15. Hafta Pazartesi Dersi

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Ör

$$A = \begin{array}{cccc} & x_1 & x_2 & x_3 \\ & \downarrow & \downarrow & \downarrow \end{array}$$
$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

A matrisinin sütun uzayı için ortonormal bir baz bulunuz.

$$A \xrightarrow{SEF} \begin{bmatrix} 1 & -1 & 4 \\ 0 & 5 & -6 \\ 0 & 5 & -2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -6/5 \\ 0 & 5 & -2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -6/5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ SEF} \checkmark$$

$$\{ \vec{x}_1 = (1,1,1,1)^T, \quad \vec{x}_2 = (-1,4,4,-1)^T, \quad \vec{x}_3 = (4,-2,2,0)^T \} \rightarrow \text{baz}$$

1) $\vec{y}_1 = \vec{x}_1 = (1,1,1,1)$

$$\left.\begin{aligned} \vec{x}_2 \cdot \vec{y}_1 &= (-1,4,4,-1)\cdot(1,1,1,1) = -1+4+4-1 = 6 \\ \vec{y}_1 \cdot \vec{y}_1 &= (1,1,1,1)\cdot(1,1,1,1) = 1+1+1+1 = 4 \end{aligned}\right\} \frac{6}{4} = \frac{3}{2}$$

2) $\vec{y}_2 = \vec{x}_2 - \dfrac{\vec{x}_2 \cdot \vec{y}_1}{\vec{y}_1 \cdot \vec{y}_1}\vec{y}_1 = (-1,4,4,-1) - \dfrac{3}{2}(1,1,1,1) = \left(-\dfrac{5}{2}, \dfrac{5}{2}, \dfrac{5}{2}, -\dfrac{5}{2}\right)$

3) $\vec{y}_3 = \vec{x}_3 - \dfrac{\vec{x}_3 \cdot \vec{y}_1}{\vec{y}_1 \cdot \vec{y}_1}\vec{y}_1 - \dfrac{\vec{x}_3 \cdot \vec{y}_2}{\vec{y}_2 \cdot \vec{y}_2}\vec{y}_2 = (4,-2,2,0) - 1.(1,1,1,1) - \left(-\dfrac{2}{5}\right)\left(-\dfrac{5}{2}, \dfrac{5}{2}, \dfrac{5}{2}, -\dfrac{5}{2}\right)$

$= (3,-3,1,-1) - (1,-1,-1,1) = (2,-2,2,-2)$

$$\left.\begin{aligned} \vec{x}_3 \cdot \vec{y}_1 &= (4,-2,2,0)\cdot(1,1,1,1) = 4 \cancel{-2} \cancel{+2} + 0 = 4 \\ \vec{y}_1 \cdot \vec{y}_1 &= 4 \end{aligned}\right\} \frac{4}{4} = 1$$

$$\left.\begin{aligned} \vec{x}_3 \cdot \vec{y}_2 &= (4,-2,2,0)\cdot\left(-\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}, -\tfrac{5}{2}\right) = -10 \cancel{-5} \cancel{+5} + 0 = -10 \\ \vec{y}_2 \cdot \vec{y}_2 &= \left(-\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}, -\tfrac{5}{2}\right)\cdot\left(-\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}, -\tfrac{5}{2}\right) = \tfrac{25}{4} + \tfrac{25}{4} + \tfrac{25}{4} + \tfrac{25}{4} = 25 \end{aligned}\right\} \frac{-10}{25} = -\frac{2}{5}$$

$$\left\{ \vec{y}_1 = (1,1,1,1), \ \vec{y}_2 = \left(-\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}, -\tfrac{5}{2}\right), \ \vec{y}_3 = (2,-2,2,-2) \right\} \rightarrow \text{ortogonal bir bazdır.}$$

$-\frac{5}{2} + \frac{5}{2} + \frac{5}{2} - \frac{5}{2} = 0 \checkmark \qquad -5-5+5+5 = 0 \checkmark$

$y_1 \cdot y_3 = 2 - 2 + 2 - 2 = 0 \checkmark$

$\sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = 1 \qquad -\cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} = 0$

$\|\vec{y}_1\| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2 \qquad \vec{y}_1 \rightarrow \dfrac{\vec{y}_1}{\|\vec{y}_1\|} = \left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}\right)$

