6. Hafta Pzt Dersi Telafisi

03 Nisan 2021 Cumartesi 11:32

$(V, +, \cdot)$ $\quad r \in \mathbb{R}$ $\quad \vec{v}_1 + \vec{v}_2 \in V$, $r\vec{v}_1 \in V$ → kapalılık

1) $\vec{v}_1 + \vec{v}_2 = \vec{v}_2 + \vec{v}_1$ (+ değişme)

2) $(\vec{v}_1 + \vec{v}_2) + \vec{v}_3 = \vec{v}_1 + (\vec{v}_2 + \vec{v}_3)$ (+birleşme)

3) $\vec{0} \in V$ ✓ $\quad \boxed{\vec{v}_1 + \vec{0} = \vec{0} + \vec{v}_1 = \vec{v}_1}$ (+ etkisiz (birim eleman))

4) $\forall \vec{v}_1 \in V \quad -\vec{v}_1 \in V \quad \vec{v}_1 + (-\vec{v}_1) = \vec{0}$ (+ ters eleman)

5) $r \in \mathbb{R} \quad r(\vec{v}_1 + \vec{v}_2) = r\vec{v}_1 + r\vec{v}_2$ (skaler çarpmanın toplama üzerine dağılma özelliği)

6) $r, s \in \mathbb{R} \quad (r+s)\vec{v}_1 = r\vec{v}_1 + s\vec{v}_2$

7) $r, s \in \mathbb{R} \quad (r.s).\vec{v}_1 = r.(s.\vec{v}_1)$

8) $1 \in \mathbb{R} \quad 1.\vec{v}_1 = \vec{v}_1$ (skaler çarpmanın birim elemanı)

Örn $(\mathbb{R}^{m\times n}, +, \cdot)$ vektör uzayı belirtir.

- $\mathbb{R}^{m\times n}$ → $m\times n$ tipinde matrisler (girdileri $\mathbb{R}$) ← vektör
- $+$ → matris toplama
- $\cdot$ → skaler çarpma

$\mathbb{R}^{2\times 2}$ 'de

$\alpha\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} d & e \\ f & g \end{bmatrix}\right)$ ✓

$A \in \mathbb{R}^{m\times n}$

1) $[\quad]_{m\times n} + [\quad]_{m\times n}$ ✓

2) $(A+B)+C = A+(B+C)$ ✓

3) $A + \vec{0} = A \quad \vec{0} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & & \end{bmatrix}_{m\times n}$ ✓

4) $\forall A \in \mathbb{R}^{m\times n}$

$A + \_ = \vec{0} \rightarrow -1.A \in \mathbb{R}^{m\times n}$ ✓

5) $\alpha(A+B) = \alpha A + \alpha B$

6) $(r+s)A = rA + sA$

7) $(r.s)A = r(sA)$

8) $1.A = A$ ✓

Örn P_n : derecesi n'den küçük polinomlar kümesi

→ $(P_n, +, \cdot)$ vektör uzayıdır.

- P_n → polinom → vektör
- $+$ → polinom toplama
- $\cdot$ → skaler çarpma

$P_3 \quad x^2+2x+1 \in P_3$ vektördür.

$x+3 \in P_3$ vektördür.

$$P_n = \{ a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} : a_i \in \mathbb{R} \}$$

1.) $p(x), q(x) \in P_n \quad p(x) + q(x) = q(x) + p(x)$ ✓

2.) $p(x), q(x), r(x) \in P_n \quad (p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$ ✓

3.) $\forall p(x) \in P_n \quad p(x) + \vec{0} = p(x) \quad \vec{0} = 0$ sabit polinomudur. ✓

4.) $\forall p(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} \quad p(x) + ? = 0 \Rightarrow -p(x) = -a_0 - a_1x - \dots - a_{n-1}x^{n-1}$

ters eleman ✓

5.) $\alpha \in \mathbb{R} \quad p(x), q(x) \in P_n \quad \alpha(p(x) + q(x)) = \alpha p(x) + \alpha q(x)$ ✓

6.) $\alpha, \beta \in \mathbb{R} \quad (\alpha + \beta)p(x) = \alpha p(x) + \beta p(x)$ ✓

7.)

7.) $\alpha, \beta \in \mathbb{R}$ $\quad (\alpha . \beta)\ p(x) = \alpha\ (\beta\ p(x))$ ✓

8.) $1 \in \mathbb{R}$ $\quad p(x) \quad 1.p(x) = p(x)$ ✓

9. $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R}, x \leq 0 \right\}$ kümesi bilinen matris toplaması ve skaler çarpımı için bir vektör uzayı olup olmadığını denetleyiniz.

