# 6th Week Monday Makeup

03 Nisan 2021 Cumartesi 14:36

1) $\underbrace{\vec{v}_1 \oplus \vec{v}_2}_{LHS} \overset{?}{=} \underbrace{\vec{v}_2 \oplus \vec{v}_1}_{RHS}$

2) closed under vector addition: $v_1, v_2 \in V \quad v_1 \oplus v_2 \in V$
closed under scalar multiplication: $\alpha \in \mathbb{R} \quad v \in V \quad \alpha v \in V$

2) $(\vec{v}_1 \oplus \vec{v}_2) \oplus \vec{v}_3 = \vec{v}_1 \oplus (\vec{v}_2 \oplus \vec{v}_3)$

3) The identity element of vector addition $\in V$

4) Inverse element of vector addition $\in V$

5) $\alpha \in \mathbb{R}$ (scalar) $\quad \alpha \odot (\vec{v}_1 \oplus \vec{v}_2) = \alpha \odot \vec{v}_1 \oplus \alpha \odot \vec{v}_2$

6) $r, s \in \mathbb{R}$ (scalars) $\vec{v} \in V$ $\quad (r+s) \odot \vec{v} = r \odot \vec{v} \oplus s \odot \vec{v}$

7) $r, s \in \mathbb{R}$ (scalars) $\vec{v} \in V$ $\quad (rs) \odot \vec{v} = r \odot (s \odot \vec{v})$

8) $1 \in \mathbb{R}$ $\vec{v} \in V$ $\quad 1 \odot \vec{v} = \vec{v}$

real number $\to$ vectors of $(\mathbb{R}, +, \cdot)$
matrices $\to$ vectors of $(\mathbb{R}^{m \times n}, +, \cdot)$
polynomials $\to$ vector of $(P_n, +, \cdot)$

**Ex** $(\underbrace{\mathbb{R}^{m \times n}}_{m \times n \text{ matrix}}, \underbrace{\oplus}_{\text{matrix addition}}, \underbrace{\odot}_{\text{scalar multiplication for matrices}})$ is a vector space over $\mathbb{R}$.

$\forall \alpha \in \mathbb{R}$

2) $A_{m \times n} + B_{m \times n} = C_{m \times n} \in \mathbb{R}^{m \times n}$ ✓
$\alpha \cdot A_{m \times n} \in \mathbb{R}^{m \times n}$ ✓

Let $A, B, C \in \mathbb{R}^{m \times n}$, $\alpha, \beta \in \mathbb{R}$

1) $A + B = B + A$ ✓

2) $A + (B + C) = (A + B) + C$ ✓

3) $\vec{0}_{\mathbb{R}^{m \times n}} = \begin{bmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix}_{m \times n}$ $\quad A + ? = A$ ✓

4) Additive inverse $\quad A + ? = \vec{0}_{\mathbb{R}^{m \times n}}$ $\quad ? = -1 . A \in \mathbb{R}^{m \times n}$ ✓

5) $\alpha (A + B) \overset{?}{=} \alpha A + \alpha B$ ✓

$\begin{bmatrix} \alpha a_{11} + \alpha b_{11} & \\ & \end{bmatrix}_{m \times n} \checkmark \begin{bmatrix} \alpha a_{11} & \\ & \ddots \end{bmatrix} + \begin{bmatrix} \alpha b_{11} & \\ & \end{bmatrix} = \begin{bmatrix} \alpha a_{11} + \alpha b_{11} & \\ & \end{bmatrix}_{m \times n}$

6) $(\alpha + \beta) A \overset{?}{=} \alpha A + \beta A$ ✓

$\begin{bmatrix} (\alpha+\beta) a_{11} & \\ & \ddots \end{bmatrix}_{m \times n} \checkmark \begin{bmatrix} \alpha a_{11} & \\ & \ddots \end{bmatrix} + \begin{bmatrix} \beta a_{11} & \\ & \ddots \end{bmatrix} = \begin{bmatrix} \alpha a_{11} + \beta a_{11} & \\ & \ddots \end{bmatrix}_{m \times n}$

