10th Week Monday

26 Nisan 2021 Pazartesi 09:43

Null Space of a matrix $A \rightarrow N(A)$

The set of all solutions of $Ax=0$

Row Space of a matrix $A_{m\times n} \rightarrow R(A) \leqslant \mathbb{R}^n$

The span of row vectors of A.

Ex
$$A = \begin{bmatrix} - r_1 - \\ - r_2 - \\ - r_m - \end{bmatrix}_{m\times n}$$
$R(A) = \text{span}\{r_1, r_2, \dots, r_m\}$

Column Space of a matrix $A_{m\times n} \rightarrow C(A) \leqslant \mathbb{R}^m$

The span of column vectors of A.

$$A = \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix}_{m\times n}$$
$C(A) = \text{span}\{c_1, c_2, \dots, c_n\}$

Ex
$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \\ 3 & -2 & 5 \\ 2 & -1 & 3 \end{bmatrix}$$

Find a basis for $N(A)$, $R(A)$, $C(A)$.

$N(A)$ → hw: Find the solutions of $Ax=0$

$\rightarrow R(A) = \text{span}\{r_1, r_2, r_3, r_4\} \leqslant \mathbb{R}^3$

$r_1 = (1,-1,2)$ $r_2 = (-2,2,-4)$
$r_3 = (3,-2,5)$ $r_4 = (2,-1,3)$

Transform A into RREF.

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \\ 3 & -2 & 5 \\ 2 & -1 & 3 \end{bmatrix}_{4\times 3} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for $R(A)$ $= \{(1,0,1), (0,1,-1)\}$

$Ax=0$

$$\xrightarrow{N(A)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_3 = 0$
$x_2 - x_3 = 0$

$x_3 = r \in \mathbb{R}$
$x_1 = -r$
$x_2 = r$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -r \\ r \\ r \end{bmatrix} \rightarrow \text{a typical vector of } N(A)$$

$$\rightarrow r\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$N(A) = \text{span}\{(-1,1,1)\}$

Basis for $N(A)$ $= \{(-1,1,1)\}$

For the column space $C(A) = \text{span}\{c_1, c_2, c_3\} \leqslant \mathbb{R}^4$

$c_1 = (1,-2,3,2)^T$ $c_2 = (-1,2,-2,-1)^T$ $c_3 = (2,-4,5,3)^T$

One way is $A^T \longrightarrow$ RREF $\rightarrow$ pick non-zero rows.

$$A^T = \begin{bmatrix} 1 & -2 & 3 & 2 \end{bmatrix} \quad \text{RREF} \quad \begin{bmatrix} 1 & -2 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & 2 \\ -1 & 2 & -2 & -1 \\ 2 & -4 & 5 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

$\rightarrow$ A basis for $C(A)$ $= \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ $\qquad \langle 1,-2,0,-1\rangle, \langle 0,0,1,1\rangle$ $\searrow$ in ww.

Another way is; $A \rightarrow$ RREF $\rightarrow$ pick the columns of A in the same position with leading 1's in RREF of A.

$$\begin{array}{c} \begin{array}{ccc} \uparrow 1\text{st} & \uparrow 2\text{nd} & \end{array} \\ A \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\rightarrow$ Pick 1st and 2nd column of A.

A basis for $C(A)$ $= \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix} \right\} \leftarrow$

Rank- Nullity Theorem $\quad A_{m\times n}$

Rank(A) = dimension of the row space of A = the dimension of the column space of A.

Null(A) = " " " null space of A

$$\text{Rank}(A) + \text{Null}(A) = n$$

Ex

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}_{3\times 4}$$

Find a basis for $R(A)$.

" " $N(A)$.

Rank(A) =? + $\underbrace{\text{Null}(A) =?}_{2}$ $\overset{?}{=}$ 4

$$A \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for $R(A)$ $= \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\} \Rightarrow$ Rank$(A) = 2$

$Ax = 0$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix}$$

$Ax = 0$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x_1 + 2x_2 - \overset{-2r}{x_3} + \overset{r}{x_4} &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned} \qquad x_4 = r \in \mathbb{R} \Rightarrow x_3 = -2r$$

$$\Rightarrow x_1 + 2\overset{s}{x_2} = -3r \qquad x_2 = s \in \mathbb{R}$$

$$\Rightarrow x_1 = -3r - 2s$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3r-2s \\ s \\ -2r \\ r \end{bmatrix} \rightarrow \text{typical vector in } N(A)$$

$$\rightarrow r \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{A basis for } N(A) = \left\{ \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \Rightarrow \text{Null}(A) = 2$$

$$\text{Rank}(A) + \text{Null}(A) = 2 + 2 = 4 \rightarrow n$$

Ex

$$A = \begin{bmatrix} -1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix}_{4\times 5}$$

Find a basis for $R(A)$.
" " " " $C(A)$.

$$A \xrightarrow{\text{REF}} \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \cancel{0} & 0 & 0 \end{bmatrix}$$

$$\text{A basis for } R(A) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\underline{\text{Rank}(A) = 3}$

$$A \xrightarrow{\text{REF}} \begin{array}{c} \begin{array}{ccccc} \downarrow 1 & \downarrow 2 & & & \downarrow 5 \end{array} \\ \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\Rightarrow$ Pick the 1st, 2nd and 5th <u>columns of A</u>.

$$\text{A basis for } C(A) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \\ 5 \end{bmatrix} \right\}$$

$\hookrightarrow$ dimension $= 3$

Change Of Basis

$\rightarrow$ There are more than 1 bases for the same vector space.

