9th Week

04 Aralık 2021 Cumartesi 12:38

1. Given

$$A=\begin{bmatrix}-1&1&2\\5&1&0\\1&1&3\end{bmatrix}\quad B=\begin{bmatrix}1&5&0\\3&-1&1\\-1&0&-1\end{bmatrix}\quad\text{and } C=\begin{bmatrix}1&0&-1\\1&1&-1\\2&0&1\end{bmatrix},$$

(a) (4 pts) $AB=?$

(b) (4 pts) Find the row echelon form (REF) of the matrix A. Write each row operation clearly.

RREF

→ (c) (8 pts) Find an LU-decomposition for C.

Lower upper

Type-3

$$E_1=\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}\quad E_1^{-1}=\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}$$

$C \xrightarrow{REF} \begin{bmatrix} & u & \\ & & \end{bmatrix}$

$E \leftarrow$

L

Type-3

$$E_2=\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\quad E_2^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}1&0&-1\\1&1&-1\\2&0&1\end{bmatrix}}_{C}\xrightarrow[-2r_1+r_3\to r_3 \;(E_2)]{-r_1+r_2\to r_2\;(E_1)}\underbrace{\begin{bmatrix}1&0&-1\\0&1&0\\0&0&3\end{bmatrix}}_{u}$$

$E_2E_1C=u$

$C=\underbrace{E_1^{-1}E_2^{-1}}_{L}\,u$

$$L=E_1^{-1}E_2^{-1}=\begin{bmatrix}1&0&0\\1&1&0\\2&0&1\end{bmatrix}$$

2. (9 pts) Determine which of the following matrices are in reduced row echelon form (RREF) and which are not. Show why.

$$A=\begin{bmatrix}1&-5&1&0&1\\0&1&2&0&1\\0&0&0&1&1\end{bmatrix}\quad B=\begin{bmatrix}1&0&1&0\\0&1&5&0\\0&0&0&0\end{bmatrix}\quad C=\begin{bmatrix}1&0&7&0&3\\0&1&2&0&-4\\0&0&1&1&0\end{bmatrix}$$

✓✓✓ ✗ ✓✓✓✓ ✓ ✓✓ ✗ ✗

3 3 3

3. Given $A=\begin{bmatrix}1&-3&3\\2&-1&7\\-1&3&-2\end{bmatrix}$,

$$\det(A)=1\begin{vmatrix}-1&7\\3&-2\end{vmatrix}+3\begin{vmatrix}2&7\\-1&-2\end{vmatrix}+3\begin{vmatrix}2&-1\\-1&3\end{vmatrix}=-19+9+15=5$$

$2-21$ $-4+7$ $6-1$

(a) (6 pts) Find $\det(A)$. ✓

(b) (6 pts) Using $\det(A)$ only, find $\det(A^{-1})$, $\det(A^3)$, $\det(2A)$.

$\Rightarrow \det(A.A^{-1})=\det(I)=1$

$=\det(A).\det(A^{-1}) \Rightarrow \det(A^{-1})=\dfrac{1}{\det(A)}=\dfrac{1}{5}$

→(c) (9 pts) Find $\text{adj}(A)$.

(d) (3 pts) Find A^{-1}. ✓

→(e) (6 pts) Assume that we apply the following 3 row operations to A and obtain the new matrix A'. Then what is $\det(A')$=?
(**Do not find the matrix A', use elementary matrices**)

$$A\xrightarrow[E_1]{3r_1+r_2\to r_2}\cdots\xrightarrow[E_2]{2r_1\to r_1}\cdots\xrightarrow[E_3]{r_2\leftrightarrow r_3}A'$$

$\det(A.A.A)=\det(A).\det(A)\det(A)=5^3$

$\det(2A)=2.2.2.\det(A)=40$

Type-1 → det = −1
Type-2 → det = k
Type-3 → det = 1

$E_3E_2E_1A=A'$

$\det(A')=\underbrace{\det(E_3)}_{-1}\underbrace{\det(E_2)}_{2}\underbrace{\det(E_1)}_{1}\underbrace{\det(A)}_{5}=-10$

Type-2

$A \to \begin{matrix}2r_1\to r_1\\2r_2\to r_2\\2r_3\to r_3\end{matrix}\begin{bmatrix} & \\ & \end{bmatrix}$

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$\begin{bmatrix}-19 & 3 & -18\end{bmatrix}$

2+2

$$\operatorname{adj}(A) = \begin{bmatrix} -19 & 3 & -18 \\ -3 & 1 & -1 \\ 5 & 0 & 5 \end{bmatrix}$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} -3 & 3 \\ 3 & -2 \end{vmatrix} = 3 \qquad A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ -1 & -2 \end{vmatrix} = 1$$

$-2+3$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & -3 \\ -1 & 3 \end{vmatrix} = 0 \qquad A_{31} = (-1)^{3+1} \begin{vmatrix} -3 & 3 \\ -1 & 7 \end{vmatrix} = -18$$

$-21+3$

$$A^{-1} = \begin{bmatrix} -19/5 & 3/5 & -18/5 \\ -3/5 & 1/5 & -1/5 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A_{32} = (-1)^{3+2} \underbrace{\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}}_{7-6=1} = -1 \qquad A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} = 5$$

$-1+6$

4. (5 pts) Let $\mathbb{R}^2$ be the set of all ordered pairs of real numbers. Define scalar multiplication and addition on $\mathbb{R}^2$ by

$$\to \lambda \odot (x_1, y_1) = (\lambda x_1, \lambda y_1) \times$$
$$(x_1, y_1) \oplus (x_2, y_2) = (\underline{x_1/x_2, 0})$$

Give at least one reason to why $\mathbb{R}^2$ is not a vector space with these operations.

