7th Week Monday

05 Nisan 2021 Pazartesi 09:27

Null Space of a matrix $N(A)$

The set of all solutions of $\boxed{A \underset{n\times 1}{x} = 0}$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$

$$N(A) = \{ x \in \mathbb{R}^n : \boxed{Ax = 0} \} \leq \mathbb{R}^n$$

$\mathbb{R}^n$: the vector space of all $n\times 1$ column vectors

* $N(A)$ is a subspace of $\mathbb{R}^n$.

1) $\vec{0}_{\mathbb{R}^n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1} \overset{?}{\in} \boxed{N(A)}$ $\quad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is the trivial solution of $\underset{m\times n}{A}\,\underset{n\times 1}{x} = \vec{0}_{m\times 1}$

2) If $\vec{x}$ and $\vec{y}$ are two solutions of $Ax = 0$

$\Rightarrow A\vec{x} =$ and $A\vec{y} = 0$

$\Rightarrow A.(\vec{x}+\vec{y}) = \underbrace{A\vec{x}}_{0} + \underbrace{A\vec{y}}_{0} = 0$

$\Rightarrow \vec{x}+\vec{y}$ is also a solution of $Ax = 0$. ✓

3) If $\vec{x}$ is a solution of $Ax=0$ and $\alpha \in \mathbb{R}$

$\Rightarrow A\vec{x} = 0$

$\Rightarrow A(\alpha\vec{x}) = \alpha \underbrace{A\vec{x}}_{0} = 0$ ✓

$\Rightarrow \alpha\vec{x}$ is also a solution of $Ax=0$. ✓

$$\Rightarrow N(A) \leq \mathbb{R}^n$$

Ex/ $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}_{2\times 4}$ What is the Null Space of A?

all solutions of $Ax = 0$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}_{2\times 4} \overset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{4\times 1}} = \overset{0}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2\times 1}}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-2r_1+r_2\to r_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-1r_2\to r_2} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{-r_2+r_1\to r_1} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right] \to \underline{\text{RREF}}$$

$$\underline{x_1} - \overset{r}{x_3} + \overset{s}{x_4} = 0$$

$$\underline{x_2} + 2\underset{r}{x_3} - \underset{s}{x_4} = 0$$

$\underline{x_3 = r \in \mathbb{R}}$ $\underline{x_4 = s \in \mathbb{R}}$

free variables

$\underline{x_1 = r - s}$

$\underline{x_2 = -2r + s}$

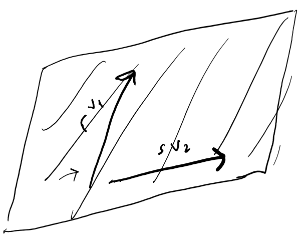
a typical solution vector

$$\vec{x} = \begin{bmatrix} r-s \\ -2r+s \\ r \\ s \end{bmatrix} : r,s \in \mathbb{R} \to \text{infinitely many solution}$$

$$r\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r-s \\ -2r+s \\ r \\ s \end{bmatrix}$$

$$N(A) = \left\{ r\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} : r,s \in \mathbb{R} \right\}$$

$N(A) \leq \mathbb{R}^n$



$r=1$
$r=4$

All linear combinations of vectors $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Span of vectors $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$$\Rightarrow \underline{\underline{N(A)}} = \underline{\text{Span}}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Linear Combinations & Span

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a set of vectors.

A linear combination is

$$\alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \dots + \alpha_n \vec{v_n} \quad \text{where} \quad \underbrace{\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}}_{\text{spesific number}}$$

All linear combinations of $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$ $= \text{Span}\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$

$$\forall \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R} \qquad \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \dots + \alpha_n \vec{v_n}$$

Ex

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{v_2} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \qquad \vec{v_3} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

a linear combination $\longrightarrow 2\vec{v_1} + 3\vec{v_2} + (-1)\vec{v_3} = \dots$

all linear combinations $= \text{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\} = \left\{ r\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} : r, s, t \in \mathbb{R} \right\}$

Spanning Set

If you can obtain all elements of V as a linear combination of a set of vectors, this set is called a spanning set of V.

