14th Week Wednesday

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Orthogonalization, Orthonormalization

Orthogonal Set: $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ if $\forall i,j$ $\underbrace{\vec{x}_i \perp \vec{x}_j}$ $(i \neq j)$ $\binom{n}{2}$

→ is an orthogonal set. → $\vec{x}_i \cdot \vec{x}_j = 0$

Orthogonal Basis: If $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\} \subseteq \mathbb{R}^n$ an orthogonal set which is linearly independent ⇒ is an orthogonal basis.

Ex/ $\{e_1, e_2, e_3\} \to \mathbb{R}^3$

$$e_1 = (1,0,0) \qquad e_1 \cdot e_2 = 1.0 + 0.1 + 0.0 = 0$$
$$e_2 = (0,1,0) \qquad e_1 \cdot e_3 = 1.0 + 0.0 + 0.1 = 0$$
$$e_3 = (0,0,1) \qquad e_2 \cdot e_3 = 0.0 + 1.0 + 0.1 = 0$$

→ is an orthogonal basis.

Ex/ $\left\{ \underset{u_1}{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \underset{u_3}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}} \right\} \subseteq \mathbb{R}^3$ → a basis of $\mathbb{R}^3$

$\vec{u}_1 \cdot \vec{u}_2 = (2,0,0) \cdot (-1,1,0) = 2.-1 + 0 + 0 = -2$

→ not an orthogonal basis.

Gram-Schmidt Orthogonalization Process

→ $\underline{\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}}$ → a non-orthogonal set of vectors.

$\{\vec{y}_1, \vec{y}_2, \vec{y}_3, \ldots$

$$\boxed{\vec{y}_1} = \vec{x}_1$$

$$\boxed{\vec{y}_2} = \vec{x}_2 - \boxed{\frac{\vec{x}_2 \cdot \vec{y}_1}{\vec{y}_1 \cdot \vec{y}_1}}\,\vec{y}_1$$

scalar

$\vec{y}_1 \cdot \vec{y}_2 = 0$ ✓

→ $\vec{y}_1 \cdot \vec{y}_3 = 0$ ✓

→ $\vec{y}_2 \cdot \vec{y}_3 = 0$ ✓

$$\boxed{\vec{y}_3} = \vec{x}_3 - \boxed{\frac{\vec{x}_3 \cdot \vec{y}_1}{\vec{y}_1 \cdot \vec{y}_1}\,\vec{y}_1} - \boxed{\frac{\vec{x}_3 \cdot \vec{y}_2}{\vec{y}_2 \cdot \vec{y}_2}\,\vec{y}_2}$$

$\vdots$

$$\boxed{\vec{y}_n} = \vec{x}_n - \sum_{i=1}^{n-1} \frac{\vec{x}_n \cdot \vec{y}_i}{\vec{y}_i \cdot \vec{y}_i}\,\vec{y}_i$$

⇒ $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n\}$ ←

is an orthogonal set.

$$\boxed{\vec{y}_n} = \vec{x}_n - \sum_{i=1} \frac{x_n' \cdot y_i}{\vec{y}_i \cdot \vec{y}_i} \vec{y}_i$$

$\Rightarrow \underline{\{ y_1, y_2, \dots, y_n \}}$

<u>is an orthogonal set.</u>

Ex
$$\Bigg\{ \underset{\vec{x}_1}{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{x}_2}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{x}_3}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}} \Bigg\} \subseteq \mathbb{R}^3$$

→ not orthogonal

use Gram-Schmidt process to obtain an orthogonal set from this.
(Observe that it is going to be still a basis)

$$\vec{y}_1 = \vec{x}_1 \rightarrow \underline{(2,0,0)} \swarrow \vec{y}_1$$

$$\rightarrow \vec{y}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{y}_1}{\vec{y}_1 \cdot \vec{y}_1} \vec{y}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{-2}{4}\right) \cdot \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow \vec{y}_2$$

$\rightarrow \vec{x}_2 \cdot \vec{y}_1 = (-1,1,0) \cdot (2,0,0) = -2$

$\vec{y}_1 \cdot \vec{y}_1 = 4$

$\vec{x}_3 \cdot \vec{y}_1 = (0,1,2) \cdot (2,0,0) = \underline{0}$

$\vec{x}_3 \cdot \vec{y}_2 = (0,1,2) \cdot (0,1,0) = 1$

$\vec{y}_2 \cdot \vec{y}_2 = (0,1,0) \cdot (0,1,0) = 1$

$$\rightarrow \vec{y}_3 = \vec{x}_3 - \overset{0}{\frac{\vec{x}_3 \cdot \vec{y}_1}{\vec{y}_1 \cdot \vec{y}_1}} \cdot \vec{y}_1 - \overset{1}{\frac{\vec{x}_3 \cdot \vec{y}_2}{\vec{y}_2 \cdot \vec{y}_2}} \vec{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - 0 - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \swarrow \vec{y}_3$$

$$\Bigg\{ \underset{\vec{y}_1}{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{y}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{y}_2}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}} \Bigg\}$$

$y_1 \cdot y_2 = 0$
$y_1 \cdot y_3 = 0$
$y_2 \cdot y_3 = 0$

→ an orthogonal set.
→ is still a basis.