$\frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} = 0$

$\|\vec{y}_2\| = \sqrt{25} = 5 \qquad \vec{y}_2 \rightarrow \dfrac{\vec{y}_2}{\|\vec{y}_2\|} = \left(-\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}, -\dfrac{1}{2}\right)$

$-\frac{1}{4} - \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 0$

$\|\vec{y}_3\| = \sqrt{2^2+2^2+2^2+2^2} = \sqrt{16} = 4 \qquad \vec{y}_3 \rightarrow \dfrac{\vec{y}_3}{\|\vec{y}_3\|} = \left(\dfrac{1}{2}, -\dfrac{1}{2}, \dfrac{1}{2}, -\dfrac{1}{2}\right)$

$$\left\{ \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right), \left(-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right), \left(\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right) \right\} \rightarrow \text{ortonormal bir bazdır.}$$

8. Use the Gram–Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $\mathbf{x}_1 = (4, 2, 2, 1)^T$, $\mathbf{x}_2 = (2, 0, 0, 2)^T$, and $\mathbf{x}_3 = (1, 1, -1, 1)^T$. lineer bağımsız ✓

$$c_3\begin{bmatrix}4\\2\\2\\1\end{bmatrix} + c_2\begin{bmatrix}2\\0\\0\\2\end{bmatrix} + c_1\begin{bmatrix}1\\1\\-1\\1\end{bmatrix} = \begin{matrix}0\\0\\0\\0\end{matrix}$$

$$\begin{matrix}1&4&2\\1&2&0\\-1&2&0\\1&1&2\end{matrix} \quad \begin{matrix}1&4&2\\0&-2&-2\\0&6&2\\0&5&0\end{matrix} \quad \begin{matrix}1&4&2\\0&1&0\\0&-2&-2\\0&6&2\end{matrix} \quad \begin{matrix}1&4&2\\0&1&0\\0&0&1\\0&0&0\end{matrix}$$

$c_2=0 \quad c_3=0$

$c_1=0$

$\vec{x_2}\cdot\vec{y_1} = (2,0,0,2)\cdot(4,2,2,1) = 8+0+0+2 = 10$

$\vec{y_1}\cdot\vec{y_1} = 16+4+4+1 = 25$ $\quad \frac{10}{25} = \frac{2}{5}$

$\vec{x_3}\cdot\vec{y_1} = (1,1,-1,1)\cdot(4,2,2,1) = 4+2-2+1 = 5 \rightarrow 5/25 = 1/5$

$\vec{x_3}\cdot\vec{y_2} = (1,1,-1,1)\cdot\left(\frac{2}{5},\frac{-4}{5},\frac{-4}{5},\frac{8}{5}\right) = \frac{2}{5}-\cancel{\frac{4}{5}}+\cancel{\frac{4}{5}}+\frac{8}{5} = 2$

$\vec{y_2}\cdot\vec{y_2} = \frac{4}{25}+\frac{16}{25}+\frac{16}{25}+\frac{64}{25} = \frac{100}{25} = 4$ $\quad \frac{2}{4} = \frac{1}{2}$

$$\vec{y_1} = \vec{x_1} = (4,2,2,1)$$

$$\vec{y_2} = \vec{x_2} - \frac{\vec{x_2}\cdot\vec{y_1}}{\vec{y_1}\cdot\vec{y_1}}\vec{y_1} = (2,0,0,2) - \frac{2}{5}(4,2,2,1) = \left(\frac{2}{5},\frac{-4}{5},\frac{-4}{5},\frac{8}{5}\right) \rightarrow \vec{y_2}$$

$$\vec{y_3} = \vec{x_3} - \frac{\vec{x_3}\cdot\vec{y_1}}{\vec{y_1}\cdot\vec{y_1}}\vec{y_1} - \frac{\vec{x_3}\cdot\vec{y_2}}{\vec{y_2}\cdot\vec{y_2}}\vec{y_2} = (1,1,-1,1) - \frac{1}{5}(4,2,2,1) - \frac{1}{2}\left(\frac{2}{5},\frac{-4}{5},\frac{-4}{5},\frac{8}{5}\right)$$

