: an operation in V → LHS $\overset{?}{=}$ RHS ← an operation in W

is a linear transformation.

* $L : V \to V$ is called a "linear operator."

Ex/ $L : \mathbb{R}^2 \to \mathbb{R} \qquad L(\vec{v}) = \sqrt{x^2+y^2}$

$\vec{v} = (x,y) \mapsto \sqrt{x^2+y^2}$ Is L a linear transformation?

1) LHS → $L\big((x_1,y_1) + (x_2,y_2)\big) = L\big((x_1+x_2, y_1+y_2)\big) = \sqrt{(x_1+x_2)^2 + (y_1+y_2)^2}$

RHS → $L(x_1,y_1) + L(x_2,y_2) = \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2+y_2^2}$ $\neq$

L does not satisfy the first rule. ⇒ L is NOT a linear transformation.

2) LHS → $L(\alpha.(x,y)) = L((\alpha x, \alpha y)) = \sqrt{(\alpha x)^2 + (\alpha y)^2} = \sqrt{\alpha^2(x^2+y^2)} = |\alpha|\sqrt{x^2+y^2}$

RHS → $\alpha . L((x,y)) = \alpha\left(\sqrt{x^2+y^2}\right) = \alpha\sqrt{x^2+y^2}$ → if $\alpha < 0$ $\neq$

L does not satisfy the second rule either.

Ex $L : \mathbb{R}^2 \to \mathbb{R}^3 \qquad L(\vec{v}) = (y, x, x+y)$

$\vec{v} = (x,y) \longmapsto (y, x, x+y)$ Is L a linear transformation?

1) LHS → $L\big((x_1,y_1)+(x_2,y_2)\big) = L\big((x_1+x_2, y_1+y_2)\big) = (y_1+y_2, x_1+x_2, x_1+x_2+y_1+y_2)$ $\overset{\checkmark}{=}$

RHS → $L((x_1,y_1)) + L((x_2,y_2)) = (y_1, x_1, x_1+y_1) + (y_2, x_2, x_2+y_2) = (y_1+y_2, x_1+x_2, x_1+y_1+x_2+y_2)$

2) LHS → $L(\alpha(x,y)) = L((\alpha x, \alpha y)) = (\alpha y, \alpha x, \alpha x + \alpha y)$ $\overset{\checkmark}{=}$

2) LHS $\to L(\alpha(x,y)) = L((\alpha x, \alpha y)) = (\alpha y, \alpha x, \alpha x + \alpha y)$

RHS $\to \alpha L((x,y)) = \alpha \cdot (y, x, x+y) = (\alpha y, \alpha x, \alpha(x+y))$ $\Big\} = \checkmark$

$\Rightarrow$ L is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.

9. Determine whether the following are linear transformations from P_2 to P_3.

✓ (a) $L(p(x)) = xp(x)$

? (b) $L(p(x)) = x^2 + p(x)$

? (c) $L(p(x)) = p(x) + xp(x) + x^2 p'(x)$

a) $L : P_2 \longrightarrow P_3$

$a_0 + a_1 x \longmapsto x(a_0 + a_1 x) = a_0 x + a_1 x^2$

1) LHS $\to L\big((a_0 + a_1 x) + (b_0 + b_1 x)\big) = L\big(a_0 + b_0 + (a_1 + b_1)x\big) = a_0 x + b_0 x + (a_1 + b_1)x^2$

RHS $\to L\big((a_0 + a_1 x)\big) + L\big((b_0 + b_1 x)\big) = (a_0 x + a_1 x^2) + (b_0 x + b_1 x^2) = a_0 x + b_0 x + (a_1 + b_1)x^2$ $= \checkmark$

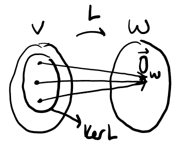
2) LHS $\to L(\alpha(a_0 + a_1 x)) = L((\alpha a_0 + \alpha a_1 x)) = \alpha a_0 x + \alpha a_1 x^2$

RHS $\to \alpha L((a_0 + a_1 x)) = \alpha(a_0 x + a_1 x^2) = \alpha a_0 x + \alpha a_1 x^2$ $\Big) = \checkmark$

$\Rightarrow$ L is a linear transformation.

Kernel and Image of a Linear Transformation

Let $L : V \to W$ be a linear transformation.

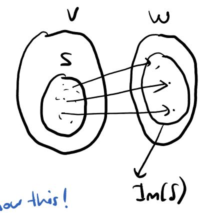


$$\boxed{Ker(L) = \{ \forall \vec{v} \in V : L(\vec{v}) = \vec{0}_W \}}$$

* At least $\vec{0}_V \in Ker(L)$, * $Ker(L) \neq \emptyset$, * $Ker(L) \leqslant V$ $\to$ subspace of V

Let $S \leqslant V$ (subspace). $\boxed{Im(S) = \{ L(s) : \forall s \in S \}}$

* $Im(S) \leqslant W$ (subspace)



Ex) $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ (ker, Im)

$\vec{v} = (x,y,z) \longmapsto (x+y, y+z)$ is a linear transformation. $\to$ Show this!

a) Find $Ker(L)$. b) $S = Span\{e_1, e_3\} \leqslant \mathbb{R}^3$ Find $Im(S)$.

a) $Ker(\vec{L}) = \{ \forall \vec{v} \in \mathbb{R}^3 : L(\vec{v}) = \overset{\vec{0}_{\mathbb{R}^2}}{(0,0)} \}$

$A\_ = 0$

$L((x,y,z)) = (x+y, y+z) = (0,0)$ $\quad \begin{cases} x+y = 0 \\ y+z = 0 \end{cases}$ $\to$ solve this system for x, y, z.

