Make Up

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1. For each of the following, compute (i) det(A), (ii) adj A and (iii) A^{-1}:

(a) $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$ (a_{11}, a_{12}, a_{13})

$A_{square} \to$ minor $|M_{ij}| \to$ delete i^{th} row j^{th} column

cofactor $A_{ij} = (-1)^{i+j} |M_{ij}|$

determinant = any row/column cofactor expansion

$\det(A) = a_{11}A_{11} + a_{12}A_{12} + \dots + a_{1n}A_{1n}$ (1st row)

$\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{21} \dots & A_{n1} \\ A_{12} & A_{22} & A_{n2} \\ \downarrow & \downarrow & \downarrow \\ A_{1n} & A_{2n} & A_{nn} \end{bmatrix}$ (cofactor 11)

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

$[A \mid I] \longrightarrow [I \mid A^{-1}]$

$\operatorname{adj}(A) = \begin{bmatrix} -3 & 5 & 2 \\ 0 & 1 & 1 \\ 6 & -8 & -5 \end{bmatrix}_{3\times3}$

Ex-c)

$A_{11} = (-1)^{1+1}|M_{11}| = 1 \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3$

$A_{12} = (-1)^{1+2}|M_{12}| = -1 \cdot \begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} = 0$ ($-2-(-2)$)

$A_{13} = (-1)^{1+3}|M_{13}| = 1 \cdot \begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} = 6$

$A_{21} = (-1)^{2+1}|M_{21}| = -1 \cdot \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = 5$

$A_{22} = (-1)^{2+2}|M_{22}| = 1 \cdot \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} = 1$ ($-1-(-2)$)

$A_{23} = (-1)^{2+3}|M_{23}| = -1 \cdot \begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix} = -8$ ($2-(-6)$)

$A_{31} = (-1)^{3+1}|M_{31}| = 1 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2$

$A_{32} = (-1)^{3+2}|M_{32}| = -1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1$

$A_{33} = (-1)^{3+3}|M_{33}| = 1 \cdot \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -5$

1st row

$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 1 \cdot (-3) + 3 \cdot 0 + 1 \cdot 6 = 3$

$A^{-1} = \begin{bmatrix} -1 & 5/3 & 2/3 \\ 0 & 1/3 & 1/3 \\ 2 & -8/3 & -1/3 \end{bmatrix}$

3. Given

$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

determine the $(2, 3)$ entry of A^{-1} by computing a quotient of two determinants.

$= \dfrac{\boxed{}\ ?}{\boxed{}\ ?}$

$A^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$

$= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \dfrac{1}{\det(A)}$

$A^{-1}_{23} \to \dfrac{A_{32}}{\det(A)} = -1 \cdot \dfrac{\det(M_{32})}{\det(A)} = -\dfrac{3}{4}$

$A_{32} = (-1)^{3+2} |M_{32}|$

$= \begin{bmatrix} A_{11}/\det A & A_{21}/\det A & A_{31}/\det A \\ A_{12}/\det A & A_{22}/\det A & A_{32}/\det A \\ A_{13}/\det A & A_{23}/\det A & A_{33}/\det A \end{bmatrix}$

$|M_{32}| = \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = 3$

2nd row cofactor exp.

$\det(A) = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$

$= \cancel{0} + 4 \cdot (-1)^{2+2}|M_{22}| + 3 \cdot (-1)^{2+3}|M_{23}|$

$= 4 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + -3 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 4 + 0 = 4$

$\left(\det(A) = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \to 1^{st} \text{ column cofactor exp} \right)$

→ Cramer's Rule

system of LE ⇄ Matrises

$Ax = b$

coeff. matrix (A); column matrix of variables (x); column matrix of result $[b_i]$ (b)

Let A be a non-singular matrix ($\to$ (A^{-1} exists))

A_i = The matrix A with i^{th} column changed to b.

coeff. matrix (square)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

column matrix of variables

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

column matrix of result

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$Ax = b$

A_i = The matrix A with i^{th} column changed to b

$$A_1 = \begin{bmatrix} b_1 & a_{12} & & a_{1n} \\ b_2 & a_{12} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & & & a_{nn} \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & & a_{nn} \end{bmatrix}$$

$$\boxed{x_i = \frac{\det(A_i)}{\det(A)}}$$

Ex

$x_1 + 2x_2 + x_3 = 5$

$2x_1 + 2x_2 + x_3 = 6$

$x_1 + 2x_2 + 3x_3 = 9$

Solve the system using Cramer's rule.

$x_1 = 1 \quad x_2 = 1 \quad x_3 = 2$

$(1, 1, 2)$

$$\rightarrow A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 6 \\ 9 \end{bmatrix}$$

coef. matrix

$$\rightarrow \det(A) = 1.(-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + 2.(-1)^{1+2}\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 1.(-1)^{1+3}\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 4 - 10 + 2 = -4 \neq 0$$

(4, 5, 2)

$$A_1 = \begin{bmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{bmatrix} \quad \det(A_1) = 5.\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} - 2.\begin{vmatrix} 6 & 1 \\ 9 & 3 \end{vmatrix} + 1.\begin{vmatrix} 6 & 2 \\ 9 & 2 \end{vmatrix} = 20 - 18 - 6 = -4$$

(4, 9, −6)

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-4}{-4} = +1$$

$$A_2 = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{bmatrix} \quad \det(A_2) = 1.\begin{vmatrix} 6 & 1 \\ 9 & 3 \end{vmatrix} - 5.\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 1.\begin{vmatrix} 2 & 6 \\ 1 & 9 \end{vmatrix} = 9 - 25 + 12 = -4$$

(18−9, 12)

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-4}{-4} = 1$$

$$A_3 = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{bmatrix} \quad \det(A_3) = 1.\begin{vmatrix} 2 & 6 \\ 2 & 9 \end{vmatrix} - 2.\begin{vmatrix} 2 & 6 \\ 1 & 9 \end{vmatrix} + 5.\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = -8$$

(6, 12, 2)

$6 - 24 + 10$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-8}{-4} = 2$$