→ Any vector in V, can be written as a linear combination of basis vectors.

Ex $\mathbb{R}^2$

$E = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$ → the standart basis $\longleftrightarrow$ $E = \begin{array}{cc} e_1 & e_2 \end{array} \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$\vec{v} = \begin{bmatrix}-2\\5\end{bmatrix} \in \mathbb{R}^2$

$\begin{bmatrix}-2\\5\end{bmatrix} = \alpha_1 \begin{bmatrix}1\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}0\\1\end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix}-2\\5\end{bmatrix}}_{\vec{v} \to b} = \underbrace{E}_{\text{basis} \to A} \cdot \begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} \to [\vec{v}]_E$

$\begin{bmatrix}-2\\5\end{bmatrix} = -2\begin{bmatrix}1\\0\end{bmatrix} + 5\begin{bmatrix}0\\1\end{bmatrix}$ $\quad \alpha_1 = -2$, $\alpha_2 = 5$

the coordinate vector of $\vec{v}$ with respect to the basis E.

$\to x = A^{-1} b$

$[\vec{v}]_E = \begin{bmatrix}-2\\5\end{bmatrix}$

$\vec{v} = \begin{bmatrix}-2\\5\end{bmatrix} \in \mathbb{R}^2$

$B = \left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix} \right\}$ → another basis for $\mathbb{R}^2$ $\longleftrightarrow$ $B = \begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}$

$\begin{bmatrix}-2\\5\end{bmatrix} = \underset{5}{\alpha_1'} \begin{bmatrix}1\\1\end{bmatrix} + \underset{7}{\alpha_2'} \begin{bmatrix}-1\\0\end{bmatrix}$ $\qquad$ $\underset{\vec{v}}{\begin{bmatrix}-2\\5\end{bmatrix}} = B \cdot \begin{bmatrix}\alpha_1'\\\alpha_2'\end{bmatrix} \to [\vec{v}]_B = ?$

the coordinate vector of $\vec{v}$ with respect to the basis B.

B^{-1} always exists for any basis.

$$\boxed{\begin{aligned} &\vec{v} = B \cdot [\vec{v}]_B \\ &B^{-1}\vec{v} = \underbrace{B^{-1} \cdot B}_{I} \cdot [\vec{v}]_B = [\vec{v}]_B \end{aligned}}$$

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$B^{-1} = \begin{bmatrix}0 & 1\\-1 & 1\end{bmatrix}$ $\qquad$ $B^{-1}\vec{v} = \begin{bmatrix}0 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}-2\\5\end{bmatrix} = \begin{bmatrix}5\\7\end{bmatrix} \Rightarrow [\vec{v}]_B = \begin{bmatrix}5\\7\end{bmatrix}$

$\vec{v} = \begin{bmatrix}-2\\5\end{bmatrix}$ $\qquad$ $\mathcal{U} = \left\{ \underset{u_1}{\begin{bmatrix}3\\5\end{bmatrix}}, \underset{u_2}{\begin{bmatrix}-1\\0\end{bmatrix}} \right\}$ → is another basis for $\mathbb{R}^2$ $\qquad$ $[\vec{v}]_{\mathcal{U}} = ?$

$2 + 3$

$[\vec{v}]_{\mathcal{U}} = \mathcal{U}^{-1}\vec{v} = \begin{bmatrix}0 & 1/5\\-1 & 3/5\end{bmatrix}\begin{bmatrix}-2\\5\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix} \to [\vec{v}]_{\mathcal{U}}$

$\mathcal{U} = \begin{bmatrix}3 & -1\\5 & 0\end{bmatrix} \Rightarrow \mathcal{U}^{-1} = \begin{bmatrix}0 & 1/5\\-1 & 3/5\end{bmatrix}$ $\qquad$ $\begin{bmatrix}-2\\5\end{bmatrix} \overset{?}{=} 1 \underset{u_1}{\begin{bmatrix}3\\5\end{bmatrix}} + 5 \underset{u_2}{\begin{bmatrix}-1\\0\end{bmatrix}}$

Let B be matrix form of any basis for a vector space V.

Let $\vec{v}$ be any vector in V.

The coordinate vector of $\vec{v}$ wrt B $= [\vec{v}]_B = B^{-1} \cdot \vec{v}$