$\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2, \quad (x_1, y_1) \oplus (x_2, y_2) = (x_1/x_2, 0)$

$(1,2) \oplus (0,1) = (\underbrace{1/0}_{\notin \mathbb{R}}, 0) \notin \mathbb{R}^2 \to$

$$x_i = \frac{\det(A_i)}{\det(A)}$$

6. (9 pts) For which values of k does the following system of linear equations have

$$\begin{cases} x_1 + -2x_2 + x_3 - 7x_4 = 5 \\ -x_1 + 2x_2 + -2x_3 + 11x_4 = -6 \\ -x_1 + 2x_2 - x_3 + (k^2-3)x_4 = k \end{cases}$$

i.) no solution? $k = \pm\sqrt{10}$

ii.) ~~unique solution?~~ never

iii.) infinitely many solutions? $k \neq \pm\sqrt{10}$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -7 & 5 \\ -1 & 2 & -2 & 11 & -6 \\ -1 & 2 & -1 & k^2-3 & k \end{array}\right] \xrightarrow[r_1+r_3 \to r_3]{r_1+r_2 \to r_2} \left[\begin{array}{cccc|c} 1 & -2 & 1 & -7 & 5 \\ 0 & 0 & -1 & 5 & -1 \\ \boxed{0} & 0 & 0 & k^2-10 & k+5 \end{array}\right]$$

$0 = \text{nonzero number} \to \text{no solution.}$

$(k-\sqrt{10})(k+\sqrt{10})\, x_4 = (k+5)$

$\times$ impossible

7. (10 pts) Find determinants of the following matrices (Not too many calculations!).

(a) $\begin{bmatrix} -1 & 0 & 7 & 5 & -2 \\ \cancel{7} & \cancel{2} & \cancel{12} & \cancel{-3} & \cancel{4} \\ -4 & 0 & -2 & 0 & 0 \\ 5 & 0 & 3 & 0 & 0 \\ 4 & 0 & -1 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -5 & 6 & 5 & -2 \\ 7 & 2 & -12 & -3 & 4 \\ -4 & 1 & -3 & -2 & 1 \\ 5 & 0 & 9 & 0 & -3 \\ 4 & -12 & 0 & -9 & 0 \end{bmatrix}$

$2.(-1)^{2+2} \begin{vmatrix} \cancel{-1} & \cancel{7} & \cancel{5} & \cancel{-2} \\ -4 & -2 & 0 & 0 \\ 5 & 3 & 0 & 0 \\ 4 & -1 & 0 & 3 \end{vmatrix} \rightarrow 5.(-1)^{1+3} \begin{vmatrix} -4 & -2 & 0 \\ 5 & 3 & 0 \\ \cancel{4} & \cancel{-1} & \cancel{3} \end{vmatrix} \rightarrow 3.(-1)^{3+3} \begin{vmatrix} -4 & -2 \\ 5 & 3 \end{vmatrix}$

$\underbrace{-12+10}_{-2}$

$\det = 2.5.3.-2 = -60$

3^{rd} column is -3 multiple of 5^{th} column

$\det = 0$

8. (9 pts) Let $\mathbb{R}^3$ (x, y, z) be the vector space of all ordered triples of real numbers. Determine whether the following subsets are subspaces of $\mathbb{R}^3$ or not, tell why.

→(a) $S = \{(x, y, z) \mid -7x - 9y - 5z = 4\}$ $(0,0,0) \notin S$ ✗

(b) $S = \{(x, y, z) \mid -3x - 2y - 6z = 0\}$ ✓

→(c) $S = \{(x, y) \mid y = 2x\}$ ✗ $S \not\subset \mathbb{R}^3$

1) $(0,0,0) \overset{?}{\in} S$
2) $\forall (x_1, y_1, z_1) + \forall (x_2, y_2, z_2) \in S$
3) $\forall \lambda \in \mathbb{R}$, $\forall (x_1, y_1, z_1) \in S$ } $\lambda(x_1, y_1, z_1) \in S$

1) $x=0, y=0, z=0$ $-3x - 2y - 6z = 0$ ✓

$(0,0,0) \in S$ ✓

2) $\underset{\in S}{(x_1, y_1, z_1)} + \underset{\in S}{(x_2, y_2, z_2)} = (x_1 + x_2, \; y_1 + y_2, \; z_1 + z_2)$

$-3(x_1 + x_2) - 2(y_1 + y_2) - 6(z_1 + z_2)$

$= -3x_1 - 3x_2 - 2y_1 - 2y_2 - 6z_1 - 6z_2 = 0$ ✓

0 0

3) $\underset{\in S}{\lambda(x_1, y_1, z_1)} = (\lambda x_1, \lambda y_1, \lambda z_1)$

$-3\lambda x_1 - 2\lambda y_1 - 6\lambda z_1 = \lambda(\underbrace{-3x_1 - 2y_1 - 6z_1}_{=0}) = \lambda . 0 = 0$ ✓