13th Week Monday

17 Mayıs 2021 Pazartesi 09:27

Inner Product Spaces – Operations on Vectors

$\mathbb{R}^n$

V, +, ·

vector addition, scalar multiplication

Dot Product (Inner Product)

$\vec{u},\vec{v} \in \mathbb{R}^n$

$$\vec{u}\cdot\vec{v} = \sum_{i=1}^{n} u_i v_i \in \mathbb{R}$$

$\vec{u} = (u_1, u_2, \dots, u_n)$

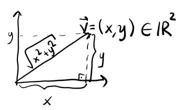
$\vec{v} = (v_1, v_2, \dots, v_n)$

$\Rightarrow \vec{u}\cdot\vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n \rightarrow$ a scalar!

Ex/ $\vec{u} = (-1, 2, -5)$, $\vec{v} = (2, -3, 6)$, $\vec{u},\vec{v} \in \mathbb{R}^3$

$\vec{u}\cdot\vec{v} = -1.2 + 2.3 + -5.6$
$= -2 - 6 - 30 = \underline{-38} \in \mathbb{R}$

Norm of A Vector (~ Length)



Norm $\rightarrow \|\vec{v}\| = \sqrt{x^2+y^2}$

In $\mathbb{R}^n$: $\vec{u} = (u_1, u_2, \dots, u_n)$

Norm of $\vec{u}$: $\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$

Ex/ $\vec{v} = (-1, 2, 3, 4) \in \mathbb{R}^4$ $\|\vec{v}\| = \sqrt{(-1)^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$

* Norm > 0 (Norm $= 0$ only if $\vec{u} = (0, 0, \dots, 0)$)

* $\vec{u} \in \mathbb{R}^n$ $\vec{u}\cdot\vec{u} = u_1 u_1 + u_2 u_2 + \dots + u_n u_n = u_1^2 + u_2^2 + \dots + u_n^2$

$$\boxed{\vec{u}\cdot\vec{u} = \|\vec{u}\|^2}$$

* **Normed Vector**: (u_0)

$\vec{u} \in \mathbb{R}^n$

$\vec{u} = (u_1, u_2, \dots, u_n)$

$$\boxed{\vec{u}_0 = \frac{1}{\|\vec{u}\|}\cdot\vec{u}}$$ $\rightarrow$ a vector with norm 1

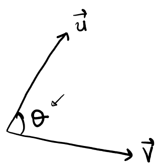
Represents the direction of $\vec{u}$.

Ex/ $\vec{u} = (-1, 2, 5) \in \mathbb{R}^3$ $\|u\| = \sqrt{(-1)^2 + 2^2 + 5^2} = \sqrt{30}$

$\vec{u}_0 = \frac{1}{\sqrt{30}}\cdot(-1, 2, 5) = \left(\frac{-1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}\right) \rightarrow$ the normed vector of u.

$\|\vec{u}_0\| = \sqrt{\frac{1}{30} + \frac{4}{30} + \frac{25}{30}} = 1$

* ## The Angle Between Two Vectors

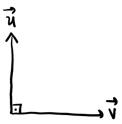


$\vec{u},\vec{v} \in \mathbb{R}^n$

$$\boxed{\cos\theta = \frac{\overbrace{\vec{u}\cdot\vec{v}}^{\text{dot product, scalar}}}{\underbrace{\|\vec{u}\|}_{\in\mathbb{R}} \cdot \underbrace{\|\vec{v}\|}_{\in\mathbb{R}}}}$$

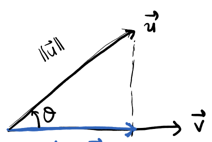
(mult.) $\rightarrow$ scalar

* $\vec{u} \perp \vec{v}$
orthogonal vectors



$\boxed{\theta = 90^\circ, \ \cos\theta = 0} \Leftrightarrow \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} \ (\neq 0) \Leftrightarrow \boxed{\vec{u}\cdot\vec{v} = 0}$

Projection:



$\text{proj}_{\vec{v}}\vec{u} \rightarrow$ the projection of the vector $\vec{u}$

$$\boxed{\underbrace{\|\vec{u}\|\cos\theta}_{\text{the norm/ the scalar component of the proj.}} \cdot \underbrace{\frac{\vec{v}}{\|\vec{v}\|}}_{\text{the direction of the proj.}} = \underbrace{\overrightarrow{\text{proj}_{\vec{v}}\vec{u}}}_{\text{Projection Vector}}}$$

the direction of this

the scalar component of the proj. | the direction of the proj. | Projection vector

