15th Week Monday

31 Mayıs 2021 Pazartesi 10:02

Ex

$$A = \begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$A=\begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

Find an orthonormal basis for the column space of the matrix A.

We should start with finding a basis for the column space.

$$\to A \xrightarrow{REF} \begin{bmatrix} 1 & -1 & 4 \\ 0 & 5 & -6 \\ 0 & 5 & -2 \\ 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -6/5 \\ 0 & 5 & -2 \\ 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -6/5 \\ 0 & 0 & 4 \\ 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -6/5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\vec{x}_1 = (1,1,1,1)^T \qquad \vec{x}_2 = (-1,4,4,-1)^T \qquad \vec{x}_3 = (4,-2,2,0)^T$

$\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is a basis for the column space of A.

$(-1,4,4,-1)\cdot(1,1,1,1) = -1+4+4-1 = 6 \qquad \frac{6}{4} = \frac{3}{2}$

$(1,1,1,1)\cdot(1,1,1,1) = 1+1+1+1 = 4$

$$\vec{y}_1 = \vec{x}_1 = (1,1,1,1)$$

$$\vec{y}_2 = \vec{x}_2 - \frac{\vec{x}_2\cdot\vec{y}_1}{\vec{y}_1\cdot\vec{y}_1}\vec{y}_1 = (-1,4,4,-1) - \frac{3}{2}(1,1,1,1) = (-5/2, 5/2, 5/2, -5/2) \to \vec{y}_2$$

$$\vec{y}_3 = \vec{x}_3 - \frac{\vec{x}_3\cdot\vec{y}_1}{\vec{y}_1\cdot\vec{y}_1}\vec{y}_1 - \frac{\vec{x}_3\cdot\vec{y}_2}{\vec{y}_2\cdot\vec{y}_2}\vec{y}_2 = (4,-2,2,0) - 1(1,1,1,1) - \frac{-2}{5}\left(-\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, -\frac{5}{2}\right)$$

$(1,-1,-1,1)$

$\frac{-10}{25} = -\frac{2}{5}$

$(4,-2,2,0)\cdot(1,1,1,1) = 4-2+2+0 = 4 \qquad 4 \qquad 4/4 = 1$

$(4,-2,2,0)\cdot\left(-\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, -\frac{5}{2}\right) = -10-5+5+0 = -10$

$\left(-\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, -\frac{5}{2}\right)\cdot\left(-\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, -\frac{5}{2}\right) = \frac{25}{4}+\frac{25}{4}+\frac{25}{4}+\frac{25}{4} = 25$

$= (3,-3,1,-1) - (1,-1,-1,1) = (2,-2,2,-2) \to \vec{y}_3$

$-5 \quad -5 \quad 5 \quad 5$

$2 \; \cancel{-2} + \cancel{2} - 2$

$\left\{ \vec{y}_1 = (1,1,1,1),\ \vec{y}_2 = \left(-\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, -\frac{5}{2}\right),\ \vec{y}_3 = (2,-2,2,-2) \right\} \to$ this set is orthogonal.

$y_1 \cdot y_2 = 0$ ✓ $\quad y_2 \cdot y_3 = 0$ ✓ $\quad y_1 \cdot y_3 = 0$ ✓

To get an orthonormal basis, we should divide each vector to its norm.

$\|\vec{y}_1\| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2$

$\|\vec{y}_2\| = \sqrt{\frac{25}{4}+\frac{25}{4}+\frac{25}{4}+\frac{25}{4}} = \sqrt{25} = 5$

$\|\vec{y}_3\| = \sqrt{4+4+4+4} = \sqrt{16} = 4$

$\vec{y}_1 \to \frac{\vec{y}_1}{\|\vec{y}_1\|} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \to \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = \sqrt{1} = 1$

$\vec{y}_2 \to \frac{\vec{y}_2}{\|\vec{y}_2\|} = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \to \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = 1$

$\vec{y}_3 \to \frac{\vec{y}_3}{\|\vec{y}_3\|} = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \to \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = 1$

If you check pairwise dot products you will see all of them are still 0.
⇒ The set is still orthogonal
+ the norms of each vector = 1
⇒ The set is orthonormal.

$\left\{ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \right\}$

is an orthonormal basis for the column space of A.