7) $\alpha (\beta . A) \overset{?}{=} (\alpha \beta) A$ ✓

$\begin{bmatrix} \alpha \beta a_{11} & \\ & \ddots \end{bmatrix} \overset{\checkmark}{=} \alpha \beta \begin{bmatrix} a_{11} & \\ & \ddots \end{bmatrix} = \begin{bmatrix} \alpha \beta a_{11} & \\ & \ddots \end{bmatrix}$

8) $1 \in \mathbb{R}$ $\quad 1 . A = A$

$\begin{bmatrix} a_{11} & \\ & \end{bmatrix} = A$ ✓

**Ex** $P_n$ : polynomial with degree less than $n$.

$P_n = \{ a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1} : a_i \in \mathbb{R} \}$

$(P_n, +, \cdot)$ is a vector space over $\mathbb{R}$.

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$P_n$ → each polynomial $\deg < n$ is a vector

$+$ → addition of polynomials.

$\cdot$ → scalar multiplication

$\alpha \in \mathbb{R}$, $p, q \in P_n$

② $p(x) + q(x) = r(x)$ $\deg(p) < n$ $\deg(q) < n$ $\underline{\deg(r(x)) < n}$ ✓

② $\alpha \cdot p(x) \in P_n$ ✓

Let $p, q, r \in P_n$, $\alpha, \beta \in \mathbb{R}$

1) $p(x) + q(x) = q(x) + p(x)$ ✓

2) $p(x) + (q(x) + r(x)) = (p(x) + q(x)) + r(x)$ ✓

→ 3) $\vec{0}_{P_n} = 0$ $\quad p(x) + ? = p(x)$ → the 0 constant poly.

4) Additive inverse for $p(x) = a_0 + a_1 x + \dots + a_{n-1} x^{n-1}$ $= -a_0 + -a_1 x + \dots + -a_{n-1} x^{n-1}$

5) $\alpha(p(x) + q(x)) = \alpha p(x) + \alpha q(x)$ ✓

6) $(\alpha + \beta) p(x) = \alpha p(x) + \beta p(x)$ ✓

7) $(\alpha\beta) \cdot p(x) = \alpha \cdot (\beta p(x))$ ✓

8) $1 \in \mathbb{R}$ $\quad 1 \cdot p(x) = p(x)$ ✓

Ex $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathbb{R},\ \underline{x \leq 0} \right\}$ does this set form a vector space under usual matrix addition and scalar multiplication.

② → is $V$ closed under addition? ✓

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

$x_1 \leq 0$ $\quad x_2 \leq 0$ $\quad \underline{x_1 + x_2 \leq 0}$ ✓

② → is $V$ closed under scalar multiplication? ✗

$\underline{\forall \alpha \in \mathbb{R}}$ $\quad \begin{bmatrix} x \\ y \end{bmatrix} \in V$ $\quad \alpha \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha y \end{bmatrix} \overset{?}{\notin} V$

$-2$ $\quad \underline{x \leq 0}$ $\quad -2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} \in V$ $\quad \begin{bmatrix} 2 \\ -6 \end{bmatrix} \notin V$ $\quad \alpha x \overset{?}{\leq} 0$

⇒ $V$ is not a vector space!

Ex Does $V = \mathbb{R}^+$ with addition and scalar multiplication defined below, form a vector space?

$(0, \infty)$

$$\begin{cases} x_1 \oplus x_2 = 2x_1 - x_2 \ ? \\ c \odot x_1 = \underline{c x_1} \leftarrow \underline{\text{usual scalar multiplication}} \end{cases}$$

$\mathbb{R}, +, \cdot$ is a vector space over itself

② → Is $\mathbb{R}^+$ closed under addition? ✗

$\forall x_1, x_2 \in \mathbb{R}^+$

$\cup$

$\forall x_1, x_2 \in \mathbb{R}^+$

$x_1 \oplus x_2 = 2x_1 - x_2 \overset{?}{\in} \mathbb{R}^+$

$x_1 = 2, \ x_2 = 5 \in \mathbb{R}^+ \qquad x_1 \oplus x_2 = 2.2 - 5 = -1 \notin \mathbb{R}^+$

$\mathbb{R}^+$ is not closed under this addition!