$\text{proj}_{\vec v}\vec u$ → the projection of the vector $\vec u$ onto the vector $\vec v$ → the direction of this projection = the direction of $\vec v$ $= \dfrac{1}{\|\vec v\|}.\vec v$

$\|\vec u\|.\cos\theta$ → scalar component of the projection

$$\cos\theta = \frac{\overbrace{\vec u\cdot\vec v}^{\text{dot product}\ \to\ \text{scalar}}}{\underbrace{\|\vec u\|}_{\in\mathbb{R}}\ \underbrace{.}_{\text{mult.}}\ \underbrace{\|\vec v\|}_{\in\mathbb{R}}}\ \to \text{scalar}$$

$$\underbrace{\|\vec u\|\cos\theta}_{\text{the norm/ the scalar component of the proj.}}\ \underbrace{\frac{\vec v}{\|\vec v\|}}_{\text{the direction of the proj.}} = \underbrace{\overrightarrow{\text{proj}_{\vec v}\vec u}}_{\text{Projection vector}}$$

$$\Rightarrow \overrightarrow{\text{proj}_{\vec v}\vec u} = \|\vec u\|\cdot\cos\theta\cdot\frac{\vec v}{\|\vec v\|} = \cancel{\|\vec u\|}\cdot\frac{\vec u\cdot\vec v}{\cancel{\|\vec u\|}\,\|\vec v\|}\cdot\frac{\vec v}{\|\vec v\|}$$
vector

$$\Rightarrow \overrightarrow{\text{proj}_{\vec v}\vec u} = \frac{\vec u\cdot\vec v}{\|\vec v\|^2}\cdot\vec v$$
vector

Ex

$\vec u = (3,4)$ a) Find the angle between $\vec u$ and $\vec v$

$\vec v = (-1,7)$ b) Find $\text{proj}_{\vec v}\vec u$

$\vec u,\vec v\in\mathbb{R}^2$

a) $\cos\theta = \dfrac{\vec u\cdot\vec v}{\|\vec u\|\,\|\vec v\|} = \dfrac{25}{5.\,5\sqrt2} = \dfrac{1}{\sqrt2} \Rightarrow \theta = \pi/4$

$\vec u\cdot\vec v = 3.-1 + 4.7 = 25$

$\|\vec u\| = \sqrt{3^2+4^2} = 5$

$\|\vec v\| = \sqrt{(-1)^2+7^2} = 5\sqrt2$

b) $\text{proj}_{\vec v}\vec u = (\|\vec u\|\cos\theta)\cdot\left(\dfrac{\vec v}{\|\vec v\|}\right)$

$= 5\cdot\dfrac{1}{\sqrt2}\cdot\left(\dfrac{-1}{5\sqrt2},\dfrac{7}{5\sqrt2}\right) = \left(-\dfrac12,\dfrac72\right)$

Ex

$\vec u = (5,2)$ Find $\overrightarrow{\text{proj}_{\vec v}\vec u}$.

$\vec v = (1,-3)$ Find the scalar component of $\overrightarrow{\text{proj}_{\vec v}\vec u}$.

$$\overrightarrow{\text{proj}_{\vec v}\vec u} = \frac{\vec u\cdot\vec v}{\|\vec v\|^2}\vec v = \frac{5.1+2.-3}{1^2+(-3)^2}\vec v = \frac{-1}{10}(1,-3) = \left(\frac{-1}{10},\frac{3}{10}\right)$$

the scalar component of the proj.

$$\|u\|\cos\theta = \cancel{\|u\|}\cdot\frac{\vec u\cdot\vec v}{\cancel{\|u\|}\,\|v\|} = \frac{5.1+2.-3}{\sqrt{1^2+(-3)^2}} = \frac{-1}{\sqrt{10}}$$ → the scalar component of $\overrightarrow{\text{proj}_v\vec u}$.

