14. Hafta Çarşamba Dersi

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Ortogonal Kümeler, Ortogonal Bazlar, Ortonormallik

$\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\} \subseteq \mathbb{R}^n \rightarrow$ lineer bağımsız bir küme

Bir vektör kümesinin ortogonal olması:

$\{\vec{y}_1, \vec{y}_2, \dots, \vec{y}_n\}$ $\quad \vec{y}_i \perp \vec{y}_j \quad (i \neq j) \quad (\vec{y}_i \cdot \vec{y}_j = 0)$

Örn $\mathbb{R}^3$'teki standart baz: $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ (e_1, e_2, e_3)

$\vec{e}_1 \cdot \vec{e}_2 = 1.0 + 0.1 + 0.0 = 0$

$\vec{e}_1 \cdot \vec{e}_3 = 1.0 + 0.0 + 0.1 = 0$

$\vec{e}_2 \cdot \vec{e}_3 = 0.0 + 1.0 + 0.1 = 0$

→ ortogonal bir baz.

Örn $\mathbb{R}^3$'te $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix} \right\}$ (u_1, u_2, u_3)

$\vec{u}_1 \cdot \vec{u}_2 = 1.1 + 0.1 + 0.0 = 1$

→ ortogonal olmayan bir bazdır.

✱ Ortogonal olmayan bir vektör kümesinden nasıl ortogonal bir küme elde edilir?

Gram-Schmidt Ortogonalleştirme

$\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\} \rightarrow$ ortogonal olmayan, lineer bağımsız bir küme

1. adım: $\vec{y}_1 = \vec{x}_1$

$y_1 \cdot y_2 = 0$

$\{\vec{y}_1, \vec{y}_2, \vec{y}_3,$

$$\vec{y}_2 = \vec{x}_2 - \underbrace{\frac{\vec{x}_2 \cdot \vec{y}_1}{\vec{y}_1 \cdot \vec{y}_1}}_{\text{skaler}} \vec{y}_1$$

$$\vec{y}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{y}_1}{\vec{y}_1 \cdot \vec{y}} \vec{y}_1 - \frac{\vec{x}_3 \cdot \vec{y}_2}{\vec{y}_2 \cdot \vec{y}_2} \vec{y}_2$$

$\vec{y}_1 \cdot \vec{y}_3 = 0$

$\vec{y}_2 \cdot \vec{y}_3 = 0$

$$\vec{y}_4 = \vec{x}_4 - \frac{\vec{x}_4 \cdot \vec{y}_1}{\vec{y}_1 \cdot \vec{y}_1} \vec{y}_1 - \frac{\vec{x}_4 \cdot \vec{y}_2}{\vec{y}_2 \cdot \vec{y}_2} \vec{y}_2 - \frac{\vec{x}_4 \cdot \vec{y}_3}{\vec{y}_3 \cdot \vec{y}_3} \cdot \vec{y}_3$$

$y_1 \cdot y_4 = 0$

$y_2 \cdot y_4 = 0$

$y_3 \cdot y_4 = 0$

$\vdots$

$$\vec{y}_n = \vec{x}_n - \sum_{i=1}^{n-1} \frac{\vec{x}_n \cdot \vec{y}_i}{\vec{y}_i \cdot \vec{y}_i} \vec{y}_i$$

$\Rightarrow \{\vec{y}_1, \vec{y}_2, \dots, \vec{y}_n\} \to$ ortogonal bir kümedir.

Ör: $\vec{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\vec{x}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ kümesinden ortogonal bir küme elde edelim

$(2,0,0)\cdot(1,1,0) = 2.1 + \cancel{0.1} + \cancel{0.0}$

$\to (2,0,0)\cdot(2,0,0) = 2.2 + 0. + 0 = 4$

$\vec{y}_1 = \vec{x}_1 = \underline{(2,0,0)}$

$$\vec{y}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{y}_1}{\vec{y}_1 \cdot \vec{y}_1} \cdot \vec{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{y}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{y}_1}{\vec{y}_1 \cdot \vec{y}_1} \cdot \vec{y}_1 - \frac{\vec{x}_3 \cdot \vec{y}_2}{\vec{y}_2 \cdot \vec{y}_2} \vec{y}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$-2 + 0$

$$\frac{(-1,2,1)\cdot(2,0,0)}{4} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \qquad \frac{(-1,2,1)\cdot(0,1,0)}{(0,1,0)\cdot(0,1,0)} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$2/1$

$$\{\vec{y}_1, \vec{y}_2, \vec{y}_3\} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\vec{y}_1 \cdot \vec{y}_2 = 0$
$\vec{y}_1 \cdot \vec{y}_3 = 0$
$\vec{y}_2 \cdot \vec{y}_3 = 0$

Ortonormalleştirme

Ortonormal Küme

$\{\vec{y}_1, \vec{y}_2, \dots, \vec{y}_n\} \to$ ortogonal kümesinde

$$\left\{ \frac{\vec{y}_1}{\|\vec{y}_1\|}, \frac{\vec{y}_2}{\|\vec{y}_2\|}, \dots, \frac{\vec{y}_n}{\|\vec{y}_n\|} \right\} \to \text{ortonormal küme}$$

her vektörün normu 1 olan ortogonal kümeye ortonormal küme denir.

$\vec{y}_i \cdot \vec{y}_j = 0$

$(a_1, a_2, \dots, a_n)$ $(b_1, b_2, \dots, b_n)$

$\Rightarrow \underline{a_1 b_1 + a_2 b_2 + \dots + a_n b_n = 0}$

$$\frac{\vec{y}_i}{\|y_i\|} \cdot \frac{\vec{y}_j}{\|y_j\|} = \frac{a_1}{\sqrt{\Sigma a_i^2}} \cdot \frac{b_1}{\sqrt{\Sigma b_i^2}} + \frac{a_2}{\sqrt{\Sigma a_i^2}} \cdot \frac{b_2}{\sqrt{\Sigma b_i^2}} + \dots + \frac{a_n}{\sqrt{\Sigma a_i^2}} \cdot \frac{b_n}{\sqrt{\Sigma b_i^2}}$$

$$= \frac{\overbrace{a_1 b_1 + a_2 b_2 + \dots + a_n b_n}^{=0}}{\quad} = 0$$

$$= \frac{a_1b_1 + a_2b_2 + \dots + a_nb_n}{\sqrt{\sum a_i^2}\sqrt{\sum b_i^2}} = 0$$

Örn

$$\left\{ \underset{y_1}{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \rightarrow \text{ortogonal küme}$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \rightarrow \text{ortonormal küme}$$

$\|y_1\| = \sqrt{4+0+0} = 2$

$\|y_2\| = 1$

$\|y_3\| = 1$