PS - Problem Session

25 Aralık 2021 Cumartesi 20:07

$\mathbb{R}^3$

$\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \right\}$ lin. indep.

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\2\\0\end{bmatrix}$

$\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$

Problem 1. ANSWERS ONLY CHECKED -- ANSWERS NOT RECORDED

(5 points) Indicate whether the following statements are ALWAYS true or false?

✓ False 1. If the set of vectors U is linearly independent in a subspace S then vectors can be added to U to create a basis for S

✓ False 2. If a set of vectors U spans a subspace S, then vectors can be removed from U to create a basis for S

✓ False 3. If the set of vectors U is linearly independent in a subspace S then vectors can be removed from U to create a basis for S.

✓ False 4. A set of $n+1$ vectors can not span a vector space of dimension n.

✓ False 5. If $S = \text{span}(u_1, u_2, u_3)$, then $dim(S) = 3$.

lin. ind?

Problem 2. ANSWERS ONLY CHECKED -- ANSWERS NOT RECORDED

(9 points) Determine whether each set $\{p_1, p_2\}$ is a linearly independent set in $\mathbb{P}_3$. Type "yes" or "no" for each answer.

$P_3 = \{ at^2 + bt + c : a, b, c \in \mathbb{R} \}$

The polynomials $p_1(t) = 1 + t^2$ and $p_2(t) = 1 - t^2$. yes ✓

The polynomials $p_1(t) = 2t + t^2$ and $p_2(t) = 1 + t$. yes ✓

The polynomials $p_1(t) = 2t - 4t^2$ and $p_2(t) = 6t^2 - 3t$. no ✓

$p_2 = -\frac{3}{2} p_1$

Note: *You can earn partial credit on this problem.*

Problem 3. ANSWERS ONLY CHECKED -- ANSWERS NOT RECORDED

(5 points)

Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a linearly dependent set of vectors.

Select the best statement.

✗ A. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ could be a linearly independent or linearly dependent set of vectors depending on the vectors chosen.

✗ B. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is a linearly independent set of vectors unless $\mathbf{u}_4 = \mathbf{0}$.

✗ C. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is always a linearly independent set of vectors.

✗ D. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is a linearly independent set of vectors unless $\mathbf{u}_4$ is a linear combination of other vectors in the set.

✓ E. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is always a linearly dependent set of vectors.

F. none of the above

Problem 4. ANSWERS ONLY CHECKED -- ANSWERS NOT RECORDED

(7 points) Determine whether or not the following set S of 2×2 matrices are linearly independent in $\mathbb{R}^{2x2}$.

~~Linearly Dependent~~ independent. 1. $S = \left\{ \begin{pmatrix}-1 & 4\\-2 & 1\end{pmatrix}, \begin{pmatrix}4 & -1\\-1 & -2\end{pmatrix}, \begin{pmatrix}4 & -2\\-1 & 0\end{pmatrix} \right\}$

Entered	Answer Preview	Correct	Result
LINEARLY_DEPENDENT	LINEARLY_DEPENDENT	LINEARLY_INDEPENDENT	incorrect

$c_1 \begin{bmatrix}-1 & 4\\-2 & 1\end{bmatrix} + c_2 \begin{bmatrix}4 & -1\\-1 & -2\end{bmatrix} + c_3 \begin{bmatrix}4 & -2\\-1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$-c_1 + 4c_2 + 4c_3 = 0$
$4c_1 - c_2 - 2c_3 = 0$
$-2c_1 - c_2 - c_3 = 0$
$c_1 - 2c_2 = 0$

$\left[\begin{array}{ccc|c} -1 & 4 & 4 & 0\\ 4 & -1 & -2 & 0\\ -2 & -1 & -1 & 0\\ 1 & -2 & 0 & 0 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_4} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0\\ 4 & -1 & -2 & 0\\ -2 & -1 & -1 & 0\\ -1 & 4 & 4 & 0 \end{array}\right]$

