7th Week Wednesday

07 Nisan 2021 Çarşamba 12:34

$P_n \rightarrow$ polynomials with degree less than n

Ex $\underbrace{P_3 = \{ a_0 + a_1 x + a_2 x^2}_{\text{a typical vector in } P_3} \; : \; a_0, a_1, a_2 \in \mathbb{R} \}$

$$\underbrace{a_0 + a_1 x + a_2 x^2}_{\checkmark} = \underset{v_1}{a_0 \cdot 1} + \underset{v_2}{a_1 \cdot x} + \underset{v_3}{a_2 x^2}$$

$\Rightarrow P_3 = \text{Span}\{1, x, x^2\}$

Ex $\vec{v_1} = 1 - x^2 \qquad \vec{v_2} = 2 + x \qquad \vec{v_3} = x^2$

Is $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ a spanning set for P_3 ?

$P_3 \rightarrow a_0 + a_1 x + a_2 x^2$

$\longrightarrow a_0 + a_1 x + a_2 x^2 = \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \alpha_3 \vec{v_3}$

$\rightarrow$ Can we find $\alpha_1, \alpha_2, \alpha_3$ in terms of a_0, a_1, a_2 ?

Can we find a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$ for any $a_0 + a_1 x + a_2 x^2$?

$$a_0 + a_1 x + a_2 x^2 = \alpha_1 \cdot (\underbrace{1 - x^2}_{v_1}) + \alpha_2 \cdot (\underbrace{2 + x}_{v_2}) + \alpha_3 \underbrace{x^2}_{v_3}$$

$$a_0 + a_1 x + a_2 x^2 = \alpha_1 - \alpha_1 x^2 + 2\alpha_2 + \alpha_2 x + \alpha_3 x^2$$

$$\begin{array}{ccc|c} \alpha_1 & \alpha_2 & \alpha_3 & \\ 1 & 2 & 0 & a_0 \\ 0 & 1 & 0 & a_1 \\ -1 & 0 & 1 & a_2 \end{array}$$

$a_0 = \alpha_1 + 2\alpha_2^{a_1}$ $\qquad \alpha_1 = ?$

$\rightarrow \boxed{a_1 = \alpha_2}$ $\qquad \alpha_2 = ?$

$\rightarrow a_2 = -\underset{a_0 - 2a_1}{\alpha_1} + \alpha_3$ $\qquad \alpha_3 = ?$

$$\rightarrow \boxed{\begin{aligned} \alpha_2 &= a_1 \\ \alpha_1 &= a_0 - 2a_1 \\ \alpha_3 &= a_2 + a_0 - 2a_1 \end{aligned}}$$

$\Rightarrow \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is a spanning set for P_3.

$\Rightarrow P_3 = \text{Span}\{1 - x^2, 2 + x, x^2\}$

Linear Independence

For $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$

If $c_1 \vec{v_1} + c_2 \vec{v_2} + \dots + c_n \vec{v_n} = \vec{0} \Rightarrow \underbrace{c_1 = c_2 = \dots = c_n = 0}_{\checkmark}$

linear independence

$\boxed{A\underset{c_1, c_2, \dots, c_n}{x} = 0}$

$\swarrow$ trivial ✓ $\qquad \searrow$ infinitely many ✗

If $\underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}}_{\uparrow} \Rightarrow \underbrace{c_1 = c_2 = \dots = c_n = 0}_{\checkmark}$

Then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent. ✓

* If you can write any vector in the set as a linear combination of some other vectors in the same set,

$\Rightarrow$ this set is linearly dependent.

Ex/ $\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$ if $v_2 = 2v_1 \Rightarrow \vec{v}_1, \vec{v}_2, \vec{v}_3 \rightarrow$ is this set dependent.

Ex/ $\underbrace{\underset{e_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{e_2}{\begin{bmatrix}0\\1\end{bmatrix}}}, \underset{v}{\begin{bmatrix}2\\3\end{bmatrix}} \in \mathbb{R}^2$

$\begin{bmatrix}2\\3\end{bmatrix}$ can be written as a linear combination of e_1 and e_2.

$\{e_1, e_2, v\}$ is a linearly dependent set.

Ex/ $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix}1\\2\end{bmatrix}$

$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0} \Rightarrow \begin{matrix}c_1 = ?\\c_2 = ?\end{matrix}$

$\rightarrow c_1 = c_2 = 0 \Rightarrow$ v_1, v_2 lin. indep

$\rightarrow$ inf. many solns. $\Rightarrow$ v_1, v_2 not lin indep.

