9th Week Monday

19 Nisan 2021 Pazartesi 09:39

Dimension : The fixed number of elements in a basis of V.
dim(V)

$\mathbb{R}^2 \to$ standard basis $\left\{ \underset{e_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{e_2}{\begin{bmatrix}0\\1\end{bmatrix}} \right\}$ $\Rightarrow \dim(\mathbb{R}^2)=2$

$\mathbb{R}^3 \to$ standard basis $\left\{ \underset{e_1}{\begin{bmatrix}1\\0\\0\end{bmatrix}}, \underset{e_2}{\begin{bmatrix}0\\1\\0\end{bmatrix}}, \underset{e_3}{\begin{bmatrix}0\\0\\1\end{bmatrix}} \right\}$ $\Rightarrow \dim(\mathbb{R}^3)=3$

... $\mathbb{R}^n \to \{e_1, e_2, \ldots, e_n\}$ $\dim(\mathbb{R}^n)=n$

$\mathbb{R}^{n\times n} \to$ standard basis $\left\{ \begin{bmatrix}1 & 0 & 0 & \cdots\\ 0 & 0 & 0 \\ 0 & & \\ \vdots \end{bmatrix}, \begin{bmatrix}0 & 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix}0 \\ 1 \end{bmatrix} \cdots \right\}$

$\Rightarrow \dim(\mathbb{R}^{n\times n}) = n \times n = n^2$

$\mathbb{R}^{2\times 2} \to \left\{ \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix} \right\}$

$\dim(\mathbb{R}^{2\times 2}) = 4$

$P^n \to a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} \to$ any element $\{1, x, x^2, \ldots, x^{n-1}\} \to$ a basis

$\dim(P^n) = n$

* You may not be able to find a finite basis for a vector space

Ex $P \to$ the vector space of all polynomials
$C \to$ the vector space of all continuous functions
} infinite dimensional vector spaces

* → If $\boxed{\dim(V)=n} > 0$

Basis → linear indep. ✓ / span ✓

- Any linearly independent set with n elements is a basis. ✓
(it is automaticly a spanning set)
- Any spanning set with n elements is a basis. ✓
(it is automatically linearly indep.)

Ex $v_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$ $v_2 = \begin{bmatrix}-2\\1\\0\end{bmatrix}$ $v_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}$ Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis for $\mathbb{R}^3$?

! $\boxed{3 \text{ elements} = \dim(\mathbb{R}^3)=3}$

$$\begin{vmatrix}1 & -2 & 1\\ 2 & 1 & 0\\ 3 & 0 & 1\end{vmatrix} = 1\begin{vmatrix}1&0\\0&1\end{vmatrix} + 2\begin{vmatrix}2&0\\3&1\end{vmatrix} + 1\begin{vmatrix}2&1\\3&0\end{vmatrix}$$
$$= 1 + 4 - 3 = 2 \neq 0$$

$\det(X) \neq 0$
$\to \{v_1, v_2, v_3\}$ is linearly independent. ✓

$\Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis!

* If $\dim(V) = n > 0$ (in any basis number of elements $=n$)

- A set with number of elements $< n$ can not be a spanning set.

ex $\mathbb{R}^3$, $\begin{bmatrix}1\\ \vdots\end{bmatrix}$ $\begin{bmatrix}0\\1\end{bmatrix}$ $\begin{bmatrix}0\\0\end{bmatrix}$

— A set with number of elements $< n$ can not be a spanning set.

ex $\mathbb{R}^3$ span. lin.ind. ✓ $\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix}$

span ✗ lin. ind. ✓ $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ 2

— If we have a linearly independent set with number of elements $< n$, we may make it a basis by adding appropriat vectors in this set.

→ — If we have a spanning set with number of elements $> n$, we may make it a basis by removing appropriate vectors from this set.

* If $\dim(V) = n$, a linearly independent set in V can have at most n elements.

a spanning set in V can have at least n elements.

Ex A set with 4 elements. Can it be linearly independent in $\mathbb{R}^3$? ✗

$\dim(\mathbb{R}^3) = 3$

v_1, v_2, v_3, v_4

A set with 3 elements. Can it be a spanning set in $\mathbb{R}^{2\times 2}$? ✗

$\dim(\mathbb{R}^{2\times 2}) = 4$

A set with 3 elements. Can it be a spanning set in $\mathbb{R}^2$?

Yes, it can. We would check.

$\begin{bmatrix}a\\b\end{bmatrix} \in \mathbb{R}^2$ $\quad \alpha_1 + 2\alpha_2 + 3\alpha_3 = a$
$\quad 0 + 0 + 0 = b$

if $\boxed{b \neq 0}$ you have no solns.

Ex → $\left\{ \overset{e_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix} \right\}$ → not a spanning set for $\mathbb{R}^2$

Ex → $\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix} \right\}$ → is a spanning set for $\mathbb{R}^2$

A set with 2 elements. Can it be linearly independent in $\mathbb{R}^3$?

