13. Hafta Pazartesi Dersi

17 Mayıs 2021 Pazartesi 11:33

İç Çarpım Uzayları ($\mathbb{R}^n$)

$V,\ +,\ \cdot$ (vektörel toplama, skaler çarpma)

İç Çarpım (Dot Product) $\quad \vec{u},\vec{v}\in\mathbb{R}^n$

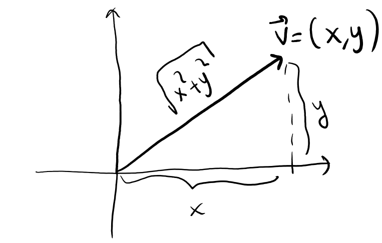
$\vec{u}=(u_1,u_2,\dots,u_n)$

$\vec{v}=(v_1,v_2,\dots,v_n)$

$$\boxed{\vec{u}\cdot\vec{v}=\sum_{i=1}^{n}u_iv_i=u_1.v_1+u_2v_2+\dots+u_nv_n\in\mathbb{R}}$$

Örn/ $\left.\begin{array}{l}\vec{u}=(1,2,3)\\ \vec{v}=(-2,3,5)\end{array}\right\}\in\mathbb{R}^3$ $\quad \vec{u}\cdot\vec{v}=1.-2+2.3+3.5=19\in\mathbb{R}$

Bir Vektörün Normu (~uzunluk)



Norm → $\|\vec{v}\|=\sqrt{x^2+y^2}$

Genel olarak; $\vec{u}\in\mathbb{R}^n$

$\vec{u}=(u_1,u_2,\dots,u_n)$

$$\boxed{\|\vec{u}\|=\sqrt{u_1^2+u_2^2+\dots+u_n^2}}$$

Örn/ $\vec{u}=(-1,2,3,4)\in\mathbb{R}^4 \quad \|\vec{u}\|=\sqrt{(-1)^2+2^2+3^2+4^2}=\sqrt{30}$

* Norm >0 $\quad$ (Norm $=0 \iff \vec{u}=\vec{0} \quad \|\vec{u}\|=\sqrt{0^2+0^2+\dots+0^2}$)

* $\vec{u}\cdot\vec{u}=u_1.u_1+u_2.u_2+\dots+u_n.u_n=u_1^2+u_2^2+\dots+u_n^2=\|\vec{u}\|^2$

$$\boxed{\vec{u}\cdot\vec{u}=\|\vec{u}\|^2}$$

Normlanmış Vektör $(\vec{u}_0)$ → Birim vektör

$\vec{u}\in\mathbb{R}^n$

$\vec{u}=(u_1,u_2,\dots,u_n)$

$$\underbrace{\vec{u}_0}_{\text{vektör}}=\underbrace{\frac{1}{\|\vec{u}\|}}_{\text{skaler}}\cdot\underbrace{\vec{u}}_{\text{vektör}}$$

→ Normu 1 olan bir vektör.

→ $\vec{u}$ vektörünün doğrultusunu temsil eden birim vektör.

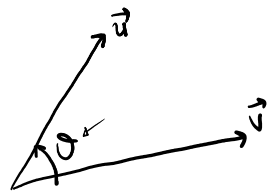
Ör: $\vec{u} = (-1, 2, 5) \in \mathbb{R}^3$

$\|\vec{u}\| = \sqrt{(-1)^2 + 2^2 + 5^2} = \sqrt{30}$

$\vec{u}_0 = \frac{1}{\sqrt{30}} \cdot (-1, 2, 5) = \left(\frac{-1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}} \right)$

$\|\vec{u}_0\| = \sqrt{\frac{1}{30} + \frac{4}{30} + \frac{25}{30}} = 1$

İki Vektör Arasındaki (dar) Açı



$\vec{u}, \vec{v} \in \mathbb{R}^n$

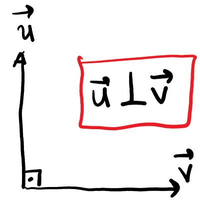
$0 \leqslant \theta \leqslant \pi$

$$\boxed{\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}}$$

iç çarpım

$\vec{u} \cdot \vec{v}$ → skaler

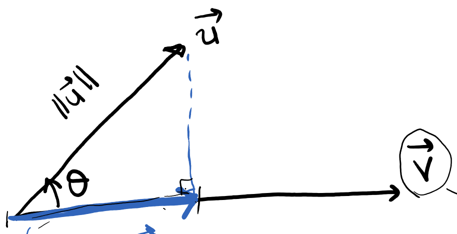
$\|\vec{u}\|\,\|\vec{v}\|$ → skaler, → $\neq 0$

