A, B: square, $n \times n$

$\underline{AB} = \underline{BA} = I_n$

Singular - Non-Singular Matrices

If A has an inverse → it is NON-singular. (A is invertible.)

The matrices which do not have a multiplicative inverse (Singular)

Elementary Matrices

$$I_n = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n \times n} \xrightarrow[!]{\text{Only 1 row operation}} \text{Elementary Matrix}$$

Identity Matrix

Row Operations

1) $r_i \leftrightarrow r_j$ $\qquad I_n \xrightarrow{r_i \leftrightarrow r_j}$ Type-I Elementary Matrix
2) $cr_i \to r_i$ $\qquad I_n \xrightarrow{cr_i \to r_i}$ Type-II " "
3) $cr_j + r_i \to r_i$ $\qquad I_n \xrightarrow{cr_j + r_i \to r_i}$ Type-III " "

Ex/

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4 \times 4} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E$$

→ is a Type-I elementary matrix

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5 \times 5} \qquad I_5 = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{bmatrix} \xrightarrow{3r_2 \to r_2} \begin{bmatrix} 1 & & & & 0 \\ & 3 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{bmatrix} = E$$

→ is a Type-II elementary matrix.

$$E = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3} \xrightarrow{2r_2 + r_1 \to r_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ is a Type III elementary matrix

** Any elementary matrix corresponds to the same type of row (column) operation.

$$A \xrightarrow{\text{row op.}} A' \quad \approx \quad EA = A'$$

Ex/

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{-3r_1 \to r_1} \begin{bmatrix} -3 & -6 & -9 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = A' \qquad E = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-3+0+0 \qquad -3 \cdot 2 + 0 + 0 \qquad -3 \cdot 3 + 0 + 0$

$$EA = A' \qquad \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} -3 & -6 & -9 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = A'$$

E →

Ex/

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{2r_1 + r_2 \to r_2} \begin{bmatrix} 1 & 2 & 3 \\ 6 & 9 & 12 \\ 7 & 8 & 9 \end{bmatrix} = A' \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

=

$$EA = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad A'$$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 9 & 12 \\ 7 & 8 & 9 \end{bmatrix} = A'$$

$2.1 + 1.4 + 0$
$2.2 + 1.5 + 0$
$2.3 + 1.6 + 0$

Multiplying any matrix A with E from the left → row operation
(" " " " " " from the right → column operation)

Ex

$$A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{3\times 3} \to \text{RREF}$$
REF

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A \text{ is not invertible.}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A \text{ is invertible}}$$

If you transform any square matrix to RREF → you either get Identity Matrix I_n (if it is invertible) or A is not invertible.

Ex

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow[E_1]{-3r_1 + r_2 \to r_2} \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}}_{=E_1 A} \xrightarrow[E_2]{-\frac{1}{2} r_2 \to r_2} \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{=E_2(E_1 A)} \to \text{REF}$$

$$\xrightarrow[E_3]{-2r_2 + r_1 \to r_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_n$$
$= E_3(E_2(E_1 A))$

$= A^{-1}$?

$E_3 E_2 E_1 A = I_n$

the multiplicative inverse of A.

$$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{E_3 E_2 E_1 I = A^{-1}}$$

$$E_2 E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3/2 & -1/2 \end{bmatrix}$$

$$E_3(E_2 E_1) = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = A^{-1}$$

* !

$$\left[\underbrace{A}_{\text{becomes RREF}} \;\middle|\; \underbrace{I}_{\text{becomes the inverse of } A} \right] \xrightarrow{E_1} \xrightarrow{E_2} \xrightarrow{E_3}$$

$$\to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow[-3r_1 + r_2 \to r_2]{E_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{2} r_2 \to r_2]{E_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

$$\xrightarrow{E_3} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \end{array}\right.$$

$$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
✓
$$A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow[]{\overset{E_3}{-2r_2+r_1\to r_1}} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

RREF of A $= I_2$

The inverse of A $= A^{-1}$

$A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Ex

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{c|c} A & I \end{array}\right] \Rightarrow$$

A → RREF $= I_n$

I → A^{-1} if possible.

