5th Week Wednesday

24 Mart 2021 Çarşamba 12:33

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A (square) $\xrightarrow[E_1, E_2, \dots, E_n]{\text{RREF}}$

- $\underline{I_n} \rightarrow \det = 1 \Rightarrow \underline{\det(A) \neq 0}$ — $\underline{A \text{ is invertible.}}$
- not I_n (RREF) $\rightarrow \underline{\det = 0} \Rightarrow \underline{\det(A) = 0}$ — $\underline{A \text{ is not invertible.}}$

$$A \xrightarrow{E_1} A' \xrightarrow{E_2} A'' \cdots \xrightarrow{E_n} A^{(n)} \begin{cases} = I_n \\ \neq I_n \end{cases}$$

$$A' = E_1 A, \quad A'' = E_2(E_1 A), \quad \dots, \quad A^{(n)} = \underline{E_n(\dots(E_2(E_1 A)))}$$

proof

$$\det(A^{(n)}) = \det(E_n \cdots E_2 E_1 A) = \underbrace{\det(E_n)}_{\neq 0} \cdots \underbrace{\det(E_2)}_{\neq 0} \underbrace{\det(E_1)}_{\neq 0} \det(A)$$

if $A^{(n)} = I^n$, $\det(A^{(n)}) = 1 \longrightarrow \det(A) = \dfrac{1}{\det(E_n) \dots \det(E_2)\det(E_1)}$

if $A^{(n)} \neq I_n$

$\det(A^{(n)}) = 0 \longrightarrow \det(A) = 0$

Adjoint Matrix adj(A)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \text{square}$$

!

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

$$A_{ij} = (-1)^{i+j} |M_{ij}|$$

$$A \cdot \text{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}_{n \times n} = \begin{bmatrix} \det(A) & 0 & 0 & \cdots & 0 \\ 0 & \det(A) & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & & \det(A) \end{bmatrix}_{n \times n} = \det(A) \cdot I_n$$

$\rightarrow a_{11}A_{11} + a_{12}A_{12} + \dots + a_{1n}A_{1n} = \det(A)$

$a_{11}A_{21} + a_{12}A_{22} + \dots + a_{1n}A_{2n} = 0$

$\vdots$

$$\Rightarrow A \cdot \text{adj}(A) = \det(A) \cdot I_n$$

$$\Rightarrow \frac{1}{\det(A)} \cdot A \, \text{adj}(A) = I_n \Rightarrow \boxed{\frac{1}{\det(A)} \cdot \text{adj}(A) = A^{-1}}$$

?

Ex

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

adj(A) = ? ✓ det(A) = ? ✓ A^{-1} = ? ✓

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$A_{11} = \underbrace{(-1)^{1+1}}_{+} \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} = 2$ (6 − 4) $A_{21} = \underbrace{(-1)^{2+1}}_{-} \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 1$ (3 − 4) $A_{31} = \underbrace{(-1)^{3+1}}_{+} \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = -2$ (2 − 4)

$A_{12} = \underbrace{(-1)^{1+2}}_{-} \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = -7$ $A_{22} = \underbrace{(-1)^{2+2}}_{+} \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4$ (6 − 2) $A_{32} = \underbrace{(-1)^{3+2}}_{-} \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = 2$ (4 − 6)

$A_{13} = \underbrace{(-1)^{1+3}}_{+} \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 4$ $A_{23} = \underbrace{(-1)^{2+3}}_{-} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3$ $A_{33} = \underbrace{(-1)^{3+3}}_{+} \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1$

$$\text{adj}(A) = \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2/5 & 1/5 & -2/5 \\ -7/5 & 4/5 & 2/5 \\ 4/5 & -3/5 & 1/5 \end{bmatrix}$$

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + \overset{2}{a_{13}}A_{13} = \underset{4}{2.2} + \underset{-7}{1.(-7)} + \underset{+8}{2.4} = 5$$

Cramer's Rule

$A x = b$ → column matrix of results

A: Coefficient matrix **(square)**

x: Column matrix of unknowns

Let A be a non-singular matrix (A^{-1} exists) ?

A_i := The matrix A with i^{th} column changed with b.

$$\boxed{x_i = \frac{\det(A_i)}{\det(A)}} \quad \det(A) \neq 0$$

Ex

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 5 \\ 2x_1 + 2x_2 + x_3 &= 6 \\ x_1 + 2x_2 + 3x_3 &= 9 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$x_1 = ?$ $x_2 = ?$ $x_3 = ?$

$$\det(A) = 1.\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 4 - 10 + 2 = -4 \neq 0$$

$$A_1 = \begin{bmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{bmatrix} \quad \det(A_1) = 5\overset{4}{\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}} - 2\overset{9}{\begin{vmatrix} 6 & 1 \\ 9 & 3 \end{vmatrix}} + 1\overset{12-18}{\begin{vmatrix} 6 & 2 \\ 9 & 2 \end{vmatrix}} = 20 - 18 - 6 = -4 \quad x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-4}{-4} = 1$$

$$A_2 = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{bmatrix} \quad \det(A_2) = 1.\overset{18-9}{\begin{vmatrix} 6 & 1 \\ 9 & 3 \end{vmatrix}} - 5\overset{5}{\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}} + 1\overset{18-6}{\begin{vmatrix} 2 & 6 \\ 1 & 9 \end{vmatrix}} = 9 - 25 + 12 = -4 \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-4}{-4} = 1$$

$$A_3 = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{bmatrix} \quad \det(A_3) = 1\overset{18-12}{\begin{vmatrix} 2 & 6 \\ 2 & 9 \end{vmatrix}} - 2\overset{18-6}{\begin{vmatrix} 2 & 6 \\ 1 & 9 \end{vmatrix}} + 5\overset{2}{\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}} = 6 \underset{-18}{-24} + 10 = -8 \quad x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-8}{-4} = 2$$

$x_1 = 1 \quad x_2 = 1 \quad x_3 = 2$