$V = \text{Span}\{v_1, v_2, \dots, v_n\}$

$\{v_1, v_2, \dots, v_n\}$ = the spanning set of V

Ex Find a spanning set for $\mathbb{R}^2$

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \checkmark$$

You can write any vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, as a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

pick $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \mathbb{R}^2$ $\quad \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ✓

$\Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ span $\mathbb{R}^2$.

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a spanning set for $\mathbb{R}^2$

$\Rightarrow \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$ is a minimum spanning set for $\mathbb{R}^2$

$\Rightarrow \text{Span}\left\{ \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{e_2} \right\} = \mathbb{R}^2$

ex $\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \overset{0}{\begin{bmatrix}2\\1\end{bmatrix}}, \overset{0}{\begin{bmatrix}1\\3\end{bmatrix}} \right\}$ may also be a spanning set for $\mathbb{R}^2$

$$\begin{bmatrix}2\\3\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix} + 0\begin{bmatrix}2\\1\end{bmatrix} + 0\begin{bmatrix}1\\3\end{bmatrix} \checkmark$$

Ex A minimum spanning set for $\mathbb{R}^3 = \left\{ \underset{\vec{e}_1}{\begin{bmatrix}1\\0\\0\end{bmatrix}}, \underset{\vec{e}_2}{\begin{bmatrix}0\\1\\0\end{bmatrix}}, \underset{\vec{e}_3}{\begin{bmatrix}0\\0\\1\end{bmatrix}} \right\}$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = x\begin{bmatrix}1\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$\mathbb{R}$ a

Ex Let $\vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a spanning set for $\mathbb{R}^3$?

a typical element of this vectorspace $= \begin{bmatrix}x\\y\\z\end{bmatrix} \in \mathbb{R}^3$

$x, y, z \in \mathbb{R}$

$$\rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \alpha_1 \overset{\vec{v}_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}} + \alpha_2 \overset{\vec{v}_2}{\begin{bmatrix}1\\1\\0\end{bmatrix}} + \alpha_3 \overset{\vec{v}_3}{\begin{bmatrix}1\\0\\0\end{bmatrix}}$$

Can you find $\underbrace{\alpha_1, \alpha_2, \alpha_3}_{\text{unknowns}}$ in terms of x, y, z ? ✓ If yes

$$\rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}\alpha_1 + \alpha_2 + \alpha_3\\ \alpha_1 + \alpha_2\\ \alpha_1\end{bmatrix}$$

$$\left\{ \begin{array}{l} \overset{z}{\alpha_1} + \overset{y-z}{\alpha_2} + \alpha_3 = x \\ \overset{z}{\alpha_1} + \alpha_2 = y \\ \alpha_1 = z \end{array} \right\}$$

solve this system

$\alpha_1 = z$ ✓

$\alpha_2 = y - z$ ✓

$\alpha_3 = x - y$ ✓

if you can find a solution $\Rightarrow$ YES!

ex $\begin{bmatrix}2\\5\\7\end{bmatrix} \in \mathbb{R}^3$

$x = 2$

$y = 5$

$z = 7$

$$\begin{bmatrix}2\\5\\7\end{bmatrix} \overset{?}{=} 7 \overset{\vec{v}_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}} + -2 \overset{\vec{v}_2}{\begin{bmatrix}1\\1\\0\end{bmatrix}} + -3 \overset{\vec{v}_3}{\begin{bmatrix}1\\0\\0\end{bmatrix}}$$

✓

$\alpha_2 = y - z$
$= 5 - 7$

$\alpha_3 = x - y$
$2 - 5$

$x=2$
$y=5$
$z=7$

✓

$\alpha_2 = y-z$
$= 5-7$

$\alpha_3 = x-y$
$2-5$

find a solution $\Rightarrow$ YES!

$\Rightarrow \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is a spanning set for $\mathbb{R}^3$.