* 

$\boxed{\vec{u} \perp \vec{v}} \Leftrightarrow \boxed{\cos\frac{\pi}{2} = 0} \Leftrightarrow \boxed{\vec{u} \cdot \vec{v} = 0}$

(dik) ortogonal vektörler

İzdüşüm (Projection)



$\vec{v}$ → doğrultusu

$\frac{\vec{v}}{\|\vec{v}\|}$ → vektör

$\mathrm{proj}_{\vec{v}}\vec{u}$ → $\vec{u}$ vektörünün $\vec{v}$ vektörü üzerine izdüşüm vektörü

$$\boxed{\begin{array}{l}\text{izdüşümün skaler bileşeni} \\ = \|\vec{u}\| \cos\theta\end{array}}$$

$$\boxed{\overrightarrow{\mathrm{proj}_{\vec{v}}\vec{u}} = \underbrace{\|\vec{u}\| . \cos\theta}_{} \cdot \frac{\vec{v}}{\|\vec{v}\|}}$$

$\|\vec{u}\| . \cos\theta$ → izdüşümün skaler bileşeni → mutlak değeri izdüşümün normunu verir.

$\frac{\vec{v}}{\|\vec{v}\|}$ → izdüşümün doğrultusu

*

$$\boxed{\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}}$$

iç çarpım

$\vec{u} \cdot \vec{v}$ → sk

$\|\vec{u}\|\,\|\vec{v}\|$ → sk, → $\neq$

$$\boxed{\overrightarrow{\mathrm{proj}_{\vec{v}}\vec{u}} = \underbrace{\|\vec{u}\| \cos\theta}_{} \cdot \frac{\vec{v}}{\|\vec{v}\|}}$$

$$\boxed{\overrightarrow{\mathrm{proj}_{\vec{v}}\vec{u}} = \|\vec{u}\| \cos\theta \; \frac{\vec{v}}{\|\vec{v}\|} = \cancel{\|\vec{u}\|} \frac{\vec{u} \cdot \vec{v}}{\cancel{\|\vec{u}\|}\|\vec{v}\|} \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \cdot \vec{v}}$$

$\vec{v}$ → vektör

Örn

$\vec{u} = (3,4)$

$\vec{v} = (-1,7)$

a) $\vec{u}$ ve $\vec{v}$ vektörleri arasındaki açı ?

b) $\overrightarrow{proj_{\vec{v}}\vec{u}} = ?$

a) $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \dfrac{3.-1 + 4.7}{5.5\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$0 < \theta < \pi$

$\theta = \arccos\left(\frac{1}{\sqrt{2}}\right)$

$\theta = \dfrac{\pi}{4}$

$\|\vec{u}\| = \sqrt{3^2 + 4^2} = 5$

$\|\vec{v}\| = \sqrt{(-1)^2 + 7^2} = 5\sqrt{2}$

b) $\overrightarrow{proj_{\vec{v}}\vec{u}} = \|\vec{u}\| \cos\theta \cdot \dfrac{\vec{v}}{\|\vec{v}\|} = 5 \cdot \dfrac{1}{\sqrt{2}} \cdot \left(\dfrac{-1}{5\sqrt{2}}, \dfrac{7}{5\sqrt{2}}\right) = \left(\dfrac{-1}{2}, \dfrac{7}{2}\right)$

Örn

$\vec{u} = (5,2)$

$\vec{v} = (1,-3)$

$\overrightarrow{proj_{\vec{v}}\vec{u}} = ?$ $\quad$ $\overrightarrow{proj_{\vec{v}}\vec{u}}$ 'nun skaler bileşenini bulunuz.

$\overrightarrow{proj_{\vec{v}}\vec{u}} = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2} \vec{v} = \dfrac{5.1 + 2.-3}{1.1 + -3.-3} \cdot (1,-3)$

$\|\vec{v}\|^2 \rightarrow \vec{v}\cdot\vec{v}$

$= \dfrac{5-6}{1+9} \cdot (1,-3) = \left(\dfrac{-1}{10}, \dfrac{3}{10}\right)$

$\|\overrightarrow{proj_{\vec{v}}\vec{u}}\| = \sqrt{\frac{1}{100} + \frac{9}{100}} = \frac{1}{\sqrt{10}}$

izdüşümün skaler bileşeni

$\|\vec{u}\| \cos\theta = \cancel{\|\vec{u}\|} \cdot \dfrac{\vec{u}\cdot\vec{v}}{\cancel{\|\vec{u}\|}\,\|\vec{v}\|} = \dfrac{5.1 + 2.-3}{\sqrt{1^2 + (-3)^2}} = -\dfrac{1}{\sqrt{10}}$

