9th Week Wednesday

21 Nisan 2021 Çarşamba 12:33

$\mathbb{R}^2 \not\leq \mathbb{R}^3$
$(-,-) \quad (-,-,-)$

Ex

$\rightarrow S = \{ (\underline{a+b}, \underline{a-b+2c}, \underline{b}, \underline{c}) : a, b, c \in \mathbb{R} \}$

$$\boxed{\begin{array}{c} S \leq \mathbb{R}^4 \\ \dim(S) \leq 4 \\ ? \end{array}}$$

Find a basis for S. $\dim(S) = ?$

- lin. indep. ✓
- spanning ✓

a typical vector in S $\rightarrow$

$$\begin{bmatrix} a+b \\ a-b+2c \\ b \\ c \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + b \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{v_2} + c \underbrace{\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{v_3}$$

$\Rightarrow \{v_1, v_2, v_3\}$ is a spanning set for S. ✓

Are they linearly independent?

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{4\times 3}}_{X}$$

$$\rightarrow c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$Xc = 0 \quad c = ?$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{ref}$$

$c_1 + c_2 = 0$
$c_1 - c_2 + 2c_3 = 0$
$\underline{c_2 = 0}$
$\underline{c_3 = 0}$ $\Big\} \Rightarrow c_1 = 0$

$c_1 = c_2 = c_3 = 0$ ✓ $\Rightarrow \{v_1, v_2, v_3\}$ is linearly independent. ✓

$\Rightarrow \{v_1, v_2, v_3\}$ is a basis for S.

$\Rightarrow \dim(S) = 3$ //

Ex We know that $\underbrace{\text{the set of all } 2\times 2 \text{ diagonal matrices}}_{S}$ form a subspace of $\mathbb{R}^{2\times 2}$.

Find a basis for S and find its dimension.

$$S = \left\{ \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} : \underline{d_1, d_2 \in \mathbb{R}} \right\} \leq \mathbb{R}^{2\times 2}$$

$\underline{\dim(\mathbb{R}^{2\times 2}) = 4}$

$\hookrightarrow$ a typical element in S.

$$\rightarrow \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} = d_1 \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}}_{v_1 \to e_1} + d_2 \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}}_{v_2 \to e_4}$$

$$\begin{bmatrix} d_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$$

$\Rightarrow \{v_1, v_2\}$ is a spanning set for S. ✓

$= \{e_1, e_4\} \rightarrow$ a subset of $\{e_1, e_2, e_3, e_4\} \Rightarrow$ lin. indep. $\Rightarrow$

Remember that, the standard basis for $\mathbb{R}^{2\times 2} = \left\{ \underset{e_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{e_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{e_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{e_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$

$\Rightarrow \{e_1, e_4\}$ is a linearly independent set. ✓

$\Rightarrow \{e_1, e_4\}$ is a basis for S. $\Rightarrow \dim(S) = 2$

Ex/ Find a basis for the supspace of P_3 spanned by;

(b) $x, x-1, x^2+1, x^2-1$

$$S = \text{Span}\{x, x-1, x^2+1, x^2-1\} \leqslant \underbrace{P_3}_{\dim(P_3)=3}$$

Any linearly independent set in P_3 can have at most 3 elements.

$$c_1 x + c_2(x-1) + c_3(x^2+1) + c_4(x^2-1) = 0$$

$$c_1 x + c_2 x - c_2 + c_3 x^2 + c_3 + c_4 x^2 - c_4 = 0$$

$\rightarrow$ $-c_2 + c_3 - c_4 = 0$ $\quad\rightarrow c_4 = r \in \mathbb{R}$

$\rightarrow c_1 + c_2 = 0$ $\quad c_3 = -r$

$c_3 + c_4 = 0$ $\quad -c_2 - r - r = 0$

$c_1 = 2r \qquad c_2 = -2r$

$$\rightarrow x^2 - 1 = 1.(x^2+1) + 1(x-1) + -2.x$$

$$ax^2 + ax + bx - b + cx = x^2 - 1$$

$a = 1$ $\qquad a + b + c = 0$

$b = 1$ $\qquad -b = -1$

$c = -2$

$\rightarrow \quad c_1\underline{(x^2+1)} + c_2(\underline{x-1}) + c_3\underline{x} = 0$

$c_1x^2 + c_1 + \underline{c_2x} - c_2 + \underline{c_3x} = 0$

$c_1 = 0 \qquad c_2 + c_3 = 0 \qquad c_1 - c_2 = 0 \Rightarrow c_2 = 0$

$\Rightarrow c_3 = 0.$

$\rightarrow \{x^2+1, x-1, x\}$ is a basis for this set.

$\dim(S) = 3.$

3. Consider the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

$\dim(\mathbb{R}^2) = 2$

$\rightarrow$ **(a)** Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis for $\mathbb{R}^2$. $\rightarrow$ a) $\{x_1, x_2\}$ in $\mathbb{R}^2$

$\rightarrow$ **(b)** Why must $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be linearly dependent?

(c) What is the dimension of Span$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

$\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 6-4 = 2 \neq 0$

basis $\rightarrow$ lin. indep., spanning

1 is enough to check.

$\Rightarrow$ lin. indep. ✓

$\Rightarrow$ basis ✓

b) $\{x_1, x_2, x_3\}$ is not linearly independent, why?

In $\mathbb{R}^2$, any linearly independent can have AT MOST 2 elements.

It has 3 elements $\Rightarrow$ it can not be lin. indep.

c) $\text{Span}\{x_1, x_2, \cancel{x_3}\} \rightarrow \{x_1, x_2\} \rightarrow$ is also a basis for this set.

$\Rightarrow \dim(\text{span}\{x_1, x_2, x_3\}) = 2.$

$\text{Span}\{x_1, x_2, x_3\} = \text{Span}\{x_1, x_2\}$