5. Hafta Çarşamba Dersi - Adjoint Matris ve Cramer Kuralı

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$$A \xrightarrow[E_1\, E_2 \dots E_n]{\text{İSEF}} \underbrace{}_{B}$$

$\to I_n \to \det = 1$ (A terslenebilir)

$\to I_n$'e ulaşmıyor $\to \det = 0$ (A terslenebilir değil)

A terslenebilir değilse

$\det(A) = 0$

$$\underbrace{E_n \dots E_2 E_1 A}_{} = B$$

$$\underbrace{\det(E_n)}_{\neq 0} \dots \underbrace{\det(E_2)}_{\neq 0}\, \underbrace{\det(E_1)}_{\neq 0}\, \underbrace{\det(A)}_{} = \det(B) \to 0$$

Adjoint Matris (EL-matris) adj (A)

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}_{n\times n}$$

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}^{T} = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{bmatrix}$$

$$\boxed{A_{ij} = (-1)^{i+j} |M_{ij}|}$$

$$A \cdot \text{adj}\, A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{22} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{bmatrix}_{n\times n} = \begin{bmatrix} \det(A) & 0 & 0 \dots 0 \\ 0 & \det(A) & 0 \dots 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & & \det(A) \end{bmatrix}_{n\times n}$$

$\to a_{11}A_{11} + a_{12}A_{12} + \dots + a_{1n}A_{1n} \to$ 1. satır kofaktör açılımı

$= \det(A) \cdot I_n$

$\to a_{11}A_{21} + a_{12}A_{22} + \dots + a_{1n}A_{2n} = 0$

$$\Rightarrow A \cdot \text{adj}(A) = \det(A)\, I_n \Rightarrow \frac{1}{\det(A)} \cdot A\, \text{adj}(A) = I_n$$

$$\Rightarrow \boxed{\frac{1}{\det(A)} \cdot \text{adj}(A) = A^{-1}}$$

Örn

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$A^{-1} = ?$ ✓ $\text{adj}(A) = ?$ ✓ $\det(A) = ?$ ✓

$$A_{11} = \underset{+}{(-1)^{1+1}} \overset{6-4}{\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix}} = 2 \qquad A_{21} = \underset{-}{(-1)^{2+1}} \overset{3-4}{\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}} = 1 \qquad A_{31} = \underset{+}{(-1)^{3+1}} \overset{2-4}{\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}} = -2$$

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} \overset{6-4}{=} 2 \qquad A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \overset{3-4}{=} 1 \qquad A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \overset{2-4}{=} -2$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} \overset{9-2}{=} -7 \qquad A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} \overset{6-2}{=} 4 \qquad A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} \overset{4-6}{=} 2$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \overset{6-2}{=} 4 \qquad A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3 \qquad A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1$$

$$\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\operatorname{adj}(A) = \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 4 - 7 + 8 = 5$$
$$2.2 + 1.(-7) + 2.4$$

$$A^{-1} = \begin{bmatrix} 2/5 & 1/5 & -2/5 \\ -7/5 & 4/5 & 2/5 \\ 4/5 & -3/5 & 1/5 \end{bmatrix}$$

Cramer Kuralı

$Ax = b \rightarrow$ sonuç sütunu

A: katsayılar matrisi (kare matris)

x: değişkenler sütunu

A singüler olmayan bir matris (A^{-1} var)

A_i : A matrisinin i. sütununun b sütunu ile değiştirilmiş hali

$$\boxed{x_i = \frac{\det(A_i)}{\det(A)}}$$

Örn

$$x_1 + 2x_2 + x_3 = 5$$
$$2x_1 + 2x_2 + x_3 = 6$$
$$x_1 + 2x_2 + 3x_3 = 9$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\det(A) = 1.\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 4 - 10 + 2 = -4 \neq 0$$

$$A_1 = \begin{bmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{bmatrix} \qquad \det(A_1) = 5\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 6 & 1 \\ 9 & 3 \end{vmatrix} + 1\begin{vmatrix} 6 & 2 \\ 9 & 2 \end{vmatrix} = 20 - 18 - 6 = -4 \qquad x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-4}{-4} = 1$$

$$A_2 = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{bmatrix} \qquad \det(A_2) = 1.\begin{vmatrix} 6 & 1 \\ 9 & 3 \end{vmatrix} - 5\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 6 \\ 1 & 9 \end{vmatrix} = 9 - 25 + 12 = -4 \qquad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-4}{-4} = 1$$

$$A_3 = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{bmatrix} \qquad \det(A_3) = 1.\begin{vmatrix} 2 & 6 \\ 2 & 9 \end{vmatrix} - 2\begin{vmatrix} 2 & 6 \\ 1 & 9 \end{vmatrix} + 5\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 6 - 24 + 10 = -8 \qquad x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-8}{-4} = 2$$