Norm of $\text{proj}_{\vec v}\vec u$ = | the scalar component of $\text{proj}_{\vec v}\vec u$ | = $1/10$

Ex

$\vec u = (6,3,2)$ $\quad \overrightarrow{\text{proj}_{\vec v}\vec u} = ?\ \neq\ \overrightarrow{\text{proj}_{\vec u}\vec v} = ?$

$\vec v = (1,-2,-2)$

$\overrightarrow{\text{proj}_{\vec v}\vec u} = \dfrac{\vec u\cdot\vec v}{\|\vec v\|^2}\cdot\vec v$ | $\overrightarrow{\text{proj}_{\vec u}\vec v} = \dfrac{\vec u\cdot\vec v}{\|\vec u\|^2}\cdot\vec u$

$\overrightarrow{\ \ }$ = $\underbrace{6.1+3.-2+2.-2}_{-4}$ (. .) | proj ... (−4)

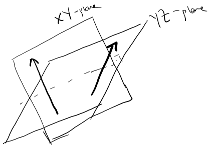
$\|\vec v\|^2$ $\qquad$ $\|\vec u\|^2$

$$\overrightarrow{proj_{\vec v}\vec u} = \frac{6.1+3.-2+2.-2}{1^2+(-2)^2+(-2)^2}\cdot(1,-2,-2) \qquad (-4)$$

$$= \frac{-4}{9}(1,-2,-2) = \left(-\frac{4}{9}, \frac{8}{9}, \frac{8}{9}\right)$$

$$proj_{\vec u}\vec v = \frac{-4}{6^2+3^2+2^2}\cdot(6,3,2)$$

$$= \frac{-4}{49}\cdot(6,3,2) = \left(\frac{-24}{49}, \frac{-12}{49}, \frac{-8}{49}\right)$$



Orthogonal Subspaces

$\vec u \perp \vec v \iff \vec u \cdot \vec v = 0$
orthogonality of two vectors

Let $S \leqslant \mathbb{R}^n$ $\quad T \leqslant \mathbb{R}^n$ $\quad$ (S, T are subspaces of $\mathbb{R}^n$)

$\iff$ All vectors in S $\perp$ All vectors in T

$\iff$ $S \perp T$ $\qquad \boxed{\forall \vec s \in S,\ \forall \vec t \in T \quad \vec s \cdot \vec t = 0}$

Orthogonal Complement: $S^\perp$ $\rightarrow$ orthogonal complement of the subspace S

$S \leqslant \mathbb{R}^n$

$$S^\perp = \{\ \vec v \in \mathbb{R}^n \ :\ \vec v \cdot \vec s = 0 \quad \forall \vec s \in S\}$$

Ex $\quad S = \text{span}\{e_1\} \leqslant \mathbb{R}^3$ $\quad$ Find a basis for the orthogonal complement of S.

$e_1 = (1,0,0)$

A typical element in S can be written as $(r, 0, 0)$ $\quad r \in \mathbb{R}$

$$S^\perp = \{\ \vec v \in \mathbb{R}^3 \ :\ \vec v \cdot \vec s = 0 \quad \forall \vec s \in S\ \}$$

$$= \{\ \vec v \in \mathbb{R}^3 \ :\ (v_1, v_2, v_3)\cdot(r,0,0) = 0\ \}$$

$$= \{\ \vec v \in \mathbb{R}^3 \ :\ v_1.r + v_2.0 + v_3.0 = 0\ \}$$

what is the solution for v_1, v_2, v_3 ?

free variables
$v_2 = a \in \mathbb{R}$
$v_3 = b \in \mathbb{R}$
$v_1 = 0$

$$= \{\ \vec v \in \mathbb{R}^3 \ :\ \underbrace{(0, a, b)}_{\text{a typical element of } S^\perp},\ a, b \in \mathbb{R}\ \}$$

$$\begin{bmatrix} 0 \\ a \\ b \end{bmatrix} = a\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

A basis for $S^\perp$ is $\left\{ \underset{e_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{e_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\} = \text{span}\{e_2, e_3\}$

Ex #5.2

2. Let S be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{x} = (1, -1, 1)^T$.
(a) Find a basis for $S^\perp$.
~~(b) Give a geometrical description of S and $S^\perp$.~~

