Linear Combinations & Span & Spanning Set

$\{v_1, v_2, \dots\}$

a typical element

some examples: $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, ax^2+bx+c, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\longrightarrow = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ → solving a system of linear equations for $\alpha_1, \alpha_2, \dots, \alpha_n$

Try to find $\alpha_1, \alpha_2, \dots, \alpha_n$ for all $a, b, c, (d)$

yes ✓ → $v_1, v_2, \dots, v_n$ → is a spanning set

no X → " is not a s.s.

Linear Independence

A set of vectors $\{v_1, v_2, \dots, v_n\}$ is called "Linear Independent" if

when $\underbrace{c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0}$ $\Rightarrow$ $c_1 = c_2 = \dots = c_n = 0$.

coefficients in this linear combination. → trivial solution

Solving a system (homogeneous $Ax=0$) of linear equations for $c_1, c_2, \dots, c_n$

Ex/ Let $v_1 = (1, 0, 2)^T$, $v_2 = (-2, 1, 0)^T$, $v_3 = (1, -1, 3)^T$ $\quad v_1, v_2, v_3 \in \mathbb{R}^3$

Are v_1, v_2, v_3 linearly independent? (Is $\{v_1, v_2, v_3\}$ " " ?)

$c_1 \vec{v_1} + c_2 \vec{v_2} + c_3 \vec{v_3} = 0 \overset{?}{\Rightarrow} c_1 = c_2 = c_3 \overset{?}{=} 0$ yes ↳ linearly indep.

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{aligned} c_1 - 2c_2 + c_3 &= 0 \\ c_2 - c_3 &= 0 \\ 2c_1 + 3c_3 &= 0 \end{aligned}$$

Kısa yol: $1 \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} + 2 \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}$, $3 + 2 = 5$, $\det(A) = 5 \neq 0$

$\Rightarrow \{v_1, v_2, v_3\}$ is linearly independent.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 0 & 3 & 0 \end{array}\right] \longrightarrow \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\neq I_3} \qquad \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]}_{=I_3} \quad \begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \end{aligned}$$

$A =$ coeffs matrix

$A \to RREF$

$\det(A) \neq 0$ ← $= I_3 \to A$ is invertible → $c_1 = c_2 = c_3 = 0$ → trivial soln. → linearly indep.

$\det(A) = 0$ ← $\neq I_3 \to A$ is not invertible → Solutions → inf. many → not linearly indep.

*** → If you get a square → A coeff. matrix from in $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$;

if $\det(A) \neq 0 \Rightarrow \{v_1, v_2, \dots, v_n\}$ is linearly independent

if $\det(A) = 0 \Rightarrow \{v_1, v_2, \dots, v_n\}$ is not linearly independent.

Solution/

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 0 & 3 & 0 \end{array}\right] \xrightarrow{-2r_1 + r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 4 & 1 & 0 \end{array}\right] \xrightarrow{-4r_2 + r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right]$$

$c_1 - 2c_2 + c_3 = 0 \Rightarrow c_1 = 0$

$c_2 - c_3 = 0 \Rightarrow c_2 = 0$

$5c_3 = 0 \Rightarrow c_3 = 0$

$\Rightarrow c_1 = c_2 = c_3 = 0$. ✓

$\Rightarrow \{v_1, v_2, v_3\}$ are linearly independent.

Ex/ $v_1 = (1, 0, 2)^T$, $v_2 = (2, -1, 3)^T$, $v_3 = (4, 0, 8)^T$

Is $\{v_1, v_2, v_3\}$ linearly independent?

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} c_1 + 2c_2 + 4c_3 &= 0 \\ -c_2 &= 0 \\ 2c_1 + 3c_2 + 8c_3 &= 0 \end{aligned}$$

Solution/ $\det(A) = 0 \Rightarrow \{v_1, v_2, v_3\}$ is not linearly independent.

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & -1 & 0 & 0 \\ 2 & 3 & 8 & 0 \end{array}\right]}_{A} \xrightarrow{-2r_1 + r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \xrightarrow{r_2 + r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c_3 =$ free $= r$

$c_2 = 0$

$c_1 + 2c_2 + 4c_3 = 0$

$c_1 = -4r$

$c_1 = -4r \quad c_2 = 0 \quad c_3 = r \quad : r \in \mathbb{R}$

inf. many solutions

$\Rightarrow \{v_1, v_2, v_3\}$ is not linearly independent

Minimal Spanning Set: A set of linearly independent vectors which has the least number of elements and which is a spanning set for V is called a "minimal spanning set".

elements of a minimum spanning set = "Dimension" of V

n

Ex/ $\mathbb{R}^3 \longrightarrow$ dimension $= 3$ $\quad \boxed{\dim(V) = n}$

$\dim(\mathbb{R}^3) = 3$

$\mathbb{R}^2 \longrightarrow \dim(\mathbb{R}^2) = 2$

$\mathbb{R}^{m\times n} \longrightarrow \dim(\mathbb{R}^{m\times n}) = m\times n$

Find a spanning set, look at linear independency. if they are linearly independent $\Rightarrow$ # elements $= \dim(V)$

Ex/ $v_1 = (1,0,1)$
$v_2 = (2,1,3)$
$v_3 = (1,-1,0)$
$v_4 = (-1,0,4)$

$\{v_1,v_2,v_3,v_4\}$

$\mathbb{R}^3$

$\dim(\mathbb{R}^3) = 3$

a) Is $\text{Span}(\{v_1,v_2,v_3,v_4\}) = V$

b) Is $\{v_1,v_2,v_3,v_4\}$ linearly independent?

a) $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & a \\ 0 & 1 & -1 & 0 & b \\ 1 & 3 & 0 & 4 & c \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & a \\ 0 & 1 & -1 & 0 & b \\ 0 & 1 & -1 & 5 & c-a \end{array}\right] \xrightarrow{-r_2+r_3\to r_3} \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & a \\ 0 & 1 & -1 & 0 & b \\ 0 & 0 & 0 & 1 & (c-a-b)/5 \end{array}\right] \end{array}$$

$\left.\begin{array}{l} \alpha_4 = \dfrac{(c-a-b)}{5} \\ \alpha_3 = \text{free} = r \\ \alpha_2 = b-r \\ \alpha_1 = \ldots\ldots a,b,c. \end{array}\right\}$ have a solution for all a,b,c.

$\Rightarrow \{v_1,v_2,v_3,v_4\}$ is a spanning set for $\mathbb{R}^3$

b) $$\begin{array}{c} \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 3 & 0 & 4 & 0 \end{array}\right] \end{array} \longrightarrow \text{underdetermined homogeneous system} \longrightarrow \text{inf. many solutions.}$$

$c_1=c_2=c_3=c_4=0$ is not the only soln.

$\Rightarrow \{v_1,v_2,v_3,v_4\}$ is not linearly independent!

Basis : $\{v_1,v_2,\ldots,v_n\}$ is a basis for V $\Leftrightarrow$ it is both linearly independent & a spanning set.

minimal spanning set

Dimension : # elements of a basis = Dimension

$\{v_1,v_2,v_3\}$