$-1-\frac{2}{5}$

$$= \left(\frac{1}{5},\frac{3}{5},\frac{-7}{5},\frac{4}{5}\right) - \left(\frac{1}{5},\frac{-2}{5},\frac{-2}{5},\frac{4}{5}\right)$$

$$= (0,1,-1,0) \rightarrow \vec{y_3}$$

$$\left\{ y_1 = (4,2,2,1),\ y_2 = \left(\frac{2}{5},\frac{-4}{5},\frac{-4}{5},\frac{8}{5}\right),\ y_3 = (0,1,-1,0) \right\} \rightarrow \text{ortogonal baz.}$$

$y_1\cdot y_3 = 0+2-2+0 = 0$

$y_1\cdot y_2 = \frac{8}{5}-\frac{8}{5}-\frac{8}{5}+\frac{8}{5} = 0$

$y_2\cdot y_3 = 0-\frac{4}{5}+\frac{4}{5}+0 = 0$

$\|\vec{y_1}\| = \sqrt{25} = 5$

$\|\vec{y_2}\| = \sqrt{4} = 2$

$\|\vec{y_3}\| = \sqrt{1+1} = \sqrt{2}$

$\vec{y_1} \rightarrow \left(\frac{4}{5},\frac{2}{5},\frac{2}{5},\frac{1}{5}\right) \rightarrow 1$

$\frac{4}{25}-\frac{4}{25}-\frac{4}{25}+\frac{4}{25} = 0$

$\vec{y_2} \rightarrow \left(\frac{1}{5},\frac{-2}{5},\frac{-2}{5},\frac{4}{5}\right) \rightarrow 1$

$\vec{y_3} = \left(0,\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}},0\right) \rightarrow 1$

ortonormal baz.

5. Let

$$A = \begin{array}{cc} x_1 & x_2 \\ \left[\begin{matrix}2&1\\1&1\\2&1\end{matrix}\right] \end{array} \quad \text{and} \quad \mathbf{b} = \cancel{\begin{bmatrix}12\\6\\18\end{bmatrix}}$$

(a) Use the Gram–Schmidt process to find an orthonormal basis for the column space of A.

$x_1 = (2,1,2) \quad x_2 = (1,1,1) \rightarrow$ Lineer bağımsız ✓

$\vec{y_1} = x_1 = (2,1,2)$

$$\vec{y_2} = \vec{x_2} - \frac{\vec{x_2}\cdot\vec{y_1}}{\vec{y_1}\cdot\vec{y_1}}y_1 = (1,1,1) - \frac{5}{9}(2,1,2)$$

$$= \left(\frac{-1}{9},\frac{4}{9},\frac{-1}{9}\right)$$

$\vec{x_2}\cdot\vec{y_1} = (1,1,1)\cdot(2,1,2) = 2+1+2 = 5$

$\vec{y_1}\cdot\vec{y_1} = 2^2+1^2+2^2 = 9$ $\quad \frac{5}{9}$

$$\left\{ \vec{y_1} = (2,1,2),\ y_2 = \left(-\frac{1}{9},\frac{4}{9},-\frac{1}{9}\right) \right\} \rightarrow \text{ortogonal baz.}$$

$-\frac{2}{9}+\frac{4}{9}-\frac{2}{9} = 0$ ✓

$\|y_1\| = \sqrt{9} = 3$

$\|\vec{y_2}\| = \sqrt{\frac{1}{81}+\frac{16}{81}+\frac{1}{81}} = \frac{\sqrt{2}}{3}$

$-\frac{1}{9}\cdot\frac{3}{\sqrt{2}}$

$$\|\vec{y_2}\| = \sqrt{\frac{1}{81}+\frac{16}{81}+\frac{1}{81}} = \frac{\sqrt{2}}{3}$$

$-\frac{1}{9}\cdot\frac{3}{\sqrt{2}}$

$\frac{4}{9}\cdot\frac{3}{\sqrt{2}}$

$-\frac{2}{9}+\frac{4}{9}-\frac{2}{9}=0$ ✓

$\left\{ \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right), \left(\frac{-1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, \frac{-1}{3\sqrt{2}}\right) \right\} \rightarrow$ ortonormal baz.

$-\frac{2}{9\sqrt{2}}+\frac{4}{9\sqrt{2}}-\frac{2}{9\sqrt{2}}=0$ ✓ $\qquad \frac{1}{18}+\frac{16}{18}+\frac{1}{18}=1$

7. Given $\mathbf{x}_1 = \frac{1}{2}(1,1,1,-1)^T$ and $\mathbf{x}_2 = \frac{1}{6}(1,1,3,5)^T$, verify that these vectors form an orthonormal set in $\mathbb{R}^4$. Extend this set to an orthonormal basis for $\mathbb{R}^4$ by finding an orthonormal basis for the null space of

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{pmatrix}$$

[*Hint*: First find a basis for the null space and then use the Gram–Schmidt process.]