$(\mathbb{R}^+, \oplus, \odot)$ is not a vector space!

! Ex/ Does $V = \mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$ with operations

$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ ✓ → usual vector addition ✓

$\alpha \odot (x_1, x_2) = (\alpha x_1, x_2)$ ← ? form a vector space?

✓ Is V closed under addition? $(x_1, x_2) \oplus (y_1, y_2) = (\underbrace{x_1 + y_1}_{\in \mathbb{R}}, \underbrace{x_2 + y_2}_{\in \mathbb{R}}) \in V$ ✓

✓ Is V closed under $\odot$? $\alpha \odot (x_1, x_2) = (\underbrace{\alpha x_1}_{\in \mathbb{R}}, \underbrace{x_2}_{\in \mathbb{R}}) \in V$ ✓

→ 1, 2, 3, 4 → will not check them first ✓

5.) $\alpha \in \mathbb{R}$

$$\alpha \odot ((x_1, y_1) \oplus (x_2, y_2)) \overset{?}{=} \alpha \odot (x_1, y_1) \oplus \alpha \odot (x_2, y_2)$$

$$= \alpha \odot (x_1 + x_2, y_1 + y_2) \qquad (\alpha x_1, y_1) \oplus (\alpha x_2, y_2)$$

$$= (\alpha (x_1 + x_2), y_1 + y_2) \ \underline{\ =✓\ }\ (\alpha x_1 + \alpha x_2, y_1 + y_2)$$

6.) $\alpha, \beta \in \mathbb{R}$

$$(\alpha + \beta) \odot (x_1, y_1) \overset{?}{=} \alpha \odot (x_1, y_1) \oplus \beta \odot (x_1, y_1)$$

$$= ((\alpha + \beta) x_1, y_1) \qquad = (\alpha x_1, y_1) \oplus (\beta x_1, y_1)$$

$$\neq \qquad = ((\alpha + \beta) x_1, 2y_1)$$

✗

⇒ V is not a vector space with these operations.

# Subspaces

$S \subseteq V$ → vector space $(V, \oplus, \odot)$

↘ a subset of V

1) $\vec{0}_V \in S$

2) $\forall \vec{s}_1, \vec{s}_2 \in S \qquad \vec{s}_1 \oplus \vec{s}_2 \in S$

3) $\forall \alpha \in \mathbb{R}, \forall \vec{s} \in S \qquad \alpha \odot \vec{s} \in S$

3) $\forall \alpha \in \mathbb{R}, \forall \vec{s} \in S \quad \alpha \cdot \vec{s} \in S$

If $S$ satisfies these 3 properties, then $S$ is a subspace of $V$.

$S \leqslant V$

subspace notation

Ex $(\mathbb{R}^2, +, \cdot)$ is a vector space $\quad \mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$

$S = \{(x,y) : \boxed{y = 2x}, \ x,y \in \mathbb{R}\} \qquad S \subseteq \mathbb{R}^2$ ✓

Is $S$ a subspace of $\mathbb{R}^2$ ?

1) $\vec{0}_{\mathbb{R}^2} = (0,0) \qquad 0 = 0.2$ ✓ $\quad \vec{0}_{\mathbb{R}^2} \in S$ ✓

2) Let $(x_1,y_1), (x_2,y_2) \in S \Rightarrow \underline{y_1 = 2x_1}$ and $\underline{y_2 = 2x_2}$

$(x_1,y_1) + (x_2,y_2) = (x_1+x_2, y_1+y_2) = (x_1+x_2, 2x_1+2x_2)$ ✓

$= (x_1+x_2, 2(x_1+x_2)) \in S$

3) $\forall \alpha \in \mathbb{R} \quad (x,y) \in S \Rightarrow \boxed{y = 2x} \rightarrow \alpha y = \alpha 2x = 2(\alpha x)$

$\alpha(x,y) = (\alpha x, \alpha y) \in S$ ✓

$\Rightarrow S$ is a subspace of $V$. $\rightarrow S \leqslant V$

Ex $(\mathbb{R}^2, +, \cdot)$ is a vector space ✓

Is $S = \{(x,1) : x \in \mathbb{R}\}$ a subspace of $\mathbb{R}^2$ ?

