# 7. Hafta Pazartesi Dersi

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Ex

$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}_{2\times 4}$ $N(A) = ?$ (A'nın sıfırlık uzayı?)

($Ax=0$ 'ın tüm çözümlerini bul)

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}_{2\times 4} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{4\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2\times 1}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-2r_1+r_2 \to r_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-1r_2 \to r_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{-r_2+r_1 \to r_1} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right] \to \text{İSEF} \checkmark$$

$x_3 = r \in \mathbb{R}$ $x_4 = s \in \mathbb{R}$

$$x_1 - \underset{r}{x_3} + \underset{s}{x_4} = 0$$
$$x_2 + 2\underset{r}{x_3} - \underset{s}{x_4} = 0$$

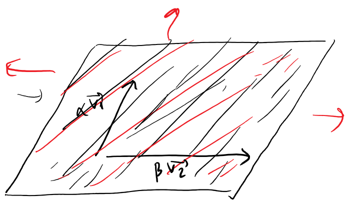
$x_1 = r - s$

$x_2 = -2r + s$

$$N(A) = \left\{ \begin{bmatrix} r-s \\ -2r+s \\ r \\ s \end{bmatrix} : r,s \in \mathbb{R} \right\}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} r-s \\ -2r+s \\ r \\ s \end{bmatrix} = r \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$N(A) = \left\{ \underbrace{r \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} : r,s \in \mathbb{R}}_{} \right\}$$




$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \leftarrow$ vektörlerinin tüm lineer kombinasyonları

N(A), bu vektörler tarafından gerilir.

Span = germek, geren küme

$$N(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

N(A)'nın bir geren küme'si $= \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

# Lineer Kombinasyon & Germe (Span)

$\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ bir vektör kümesi olsun.

Bu vektörlerin **bir** lineer kombinasyonu;

$$\alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \ldots + \alpha_n\vec{v_n} \qquad \underbrace{\alpha_1, \alpha_2, \ldots, \alpha_n}_{\text{belirli}} \in \mathbb{R}$$

Bu vektörlerin **tüm** lineer kombinasyonları; $= \text{Span}\{\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}\}$

$$\alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \ldots + \alpha_n\vec{v_n} \qquad \forall\, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$$

$\text{Span}\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\} = \vec{v_1}, \vec{v_2}, \ldots \vec{v_n}$ vektörleri tarafından geren küme

## Geren Küme (Spanning Set)

$V$ vektör uzayının tüm elemanlarını oluşturabildiğimiz vektör kümesi "Geren Küme" dir.

o vektörlerin bir lineer kombinasyonu şeklinde yazabiliyorsak.

Örn

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

örn/ $\vec{v_1}, \vec{v_2}, \vec{v_3}$'ün **bir** lineer kombinasyonu

$$2\vec{v_1} + 3\vec{v_2} + (-1)\cdot\vec{v_3} = 2\cdot\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ 16 \\ 23 \end{bmatrix}$$

$\rightarrow$ $\vec{v_1}, \vec{v_2}, \vec{v_3}$'ün Tüm lineer kombinasyonları

$$= \text{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\} = \left\{ \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} ; \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}$$

Örn/ $\mathbb{R}^2$ için **bir** geren küme bulunuz.

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} ; x, y \in \mathbb{R}$$

$$\mathbb{R}^2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ \phantom{0} \end{bmatrix}, \begin{bmatrix} 0 \\ \phantom{1} \end{bmatrix} \right\} \qquad \text{7} \sim x \begin{bmatrix} 1 \\ \phantom{0} \end{bmatrix} + y \begin{bmatrix} 0 \\ \phantom{1} \end{bmatrix} + 0 \begin{bmatrix} 1 \\ \phantom{1} \end{bmatrix}$$

$$\mathbb{R}^2 = \text{Span}\left\{ \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{e_2} \right\}$$

$\searrow$ minimal geren küme

$$\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix} + 0\begin{bmatrix}1\\2\end{bmatrix} \checkmark$$

* Geren kümeye vektör ilave edersek germe özelliği bozulmaz.