$(x,y) \quad x \leq 0$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \boxed{x_1 + x_2} \\ y_1 + y_2 \end{bmatrix} \in V \checkmark$$

$x_1 \leq 0 \quad x_2 \leq 0 \quad x_1 + x_2 \leq 0$

$\alpha \in \mathbb{R}$

$\alpha = -2$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \alpha \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \end{bmatrix} \notin V$$

$x_1 \leq 0 \quad \underset{6}{\alpha x_1} \leq 0\ ?$

-3

$\Rightarrow$ Skaler çarpma altında kapalı değildir.

8. $V = \mathbb{R}^+$ kümesi için $x_1 \oplus x_2 = 2x_1 - x_2$ ve $c \odot x_1 = cx_1$ işlemleri tanımlansın. V kümesinin bir vektör uzayı olup olmadığını denetleyiniz.

$V = (0, \infty) \rightarrow x_1 \oplus x_2 = 2x_1 - x_2$

$c \odot x_1 = cx_1$ ✓ (bildiğimiz skaler çarpma)

$x_1, x_2 \in \mathbb{R}^+$

$x_1 \oplus x_2 = 2x_1 - x_2 \in \mathbb{R}^+$?

$x_1 = 1 \quad x_2 = 8$

$1 \oplus 8 = 2.1 - 8 = -6 \notin \mathbb{R}^+$

$\Rightarrow$ Vektörel toplama kapalı değildir.

Let V be the set of all ordered pairs of real numbers with addition defined by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$ (bildiğimiz vektörel toplama)

and scalar multiplication defined by

$$\alpha \circ (x_1, x_2) = \boxed{(\alpha x_1, x_2)}$$!

$V = \{ (x_1, y_1) : x_1, y_1 \in \mathbb{R} \}$

$\alpha.(x_1, x_2) = (\underset{\in \mathbb{R}}{\alpha x_1}, \underset{\in \mathbb{R}}{x_2}) \in V$ ✓

1, 2, 3, 4 ✓

5) $\alpha \in \mathbb{R}$ $\quad \alpha\left((x_1, y_1) + (x_2, y_2)\right) = \alpha\ (x_1 + x_2, y_1 + y_2) = (\alpha x_1 + \alpha x_2, y_1 + y_2)$

$\checkmark = ($

$\alpha (x_1, y_1) + \alpha (x_2, y_2) = (\alpha x_1, y_1) + (\alpha x_2, y_2) = (\alpha x_1 + \alpha x_2, y_1 + y_2)$

6) $\alpha, \beta \in \mathbb{R}$ $\quad \rightarrow (\alpha + \beta)\ (x_1, y_1) = ((\alpha + \beta) x_1, y_1) = (\alpha x_1 + \beta x_1, y_1)$

✗ $\quad ? = ($

$\alpha (x_1, y_1) + \beta (x_1, y_1) = (\alpha x_1, y_1) + (\beta x_1, y_1) = (\alpha x_1 + \beta x_1, 2y_1)$

normal skaler çarpma

$(\alpha + \beta)\ (x_1, y_1) = ((\alpha + \beta) x_1, (\alpha + \beta) y_1)$ $= \checkmark$

$\alpha (x_1, y_1) + \beta (x_1, y_1) = (\alpha x_1, \alpha y_1) + (\beta x_1, \beta y_1) = (\alpha x_1 + \beta x_1, \alpha y_1 + \beta y_1)$

V, bu işlemlerle birlikte bir vektör uzayı belirtmez.

8+2 ✓

$8+2$ ✓

Altuzaylar

$S \subseteq V \rightarrow$ vektör uzayı $\quad (V, +, \cdot)$ ✓

alt küme $\quad \vec{0}$

0) $\vec{0}_V \in S$

I) $\forall \alpha \in \mathbb{R}$, $\forall \vec{s} \in S$ için $\alpha \vec{s} \in S$

II) $\forall \vec{s_1}, \vec{s_2} \in S$ için $\vec{s_1} + \vec{s_2} \in S$

şartları var ise S, V'nin altuzayıdır. $\quad S \leqslant V$

Örn $\mathbb{R}^2 = \{ (x_1, y_1) : x_1, y_1 \in \mathbb{R} \}$ vektör uzayı için $(\mathbb{R}^2, +, \cdot)$

$S = \{ (x_1, y_1) : \boxed{y_1 = 2x_1}, \; x_1, y_1 \in \mathbb{R} \} \qquad S \subseteq \mathbb{R}^2$