$\xrightarrow[\substack{2r_1 + r_3 \to r_3 \\ r_1 + r_4 \to r_4}]{-4r_1 + r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0\\ 0 & 7 & -2 & 0\\ 0 & -5 & -1 & 0\\ 0 & 2 & 4 & 0 \end{array}\right] \xrightarrow[\frac{1}{2}r_2 \to r_2]{r_2 \leftrightarrow r_4} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0\\ 0 & 1 & 2 & 0\\ 0 & -5 & -1 & 0\\ 0 & 7 & -2 & 0 \end{array}\right] \xrightarrow[-7r_2 + r_4 \to r_4]{5r_2 + r_3 \to r_3} \begin{array}{c} c_1 \quad c_2 \quad c_3 \\ \left[\begin{array}{cccc} 1 & -2 & 0 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 9 & 0\\ 0 & 0 & -16 & 0 \end{array}\right] \end{array}$

$c_1 = 0$
$c_2 = 0$

$\left.\begin{array}{l} 9c_3 = 0 \\ -16c_3 = 0 \end{array}\right\} \Rightarrow c_3 = 0$

$A = A^T$

(8 points) Let V be the vector space of symmetric 2×2 matrices and W be the subspace

$$W = \text{span}\left\{ \underbrace{\begin{bmatrix}-5 & -1\\-1 & 5\end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix}-2 & 4\\4 & -2\end{bmatrix}}_{\vec{v}_2} \right\}. \leqslant V \to 2x2 \text{ symmetric matrices}$$

$3\vec{v}_1 + 2\vec{v}_2 \in W$

a. Find a nonzero element X in W.

$X = \begin{bmatrix} -7 & 3 \\ 3 & 3 \end{bmatrix}$

b. Find an element Y in V that is not in W.

$Y = \begin{bmatrix} 3 & 7 \\ 7 & -1 \end{bmatrix}$

$\begin{bmatrix}a & b\\b & c\end{bmatrix} = \alpha_1 \begin{bmatrix}-5 & -1\\-1 & 5\end{bmatrix} + \alpha_2 \begin{bmatrix}-2 & 4\\4 & -2\end{bmatrix}$

Note: *You can earn partial credit on this problem.*

$-5 \quad -4 \qquad -9$

$\begin{bmatrix}3 & 7\\7 & -1\end{bmatrix} \in W \quad \left\{ \begin{array}{l} -5\alpha_1 - 2\alpha_2 = a \\ -\alpha_1 + 4\alpha_2 = b \to 7 \\ -\alpha_1 + 4\alpha_2 = b \to 7 \\ 5\alpha_1 - 2\alpha_2 = c \end{array} \right\}$

$2 \quad 1 \qquad b = 1$
$1 \quad 2$

$-\alpha_1 + 4\alpha_2 = 1$
$-5\alpha_1 - 2\alpha_2 = 2$
$-10\alpha_1 - 4\alpha_2 = 4$
$-11\alpha_1 = 5 \qquad \alpha_1 = -\frac{5}{11}$

$\alpha_2 = \ldots$

Problem 6. ANSWERS ONLY CHECKED -- ANSWERS NOT RECORDED

$H \subseteq \mathbb{R}^2$

(12 points)

Let H be the set of all points in the second quadrant in the plane $V = \mathbb{R}^2$. That is, $H = \{(x, y) \mid x \leq 0, y \geq 0\}$. Is H a subspace of the vector space V?

$x_1 \leq 0 \quad x_2 \geq 0$

$x_3 \leq 0 \quad x_4 \geq 0$

1. Does H contain the zero vector of V?

H contains the zero vector of V ✓

1) $(0,0) \overset{?}{\in} H$

2) $\left.\begin{array}{l} \forall (x_1, x_2) \in H \\ \forall (x_3, x_4) \in H \end{array}\right\} \Rightarrow (x_1, x_2) + (x_3, x_4) \overset{?}{\in} H$

$(\underbrace{x_1 + x_3}_{\leq 0}, \underbrace{x_2 + x_4}_{\geq 0})$

2. Is H closed under addition? If it is, enter *CLOSED*. If it is not, enter two vectors in H whose sum is not in H, using a comma separated list and syntax such as `<1,2>, <3,4>`.