EX

Let $\vec{v_1} = \begin{bmatrix}1\\2\\4\end{bmatrix}$ $\vec{v_2} = \begin{bmatrix}2\\1\\3\end{bmatrix}$ $\vec{v_3} = \begin{bmatrix}4\\-1\\1\end{bmatrix}$

Is $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ a spanning set for $\mathbb{R}^3$? $\begin{bmatrix}x\\y\\z\end{bmatrix}$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \alpha_1 \overset{\vec{v_1}}{\begin{bmatrix}1\\2\\4\end{bmatrix}} + \alpha_2 \overset{\vec{v_2}}{\begin{bmatrix}2\\1\\3\end{bmatrix}} + \alpha_3 \overset{\vec{v_3}}{\begin{bmatrix}4\\-1\\1\end{bmatrix}}$$

Can you find $\underbrace{\alpha_1, \alpha_2, \alpha_3}_{\text{unknowns}}$ in terms of x, y, z?

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}\alpha_1 + 2\alpha_2 + 4\alpha_3\\ 2\alpha_1 + \alpha_2 - \alpha_3\\ 4\alpha_1 + 3\alpha_2 + \alpha_3\end{bmatrix}$$

$$\alpha_1 + 2\alpha_2 + 4\alpha_3 = x$$
$$2\alpha_1 + \alpha_2 - \alpha_3 = y$$
$$4\alpha_1 + 3\alpha_2 + \alpha_3 = z$$

the augmented matrix

$$\left[\begin{array}{ccc|c}1 & 2 & 4 & x\\ 2 & 1 & -1 & y\\ 4 & 3 & 1 & z\end{array}\right] \xrightarrow[-4r_1+r_3 \to r_3]{-2r_1+r_2 \to r_2} \left[\begin{array}{ccc|c}1 & 2 & 4 & x\\ 0 & -3 & -9 & -2x+y\\ 0 & -5 & -15 & -4x+z\end{array}\right]$$

$$\xrightarrow{-1/3\, r_2 \to r_2} \left[\begin{array}{ccc|c}1 & 2 & 4 & x\\ 0 & 1 & 3 & (2x-y)/3\\ 0 & -5 & -15 & -4x+z\end{array}\right] \xrightarrow{5r_2+r_3 \to r_3} \left[\begin{array}{ccc|c}1 & 2 & 4 & x\\ 0 & 1 & 3 & (2x-y)/3\\ 0 & 0 & 0 & \frac{5}{3}(2x-y)-4x+z\end{array}\right]$$

if $\neq 0$
there is no solution.

We should be finding a solution for all x,y,z
But we couldn't.

$\Rightarrow \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is not a spanning set for $\mathbb{R}^3$.

11. Determine whether the following are spanning sets for $\mathbb{R}^2$:

$\begin{bmatrix}x\\y\end{bmatrix} = \alpha_1 \begin{bmatrix}2\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}3\\2\end{bmatrix}$

Can you find α_1, α_2 in terms of x, y?

(a) $\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}4\\6\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\}$

(d) $\left\{\begin{bmatrix}-1\\2\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}2\\-4\end{bmatrix}\right\}$

(e) $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$

12. Which of the sets that follow are spanning sets for $\mathbb{R}^3$? Justify your answers.

(a) $\{(1,0,0)^T, (0,1,1)^T, (1,0,1)^T\}$

(b) $\{(1,0,0)^T, (0,1,1)^T, (1,0,1)^T, (1,2,3)^T\}$

(c) $\{(2,1,-2)^T, (3,2,-2)^T, (2,2,0)^T\}$

(d) $\{(2,1,-2)^T, (-2,-1,2)^T, (4,2,-4)^T\}$

(e) $\{(1,1,3)^T, (0,2,1)^T\}$

x, y :

13. Given

$$\mathbf{x}_1 = \begin{bmatrix}-1\\2\\3\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix}3\\4\\2\end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix}2\\6\\6\end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix}-9\\-2\\5\end{bmatrix}$$

(a) Is $\mathbf{x} \in \mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2)$?

(b) Is $\mathbf{y} \in \mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2)$?

Prove your answers.

a) $\begin{bmatrix}2\\6\\6\end{bmatrix} = \alpha_1\begin{bmatrix}-1\\2\\3\end{bmatrix} + \alpha_2\begin{bmatrix}3\\4\\2\end{bmatrix}$

$\alpha_1 = ?\quad \alpha_2 = ?$