Yes, it can we would check.

Ex $\left\{ \overset{v_1}{\begin{bmatrix}1\\2\\3\end{bmatrix}}, \overset{v_2}{\begin{bmatrix}2\\4\\6\end{bmatrix}} \right\}$ → not linearly independent. $\quad 2v_1 = v_2$

Ex $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$ → is linearly independent.

Ex $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\} \rightarrow$ is linearly independent.

* Any subset $(\neq \emptyset)$ of a linearly independent set is a linearly independent set.

! removing vectors from a linearly independent set does not violate linear independence.

* We can not be sure about the subsets of a linearly dependent set.

Ex $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix} \right\}$ is linearly dependent.

$\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix} \right\}$ subset is linearly dependent.

$\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$ subset is linearly independent.

Ex 3.4 #10

The vectors

$$\mathbf{x}_1 = \begin{bmatrix}1\\2\\2\end{bmatrix}, \quad \cancel{\mathbf{x}_2 = \begin{bmatrix}2\\5\\4\end{bmatrix}}, \quad \mathbf{x}_3 = \begin{bmatrix}1\\3\\2\end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix}2\\7\\4\end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

$x_2 = x_1 + x_3$

span $\mathbb{R}^3$. Pare down the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$ to form a basis for $\mathbb{R}^3$.

→ can not be linearly independent.

$\{\underbrace{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, \vec{x}_5}_{5}\}$ is given as a spanning set for $\mathbb{R}^3$.

$\dim \mathbb{R}^3 = 3$

$5-3=2$ vectors should be removed in order to form a basis.

I need 3 vectors which form a linearly independent set.

$\binom{5}{3} = 10$

x_1, x_2, x_3
$$\begin{vmatrix}1&2&1\\2&5&3\\2&4&2\end{vmatrix} = 1\underbrace{\begin{vmatrix}5&3\\4&2\end{vmatrix}}_{\substack{10-12\\-2}} - 2\underbrace{\begin{vmatrix}2&3\\2&2\end{vmatrix}}_{\substack{4-6\\+4}} + 1\underbrace{\begin{vmatrix}2&5\\2&4\end{vmatrix}}_{\substack{8-10\\-2}} = 0$$

x_1, x_2, x_4
$$\begin{vmatrix}1&2&2\\2&5&7\\2&4&4\end{vmatrix} = 1\underbrace{\begin{vmatrix}5&7\\4&4\end{vmatrix}}_{20-28} - 2\underbrace{\begin{vmatrix}2&7\\2&4\end{vmatrix}}_{8-14\ -6} + 2\underbrace{\begin{vmatrix}2&5\\2&4\end{vmatrix}}_{8-10} = -8+12-4=0$$

x_1, x_3, x_4
$$\begin{vmatrix}1&1&2\\2&3&7\\2&2&4\end{vmatrix} = 1\underbrace{\begin{vmatrix}3&7\\2&4\end{vmatrix}}_{\substack{12-14\\-2}} - 1\underbrace{\begin{vmatrix}2&7\\2&4\end{vmatrix}}_{\substack{8-14\\+6}} + 2\underbrace{\begin{vmatrix}2&3\\2&2\end{vmatrix}}_{\substack{4-6\\-2}} = -2+6-4=0$$

$$\begin{array}{c} x_1\\ x_3\\ x_4\\ x_5 \end{array}\begin{bmatrix}1&2&2\\1&3&2\\2&7&4\\1&1&0\end{bmatrix} \xrightarrow[\substack{-2r_1+r_3\to r_3\\ -r_1+r_4\to r_4}]{-r_1+r_2\to r_2} \begin{bmatrix}1&2&2\\ \boxed{\begin{matrix}0&1&0\\0&3&0\end{matrix}}\\0&-1&-2\end{bmatrix}$$

but don't exchange rows

shows dependence

pick either of x_3 or x_4

$(-x_1+x_3)3 = (-2x_1+x_4)$

this is the dependence

exchange rows [illegible]

$(-x_1+x_3)3 = (-2x_1+x_4)$

this is the dependence relationship.

$\{x_1, x_3, x_5\}$ or $\{x_1, x_4, x_5\}$ or $\{x_3, x_4, x_5\}$?

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 2 & 2 & 0 \end{vmatrix} = 1\underbrace{\begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix}}_{-2} - 1\underbrace{\begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix}}_{+2} + 1\underbrace{\begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix}}_{4-6} = -2+2-2 \neq 0 \quad ✓$$

$\{x_1, x_3, x_5\}$ is a basis for $\mathbb{R}^3$. $\quad x_2 \ \overline{\text{and}}^{\text{remove}}\ x_4$

Ex

$S = \{ (a+b,\ a-b+2c,\ b,\ c) : a,b,c \in \mathbb{R}\} \leq \mathbb{R}^4$

Find a basis for S and $\dim(S)=?$