CLOSED ✓

3) $\left.\begin{array}{l} \forall \alpha \in \mathbb{R} \\ \forall (x_1, x_2) \in H \end{array}\right\} \Rightarrow \alpha (x_1, x_2) \overset{?}{\in} H$

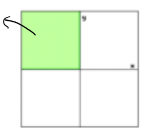
$x_1 \leq 0 \quad x_2 \geq 0 \qquad \alpha x_1 \leq 0 \quad \alpha x_2 \geq 0$

$\alpha < 0$

3. Is H closed under scalar multiplication? If it is, enter *CLOSED*. If it is not, enter a scalar in $\mathbb{R}$ and a vector in H whose product is not in H, using a comma separated list and syntax such as `2, <3,4>`.

-2,<-1,2> $\quad (2, -4) \notin H$

$\langle -1, 2 \rangle \in H \qquad \alpha \in \mathbb{R}$



CLOSED ✓

$\forall (x_1,x_2)\in H \} \Rightarrow \alpha(x_1,x_2)\in H$

$x_1 \le 0 \quad x_2 \ge 0 \qquad \alpha x_1 \le 0 \quad \alpha x_2 \ge 0$

3. Is H closed under scalar multiplication? If it is, enter CLOSED. If it is not, enter a scalar in $\mathbb{R}$ and a vector in H whose product is not in H, using a comma separated list and syntax such as 2, <3,4>.

$\alpha < 0$

-2,<-1,2> $\quad (2,-4) \notin H \qquad \langle -1,2\rangle \in H \qquad \alpha\in\mathbb{R}$

$\alpha\in\mathbb{R} \quad \ge 0 \quad \le 0$

4. Is H a subspace of the vector space V? You should be able to justify your answer by writing a complete, coherent, and detailed proof based on your answers to parts 1-3.

H is not a subspace of V

Problem 7. ANSWERS ONLY CHECKED -- ANSWERS NOT RECORDED

(5 points)

Let $\mathbf{u}_4$ be a linear combination of $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$.

$u_4 \in \text{Span}\{u_1,u_2,u_3\}$

lin. indep ?

Select the best statement.

A. There is no obvious relationship between span$\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ and span$\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$.

$\text{span}\{u_1,u_2,u_3\} = \text{span}\{u_1,u_2,u_3,u_4\}$

$\subseteq$

$\supseteq$

B. We only know that span$\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}\subseteq$ span$\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$.

C. span$\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}=$ span$\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$ when none of $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ is a linear combination of the others.

no need to restrict !

D. span$\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}=$ span$\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$.

E. We only know that span$\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}\subseteq$ span$\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$.

F. none of the above

Problem 8. ANSWERS ONLY CHECKED -- ANSWERS NOT RECORDED

$f'(x) = 2ax+b$

(8 points) Let $\mathbb{P}_3[x]$ be the vector space of polynomials in x with degree less than 3. Find a basis $\{p(x),q(x)\}$ for the subspace of $\mathbb{P}_3[x]$ defined as below:

ax^2+bx+c :

$-8a+\cancel{b} = a+\cancel{b}+c$

$$S=\{f(x)\in\mathbb{P}_3[x] : f'(-4)=f(1)\}$$

$S = \{ ax^2+bx+c : \boxed{c=-9a} \}$

Note that the basis polynomials should also have degrees less than 3, and the order is not important.