Örn

$\vec{u} = (6,3,2)$

$\vec{v} = (1,-2,-2)$

$\overrightarrow{proj_{\vec{v}}\vec{u}} = ? \neq \overrightarrow{proj_{\vec{u}}\vec{v}} = ?$

$\overrightarrow{proj_{\vec{v}}\vec{u}} = \dfrac{\overset{-4}{\vec{u}\cdot\vec{v}}}{\|\vec{v}\|^2} \cdot \vec{v}$

$= \dfrac{-4}{9} (1,-2,-2)$

$\overrightarrow{proj_{\vec{u}}\vec{v}} = \dfrac{\overset{-4}{\vec{u}\cdot\vec{v}}}{\|\vec{u}\|^2} \cdot \vec{u}$

$= \dfrac{-4}{49} \cdot (6,3,2)$

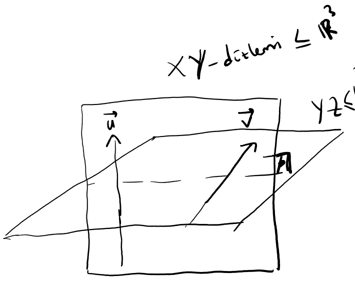
$$= \frac{-4}{9}\cdot(1,-2,-2) \qquad\qquad = \frac{-4}{49}\cdot(6,3,2)$$

$$= \left(\frac{-4}{9}, \frac{8}{9}, \frac{8}{9}\right) \qquad\qquad = \left(\frac{-24}{49}, \frac{-12}{49}, \frac{-8}{49}\right)$$

$$\vec{u}\cdot\vec{v} = 6.1 + 3.-2 + 2.-2 = -4$$

$$\|\vec{u}\|^2 = \vec{u}\cdot\vec{u} = 6^2+3^2+2^2 = 49$$

$$\|\vec{v}\|^2 = \vec{v}\cdot\vec{v} = 1^2 + (-2)^2 + (-2)^2 = 9$$



$\vec{u}\perp\vec{v} \quad \forall\vec{v}\in YZ$

$\vec{u}\perp\vec{v} \quad \forall\vec{u}\in XY$

(Dik) Ortogonal Altuzaylar

$\vec{u}\perp\vec{v} \Leftrightarrow \vec{u}\cdot\vec{v}=0$

$S \leqslant \mathbb{R}^n$, $T \leqslant \mathbb{R}^n$ (S,T $\mathbb{R}^n$'in altuzayları)

S'deki tüm vektörler $\perp$ T'deki tüm vektörler

$\Leftrightarrow \quad S \perp T$

$\Leftrightarrow \quad \boxed{\forall\vec{s}\in S,\ \forall\vec{t}\in T,\ \vec{s}\cdot\vec{t}=0}$

Dik Tümleyen : $S^{\perp} \rightarrow$ S altuzayının dik tümleyeni

(Ortogonal Tümleyen)

$S \leqslant \mathbb{R}^n$

$$\boxed{S^{\perp} = \{\ \vec{v}\in\mathbb{R}^n \ :\ \vec{v}\cdot\vec{s}=0,\ \forall\vec{s}\in S\ \}}$$

Ör/ $S = \text{span}\{e_1\} \leqslant \mathbb{R}^3$ veriliyor. $S^{\perp}$ için bir baz bulunuz.

$e_1 = (1,0,0)$

S'nin tipik bir elemanı $\rightarrow$ $(r,0,0)$ $\quad r\in\mathbb{R}$

$S^{\perp} = \{\ \vec{v}\in\mathbb{R}^3 \ :\ \vec{v}\cdot\vec{s} = 0,\ \forall\vec{s}\in S\ \}$

$S^{\perp} = \{ \vec{v} \in \mathbb{R}^3 \; : \; \vec{v} \cdot \vec{s} = 0 \; , \forall \vec{s} \in S \}$

$\Rightarrow S^{\perp} = \{ \vec{v} \in \mathbb{R}^3 \; : \; (v_1, v_2, v_3) \cdot (r, 0, 0) = 0 \; , \forall r \in \mathbb{R} \}$

$v_1 r + v_2 . 0 + v_3 . 0 = 0 \quad \rightarrow (v_1, v_2, v_3) = ?$

$v_1 r = 0$

$\Rightarrow v_1 = 0 \qquad v_2 = a \in \mathbb{R}$

$v_3 = b \in \mathbb{R}$

$(0, a, b) \qquad a, b \in \mathbb{R}$

$\Rightarrow S^{\perp} = \{ (0, a, b) \; : \; a, b \in \mathbb{R} \}$

$S^{\perp}$'nin bazı $= \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\begin{bmatrix} 0 \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$S^{\perp} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$S^{\perp} = \text{span} \{ e_2, e_3 \}$

$S \perp S^{\perp}$

Örn/

2. Let S be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{x} = (1, -1, 1)^T$.
   (a) Find a basis for $S^{\perp}$.