$\Rightarrow$ it is an orthogonal basis.

Orthonormal Sets and Orthonormalization

If $\{ \vec{y}_1, \vec{y}_2, \dots, \vec{y}_n \}$ is an orthogonal set in which each vector has norm $= 1$
$\Rightarrow$ this set is an <u>orthonormal</u> set.

How can we transform any orthogonal set to an orthonormal set?

→ Let $\{ \vec{y}_1, \vec{y}_2, \dots, \vec{y}_n \}$ be <u>an orthogonal</u> set.

Let $y_i = (a_1, a_2, \dots, a_n)$
$y_j = (b_1, b_2, \dots, b_n)$

$$\Rightarrow y_i \cdot y_j = (a_1, a_2, \dots, a_n) \cdot (b_1, b_2, \dots, b_n) = 0$$
$$= a_1 b_1 + a_2 b_2 + \dots + a_n b_n = 0. \leftarrow$$

$$\longrightarrow \left\{ \frac{\vec{y}_1}{\|\vec{y}_1\|}, \frac{\vec{y}_2}{\|\vec{y}_2\|}, \dots, \frac{\vec{y}_n}{\|\vec{y}_n\|} \right\} \rightarrow \text{is still orthogonal.}$$

$$\longrightarrow \quad \left\{ \underbrace{\frac{\vec{y}_1}{\|\vec{y}_1\|}}_{\substack{\downarrow \\ \text{normed vector.} \\ \text{norm} = 1.}}, \frac{\vec{y}_2}{\|\vec{y}_2\|}, \dots, \frac{\vec{y}_n}{\|\vec{y}_n\|} \right\} \rightarrow \text{is still orthogonal.}$$

$$\frac{\vec{v}}{\|\vec{v}\|} \rightarrow \text{normed vector}$$

$$\frac{\vec{y}_i}{\|\vec{y}_i\|} \cdot \frac{\vec{y}_j}{\|\vec{y}_j\|} = \left(\frac{a_1}{\sqrt{\Sigma a_i^2}}, \frac{a_2}{\sqrt{\Sigma a_i^2}}, \dots, \frac{a_n}{\sqrt{\Sigma a_i^2}}\right) \cdot \left(\frac{b_1}{\sqrt{\Sigma b_i^2}}, \frac{b_2}{\sqrt{\Sigma b_i^2}}, \dots, \frac{b_n}{\sqrt{\Sigma b_n^2}}\right)$$

$$\|\vec{y}_i\| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2} = \sqrt{\Sigma a_i^2}$$

$$\|\vec{y}_j\| = \sqrt{b_1^2 + b_2^2 + \dots + b_n^2} = \sqrt{\Sigma b_i^2}$$

$$= \frac{a_1 b_1}{\sqrt{\Sigma a_i^2}\sqrt{\Sigma b_i^2}} + \frac{a_2 b_2}{\sqrt{\Sigma a_i^2}\sqrt{\Sigma b_i^2}} + \dots + \frac{a_n b_n}{\sqrt{\Sigma a_i^2}\sqrt{\Sigma b_i^2}}$$

$$= \frac{\overbrace{a_1 b_1 + a_2 b_2 + \dots + a_n b_n}^{=0}}{\sqrt{\Sigma a_i^2}\sqrt{\Sigma b_i^2}} = 0$$

EX

$$\left\{ \underset{y_1}{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}, \underset{y_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{y_3}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}} \right\} \rightarrow \text{this was an orthogonal basis of } \mathbb{R}^3$$

Let's orthonormalize this.

$$\|\vec{y}_1\| = \sqrt{4+0+0} = 2$$

$$\frac{\vec{y}_1}{\|\vec{y}_1\|} = \frac{1}{2} \cdot \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left\{ \frac{\vec{y}_1}{\|\vec{y}_1\|}, \frac{\vec{y}_2}{\|\vec{y}_2\|}, \frac{\vec{y}_3}{\|\vec{y}_3\|} \right\} = \underbrace{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\substack{\downarrow \\ \text{orthonormal basis.}}}$$