$y = r \in \mathbb{R} \Rightarrow x = -r \quad z = -r$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \begin{bmatrix} -r \\ r \\ -r \end{bmatrix} \in Ker(L)$ $\to$ a typical vector in $Ker(L)$.

$r \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow Ker(L) = span\left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$

! A basis for $Ker(L) = \left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$

$\dim(Ker(L)) = 1$

b) $S = span\{e_1, e_3\}$ $\quad s = r_1 \begin{bmatrix} 1 \\ 0 \\ \cdot \end{bmatrix} + r_2 \begin{bmatrix} 0 \\ 0 \\ \cdot \end{bmatrix} = \begin{bmatrix} r_1 \\ 0 \\ \cdot \end{bmatrix}$, $r_1, r_2 \in \mathbb{R}$

$\to$ a typical vector of S

b) $S = \text{span}\{e_1, e_3\}$ — a subspace of $\mathbb{R}^3$

$s = r_1 \underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{e_1} + r_2 \underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{e_2} = \begin{bmatrix}r_1\\0\\r_2\end{bmatrix}$, $r_1, r_2 \in \mathbb{R}$ → a typical vector of S

$r_1, r_2 \in \mathbb{R}$

$$\text{Im}(S) = \{ L(s) : \forall s \in S \} = \{ L((r_1, 0, r_2)) : r_1, r_2 \in \mathbb{R} \}$$

$$= \{ (r_1 + 0, 0 + r_2) : r_1, r_2 \in \mathbb{R} \}$$

$$= \{ (r_1, r_2) : r_1, r_2 \in \mathbb{R} \} \leqslant \mathbb{R}^2$$

$$\begin{bmatrix}r_1\\r_2\end{bmatrix} = r_1\begin{bmatrix}1\\0\end{bmatrix} + r_2\begin{bmatrix}0\\1\end{bmatrix} \qquad = \mathbb{R}^2$$

$\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$ is a basis for $\text{Im}(S)$.

Ex

Soru: $T(x,y) = (x, y, x+y+3)$ → not even a linear transformation

$\mathbb{R}^2 \to \mathbb{R}^3$ Çekirdek = ?

$T(x,y) = (0,0,0)$

$x = 0$

$y = 0$

$x + y + 3 \neq 0$

~~Çekirdek = $\emptyset$~~

~~Ker(T) = $\emptyset$~~

we can not talk about its kernel!

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$(x,y) \mapsto (x, y, x+y+3)$$

1) LHS → $T((x_1,y_1) + (x_2,y_2)) = T((x_1+x_2, y_1+y_2)) = (x_1+x_2, y_1+y_2, x_1+x_2+y_1+y_2+3)$ $\neq$

RHS → $T((x_1,y_1)) + T((x_2,y_2)) = (x_1, y_1, x_1+y_1+3) + (x_2, y_2, x_2+y_2+3) = (x_1+x_2, y_1+y_2, x_1+y_1+x_2+y_2+6)$

⇒ T is not a linear transformation.

* A linear transformation L is one-to-one ⟺ $\text{Ker}(L) = \{\vec{0}\}$

* Range of L ⇒ $\text{Im}(V)$

19. Find the kernel and range of each of the following linear operators on P_3:

→ (a) $L(p(x)) = xp'(x)$? (b) $L(p(x)) = p(x) - p'(x)$

? (c) $L(p(x)) = p(0)x + p(1)$

a) $L : P_3 \longrightarrow P_3$

$p(x) = a_0 + a_1x + a_2x^2 \longrightarrow a_1x + 2a_2x^2$

$p'(x) = a_1 + 2a_2x$

$x\,p'(x) = a_1x + 2a_2x^2$

$$\text{Ker}(L) = \{ p(x) \in P_3 : L(p(x)) = 0 \}$$

$$= \{ a_0 + a_1x + a_2x^2 : a_1x + 2a_2x^2 = 0 \} \Rightarrow$$

$a_1 = 0$

$2a_2 = 0$

$a_0 = r \in \mathbb{R}$

$a_0 + a_1x + a_2x^2$

$r + 0 + 0$ → a typical vector in Ker(L)

$r.1$ → $\{1\}$ → is a basis for Ker(L)

$$\text{Range}(L) = \{ L(p(x)) : \forall p(x) \in P_3 \}$$

$$= \{ a_1x + 2a_2x^2 : \forall a_0 + a_1x + a_2x^2 \in P_3 \}$$

$\{x, 2x^2\}$ is a basis for Range(L).

$\{x, 2x^2\}$ is a basis for Range(L).

$\ast$ $\boxed{\dim(\operatorname{Ker}(L)) + \dim(\operatorname{Range}(L)) = \dim(V)}$ $L : V \to V$

where L is a linear operator on V.