$ax^2+bx-9a \to$ typical element

$p(x)=$ x^2-9 , $q(x)=$ ~~2x^2-18~~

$= a(\underbrace{x^2-9}_{v_1}) + b(\underbrace{x}_{v_2})$

$\{x^2-9,\ x\}$

Entered	Answer Preview	Correct	Result
(x^2)-9; 2*(x^2)-18	$x^2-9;\ 2x^2-18$	x^2-9; x	correct

$$E=\left\{\begin{bmatrix}1\\4\\5\end{bmatrix},\begin{bmatrix}0\\1\\-2\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

$A^{-1} = \dfrac{\text{Adj}(A)}{\det(A)}$

$\det(E)=1$

$E^{-1} = \begin{bmatrix}1&0&0\\-4&1&0\\-13&2&1\end{bmatrix}\begin{bmatrix}-1\\-4\\3\end{bmatrix} = \begin{bmatrix}-1\\0\\8\end{bmatrix}$

$-8-5 \qquad A_{11} \quad (-1)$

$-1.$

$1.$

$\begin{bmatrix}A_{11}&A_{21}&A_{31}\\A_{12}&A_{22}&A_{32}\\A_{13}&A_{23}&A_{33}\end{bmatrix}$

$4-4$

$13-8+3$

iven by

$$F_1=\left\{\begin{bmatrix}5\\-2\end{bmatrix},\begin{bmatrix}-1\\3\end{bmatrix}\right\}$$

y

$F_2^{-1}F_1 = ?$

$$F_2=\left\{\begin{bmatrix}-2\\3\end{bmatrix},\begin{bmatrix}1\\-2\end{bmatrix}\right\}\quad \begin{bmatrix}-2\\-3\end{bmatrix}$$

that $[x]_{F_2}=P_{F_2\leftarrow F_1}[x]_{F_1}$ for all x in $\mathbb{R}^2$.

$4-3$

$F_2^{-1} = \begin{bmatrix}-2&-1\\-3&-2\end{bmatrix}$

$\begin{bmatrix}-2&-1\\-3&-2\end{bmatrix}\begin{bmatrix}5&-1\\-2&3\end{bmatrix} = \begin{bmatrix}-8&-1\\-11&-3\end{bmatrix}$

$-15 \qquad -15+4$

$3-6$

Problem 10. ANSWERS ONLY CHECKED -- ANSWERS NOT RECORDED

(8 points) Suppose $S=\{r,u,d\}$ is a set of linearly independent vectors.

If $x=r+3u+2d$, determine whether $T=\{r,u,x\}$ is a linearly independent set.

Linearly Independent 1. Is T linearly independent or dependent?

$c_1 r + c_2 u + c_3 x = 0$

$c_1 r + c_2 u + c_3(r+3u+2d) = 0$

Depending on your answer, find the scalars for the following linear combination.

$c_1 r + c_2 u + c_3 r + 3c_3 u + 2c_3 d = 0$

0 $r+$ 0 $u+$ 0 $x=0$.

Note: *You can earn partial credit on this problem.*

$\underbrace{(c_1+c_3)}_{0}r + \underbrace{(c_2+3c_3)}_{0}u + \underbrace{2c_3}_{0}d = 0$

$c_1=0 \qquad c_2=0 \qquad c_3=0$

$\{1,x,x^2\}$

$H \le \mathbb{P}_2$

Problem 11. ANSWERS ONLY CHECKED -- ANSWERS NOT RECORDED

(9 points) Let $\mathcal{P}_2$ be the vector space of all polynomials of degree 2 or less, and let H be the subspace spanned by $7x^2-14x+1$, $4x^2-6x+1$ and $10x-9x^2-3$.

$ax^2+bx+c : a,b,c\in\mathbb{R}$

a. The dimension of the subspace H is 2

b. Is $\{7x^2-14x+1, 4x^2-6x+1, 10x-9x^2-3\}$ a basis for $\mathcal{P}_2$? not a basis for P_2 ✓ Be sure you can explain and justify your answer.

c. A basis for the subspace H is $\{$ 7x^2-14x+1,10x-9x^2-3 $\}$. Enter a polynomial or a comma separated list of polynomials.

