Determinant

sadece Kare matrislerde

det : $A \longmapsto$ sayı (Matris)

$$\det(A) = |A| \in \mathbb{R}$$

$A \rightarrow$ 1x1'lik matris $\rightarrow \det(A) = a_{11}$

$A = [a_{11}]$

ex/ $A = [3]$ $\quad \det(A) = 3$

$A \rightarrow$ 2x2'lik matris $\rightarrow \det(A) = a_{11}a_{22} - a_{12}a_{21}$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

ex/ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad \det(A) = 1.4 - 2.3 = -2$

$A \rightarrow$ nxn'lik matris $(n \geq 3)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n}$$

Minör i,j $\rightarrow |M_{ij}| \rightarrow$ A matrisinin i. satır ve j. sütun silinerek elde edilen $(n-1)\times(n-1)$ matrisin determinantıdır.

ex/ $|M_{21}| = \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = 18 - 24 = -6$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$|M_{13}| = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 32 - 35 = -3$

Kofaktör i,j $\rightarrow A_{ij} \rightarrow \underbrace{(-1)}^{i+j} |M_{ij}|$

ex/ $A_{21} = \underbrace{(-1)^{2+1}}_{-} . \underbrace{|M_{21}|}_{(-6)} = 6$

$A_{13} = \underbrace{(-1)^{1+3}}_{+} . \underbrace{|M_{13}|}_{(-3)} = -3$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n}$$

Kofaktör Açılımı $\rightarrow \det(A)$

bir satır / bir sütun

$\det(A) = a_{11} . A_{11} + a_{12} . A_{12} + \dots + a_{1n} A_{1n} \rightarrow$ 1. satır kofaktör açılımı

$= a_{21} . A_{21} + a_{22} . A_{22} + \dots + a_{2n} A_{2n} \rightarrow$ 2. satır kofaktör açılımı

$\vdots$

$= a_{12} . A_{12} + a_{22} . A_{22} + \dots + a_{n2} A_{n2} \rightarrow$ 2. sütun kofaktör açılımı

i. satır Kofaktör Açılımı : $= a_{i1} A_{i1} + a_{i2} A_{i2} + \dots + a_{in} A_{in}$

j. sütun Kofaktör Açılımı : $= a_{1j} A_{1j} + a_{2j} A_{2j} + \dots + a_{nj} A_{nj}$

A matrisinin determinantı :

1) Herhangi bir satır / sütun seç. (en çok sıfırı olan)

2) Kofaktör açılımı yap = det (A)

$$\to \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ - & - & - \\ - & - & - \end{bmatrix} \to \det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$\det(A)$ ↓ 3 tane 2x2

$A_{11} \to (-1)^{1+1}|M_{11}|$, $A_{12} \to (-1)^{1+2}|M_{12}|$, $A_{13} \to (-1)^{1+3}|M_{13}|$

$$\begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}_{4\times4}$$

$$\det(A) = a_{11}\cancel{A_{11}} + a_{12}\cancel{A_{12}} + a_{13}A_{13} + a_{14}A_{14}$$

$a_{11} \to 0$, $a_{12} \to 0$

$\det(A)$ ↓ 4 tane 3x3

12 tane 2x2

$A_{11} \to (-1)^{1+1}|M_{11}|$ ↑ 3 tane 2x2

$A_{12} \to (-1)^{1+2}|M_{12}|$

Ex/

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 4 & 8 & 9 \end{bmatrix}$$

($1 = a_{11}$, $2 = a_{12}$, $3 = a_{13}$)

1.satır kofaktör açılımı ile determinantı bulalım.

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= 1.\underbrace{A_{11}}_{-2} + 2.\underbrace{A_{12}}_{-17} + 3.\underbrace{A_{13}}_{16}$$

$$\det(A) = -2 - 34 + 48 = \underline{12}$$

$$A_{11} = \underbrace{(-1)^{1+1}}_{+} \overbrace{\begin{vmatrix} 6 & 7 \\ 8 & 9 \end{vmatrix}}^{M_{11}} = -2$$

$6.9 - 7.8$

$$A_{12} = \underbrace{(-1)^{1+2}}_{-} \begin{vmatrix} 5 & 7 \\ 4 & 9 \end{vmatrix} = -17$$

$5.9 - 4.7$

$$A_{13} = \underbrace{(-1)^{1+3}}_{+} \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$

$$\left(-5 \cdot \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 6 \begin{vmatrix} 1 & 3 \\ 4 & 9 \end{vmatrix} - 7 \begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} \right) \to 2.\text{satır}$$

-6 $\quad$ $9-12$ $\quad$ $8-8$

-3 $\quad$ 0

$30 \quad -18 \quad = 12$

Ex/

$$A = \begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{bmatrix}$$

($0 \to a_{11}$, $0 \to a_{21}$, $0 \to a_{31}$, $2 \to a_{41}$)

$$\det(A) = a_{11}A_{11} + a_{21}.A_{21} + a_{31}A_{31} + a_{41}A_{41}$$

$$= \cancel{0.A_{11}} + \cancel{0.A_{21}} + \cancel{0.A_{31}} + 2.\underbrace{A_{41}}_{6} = \underline{12}$$

$$A_{41} = (-1)^{4+1} \overbrace{\begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix}}^{M_{41}} = (-1).(-6) = 6$$