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1 + c_2\\c_1 + 2c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\boxed{\begin{aligned}c_1 + c_2 &= 0\\ - \quad c_1 + 2c_2 &= 0\end{aligned}}$$

$c_2 = 0 \Rightarrow c_1 = 0$

$\boxed{c_1 = c_2 = 0}$ ✓

$\rightarrow \left[\begin{matrix}1 & 1\\1 & 2\end{matrix} \,\middle|\, \begin{matrix}0\\0\end{matrix}\right] \xrightarrow{REF}$ $\boxed{Ax = 0}$ $\rightarrow$ trivial $\rightarrow$ ~~inf. many~~

$\Rightarrow \{v_1, v_2\}$ is a linearly independent set.

Ex/ $\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ $\in \mathbb{R}^3$

$\rightarrow \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{v}\}$

$\underline{c_1\vec{e} + c_2\vec{e}_2 + c_3\vec{e}_3 + c_4\vec{v} = \vec{0}}$

$c_1 = ?$
$c_2 = ?$
$c_3 = ?$
$c_4 = ?$

$$c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix} + c_4\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix} c_1 + c_4 \\ c_2 + 2c_4 \\ c_3 + 3c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 + c_4 = 0$
$c_2 + 2c_4 = 0$
$c_3 + 3c_4 = 0$

$c_4 = r \in \mathbb{R}$
$c_1 = -r$
$c_2 = -2r$
$c_3 = -3r$

$\longrightarrow \left[\quad \mid \quad \right] \rightarrow$ REF

$\Rightarrow$ inf. many solutions.

$\Rightarrow \{\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{v}\}$ is not a linearly independent set.

Ex

$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \quad \vec{v_3} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Are $\vec{v_1}, \vec{v_2}, \vec{v_3}$ linearly independent?

$$c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{0}$$

$c_1 = ?$
$c_2 = ?$
$c_3 = ?$

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 - c_2 + 2c_3 \\ 2c_1 + c_3 \\ 3c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} c_1 - c_2 + 2c_3 &= 0 \\ 2c_1 + c_3 &= 0 \\ 3c_1 + 2c_2 &= 0 \end{aligned}}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ \textcircled{2} & 0 & 1 & 0 \\ \textcircled{3} & 2 & 0 & 0 \end{array}\right] \xrightarrow[-3r_1 + r_3 \to r_3]{-2r_1 + r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & \textcircled{2} & -3 & 0 \\ 0 & 5 & -6 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & \textcircled{5} & -6 & 0 \end{array}\right] \xrightarrow{-5r_2 + r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & \textcircled{3/2} & 0 \end{array}\right]$$

$+\frac{15}{2} - 6$

$$\xrightarrow{\frac{2}{3}r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & -1 \checkmark & 2 \checkmark & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$1c_3 = 0 \Rightarrow c_3 = 0$
$c_2 - 3/2 \underset{0}{c_3} = 0 \Rightarrow c_2 = 0$
$\Rightarrow c_1 = 0$ ✓

$\Rightarrow \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is a linearly independent set.

2. Determine whether the following vectors are linearly independent in $\mathbb{R}^3$:

(a) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $c_1, c_2\ c_3$?

(b) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}$

(c) $\begin{bmatrix}2\\1\\-2\end{bmatrix}, \begin{bmatrix}3\\2\\-2\end{bmatrix}, \begin{bmatrix}2\\2\\0\end{bmatrix}$

(d) $\begin{bmatrix}2\\1\\-2\end{bmatrix}, \begin{bmatrix}-2\\-1\\2\end{bmatrix}, \begin{bmatrix}4\\2\\-4\end{bmatrix}$

(e) $\begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}0\\2\\1\end{bmatrix}$

$$\begin{tikzcd} & Ax=0 \arrow[dl] \arrow[dr] & \\ \text{trivial} & & \text{infmany} \\ \checkmark & & \times \end{tikzcd}$$

$c_1v_1 + c_2v_2 + c_3v_3 = 0$

8. Determine whether the following vectors are linearly independent in P_3:

(a) $1, x^2, x^2 - 2$ $= 0$

(b) $2, x^2, x, 2x + 3$

(c) $x + 2, x + 1, x^2 - 1$

(d) $x + 2, x^2 - 1$

4. Determine whether the following vectors are linearly independent in $\mathbb{R}^{2\times 2}$:

(a) $c_1\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} + c_2, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ $c_1 = ?$ $c_2 = ?$

(b) $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$

(c) $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}2 & 3\\0 & 2\end{bmatrix}$