0) $\vec{0}_{\mathbb{R}^2} = (0,0) \qquad 0 = 2.0$ ✓. $\Rightarrow \vec{0}_{\mathbb{R}^2} \in S$ ✓

I) $\forall \alpha \in \mathbb{R} \qquad (x_1, y_1) \in S \Rightarrow \boxed{y_1 = 2x_1}$

$\alpha (x_1, y_1) = (\alpha x_1, \alpha y_1) \in S \qquad \cancel{\alpha} y_1 \overset{\checkmark}{=} 2 \cancel{\alpha} x_1$

II) $(x_1, y_1) \in S \Rightarrow y_1 = 2x_1$

$(x_2, y_2) \in S \Rightarrow y_2 = 2x_2$

$y_1 + y_2 = 2x_1 + 2x_2$

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$= (x_1 + x_2, 2x_1 + 2x_2) \in S$ ✓

Sonuç olarak $S \leqslant V$ (alt uzaydır)

Örn $\mathbb{R}^2 = \{ (x_1, y_1) : x_1, y_1 \in \mathbb{R} \}$ vektör uzayı için $(+, \cdot)$ ✓

$S = \{ (x, 1) : x \in \mathbb{R} \}$

✗ $\rightarrow \vec{0} \notin S \qquad (0,0) \notin S \qquad S$, $\mathbb{R}^2$'nin bir alt uzayı değildir.

✗ $\rightarrow \alpha (x, 1) = (\alpha x, \alpha) \notin S \quad \alpha \neq 1$ ise

✗ $\rightarrow (x,1) + (y,1) = (x+y, 2) \notin S$

3. Determine whether the following are subspaces of $\mathbb{R}^{2\times 2}$:
   (a) The set of all 2×2 diagonal matrices
   (b) The set of all 2×2 triangular matrices
   (c) The set of all 2×2 lower triangular matrices
   (d) The set of all 2×2 matrices A such that $a_{12} = 1$
   (e) The set of all 2×2 matrices B such that $b_{11} = 0$
   (f) The set of all symmetric 2×2 matrices
   (g) The set of all singular 2×2 matrices

$\rightarrow \mathbb{R}^{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\} \qquad , +, \cdot$

vektörel toplama $\quad$ skaler çarpma

$S_1 = \left\{ \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} : d_1, d_2 \in \mathbb{R} \right\}$

0) $\vec{0}_{\mathbb{R}^{2\times 2}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S_1$ ✓

a)

(g) The set of all singular 2 × 2 matrices

g)

0) $\vec{0}_{\mathbb{R}^{2\times 2}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S_1$ ✓

1) $\forall \alpha \in \mathbb{R}$ $\forall \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \in S$ $\alpha \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} \alpha d_1 & 0 \\ 0 & \alpha d_2 \end{bmatrix} \in S_1$ ✓

2) $\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} \in S$ $\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} + \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} = \begin{bmatrix} d_1+e_1 & 0 \\ 0 & d_2+e_2 \end{bmatrix} \in S_1$ ✓

$S_1 \leqslant \mathbb{R}^{2\times 2}$ 'nin alt uzayıdır.

e) $S = \left\{ \begin{bmatrix} 0 & a \\ b & c \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$ S, $\mathbb{R}^{2\times 2}$ 'nin alt uzayı mıdır?

0) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ ✓

1) $\alpha \begin{bmatrix} 0 & a \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & \alpha a \\ \alpha b & \alpha c \end{bmatrix} \in S$ ✓

2) $\begin{bmatrix} 0 & a \\ b & c \end{bmatrix} + \begin{bmatrix} 0 & d \\ e & f \end{bmatrix} = \begin{bmatrix} 0 & a+d \\ b+e & c+f \end{bmatrix} \in S$ ✓

$S \leqslant \mathbb{R}^{2\times 2}$ ✓

2. Determine whether the following sets form subspaces of $\mathbb{R}^3$:

(a) $\{(x_1, x_2, x_3)^T \mid x_1 + x_3 = 1\}$

(b) $\{(x_1, x_2, x_3)^T \mid x_1 = x_2 = x_3\}$

→ (c) $\{(x_1, x_2, x_3)^T \mid x_3 = x_1 + x_2\}$

(d) $\{(x_1, x_2, x_3)^T \mid x_3 = x_1 \text{ or } x_3 = x_2\}$

$\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x,y,z \in \mathbb{R} \right\}$ vektör uzayı ✓

a) $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x+z=1 \right\}$

0) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin S$ $S \not\leqslant \mathbb{R}^3$

b) $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x=y=z \right\}$

0) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S$ $0=0=0$ ✓

I) $\forall \alpha \in \mathbb{R}$ $\begin{bmatrix} x \\ x \\ x \end{bmatrix} \in S$ $\alpha \begin{bmatrix} x \\ x \\ x \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha x \\ \alpha x \end{bmatrix} \in S$ $\alpha x = \alpha x = \alpha x$ ✓

