12th Week

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Representation Matrices for Linear Transformations

1st Type

$L: V \to W$ $\quad A_{E_1,E_2} = ?$

V: standard basis, W: standard basis

$E_1 = \{e_1 \ldots\}$ $\quad E_2 = \{\ldots\}$

$L(e_1)$

$$A = \begin{bmatrix} | & \\ L(e_1) & \cdots \\ | & \end{bmatrix}$$

2nd Type

$L: V \to W$ $\quad A_{E,B} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

V: standard, $W \to B$

$(x,y) \longmapsto x\vec{b_1} + (x-y)\vec{b_2}$

$L(e_1) = \alpha \cdot \vec{b_1} + \beta \cdot \vec{b_2} \to \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

3rd Type

$L: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \mathcal{U} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{u_2} \right\}$ $\quad A_{u,u} = ?$

$v = x_1\vec{u_1} + x_2\vec{u_2} \longmapsto (x_1+x_2)\vec{u_1} + 2x_2\vec{u_2}$

$v \longmapsto w$ $\qquad e_1 \quad 1e_1 + 0e_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$[v]_E$: $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (standard basis)

$2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{5}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow [v]_u = \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix} \xrightarrow{L} 3\vec{u_1} + 1\vec{u_2}$, $\quad [w]_u = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$x_1 - x_2 = 2$

$x_1 + x_2 = 3$

$[w]_E$

$w = 3\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{u_1} + 1 \cdot \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{u_2} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$L(u_1) = L(1\vec{u_1} + 0\vec{u_2}) = (1+0)\vec{u_1} + 2 \cdot 0 \cdot \vec{u_2} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$L(u_2) = L(0\vec{u_1} + 1\vec{u_2}) = \underbrace{(0+1)}_{1}\vec{u_1} + \underbrace{2 \cdot 1}_{2}\vec{u_2} \to \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\boxed{A_{u,u} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}$$

Ex/ $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix}}_{[v]_u} = \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{[w]_u}$

$L: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \mathcal{U} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{u_2} \right\}$

$v = x_1\vec{u_1} + x_2\vec{u_2} \longmapsto (x_1+x_2)\vec{u_1} + 2x_2\vec{u_2}$

$x_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} \to [v]_E \xRightarrow{L} (x_1+x_2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + 3x_2 \end{bmatrix}$

$L: (\underbrace{x_1 - x_2}_{1}, \underbrace{x_1 + x_2}_{0}) \longmapsto (x_1 - x_2, x_1 + 3x_2)$ $\to$ in standard basis $\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{e_2} \right\}$

$x_1 - x_2 = 1$

$x_1 + x_2 = 0$

$x_1 = 1/2 \quad x_2 = -1/2$

$L(e_1) = \left(\frac{1}{2} - \left(-\frac{1}{2}\right), \frac{1}{2} + 3 \cdot \frac{-1}{2}\right) = (1, -1)$

$L(e_2) = \left(\frac{1}{2} - \frac{1}{2}, \frac{1}{2} + 3 \cdot \frac{1}{2}\right) = (0, 2)$

$$\boxed{A_{E,E} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}}$$

$x_1 - x_2 = 0 \quad x_1 + x_2 = 1$

$x_1 = \frac{1}{2} \quad x_2 = \frac{1}{2}$

Ex/ $Av \stackrel{?}{=} w$

$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad w = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

4th Type

4th Type

$L: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$(x_1, x_2) \longmapsto (x_2, x_1+x_2, x_1-x_2)$

$A_{E,E} = ? \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$ ✓

$L(e_1) = (0, 1, 1)$
$e_1 = (1,0)$

$L(e_2) = (1, 1, -1)$
$e_2 = (0,1)$

$U = \left\{ \underset{u_1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} \right\}$ $\quad V = \left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \right\}$

$A_{U,V} = ?$

$A_{U,V} = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix}$ //

$L(u_1) = \ldots \quad [L(u_1)]_V$

$\underset{(1,2)}{L(u_1)} = (\underbrace{2, 3, -1}_{E}) \longrightarrow$ convert this to the coordinates wrt the basis V