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1/2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}r_1\to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1/2 & 0 & 0 \\ 1 & -1 & 1/2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-r_1+r_2\to r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & -1 & 0 & -1/2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-1r_2\to r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-1r_2+r_3\to r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1 & 0 \\ 0 & 0 & 0 & -1/2 & 1 & 1 \end{array}\right]$$

RREF ✓ $\neq I_3$

$\Rightarrow$ A is not invertible!

$\equiv$ A is singular!

$\equiv$ A^{-1} does not exist!

Ex

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} \qquad A^{-1} = ?$$

$= I_3$

$\neq I_3 \Rightarrow A^{-1}$ does not exist

A^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2r_1+r_3\to r_3]{-2r_1+r_2\to r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -2 & 1 & 0 \\ 0 & 0 & -3 & -2 & 0 & 1 \end{array}\right]$$

$$-\frac{1}{3}r_3 \to r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2/3 & 0 & -1/3 \end{array}\right]$$

$-\frac{4}{3}+1$

$0 \quad 0 \quad 5 \quad 10/3 \quad 0 \quad -5/3$
$0 \quad 1 \quad -5 \quad -2 \quad 1 \quad 0$

$$\begin{array}{l} -2r_3 + r_1 \to r_1 \\ 5r_3 + r_2 \to r_2 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 0 & 2/3 \\ 0 & 1 & 0 & 4/3 & 1 & -5/3 \\ 0 & 0 & 1 & 2/3 & 0 & -1/3 \end{array}\right]$$

$= I_3$ ✓ $\qquad = A^{-1}$

$$\Rightarrow A^{-1} = \begin{bmatrix} -1/3 & 0 & 2/3 \\ 4/3 & 1 & -5/3 \\ 2/3 & 0 & -1/3 \end{bmatrix} \checkmark$$

$\Rightarrow$ A is invertible.
A is non-singular.

$-1/3 + 0 + 2\cdot\frac{2}{3} = -\frac{1}{3}+\frac{4}{3} = \frac{3}{3} = 1$

$\underline{A}A^{-1} \overset{?}{=} I_3$

$$AA^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1/3 & 0 & 2/3 \\ 4/3 & 1 & -5/3 \\ 2/3 & 0 & -1/3 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \checkmark$$

$-\frac{2}{3} + \frac{4}{3} - \frac{2}{3} = 0$

$2\cdot 0 + 1\cdot 1 + -1\cdot 0$

$2\cdot\frac{2}{3} + 1\cdot -\frac{5}{3} + -1\cdot -\frac{1}{3}$

$\frac{4}{3} + \frac{-5}{3} + \frac{1}{3} = 0$

$1\cdot\frac{2}{3} + 0 + 2\cdot -\frac{1}{3} = 0$

$-\frac{2}{3} + 0 + \frac{2}{3} = 0$

$2\cdot\frac{2}{3} + 0 + 1\cdot -\frac{1}{3}$

$\frac{4}{3} - \frac{1}{3} = \frac{3}{3} = 1$

HW

SECTION 1.5 EXERCISES

1. Which of the matrices that follow are elementary matrices? Classify each elementary matrix by type.

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. Find the inverse of each matrix in Exercise 1. For each elementary matrix, verify that its inverse is an elementary matrix of the same type.

3. For each of the following pairs of matrices, find an elementary matrix E such that $EA = B$.

(a) $A = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 2 \\ 5 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 3 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ -2 & 4 & 5 \end{bmatrix}$

(c) $A = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ -2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ 0 & 3 & 5 \end{bmatrix}$

4. For each of the following pairs of matrices, find an elementary matrix E such that $AE = B$.

(a) $A = \begin{bmatrix} 4 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & 4 \\ 4 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$