II) $\begin{bmatrix} x \\ x \\ x \end{bmatrix}, \begin{bmatrix} y \\ y \\ y \end{bmatrix} \in S$ $\begin{bmatrix} x \\ x \\ x \end{bmatrix} + \begin{bmatrix} y \\ y \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \\ x+y \end{bmatrix} \in S$ $x+y = x+y = x+y$ ✓

$S \leqslant \mathbb{R}^3$

c) $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : z = x+y \right\}$

0) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S$ $0=0+0$ ✓

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

I) $\forall \alpha \in \mathbb{R}$ $\quad \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} \in S$ $\quad \alpha \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha y \\ \alpha x + \alpha y \end{bmatrix} \in S$ $\quad \alpha(x+y) = \alpha x + \alpha y$ ✓ ✓

II) $\begin{bmatrix} x \\ y \\ x+y \end{bmatrix}, \begin{bmatrix} a \\ b \\ a+b \end{bmatrix} \in S$ $\quad \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} + \begin{bmatrix} a \\ b \\ a+b \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ x+y+a+b \end{bmatrix} \in S$ ✓ $\Rightarrow x+y+a+b = x+a+y+b$ ✓

$S \leqslant \mathbb{R}^3$

d) $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : z = x \text{ veya } z = y \right\}$ $\quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S$

$= \left\{ \underbrace{\begin{bmatrix} x \\ y \\ x \end{bmatrix}}_{1}, \underbrace{\begin{bmatrix} a \\ b \\ b \end{bmatrix}}_{2} : x, y, a, b \in \mathbb{R} \right\}$

I) $\alpha \begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha y \\ \alpha x \end{bmatrix} \in S$

$\alpha \begin{bmatrix} x \\ y \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha y \\ \alpha y \end{bmatrix} \in S$

II) $\begin{bmatrix} x \\ y \\ x \end{bmatrix} + \begin{bmatrix} a \\ b \\ b \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ x+b \end{bmatrix} \notin S$ $\quad \Rightarrow S \not\leqslant \mathbb{R}^3$

Sıfırlık Uzayı

$N(A)$: $A_{m \times n}$ matrisinin sıfırlık uzayı
Null Space

$N(A) = \underset{m \times n}{A} \underset{n \times 1}{x} = 0$ 'ın tüm çözümlerinden oluşan küme

$$N(A) = \{ x \in \mathbb{R}^n : \underline{Ax = 0} \} \qquad N(A) \leqslant \mathbb{R}^n$$

0) $\vec{0}_{\mathbb{R}^n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1}$ $\quad \left. \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ \vdots \\ x_n = 0 \end{matrix} \right\} \rightarrow$ $Ax = 0$ 'ın trivial çözümüdür. ✓ $\quad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \in N(A)$

1) $\forall \alpha \in \mathbb{R}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in N(A)$ $\quad \underline{Ax = 0}$ ✓

$A(\alpha x) = \alpha (Ax) = 0$ ✓

$\alpha x \in N(A)$ ✓

2) $x \in N(A)$ $\quad y \in N(A)$
$Ax = 0$ $\quad Ay = 0$ $\quad \Big\} \Rightarrow A(x+y) = Ax + Ay = 0_n$ ✓

Algebraic Rules

The following theorem provides some useful rules for doing matrix algebra.

Each of the following statements is valid for any scalars α and β and for any matrices A, B, and C for which the indicated operations are defined.

1. $A + B = B + A$ ✓
2. $(A + B) + C = A + (B + C)$ ✓
3. $(AB)C = A(BC)$
4. $A(B + C) = AB + AC$
5. $(A + B)C = AC + BC$
6. $(\alpha\beta)A = \alpha(\beta A)$ ✓
7. $\alpha(AB) = (\alpha A)B = A(\alpha B)$
8. $(\alpha + \beta)A = \alpha A + \beta A$ ✓
9. $\alpha(A + B) = \alpha A + \alpha B$ ✓

Algebraic Rules for Transposes

1. $(A^T)^T = A$
2. $(\alpha A)^T = \alpha A^T$
3. $(A + B)^T = A^T + B^T$ ✓
4. $(AB)^T = B^T A^T$ ✓

$(A^{-1})^{-1} = A$

$(AB)^{-1} = B^{-1} A^{-1}$