$\underset{(3,1)}{L(u_2)} = (\underbrace{1, 4, 2}_{E}) \longrightarrow$ //

Transition matrix from E to V $= V^{-1}E = V^{-1}$

$V^{-1}\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$

$V^{-1}\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$

$$A_{U,V} = \begin{bmatrix} | & | & \\ [L(u_1)]_V & [L(u_2)]_V & \cdots \\ | & | & \end{bmatrix}$$

Ex

$L: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$(x_1, x_2, x_3) \longmapsto (2x_2, -x_1)$

$A_{U,V} = ?$

$U = \left\{ \underset{u_1}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}, \underset{u_3}{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}} \right\}$

$V = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$ $\quad \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$

$$A = \begin{bmatrix} | & | & | \\ [L(u_1)]_V & [L(u_2)]_V & [L(u_3)]_V \\ | & | & | \end{bmatrix}$$

$V^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$

$\det = -1+2 = 1$

$\underset{1,0,-1}{L(u_1)} = (2 \cdot 0, -1) = (\underbrace{0, -1}_{\text{standard}}) \longrightarrow V$ bazına geçirmeliyim

$\underset{1,2,1}{L(u_2)} = (\underbrace{4, -1}_{\text{standard}})$

$\underset{-1,1,1}{L(u_3)} = (\underbrace{2, 1}_{\text{standard}})$

$\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

$A_{U,V} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 3 \end{bmatrix}$

6. Let

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and let L be the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ defined by

$$L(\mathbf{x}) = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + (x_1 + x_2)\mathbf{b}_3$$

Find the matrix A representing L with respect to the

$L: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$\underset{\text{standard}}{(x_1, x_2)} \longmapsto \underbrace{x_1\vec{b_1} + x_2\vec{b_2} + (x_1+x_2)\vec{b_3}}_{B}$

$A_{E,B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ //

$L(e_1) = \underline{1} \cdot \vec{b_1} + \underline{0} \cdot \vec{b_2} + \underline{1} \cdot \vec{b_3} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

and let L be the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ defined by

$$L(\mathbf{x}) = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + (x_1 + x_2)\mathbf{b}_3$$

Find the matrix A representing L with respect to the ordered bases $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

$L(e_1) = 1.\vec{b_1} + 0.\vec{b_2} + 1.\vec{b_3} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$(1,0)$

$L(e_2) = 0.\vec{b_1} + 1.\vec{b_2} + 1.\vec{b_3} \rightarrow \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$(0,1)$

18. Let $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\mathbf{b}_1 = (1, -1)^T, \qquad \mathbf{b}_2 = (2, -1)^T$$

$b_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $b_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

For each of the following linear transformations L from $\mathbb{R}^3$ into $\mathbb{R}^2$, find the matrix representing L with respect to the ordered bases $\mathcal{U}$ and $\mathcal{B}$:

$B = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$

$B^{-1} =$

(a) $L(\mathbf{x}) = (x_3, x_1)^T$

(b) $L(\mathbf{x}) = (x_1 + x_2, x_1 - x_3)^T$

(c) $L(\mathbf{x}) = (2x_2, -x_1)^T$

$L: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ $\qquad A_{\mathcal{U},B} = ?$

$(x_1, x_2, x_3) \longmapsto (x_3, x_1)$

$$A_{\mathcal{U},B} = \begin{bmatrix} | & | & | \\ [L(u_1)]_B & [L(u_2)]_B & [L(u_3)]_B \\ | & | & | \end{bmatrix} = \begin{bmatrix} -1 & -3 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

$L(u_1) = (-1, 1) \rightarrow$ standard $\xrightarrow{B}$ $\qquad B^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$ $\qquad \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$1,0,-1$

$L(u_2) = (1, 1)$ $\quad$ " $\longrightarrow$ $\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$1,2,1$

$L(u_3) = (1, -1)$ $\quad$ " $\longrightarrow$ $\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$-1,1,1$