4th Week Wednesday

17 Mart 2021 Çarşamba 12:35

A^{-1} → square matrices (#rows = # columns)

→ if A^{-1} exists (A is non-singular) $\left(A \xrightarrow{RREF} \underline{\underline{I}} \right.$ is possible. $\left. \right)$

→ if A^{-1} does not exist (A is singular) $\left(A \xrightarrow{RREF} \right.$ we can not obtain $\left. I \right)$

System of Linear Equations ;

$$\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_nx_n = b_m \end{array}}$$

Augmented Matrix

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} \cdots & a_{1n} & b_1 \\ \vdots & & & b_2 \\ & & & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} & b_m \end{array}\right]_{m\times(n+1)}$$

$$\boxed{Ax = b}$$

A → coefficient matrix; x → column matrix of variables; b → column matrix of results

The coefficient matrix

$$A \to \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

$n = n$ ✓

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

If the system is homogeneous $(b_1 = b_2 = \dots = b_m = 0) \to \boxed{Ax = 0}$ $\to \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$

Using A^{-1}, we will make some generalizations, only when the coefficient matrix ← A is square!

$$A^{-1}A = AA^{-1} = I$$

$$\boxed{Ax = b} \quad \text{if } A^{-1} \text{ exists;} \quad \underbrace{A^{-1}A}_{I}x = A^{-1}b \Rightarrow \boxed{x = A^{-1}b} \Rightarrow$$

you have a unique soln.

$$Ax = 0 \quad \text{if } A^{-1} \text{ exists;} \quad \underbrace{A^{-1}A}_{I}x = \underbrace{A^{-1}0}_{0} \Rightarrow \boxed{x = 0} \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}$$

⇒ You only have the trivial solution: $(x_1 = x_2 = \dots = x_n = 0)$

A is invertible;

$$\left[\begin{array}{c|c} A & b \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & & 0 & & \bigcirc \\ & 1 & & & \bigcirc \\ & & 1 & & \bigcirc \\ 0 & & & \ddots \; 1 & \bigcirc \end{array}\right] \begin{array}{l} = x_1 \\ = x_2 \\ \\ \vdots \\ = x_n \end{array}$$

→ a unique soln.

A is invertible;

$$\left[\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & & 0 & 0 \\ & 1 & & 0 \\ 0 & & \ddots \; 1 & \vdots \\ & & & 0 \end{array}\right] \begin{array}{l} \to x_1 = 0 \\ \to x_2 = 0 \\ \vdots \\ x_n = 0 \end{array}$$

→ the trivial soln.

Ex

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 12 \\ -x_1 - 2x_2 &= -12 \\ 2x_1 + 2x_2 + 3x_3 &= 8 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$$ → the coefficient matrix

$$x = A^{-1}b = ?$$

$$[A \mid I] \to [I \mid A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{-2r_1 + r_3 \to r_3 \\ E_2}]{\substack{E_1 \\ r_1 + r_2 \to r_2}} \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 2 & 2 & 3 \end{bmatrix}}_{A} \left| \begin{matrix} 0 & 0 & 1 \end{matrix} \right. \quad \hookrightarrow E_2 \quad \begin{bmatrix} 0 & -6 & -3 & | & -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[E_3]{\frac{1}{2}r_2 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right]$$

$A \longrightarrow U$

$$\cancel{E_5}\cancel{E_4}\cancel{E_3}\cancel{E_2}\cancel{E_1}A = U$$

$$\boxed{A = \underbrace{E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}}_{\text{the required lower triangular matrix } = L} \overset{\text{upper}}{U}}$$

$$\xrightarrow[E_4]{6r_2 + r_3 \to r_3} \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right]$$

upper

$$\xrightarrow[E_5]{1/6\, r_3 \to r_3} \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/6 & 1/2 & 1/6 \end{array}\right]$$

REF → upper $= U$

$$\xrightarrow[-3r_3 + r_1 \to r_1]{-\frac{3}{2}r_3 + r_2 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1/2 & -3/2 & -1/2 \\ 0 & 1 & 0 & 1/4 & -1/4 & -1/4 \\ 0 & 0 & 1 & 1/6 & 1/2 & 1/6 \end{array}\right]$$

$$\xrightarrow{-4r_2 + r_1 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/4 & -1/4 & -1/4 \\ 0 & 0 & 1 & 1/6 & 1/2 & 1/6 \end{array}\right]$$

$$\underbrace{\hphantom{xxxxxxxxxx}}_{A^{-1}}$$

$$x = A^{-1}b = \begin{bmatrix} -1/2 & -1/2 & 1/2 \\ 1/4 & -1/4 & -1/4 \\ 1/6 & 1/2 & 1/6 \end{bmatrix} \begin{bmatrix} 12 \\ -12 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -8/3 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} x_1 = 4 \\ x_2 = 4 \\ x_3 = -8/3 \end{matrix}}$$

$-6 + 6 + 4$

$3 + 3 - 2$

$2 - 6 + \frac{8}{6}$ $\frac{4}{3} - 4$

LU- Factorization

$$\text{square} \leftarrow A = LU$$

L: lower triangular $\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ & & d_n \end{bmatrix}$

U: upper triangular $\begin{bmatrix} d_1 & & \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$

$$A \xrightarrow{\text{REF}} U$$

Inverses of Elementary Matrices

If E is of type-I $(r_i \leftrightarrow r_j)$ $E^{-1} = E$

If E is of type-II $(kr_i \to r_i)$ $E = \begin{bmatrix} 1 & & & 0 \\ & k & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$ $E^{-1} = \begin{bmatrix} 1 & & & 0 \\ & 1/k & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$ write $1/k$ in place of k.

If E is of type-III $(kr_j + r_i \to r_i)$ ex $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$ $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -k & 1 \end{bmatrix}$ write $-k$ in place of k.

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$$

y row operation III to carry out the reduction process. At th
nes the first row from the second and then we subtract twi
d.

$$\begin{bmatrix} \bigcirc\!\!\!\!2 & 4 & 2 \\ \boxed{1} & 5 & 2 \\ \boxed{4} & -1 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix}$$

$-\frac{1}{2}r_1 + r_2 \rightarrow r_2$ E_1

$-2r_1 + r_3 \rightarrow r_3$ E_2

f the multiples of the first row that were subtracted, we
. We complete the elimination process by eliminating
.

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & \boxed{-9} & 5 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix}}_{\boxed{U}}$$

$3r_2 + r_3 \rightarrow r_3$ E_3

(ref)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix}$$

$LU = A$ ✓✓

$$\cancel{E_3}\cancel{E_2}\cancel{E_1}A = U$$

$$A = \underbrace{E_1^{-1}E_2^{-1}E_3^{-1}}_{\boxed{L}}U$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{1/2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} = L$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \boxed{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{-3} & 1 \end{bmatrix}$$