$S = \text{span}\{(1,-1,1)\} \leqslant \mathbb{R}^3$

a typical element of S

$\vec s = (r, -r, r) \quad r \in \mathbb{R}$

$$S^\perp = \{\ \vec v \in \mathbb{R}^3 \ :\ \vec v \cdot \vec s = 0, \forall s \in S\ \}$$

$$= \{\ \vec v \in \mathbb{R}^3 \ :\ (v_1, v_2, v_3)\cdot(r, -r, r) = 0\ \}$$

$$S^{\perp} = \{ \vec{v} \in \mathbb{R}^{\cdot} \; : \; \vec{v} \cdot \vec{s} = 0, \forall s \in S \}$$

$$= \{ \vec{v} \in \mathbb{R}^3 \; : \; \underbrace{(v_1, v_2, v_3) \cdot (r, -r, r) = 0} \}$$

$v_1 r - v_2 r + v_3 r = 0$ → solve this for v_1, v_2, v_3

$\Rightarrow \; v_1 - v_2 + v_3 = 0$ free $v_2 = a$, $v_3 = b$ $\Rightarrow v_1 = a - b$

$$= \{ \vec{v} \in \mathbb{R}^3 : \underline{(a-b, \; a, \; b)} \; : \; \underline{a, b \in \mathbb{R}} \}$$

$$\begin{bmatrix} a-b \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

A basis for $S^{\perp} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$S^{\perp} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Fundamental Spaces of a matrix $A_{m \times n}$

Range of A : $R(A)$ = Column space of A ✓

Range of A^T : $R(A^T)$ = Row space of A ✓

$\perp$ $\perp$

Null Space of A : $N(A)$ = Solution space of $A\vec{x} = 0$ ✓

Null Space of A^T : $N(A^T)$ = 〃 〃 〃 $A^T \vec{x} = 0$ ✓

Row Space of A $\perp$ $N(A)$ Column Space of A $\perp$ $N(A^T)$

$\underline{N(A)} = \underbrace{R(A^T)}_{\text{row space of } A}{}^{\perp}$ → $\underline{N(A^T)} = \underbrace{R(A)}_{\text{column space of } A}{}^{\perp}$

Ex

$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ $R(A)$ = column space of A = span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

↘ a typical element of $R(A)$ → $(r, 2r)$ $r \in \mathbb{R}$

$A^T = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $N(A^T) = ?$ $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_1 + 2x_2 = 0$, $x_2 = s \in \mathbb{R}$, $x_1 = -2s$ → $\begin{bmatrix} -2s \\ s \end{bmatrix}$

$N(A^T) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ → a typical element of $N(A^T)$ → $(-2s, s)$ $s \in \mathbb{R}$

$$(r, 2r) \cdot (-2s, s) = r.-2s + 2r.s = -2rs + 2rs = 0$$

$$\Rightarrow R(A) \perp N(A^T)$$

Let's check $N(A) \underset{?}{\perp} \underbrace{R(A^T)}_{\text{row space}}$ of A

$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ $N(A) = ?$ $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_1 = 0$, $2x_1 = 0$ $x_2 = r \in \mathbb{R}$ $\begin{bmatrix} 0 \\ r \end{bmatrix}$

$N(A) = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ → a typical element of $N(A)$ → $\begin{bmatrix} 0 \\ r \end{bmatrix}$, $r \in \mathbb{R}$

a typical element

$N(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \rightarrow$ a typical element of $N(A)$ $\rightarrow \begin{bmatrix} 0 \\ r \end{bmatrix}, r \in \mathbb{R}$

$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \underset{\text{Row space of } A}{R(A^T)} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \rightarrow$ a typical element of $R(A^T)$ $\rightarrow \begin{bmatrix} s \\ 0 \end{bmatrix}, s \in \mathbb{R}$

$$\Rightarrow \quad (0, r) \cdot (s, 0) = 0.s + r.0 = \underline{\underline{0}}$$

$$\Rightarrow \quad N(A) \perp R(A^T)$$

HW

1. For each of the following matrices, determine a basis for each of the subspaces $\underbrace{R(A^T)}_{\text{rowspace of } A}$, $N(A)$, $\underbrace{R(A)}_{\text{columnspace of } A.}$, and $N(A^T)$:

(a) $A = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 4 & -2 \\ 1 & 3 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}$ (d) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix}$

Ex

$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}$

Find basis for $N(A)$, rowspace of A

→ show that they are orthogonal.

Find basis for $N(A^T)$, column space of A

→ show that they are orthogonal.