$Ax=0$

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & 2 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1+x_2+\underset{-3r}{x_3}-\underset{r}{x_4}=0$ $\qquad x_4=r\in\mathbb{R}$

$x_3+3x_4=0$ $\qquad x_3=-3r$

$x_2=s\in\mathbb{R}$

$x_1+\underset{s}{x_2}-4r=0.$ $\qquad x_1=4r-s$

$$\begin{bmatrix} 4r-s \\ s \\ -3r \\ r \end{bmatrix} = r\begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$x_2=(4,0,-3,1)$ $\qquad x_1=(-1,1,0,0)$

$y_1=x_1=(-1,1,0,0)$ $\qquad \left.\begin{array}{l}(x_2\cdot y_1)=-4+0+0+0=-4 \\ y_1\cdot y_1 = 1+1=2\end{array}\right\}-2$

$$y_2 = x_2 - \frac{x_2\cdot y_1}{y_1\cdot y_1}y_1 = (4,0,-3,1)-(-2).(-1,1,0,0)$$

$$= (4,0,-3,1)-(2,-2,0,0)=(2,2,-3,1)$$

sıfırlık uzayının ortogonal bazı

$\rightarrow \{\underbrace{(-1,1,0,0)}_{y_1}, \underbrace{(2,2,-3,1)}_{y_2}\} \rightarrow$ ortogonal

$x_1\cdot y_1 = -\frac{1}{2}+\frac{1}{2}+0+0=0$ ✓

$x_2 y_2 = \frac{2}{6}+\frac{2}{6}-\frac{9}{6}+\frac{5}{6}=0$ ✓

$\mathbf{x}_1 = \frac{1}{2}(1,1,1,-1)^T$ and $\mathbf{x}_2 = \frac{1}{6}(1,1,3,5)^T$

$x_1=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2}\right)$ $\qquad x_2=\left(\frac{1}{6},\frac{1}{6},\frac{3}{6},\frac{5}{6}\right)$

$\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=1$ $\qquad \frac{1}{36}+\frac{1}{36}+\frac{9}{36}+\frac{25}{36}=\frac{36}{36}=1$

$y_1\cdot x_2=0$ ✓

$x_1\cdot y_2 = 1+1-\frac{3}{2}-\frac{1}{2}=0$ ✓

$x_1\cdot x_2 = \frac{1}{12}+\frac{1}{12}+\frac{3}{12}-\frac{5}{12}=0$

$$\left\{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2}\right),\left(\frac{1}{6},\frac{1}{6},\frac{3}{6},\frac{5}{6}\right),\left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right),\left(\frac{2}{3\sqrt{2}},\frac{2}{3\sqrt{2}},\frac{-3}{3\sqrt{2}},\frac{1}{3\sqrt{2}}\right)\right\}$$

$\mathbb{R}^4$ için ortonormal bir bazdır.

ortonormalize et.

$\|y_1\|=\sqrt{1+1}=\sqrt{2}$ $\qquad y_1\rightarrow\left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right)$ ✓

$\|y_2\|=\sqrt{4+4+9+1}=3\sqrt{2}$ $\qquad y_2\rightarrow\left(\frac{2}{3\sqrt{2}},\frac{2}{3\sqrt{2}},\frac{-3}{3\sqrt{2}},\frac{1}{3\sqrt{2}}\right)$ ✓

1. For each of the following, use the Gram–Schmidt process to find an orthonormal basis for $R(A)$. → sütun uzayı

(a) $A=\begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix}$ $\qquad$ **(b)** $A=\begin{bmatrix} 2 & 5 \\ 1 & 10 \end{bmatrix}$

3. Given the basis $\{(1,2,-2)^T,(4,3,2)^T,(1,2,1)^T\}$ for $\mathbb{R}^3$, use the Gram–Schmidt process to obtain an orthonormal basis.

6. Repeat Exercise 5 using

$$A=\begin{bmatrix} 3 & -1 \\ 4 & 2 \\ 0 & 2 \end{bmatrix}$$