* Kendisinden vektör eksilttiğimizde germe özelliğini kaybeden küme = minimal geren küme'dir.

**Ör/** $\mathbb{R}^3$ için bir geren küme bulalım.

$$\mathbb{R}^3 = \left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} : x,y,z \in \mathbb{R} \right\}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = x\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\vec{e}_1} + y\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\vec{e}_2} + z\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\vec{e}_3}$$

$$\mathbb{R}^3 = \text{Span}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\} \to \text{minimal geren küme} \qquad \to \mathbb{R}^n$$

**Ör/**

$$\vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, $\mathbb{R}^3$ için bir geren küme midir?

$\searrow$ tipik bir elemanı $= \begin{bmatrix}x\\y\\z\end{bmatrix}$ $\quad x,y,z \in \mathbb{R}$

$$\to \begin{bmatrix}x\\y\\z\end{bmatrix} = \alpha_1 \overset{\vec{v}_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}} + \alpha_2 \overset{\vec{v}_2}{\begin{bmatrix}1\\1\\0\end{bmatrix}} + \alpha_3 \overset{\vec{v}_3}{\begin{bmatrix}1\\0\\0\end{bmatrix}} = \begin{bmatrix}\alpha_1\\\alpha_1\\\alpha_1\end{bmatrix} + \begin{bmatrix}\alpha_2\\\alpha_2\\0\end{bmatrix} + \begin{bmatrix}\alpha_3\\0\\0\end{bmatrix}$$

Her $x,y,z$ için $\alpha_1, \alpha_2, \alpha_3$ bulunabilir mi? ✓

$\equiv$ $\underbrace{\alpha_1, \alpha_2, \alpha_3}_{\text{bilinmeyen}}$ $x,y,z$ cinsinden yazılabilir mi? ✓

$$\to \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}\alpha_1 + \alpha_2 + \alpha_3\\ \alpha_1 + \alpha_2 \\ \alpha_1\end{bmatrix}$$

$$\begin{aligned} \overset{z}{\alpha_1} + \overset{y-z}{\alpha_2} + \alpha_3 &= x \\ \overset{z}{\alpha_1} + \alpha_2 &= y \\ \boxed{\alpha_1 = z} \end{aligned} \qquad \searrow \qquad \boxed{\begin{aligned} \alpha_1 &= z \\ \alpha_2 &= y - z \\ \alpha_3 &= x - y \end{aligned}}$$

anahtar

$\Rightarrow$ $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ $\mathbb{R}^3$ için bir geren kümedir //

ör/ $\searrow \begin{bmatrix}2\\ \cdots\end{bmatrix} \checkmark = -7.\begin{bmatrix}1\\ \cdots\end{bmatrix} + 12.\begin{bmatrix}1\\ \cdots\end{bmatrix} + -3.\begin{bmatrix}1\\ \cdots\end{bmatrix}$ $\quad -7+12-3 = 2$

örn $\begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix} = -7. \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 12. \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + -3. \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{v_1} \quad \vec{v_2} \quad \vec{v_3}$

anahtar

$-7+12-3=2$

$-7+12=5$

$x=2$
$y=5$
$z=-7$

$\alpha_2 = y-z = 5-(-7)=12$
$\alpha_3 = x-y = 2-5$

örn $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad \vec{v_3} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$

$\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ $\mathbb{R}^3$ için bir geren küme midir?

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x,y,z \in \mathbb{R}$

$\longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \alpha_3 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$

Her $x,y,z$ için $\underline{\alpha_1, \alpha_2, \alpha_3}$ bulabilir miyim? ($x,y,z$ cinsinden)

$\longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 + 4\alpha_3 \\ 2\alpha_1 + \alpha_2 - \alpha_3 \\ 4\alpha_1 + 3\alpha_2 + \alpha_3 \end{bmatrix}$

$$\begin{aligned} \alpha_1 + 2\alpha_2 + 4\alpha_3 &= x \\ 2\alpha_1 + \alpha_2 - \alpha_3 &= y \\ 4\alpha_1 + 3\alpha_2 + \alpha_3 &= z \end{aligned}$$

bu lineer denklem sistemini $\alpha_1, \alpha_2, \alpha_3$ için çözmek!

