8th Week Wednesday

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Basis & Dimension

Basis = Spanning Property + Linear Independence

✓ ✓

$\{v_1, v_2, \dots\}$ is a basis for V. $\Leftrightarrow$

Spanning Set $\rightarrow$ inf. many soln. $\rightarrow$ ✗ not a basis

unique soln. $\rightarrow$ ✓ a basis

$\{v_1, v_2, \dots\}$

Ex In $\mathbb{R}^2$, $(-,-)$

$\det\begin{vmatrix}1&0\\0&1\end{vmatrix} = 1 \neq 0$

$\rightarrow \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$ $\rightarrow$ is a spanning set, ✓ linearly independent $\Rightarrow$ **basis** ✓

$\rightarrow \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix} \right\}$ $\rightarrow$ is also a spanning set. $\rightarrow$ NOT linearly independent $\Rightarrow$ NOT a basis!

$\begin{bmatrix}2\\3\end{bmatrix} \in \mathbb{R}^2 \rightarrow$ can be written as a linear combination of $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

$\{v_1, v_2, \dots\}$ is a basis for V $\Leftrightarrow$

* $\{v_1, v_2, \dots\}$ is a spanning set for V.
* $\{v_1, v_2, \dots\}$ is linearly independent.

* Adding / removing vectors from a basis makes it no more a basis!

Adding $\rightarrow$ violates linear independence

removing $\rightarrow$ violates the spanning property

* Basis of a vector space is not unique.

Bases $\rightarrow$ plural form of basis.

Ex $\left\{ \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix} \right\}$ is also a basis for $\mathbb{R}^2$.

$\left\{ \underset{e_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{e_2}{\begin{bmatrix}0\\1\end{bmatrix}} \right\}$ $\rightarrow$ standard basis for $\mathbb{R}^2$.

Linear independence: $\begin{vmatrix}2&1\\3&2\end{vmatrix} = 4 - 3 = 1 \neq 0$ linearly independent ✓

Spanning set: $\alpha_1 \begin{bmatrix}1\\2\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$

$\left[\begin{array}{cc|c}1&2&a\\2&3&b\end{array}\right] \rightarrow$ solve this for α_1 and α_2

$\left[\begin{array}{cc|c}1&2&a\\ \cdot & \cdot & \cdot\end{array}\right] \xrightarrow{-2r_1+r_2 \to r_2} \left[\begin{array}{cc|c}1&2&a\\ \cdot & \cdot & \cdot\end{array}\right] \xrightarrow{-1r_2 \to r_2} \left[\begin{array}{cc|c}1&2&a\\ \cdot & \cdot & \cdot\end{array}\right]$ REF

$$\left[\begin{array}{cc|c} 1 & 2 & a \\ 2 & 3 & b \end{array}\right] \xrightarrow{-2r_1+r_2\to r_2} \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & -1 & -2a+b \end{array}\right] \xrightarrow{-1r_2\to r_2} \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & 2a-b \end{array}\right] \text{ REF}$$

is a spanning set ✓

$\alpha_1 + 2\overset{2a-b}{\alpha_2} = a \Rightarrow \alpha_1 = 2b-3a$

$\alpha_2 = 2a-b$

→ $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$, P_3, P_4, $\mathbb{R}^{2\times 2}$

$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$. → standard basis.

Ex

$v_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$ $v_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$ $v_3 = \begin{bmatrix}2\\0\\1\end{bmatrix}$

Is $\{v_1, v_2, v_3\}$ a basis for $\underline{\mathbb{R}^3}$? → $\begin{bmatrix}a\\b\\c\end{bmatrix}$

Linear independence

$$\begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 0 + 2\underbrace{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}_{0} = 1 - 0 + 0 = 1 \neq 0$$

$\{v_1, v_2, v_3\}$ is linearly independent. ✓

Spanning property

$$\alpha_1\begin{bmatrix}1\\1\\1\end{bmatrix} + \alpha_2\begin{bmatrix}0\\1\\1\end{bmatrix} + \alpha_3\begin{bmatrix}2\\0\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 1 & 1 & 0 & b \\ 1 & 1 & 1 & c \end{array}\right]$$

$$\xrightarrow[-r_1+r_3\to r_3]{-r_1+r_2\to r_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & -2 & b-a \\ 0 & 1 & -1 & c-a \end{array}\right] \xrightarrow{-r_2+r_3\to r_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & -2 & b-a \\ 0 & 0 & 1 & c-b \end{array}\right] \text{ REF ✓}$$

$c - \cancel{a} - b + \cancel{a}$

$$\left.\begin{aligned} \alpha_1 + 2\overset{c-b}{\alpha_3} &= a \\ \alpha_2 - 2\overset{c-b}{\alpha_3} &= b-a \\ \alpha_3 &= c-b \end{aligned}\right\} \qquad \begin{aligned} \alpha_3 &= c-b \\ \alpha_2 &= 2c-b-a \\ \alpha_1 &= a+2b-2c \end{aligned}$$

$\alpha_2 - 2c + 2b = b - a$

$\alpha_1 + 2c - 2b = a$

$\Rightarrow \{v_1, v_2, v_3\}$ spans $\mathbb{R}^3$. ✓

Therefore, $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

Ex Check the sets below to be bases for P_3.

→ (b) $x, x-1, x^2+1, x^2-1$

→ (c) $\underbrace{x^2, x^2-x-1, x+1}$ (d) $2x, x-2$

× $x^2 - x - 1 = 1 \cdot x^2 + -1(x+1)$

→ $\{1, x, x^2\}$ spanning ✓

$c_1 \cdot 1 + c_2 x + c_3 x^2 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$

linearly indep. ✓

→ standard basis for P_3.

(c) $x^2, x^2 - x - 1, x + 1$ (d) $2x, x - 2$

* standard basis for P_3.

X $x^2 - x - 1 = 1 \cdot x^2 + -1(x+1)$

b) Linear independence?

$$c_1 x + c_2(x-1) + c_3(x^2+1) + c_4(x^2-1) = 0$$

$$c_1 x + c_2 x - c_2 + c_3 x^2 + c_3 + c_4 x^2 - c_4 = 0$$

$$-c_2 + c_3 - c_4 = 0$$

$$\rightarrow c_1 + c_2 = 0$$

$$c_3 + c_4 = 0$$

NOT linear independent!

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-1r_2 \to r_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & \end{array}\right]$$

$c_4 =$ free $\rightarrow$ inf X

$\Rightarrow$ Not a basis!

c)

$v_1 = 2x$, $v_2 = x - 2$

$$c_1 \cdot 2x + c_2(x-2) = 0$$

$$2c_1 x + c_2 x - 2c_2 = 0$$

$$\left.\begin{array}{r} 2c_1 + c_2 = 0 \\ -2c_2 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 0 \end{array}$$

they are linearly independent! ✓

$$\alpha_1 \cdot 2x + \alpha_2 (x-2) = ax^2 + bx + c \in P_3 \qquad 2x^2 - 2x + 3$$

$$2\alpha_1 x + \alpha_2 x - 2\alpha_2 = ax^2 + bx + c$$

$$0 = a$$

$$2\alpha_1 + \alpha_2 = b$$

$$-2\alpha_2 = c$$

$\alpha_1 =$

$\alpha_2 =$

the soln. can be written only if $a = 0$

if $a \neq 0$ we have no soln.

$$\left[\begin{array}{ccc|c} - & - & & \\ - & - & - & \\ - & - & & \\ 0 & 0 & 0 & b^2 - 4ac \end{array}\right] \neq 0$$

It fails spanning property X

$\Rightarrow$ not a basis!