Make Up -2

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Fundamental Spaces

- Null space of a matrix → $N(A)$ (matrix)
  - Set of all solutions to $Ax=0$
- Row space of a matrix → $R(A)$
- Column space of a matrix → $C(A)$

Örn $S = \{ \underbrace{(a+b, a-b, 2a)}_{\text{typical element}} : a,b \in \mathbb{R} \} \leq \mathbb{R}^3$

To find a basis for the subspace S,

$$\begin{bmatrix} a+b \\ a-b \\ 2a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$S = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

→ lin. indep.?

span + lin. ind. ⇒ basis for S ✓

A: matrix

Null Space of A → N(A)

The set of all solutions of $Ax=0$

Row Space of A → R(A)

The span of row vectors of A

$$A = \begin{bmatrix} - & r_1 & - \\ - & r_2 & - \\ & \vdots & \\ - & r_n & - \end{bmatrix}$$

$R(A) = \text{Span}\{r_1, r_2, \dots, r_n\}$

Column Space of A → C(A)

The span of column vectors of A

$$A = \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix}$$

$C(A) = \text{Span}\{c_1, c_2, \dots, c_n\}$

Ex $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \\ 3 & -2 & 5 \\ 2 & -1 & 3 \end{bmatrix}_{4\times 3}$ Find a basis for $N(A)$, $R(A)$, $C(A)$.

$N(A)$: $Ax=0$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ -2 & 2 & -4 & 0 \\ 3 & -2 & 5 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right] \end{array}$$

$x_3 = r \in \mathbb{R}$ → free.

$x_2 - x_3 = 0 \Rightarrow x_2 = r$

$x_1 + x_3 = 0 \Rightarrow x_1 = -r$

$$(x_1, x_2, x_3) \to (-r, r, r)$$

$$N(A) = \{ (-r, r, r) : r \in \mathbb{R} \} \to \text{solution set.}$$

$$\begin{bmatrix} -r \\ r \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \qquad N(A) = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

lin. indep.

$\Rightarrow$ basis for $N(A)$

$$= \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Null$(A) = 1$

R(A) :

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \\ 3 & -2 & 5 \\ 2 & -1 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

represents the row space

Rank$(A) = 2$

$$\text{A basis for } R(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

C(A) : $A^T \to R(A) \to$ is a hard way ✓

Easy way

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \\ 3 & -2 & 5 \\ 2 & -1 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{array}{cc} \text{1st} & \text{2nd} \\ \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\to$ Pick 1st and 2nd columns from the original matrix A.

they represent the column space of A.

$c_1v_1 + c_2v_2 + c_3v_3 \cdots = 0$

$c_1 = c_2 = c_3 = 0$?

$$C(A) = \text{Span}\left\{ \begin{bmatrix} -1 \\ -2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix} \right\}$$

lin. indep ✓

$$\text{A basis for } C(A) = \left\{ \begin{bmatrix} -1 \\ -2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix} \right\}$$

Rank – Nullity Thm : ($A_{m \times n}$)

$$\boxed{\text{Rank}(A) + \text{Null}(A) = n}$$

Rank: dim of the row space (column space)

Nullity: dim of the null space.

Ex/

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$$

Find a basis for $N(A)$, $R(A)$, $C(A)$.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 2 & 4 & -3 & 0 & 0 \\ 1 & 2 & 1 & 5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{RREF} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} \boxed{x_2 = r} \in \mathbb{R} \\ \boxed{x_4 = s} \in \mathbb{R} \end{array}\right\} \text{free.} \qquad x_3 + 2\underset{s}{x_4} = 0 \quad \boxed{x_3 = -2s}$$

$$x_1 + 2\underset{r}{x_2} + 3\underset{s}{x_4} = 0 \Rightarrow \boxed{x_1 = -2r - 3s}$$

N(A)

$$(x_1, x_2, x_3, x_4) = (\underbrace{-2r-3s, r, -2s, s}_{}) : r, s \in \mathbb{R}$$

a typical element of the solution set (Null Space)

$$\begin{bmatrix} -2r-3s \\ r \\ -2s \\ s \end{bmatrix} = r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

A basis for $N(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} \rightarrow \text{Null}(A) = 2$

R(A) :

$$A \xrightarrow{RREF} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for $R(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ 4

Rank (A) = 2

C(A) :

$$A \xrightarrow{RREF} \begin{array}{cccc} 1^{st} & & 3^{rd} & \\ \downarrow & & \downarrow & \end{array}$$
$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pick 1^{st} and 3^{rd} column from the original matrix A.

A basis for $C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$$