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & x \\ 2 & 1 & -1 & y \\ 4 & 3 & 1 & z \end{array}\right] \xrightarrow[-4r_1+r_3 \to r_3]{-2r_1+r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & 2 & 4 & x \\ 0 & -3 & -9 & -2x+y \\ 0 & -5 & -15 & -4x+z \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & 2 & 4 & x \\ 0 & 1 & 3 & (2x-y)/3 \\ 0 & -5 & -15 & -4x+z \end{array}\right] \xrightarrow{5r_2+r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & 2 & 4 & x \\ 0 & 1 & 3 & (2x-y)/3 \\ 0 & 0 & 0 & \frac{5}{3}(2x-y)-4x+z \end{array}\right] \to \text{SEF}$$

eğer $\neq 0$

çözüm yoktur.

Her $x,y,z$ için çözüm bulabilmeliydik.

Ama bulamadık!

$\Rightarrow \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ , $\mathbb{R}^3$ için bir geren küme değildir.

**12.** Which of the sets that follow are spanning sets for $\mathbb{R}^3$? Justify your answers. $\mathbb{R}^3$ için geren küme midir?

→ **(a)** $\{(1,0,0)^T, (0,1,1)^T, (1,0,1)^T\}$

→ **(b)** $\{(1,0,0)^T, (0,1,1)^T, (1,0,1)^T, (1,2,3)^T\}$ → $\alpha_1$ $\alpha_2$ $\alpha_3$ $\alpha_4$

**(c)** $\{(2,1,-2)^T, (3,2,-2)^T, (2,2,0)^T\}$

**(d)** $\{(2,1,-2)^T, (-2,-1,2)^T, (4,2,-4)^T\}$

→ **(e)** $\{(1,1,3)^T, (0,2,1)^T\}$ $\alpha_1$ $\alpha_2$ $x$ $y$ $z$

**11.** Determine whether the following are spanning sets for $\mathbb{R}^2$:

**(a)** $\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}\right\}$ ✓ **(b)** $\left\{\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}4\\6\end{bmatrix}\right\}$

**(c)** $\left\{\begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\}$

**(d)** $\left\{\begin{bmatrix}-1\\2\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}2\\-4\end{bmatrix}\right\}$

**(e)** $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$

a) $\begin{bmatrix}x\\y\end{bmatrix} = \alpha_1\begin{bmatrix}2\\1\end{bmatrix} + \alpha_2\begin{bmatrix}3\\2\end{bmatrix}$

$2\alpha_1 + 3\alpha_2 = x$
$\alpha_1 + 2\alpha_2 = y$.

$\left[\begin{array}{cc|c}2 & 3 & x\\1 & 2 & y\end{array}\right] \rightarrow \begin{array}{ccc}1 & 3/2 & x/2\\1 & 2 & y\end{array}$

$\rightarrow \begin{array}{ccc}1 & 3/2 & x/2\\0 & 1/2 & -\frac{x}{2}+y\end{array}$

$\alpha_2 = \left(-\frac{x}{2}+y\right)2$
$\alpha_1 = \frac{x}{2} - 3\left(-\frac{x}{2}+y\right)$

**13.** Given

$$\mathbf{x}_1 = \begin{bmatrix}-1\\2\\3\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix}3\\4\\2\end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix}2\\6\\6\end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix}-9\\-2\\5\end{bmatrix}$$

**(a)** Is $\mathbf{x} \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$?

**(b)** Is $\mathbf{y} \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$?

Prove your answers.

a) $\begin{bmatrix}2\\6\\6\end{bmatrix} = \alpha_1\begin{bmatrix}-1\\2\\3\end{bmatrix} + \alpha_2\begin{bmatrix}3\\4\\2\end{bmatrix}$

$-\alpha_1 + 3\alpha_2 = 2$
$2\alpha_1 + 4\alpha_2 = 6$
$3\alpha_1 + 2\alpha_2 = 6$

Sistemi çöz.