1) $\vec{0}_{\mathbb{R}^2} = (0,0) \notin S$ ✗ $\quad \rightarrow S$ is not a supspace of $\mathbb{R}^2$

( 2) $(x,1) + (y,1) = (x+y, 2) \notin S$ ✗

3) $\alpha(x,1) = (\alpha x, \alpha)$ if $\alpha \neq 1 \quad \notin S$ ✗ )

Ex **3.** Determine whether the following are subspaces of $\mathbb{R}^{2\times 2}$:

**(a)** The set of all $2\times 2$ diagonal matrices

**(b)** The set of all $2\times 2$ triangular matrices

**(c)** The set of all $2\times 2$ lower triangular matrices

$\mathbb{R}^{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}$

a) $S = \left\{ \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} : d_1, d_2 \in \mathbb{R} \right\}$

(a) The set of all $2 \times 2$ diagonal matrices
(b) The set of all $2 \times 2$ triangular matrices
(c) The set of all $2 \times 2$ lower triangular matrices
(d) The set of all $2 \times 2$ matrices $A$ such that $a_{12} = 1$
(e) The set of all $2 \times 2$ matrices $B$ such that $b_{11} = 0$
(f) The set of all symmetric $2 \times 2$ matrices
(g) The set of all singular $2 \times 2$ matrices

$A^{-1}$ does not exist

$\det(A) = 0$

a) $S = \left\{ \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} : d_1, d_2 \in \mathbb{R} \right\}$

Is $S$ a subspace of $\mathbb{R}^{2\times 2}$ ?

1) $\vec{0}_{\mathbb{R}^{2\times2}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ ✓

2) $\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} + \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} = \begin{bmatrix} d_1+e_1 & 0 \\ 0 & d_2+e_2 \end{bmatrix} \in S$ ✓

3) $\alpha \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} \alpha d_1 & 0 \\ 0 & \alpha d_2 \end{bmatrix} \in S$ ✓

$\Rightarrow S \leqslant \mathbb{R}^{2\times 2}$ ✓

g) $S = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ \underline{\det(A) = 0} \right\}$ $\qquad \det(A) = 0 \Leftrightarrow ad = bc$

1) $\det\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0 \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is singular $\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ ✓

2) $A, B \in S$ $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ $\qquad ad = bc \qquad eh = fg$

$A + B = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$ $\qquad (a+e)(d+h) - (b+f)(c+g)$

$= \cancel{ad} + ed + ah + \cancel{eh} - \cancel{bc} - fc - bg - \cancel{fg}$

$= ed + ah - fc - bg \overset{?}{=} 0$

a counter example

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \to$ singular $\qquad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \to$ singular

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \det = 1 \to$ not singular !

2. Determine whether the following sets form subspaces of $\mathbb{R}^3$:

(a) $\{(x_1, x_2, x_3)^T \mid x_1 + x_3 = 1\}$
(b) $\{(x_1, x_2, x_3)^T \mid x_1 = x_2 = x_3\}$ ✓
(c) $\{(x_1, x_2, x_3)^T \mid x_3 = x_1 + x_2\}$ ✓
(d) $\{(x_1, x_2, x_3)^T \mid x_3 = x_1 \text{ or } x_3 = x_2\}$

$\mathbb{R}^3 = \{ (x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R} \}$

a) $S = \{ (x_1, x_2, x_3) : x_1 + x_3 = 1 \}$

1) $\vec{0}_{\mathbb{R}^3} = (0, 0, 0) \notin S$ $\qquad 0 + 0 = 0 \neq 1$ $\qquad \to S$ is not a subspace of $\mathbb{R}^3$

d) $S = \{ (x_1, x_2, x_3) : \underline{x_3 = x_1} \text{ or } \underline{x_3 = x_2} \}$

1) $(0,0,0) \in S$ ✓

2) a counter example ; $\quad (x, y, x) \in S \qquad (a, b, b) \in S$

$$(x, y, x) + (a, b, b) = (x+a,\ y+b,\ x+b) \notin S$$

$\Rightarrow S$ is not a subspace of $\mathbb{R}^3$.

Null Space.