7. Hafta Çarşamba Dersi

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$P_n \to$ derecesi n'den küçük polinomlar $\searrow$ vektör

Örn

$$P_3 = \{ \underbrace{a_0 + a_1 x + a_2 x^2}_{\text{tipik bir eleman}} : a_0, a_1, a_2 \in \mathbb{R} \}$$

$$a_0 + a_1 x + a_2 x^2 = \overset{\alpha_0}{a_0} \cdot 1 + \overset{\alpha_1}{a_1} \cdot x + \overset{\alpha_2}{a_2} \cdot x^2$$

$\Rightarrow \text{Span}\{1, x, x^2\} = P_3$

Örn

$\vec{v}_1 = 1 - x^2 \qquad \vec{v}_2 = 2 + x \qquad \vec{v}_3 = x^2$

P_3

$\mathbb{R}^{2\times 2}$

$\{v_1, v_2, v_3\}$, P_3 için bir geren küme midir?

$$a_0 + a_1 x + a_2 x^2 = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3$$

$\alpha_1, \alpha_2, \alpha_3$'ü a_0, a_1, a_2 cinsinden yazabiliyor muyuz?

Her $a_0 + a_1 x + a_2 x^2$ için $\alpha_1, \alpha_2, \alpha_3$ bulunabilir mi?

$$\to \quad a_0 + a_1 x + a_2 x^2 = \alpha_1 \cdot (1 - x^2) + \alpha_2 (2 + x) + \alpha_3 \cdot x^2$$

$$a_0 + a_1 x + a_2 x^2 = \alpha_1 - \alpha_1 x^2 + \alpha_2 \cdot 2 + \alpha_2 x + \alpha_3 x^2$$

$$\left.\begin{aligned} a_0 &= \alpha_1 + 2\alpha_2 \\ a_1 &= \alpha_2 \\ a_2 &= \alpha_3 - \alpha_1 \end{aligned}\right\}$$

$$\boxed{\begin{aligned} \alpha_1 + 2\overset{a_1}{\alpha_2} &= a_0 \\ \alpha_2 &= a_1 \\ \alpha_3 - \overset{a_0 - 2a_1}{\alpha_1} &= a_2 + a_0 \end{aligned}}$$

$\to \alpha_2 = a_1$

$\to \alpha_1 = a_0 - 2a_1$ ✓

$\to \alpha_3 = a_2 + a_0 - 2a_1$

$[\quad \vdots \quad] \to$

$\Rightarrow P_3 = \text{Span}\{1 - x^2, 2 + x, x^2\}$

Lineer Bağımsızlık

$\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ vektörleri için

$$\left[\ \underline{c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_n\vec{v_n} = \vec{0}} \text{ iken } \quad \underline{c_1 = c_2 = \ldots = c_n = 0} \text{ olmak zorunda ise}\ \right]$$

$\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ lineer bağımsızdır. (Tüm vektörlerin birbirinden bağımsız olması)

* Kümede birbirleri cinsinden yazılabilen vektörler varsa küme lineer bağımlıdır.

Örn

$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \vec{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \vec{v_3} = [$

$\{\vec{v_1}, \vec{v_2}\}$ lineer bağımsız mıdır?

$$\underbrace{\overset{?}{c_1}\vec{v_1} + \overset{?}{c_2}\vec{v_2} = \boxed{\vec{0}}} \Rightarrow \overset{?}{c_1 = c_2 = 0}$$

$$c_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ olsun.

işlemi yapın

$$\begin{bmatrix} c_1 + c_2 \\ c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_1 + c_2 = 0$$
$$c_1 + 2c_2 = 0$$
$$c_2 = 0 \quad c_1 = 0$$

$\Rightarrow \{\vec{v_1}, \vec{v_2}\}$ lineer bağımsızdır.

Örn

$\vec{e_1} = (1,0,0)^T$, $\vec{e_2} = (0,1,0)^T$, $\vec{e_3} = (0,0,1)^T$, $\vec{v_1} = (1,2,3)^T$

$$c_1\vec{e_1} + c_2\vec{e_2} + c_3\vec{e_3} + c_4\vec{v_1} = \vec{0}$$

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix} + c_4\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}c_1 + c_4\\c_2 + 2c_4\\c_3 + 3c_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \leftarrow \mathbb{R}^3$$

$$\boxed{\begin{aligned}c_1 + c_4 &= 0\\c_2 + 2c_4 &= 0\\c_3 + 3c_4 &= 0\end{aligned}}$$

$c_4 = r \in \mathbb{R}$
$c_1 = -r$
$c_2 = -2r$
$c_3 = -3r$

sonsuz çözüm

$\Rightarrow \{\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{v_1}\}$ lineer bağımsız değildir.

Örn

$$\vec{v_1} = \begin{bmatrix}1\\2\\3\end{bmatrix} \quad \vec{v_2} = \begin{bmatrix}-1\\0\\2\end{bmatrix} \quad \vec{v_3} = \begin{bmatrix}2\\1\\0\end{bmatrix}$$

$\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ lineer bağımsız mıdır?

$$c_1\begin{bmatrix}1\\2\\3\end{bmatrix} + c_2\begin{bmatrix}-1\\0\\2\end{bmatrix} + c_3\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}c_1 - c_2 + 2c_3\\2c_1 + c_3\\3c_1 + 2c_2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{aligned}c_1 - c_2 + 2c_3 &= 0\\2c_1 + c_3 &= 0\\3c_1 + 2c_2 &= 0\end{aligned}$$

$$\left[\begin{array}{ccc|c}1 & -1 & 2 & 0\\2 & 0 & 1 & 0\\3 & 2 & 0 & 0\end{array}\right] \xrightarrow[-3r_1+r_3\to r_3]{-2r_1+r_2\to r_2} \left[\begin{array}{ccc|c}1 & -1 & 2 & 0\\0 & 2 & -3 & 0\\0 & 5 & -6 & 0\end{array}\right]$$

$$\xrightarrow{1/2 r_2\to r_2} \left[\begin{array}{ccc|c}1 & -1 & 2 & 0\\0 & 1 & -3/2 & 0\\0 & 5 & -6 & 0\end{array}\right] \xrightarrow{-5r_2+r_3\to r_3} \left[\begin{array}{ccc|c}1 & -1 & 2 & 0\\0 & 1 & -3/2\,c_3 & 0\\0 & 0 & 3/2\,c_3 & 0\end{array}\right] \begin{array}{l}\rightarrow c_1 = 0\\\rightarrow c_2 = 0\\\rightarrow c_3 = 0\end{array}$$

$Ax = 0$

0 00

~~sonsuz~~ çözüm

$\Rightarrow \{v_1, v_2, v_3\}$ lineer bağımsızdır.

2. Determine whether the following vectors are linearly independent in $\mathbb{R}^3$:

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

4. Determine whether the following vectors are linearly independent in $\mathbb{R}^{2\times 2}$:

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

8. Determine whether the following vectors are linearly independent in P_3:

(a) $1, x^2, x^2 - 2$ **(b)** $2, x^2, x, 2x + 3$

(c) $x + 2, x + 1, x^2 - 1$ **(d)** $x + 2, x^2 - 1$