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -6/5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the row space is

$\{ \underset{\vec{x}_1}{(1,-1,4)},\ \underset{\vec{x}_2}{(0,1,-6/5)},\ \underset{\vec{x}_3}{(0,0,1)} \}$

$\vec{y_1} = \vec{x_1} = (1,-1,4)$ $(0,1,-\frac{6}{5})\cdot(1,-1,4) = 0-1-\frac{24}{5} = -\frac{29}{5}$ $\overline{5.18}$

$\vec{y_2} = \vec{x_2} - \frac{\vec{x_2}\cdot\vec{y_1}}{\vec{y_1}\cdot\vec{y_1}} y_1 = (0,1,-\frac{6}{5}) + \frac{29}{5.18}(1,-1,4) = (0,1,-\frac{6}{5}) + (\frac{29}{5.18}, \frac{-29}{5.18}, \frac{29.2}{5.9})$

$\vec{y_1}\cdot\vec{y_1} \rightarrow (1,-1,4)\cdot(1,-1,4) = 1+1+16 = 18$

$= (\frac{29}{5.18}$ ....

$\vec{y_3} = \vec{x_3} - \frac{x_3\cdot y_1}{y_1\cdot y_1}\vec{y_1} - \frac{x_3\cdot y_2}{y_2\cdot y_2}\vec{y_2}$

3. Given the basis $\{\overset{x_1}{(1,2,-2)^T}, \overset{x_2}{(4,3,2)^T}, \overset{x_3}{(1,2,1)^T}\}$ for $\mathbb{R}^3$, use the Gram–Schmidt process to obtain an orthonormal basis.

$x_2\cdot y_1 = (4,3,2)\cdot(1,2,-2) = 4+6-4 = 6$ $6/9 = \frac{2}{3}$

$y_1\cdot y_1 = (1,2,-2)\cdot \quad = 1+4+4 = 9$

$x_3\cdot y_1 = (1,2,1)\cdot(1,2,-2) = 1+4-2 = 3 \rightarrow \frac{3}{9} = \frac{1}{3}$

$x_3\cdot y_2 = (1,2,1)\cdot(\frac{10}{3},\frac{5}{3},\frac{10}{3}) = \frac{10}{3}+\frac{10}{3}+\frac{10}{3} = 10 \Big) \frac{10}{25} = \frac{2}{5}$

$y_2\cdot y_2 = (\frac{10}{3},\frac{5}{3},\frac{10}{3})\cdot \quad = \frac{100}{9}+\frac{25}{9}+\frac{100}{9} = 25$

$\vec{y_1} = \vec{x_1} = (1,2,-2)$

$\vec{y_2} = \vec{x_2} - \frac{x_2\cdot y_1}{y_1\cdot y_1} y_1 = (4,3,2) - \frac{2}{3}(1,2,-2) = (\frac{10}{3},\frac{5}{3},\frac{10}{3})$ $\uparrow \vec{y_2}$

$\vec{y_3} = \vec{x_3} - \frac{x_3\cdot y_1}{y_1\cdot y_1} y_1 - \frac{x_3\cdot y_2}{y_2\cdot y_2} y_2$

$= (1,2,1) - \frac{1}{3}(1,2,-2) - \frac{2}{5}(\frac{10}{3},\frac{5}{3},\frac{10}{3})$ $1+\frac{2}{3}$

$= (\frac{2}{3},\frac{4}{3},\frac{5}{3}) - (\frac{4}{3},\frac{2}{3},\frac{4}{3}) = (-\frac{2}{3},\frac{2}{3},\frac{1}{3})$

$y_1 = (1,2,-2)$, $y_2 = (\frac{10}{3},\frac{5}{3},\frac{10}{3})$, $y_3 = (-\frac{2}{3},\frac{2}{3},\frac{1}{3}) \rightarrow \{\vec{y_1},\vec{y_2},\vec{y_3}\}$ is an orthogonal basis

$\|\vec{y_1}\| = \sqrt{1+4+4} = 3$ $y_1 \rightarrow (\frac{1}{3},\frac{2}{3},-\frac{2}{3})$ →1

$\|\vec{y_2}\| = \sqrt{25} = 5$ $y_2 \rightarrow (\frac{2}{3},\frac{1}{3},\frac{2}{3})$ →1

$\|\vec{y_3}\| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}} = 1$ $y_3 \rightarrow (-\frac{2}{3},\frac{2}{3},\frac{1}{3})$ →1

0✓ 0✓ 0✓

→ is an orthonormal set.

5. Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \cancel{\begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}}$$

(a) Use the Gram–Schmidt process to find an orthonormal basis for the column space of A.

$x_1 = (2,1,2)$ $x_2 = (1,1,1)$

$\vec{y_1} = \vec{x_1} = (2,1,2)$

$\vec{y_2} = \vec{x_2} - \frac{x_2\cdot y_1}{y_1\cdot y_1}\vec{y_1} = (1,1,1) - \frac{5}{9}(2,1,2)$

$= (-\frac{1}{9},\frac{4}{9},-\frac{1}{9})$

$x_2\cdot y_1 = (1,1,1)\cdot(2,1,2) = 2+1+2 = 5 \Big) \frac{5}{9}$

$y_1\cdot y_1 = (2,1,2)\cdot(2,1,2) = 4+1+4 = 9$

$\{\underbrace{(2,1,2)}_{y_1}, \underbrace{(-\frac{1}{9},\frac{4}{9},-\frac{1}{9})}_{y_2}\} \rightarrow$ orthogonal

$\|y_1\| = \sqrt{9} = 3$

$\|y_2\| = \frac{\sqrt{1+16+1}}{9} = \frac{3\sqrt{2}}{9} = \frac{\sqrt{2}}{3}$

$-\frac{1}{9}\cdot\frac{3}{\sqrt{2}}$

$\frac{4}{9}\cdot\frac{3}{\sqrt{2}}$

$\{(\frac{2}{3},\frac{1}{3},\frac{2}{3}), (-\frac{1}{3\sqrt{2}},\frac{4}{3\sqrt{2}},-\frac{1}{3\sqrt{2}})\} \rightarrow$ orthonormal ✓

$\frac{4}{9}+\frac{1}{9}+\frac{4}{9} = 1$ ✓ $\frac{1}{18}+\frac{16}{18}+\frac{1}{18} = 1$ ✓

$-\frac{2}{9\sqrt{2}} + \frac{4}{9\sqrt{2}} - \frac{2}{9\sqrt{2}} = 0$

Another one: $x_1 = (1,1,1)$, $x_2 = (2,1,2)$

$y_1 = (1,1,1)$

$2-\frac{5}{3}$

$y_1 = (1,1,1)$ $\quad 2-\frac{5}{3}$

$$y_2 = x_2 - \frac{x_2 \cdot y_1}{y_1 \cdot y_1} y_1 = (2,1,2) - \frac{5}{3}(1,1,1) = \left(\frac{1}{3}, \frac{-2}{3}, \frac{1}{3}\right)$$

$x_2 \cdot y_1 = 2+1+2 = 5$

$y_1 \cdot y_1 = 1+1+1 = 3$

$\|y_1\| = \sqrt{3}$

$\|y_2\| = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{1}{9}} = \frac{\sqrt{2}}{\sqrt{3}}$

$\left\{ (1,1,1),\ \left(\frac{1}{3}, \frac{-2}{3}, \frac{1}{3}\right) \right\} \rightarrow$ orthogonal.

$\downarrow$

$\left\{ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right),\ \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \right\} \rightarrow$ orthonormal.

6. Repeat Exercise 5 using

$$A = \begin{pmatrix} 3 & -1 \\ 4 & 2 \\ 0 & 2 \end{pmatrix}$$

7. Given $\mathbf{x}_1 = \frac{1}{2}(1,1,1,-1)^T$ and $\mathbf{x}_2 = \frac{1}{6}(1,1,3,5)^T$, verify that these vectors form an orthonormal set in $\mathbb{R}^4$. Extend this set to an orthonormal basis for $\mathbb{R}^4$ by finding an orthonormal basis for the null space of

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{pmatrix}$$

[*Hint*: First find a basis for the null space and then use the Gram–Schmidt process.]

8. Use the Gram–Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $\mathbf{x}_1 = (4,2,2,1)^T$, $\mathbf{x}_2 = (2,0,0,2)^T$, and $\mathbf{x}_3 = (1,1,-1,1)^T$.

1. For each of the following, use the Gram–Schmidt process to find an orthonormal basis for $R(A)$.

(a) $A = \begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 2 & 5 \\ 1 & 10 \end{pmatrix}$