$S = \text{span} \{ (1, -1, 1) \}$

S'nin tipik elemanı $\rightarrow (r, -r, r) \quad r \in \mathbb{R}$

$S^{\perp} = \{ \vec{v} \in \mathbb{R}^3 \; : \; \vec{v} \cdot \vec{s} = 0 \; , \forall \vec{s} \in S \}$

$= \{ \vec{v} \in \mathbb{R}^3 \; : \; \underbrace{(v_1, v_2, v_3) \cdot (r, -r, r) = 0}_{} \; , \forall r \in \mathbb{R} \}$

$v_1 r - v_2 r + v_3 r = 0$

$\Rightarrow v_1 - v_2 + v_3 = 0$

$(v_1, v_2, v_3) = ?$

$v_2 = a \in \mathbb{R}$

$v_3 = b \in \mathbb{R}$

$v_1 = a - b$

$(a-b, a, b)$

$S^{\perp} = \{ (a-b, a, b) \; : \; a, b \in \mathbb{R} \}$

$\begin{bmatrix} a-b \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\rightarrow S^{\perp}$'in bazı $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$S^{\perp} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Temel Uzaylar (Bir Matrisin Temel Uzaylar)

- → $R(A)$ → A'nın range'i = A'nın sütun uzayı
- → $R(A^T)$ → A^T'un range'i = A'nın satır uzayı
- → $N(A)$ → $A\vec{x}=0$'ın çözüm uzayı = A'nın sıfırlık uzayı
- → $N(A^T)$ → $A^T\vec{x}=0$'ın çözüm uzayı = A^T'un sıfırlık uzayı

$\perp$ $\perp$

A'nın satır uzayı ($R(A^T)$) $\perp$ $N(A)$

A'nın sütun uzayı ($R(A)$) $\perp$ $N(A^T)$

$$* \quad N(A) = R(A^T)^{\perp} \qquad N(A^T) = R(A)^{\perp}$$

Ör:

$$A = \begin{array}{cc} \text{1.sütun} & \\ \downarrow & \downarrow \\ \end{array} \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$A \rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \xrightarrow{\text{1.sat}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (1.sat)

$R(A) \perp N(A^T)$ olduğunu görelim.

$R(A)$ → sütun uzayı

$R(A)$: A'nın sütun uzayı $= \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ → tipik eleman : $\begin{bmatrix} r \\ 2r \end{bmatrix}$ $r \in \mathbb{R}$

$N(A^T)$: $A^T\vec{x}=0$'ın çözümleri $= \text{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$ → tipik eleman : $\begin{bmatrix} -2s \\ s \end{bmatrix}$ $s \in \mathbb{R}$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 = 0$ $\quad$ $x_2 = s \in \mathbb{R}$, $x_1 = -2s$

$$\forall r,s \in \mathbb{R} \quad (r, 2r) \cdot (-2s, s) = \underline{r.(-2s)} + \underline{2r.s} = 0$$

$$\Rightarrow R(A) \perp N(A^T)$$

Şimdi, $R(A^T) \perp N(A)$ olduğunu gözlemleyelim:

$R(A^T)$: A'nın satır uzayı $= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \longrightarrow$ tipik elemanı : $(r, 0)$ $\quad r \in \mathbb{R}$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \xrightarrow{SEF} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$N(A)$: A'nın sıfırlık uzayı $= \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \rightarrow$ tipik eleman : $(0, s)$ $\quad s \in \mathbb{R}$

$$A\vec{x} = 0 \qquad \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 = 0 \qquad x_2 = s \in \mathbb{R}$

$2x_1 = 0 \qquad x_1 = 0$

$$(r, 0) \cdot (0, s) = r.0 + 0.s = \underline{\underline{0}}$$

$$\Rightarrow R(A^T) \perp N(A)$$

Ödev

1. For each of the following matrices, determine a basis for each of the subspaces $R(A^T)$, $N(A)$, $R(A)$, and $N(A^T)$:

satır   sütun

(a) $A = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 4 & -2 \\ 1 & 3 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}$ (d) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix}$

HW

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}$$

$\left.\begin{array}{l} R(A^T) \text{ için baz bul.} \\ N(A) \text{ için baz bul} \end{array}\right\} \rightarrow$ dik olduklarını göster.

$\left.\begin{array}{l} R(A) \text{ bul.} \\ N(A^T) \text{ bul.} \end{array}\right\} \rightarrow$ dik olduklarını göster.