Note: *In order to get credit for this problem all answers must be correct.*

$c_1(7x^2-14x+1)+c_2(4x^2-6x+1)+c_3(10x-9x^2-3)=0$

$\begin{vmatrix}7&4&-9\\-14&-6&10\\1&1&-3\end{vmatrix}$

$7c_1+4c_2-9c_3=0$

$-14c_1-6c_2+10c_3=0$

$c_1+c_2-3c_3=0$

$$\begin{vmatrix} 7 & 4 & -9 \\ -14 & -6 & 10 \\ 1 & 1 & -3 \end{vmatrix}$$

$$\begin{aligned} 7c_1 + 4c_2 - 9c_3 &= 0 \\ -14c_1 - 6c_2 + 10c_3 &= 0 \\ c_1 + c_2 - 3c_3 &= 0 \end{aligned}$$

$$7\begin{vmatrix} -6 & 10 \\ 1 & -3 \end{vmatrix} - 4\begin{vmatrix} -14 & 10 \\ 1 & -3 \end{vmatrix} - 9\begin{vmatrix} -14 & -6 \\ 1 & 1 \end{vmatrix} = 56 - 128 + 72 = 0$$

$18 - 10$ $\quad 42 - 10$ $\quad -14 + 6$

8 $\quad 32$ $\quad -8$

det $= 0$

$\Rightarrow$ lin. dep.

Problem 12. ANSWERS ONLY CHECKED – ANSWERS NOT RECORDED

(8 points) Are the vectors $\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 10 \\ 5 \\ 25 \end{bmatrix}$ linearly independent?

linearly dependent ✓

Depending on your answer, find the scalars for the following linear combination.

$-5x \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} + -5x \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} + x \begin{bmatrix} 10 \\ 5 \\ 25 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Note: *You can earn partial credit on this problem.*

$$\begin{vmatrix} 0 & 2 & 10 \\ -2 & 3 & 5 \\ 3 & 2 & 25 \end{vmatrix} = 0 - 2\begin{vmatrix} -2 & 5 \\ 3 & 25 \end{vmatrix} + 10\begin{vmatrix} -2 & 3 \\ 3 & 2 \end{vmatrix}$$

$-50 - 15$ $\quad -4 - 9$

$2 \cdot 65 + -130 = 0$

$\Rightarrow$ lin. dep.

-80

$c_3 = r$

$c_2 = -5r$

$c_1 = -5r$

$r =$

3 Vector Spaces
- 3.1 Definition and Examples ✓
- 3.2 Subspaces ✓ 1
- 3.3 Linear Independence ✓ 1
- 3.4 Basis and Dimension ✓
- 3.5 Change of Basis ✓
- 3.6 Row Space and Column Space ✓ 1

Null Space

4 Linear Transformations
- 4.1 Definition and Examples ✓ 1
- 4.2 Matrix Representations of Linear Transformations ✓ 1
- 4.3 Similarity ✗ 1
- MATLAB: Exercises
- Chapter Test A—True or False
- Chapter Test B

5 Orthogonality

$\rightarrow$ 2nd Mdt

Final'de

5 Orthogonality
- 5.1 The Scalar Product in R[sup(n)] ✗
- 5.2 Orthogonal Subspaces ✓
- 5.3 Least Squares Problems ✗
- 5.4 Inner Product Spaces ~
- 5.5 Orthonormal Sets ✓
- 5.6 The Gram–Schmidt Orthogonalization Process ✓
- 5.7 Orthogonal Polynomials ✗

6 Eigenvalues
- 6.1 Eigenvalues and Eigenvectors ✓
- ~~6.2 Systems of Linear Differential Equations~~
- 6.3 Diagonalization ✓
- ~~6.4 Hermitian Matrices~~
- ~~6.5 The Singular Value Decomposition~~
- ~~6.6 Quadratic Forms~~
- ~~6.7 Positive Definite Matrices~~
- ~~6.8 Nonnegative Matrices~~