($0 \to b_{13}$, $0 \to b_{23}$, $3 \to b_{33}$; B)

$$= b_{..}.(-1)^{1+3}\cancel{|4\ 5|} + b_{..}.(-1)^{2+3}\cancel{|2\ 3|} + b_{..}.(-1)^{3+3}.|2\ 3|$$

$$= b_{13}\cdot(-1)^{1+3}\cdot\begin{vmatrix}4&5\\1&0\end{vmatrix} + b_{23}\cdot(-1)^{2+3}\cdot\begin{vmatrix}2&3\\1&0\end{vmatrix} + b_{33}\cdot(-1)^{3+3}\cdot\underbrace{\begin{vmatrix}2&3\\4&5\end{vmatrix}}_{10-12}$$

$$= 3.(-2) = -6$$

* A diyagonal matris ise ;

$$A = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_n \end{bmatrix} \Rightarrow \det(A) = d_1 . d_2 . \dots . d_n$$

Örn

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\det(A) = 2.(-1)^{1+1}.\underbrace{|M_{11}|}_{} + 0 + 0 + 0$$

$$\begin{vmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 3.(-1)^{1+1} \underbrace{\begin{vmatrix} 4 & 0 \\ 0 & 5 \end{vmatrix}}_{4.5}$$

$$= 2.\ 3.\ 4.\ 5 \quad //$$

* A üçgensel matris ise ;

(üst/alt farketmez)

$$A = \begin{bmatrix} d_1 & - & - \\ & d_2 & - \\ 0 & & \ddots \; d_n \end{bmatrix} \qquad A = \begin{bmatrix} d_1 & & 0 \\ - & d_2 & \\ - & - & \ddots \; d_n \end{bmatrix} \Rightarrow \boxed{\det(A) = d_1 . d_2 \dots d_n}$$

Örn

$$A = \begin{bmatrix} 2 & 99 & -11 & -1 \\ 0 & 3 & 22 & 102 \\ 0 & 0 & 4 & 33 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

a_{41} a_{42} a_{43} a_{44}

$$\det(A) = 0 + 0 + 0 + 5 . A_{44}$$

$$A_{44} = (-1)^{4+4} \overbrace{\begin{vmatrix} 2 & 99 & -11 \\ 0 & 3 & 22 \\ 0 & 0 & 4 \end{vmatrix}}^{M_{44}}$$

$$= 0 + 0 + 4 . (-1)^{3+3} \underbrace{\begin{vmatrix} 2 & 99 \\ 0 & 3 \end{vmatrix}}_{2.3}$$

$$= 5.\ 4.\ 2.\ 3 \quad //$$

Örn

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -1 & 5 & 0 & 0 \\ 2 & 3 & 7 & 0 \\ 4 & 5 & 6 & 10 \end{bmatrix}$$

$$\det(A) = 3.(-1)^{1+1} \begin{vmatrix} 5 & 0 & 0 \\ 3 & 7 & 0 \\ 5 & 6 & 10 \end{vmatrix}$$

$$= 5.(-1)^{1+1} \underbrace{\begin{vmatrix} 7 & 0 \\ 6 & 10 \end{vmatrix}}_{7.10}$$

$$= 3.\ 5.\ 7.\ 10 \quad //$$

Determinantın Özellikleri

* $\det(A) = \det(A^T)$

* Eğer A matrisinin tamamen 0 olan bir satır (veya sütun) varsa; $\det(A) = 0$

! → * $\det(AB) = \det(A).\det(B)$

→ * E_1 : 1. tip elementer matris $(r_i \leftrightarrow r_j)$ ⇒ $\boxed{\mathrm{Det}(E_1) = -1}$

E_2 : 2. tip elementer matris $(kr_i \to r_i)$ ⇒ $\boxed{\mathrm{Det}(E_2) = k}$

E_3 : 3. tip elementer matris $(k.r_j + r_i \to r_i)$ ⇒ $\boxed{\mathrm{Det}(E_3) = 1}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ -1

$\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & k \end{bmatrix}$

$\begin{bmatrix} 1 & & k & \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$

* → Eğer A matrisine 1. tip satır operasyonu yapılırsa; → E_1A

$$\det(E_1A) = \underbrace{\det(E_1)}_{-1}.\det(A) = -\det(A)$$

→ Eğer A matrisine 2. tip satır operasyonu yapılırsa; → E_2A

$$\det(E_2A) = \underbrace{\det(E_2)}_{k}.\det(A) = k.\det(A)$$

→ Eğer A matrisine 3. tip satır operasyonu yapılırsa; → E_3A

$$\det(E_3A) = \underbrace{\det(E_3)}_{1}.\det(A) = \det(A)$$

ödev

Rastgele bir kare matris alın. $= A$

$\det(A) = ?$

Rastgele bir satır op. uygulayın. $= A'$ (↳ tip ?)

$A \xrightarrow{E} A'$, $\det(A') = ?$

$A' \xrightarrow{E'} A''$, $\det(A'') =$

$$A \xrightarrow{E} \underset{EA}{A'} \xrightarrow{E'} \underset{E'(EA)}{A''} \xrightarrow{E''} \underset{E''(E'(EA))}{A'''}$$

$$\det(A''') = \det(E'')\underbrace{\det(E')\underbrace{\det(E)\det(A)}_{\det(A')}}_{\det(A'')}$$

(A) A (kare) - - - - - - → I_n / I_n olmazsa $\begin{bmatrix} 1 & 0 & 0 \\ & & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\det(A) = ✓$; $\det = 1$; $\det = 0$

A^{-1}

(kare)

$\underline{\det = 1}$ $\boxed{\det = 0}$

$\det(A) = \checkmark$