5th Week Monday

22 Mart 2021 Pazartesi 09:33

The Determinant of a Matrix

for square matrices only

$\det(M) = |M|$ $\to \in \mathbb{R}$

$\det(M) = d$

$M \mapsto d$

matrix number

$M \to 1\times 1$ matrix $\to$ a number consisting only $\to$ $\det(M) = M_{11}$

ex/ $[3] \to \det([3]) = 3$

$M \to 2\times 2$ matrix $\to$ $M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$

ex/ $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\det(M) = 1.4 - 2.3 = -2$

$A \to n\times n$ matrix $(n \geq 3)$

ex/ $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

minor : $|M_{ij}| \to$ determinant of the $(n-1)\times(n-1)$ submatrix obtained from deleting the ith row and jth column of the matrix A.

ex/ $|M_{21}| = \det\left(\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}\right) = 2.9 - 3.8 = -6$

delete the 2^{nd} row 1^{st} column of A. 2×2

cofactor : $A_{ij} \longrightarrow (-1)^{i+j} \cdot |M_{ij}|$

ex/ $A_{21} = (-1)^{2+1} \cdot |M_{21}| = -1.-6 = 6$

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$

cofactor expansion :

$\det(A) = a_{11} \cdot A_{11} + a_{12} A_{12} + \dots + a_{1n} A_{1n} \to$ 1st row cofactor expansion

$= a_{21} \cdot A_{21} + a_{22} A_{22} + \dots + a_{2n} A_{2n} \to$ 2^{nd} row cofactor expansion

$\vdots$

$= a_{1n} A_{1n} + a_{2n} A_{2n} + \dots + a_{nn} A_{nn} \to$ n^{th} column cofactor expansion

i^{th} row cofactor expansion : $a_{i1} A_{i1} + a_{i2} A_{i2} + \dots + a_{in} A_{in}$ $(= \det(A))$

j^{th} column cofactor expansion : $a_{1j} A_{1j} + a_{2j} A_{2j} + \dots + a_{nj} A_{nj}$ $(= \det(A))$

wrong cofactor

$a_{11}\ a_{12}\ \dots\ a_{1n}$

$a_{11} A_{21} + a_{12} A_{22} \dots = 0$

1st row cofactors from 2^{nd} row $= 0$

determinant of A : 1) Choose any row or column

2) Find the cofactor expansion. $= \det(A)$

ex/ $A = \begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

I choose the 1^{st} row cofactor expansion:

$\det(A) = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$

$(-1)^{1+1} |M_{11}|$ $\quad (-1)^{1+2} |M_{12}|$ $\quad (-1)^{1+3} |M_{13}|$

$45 - 48$

$A_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 6 \qquad (36 - 42)$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3 \qquad (32 - 35)$$

$$(-1)^{1+1}|M_{11}| \qquad (-1)^{1+2}|M_{12}| \qquad (-1)^{1+3}|M_{13}|$$

$$= 1.\underbrace{A_{11}}_{-3} + 2.\underbrace{A_{12}}_{6} + 3.\underbrace{A_{13}}_{-3}$$

$$= -3 + 12 - 9 = 0$$

Break HW : Try choosing another row / column and see it will give the same result.

Ex/

1st column

$$A = \begin{array}{c} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{array}\begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{bmatrix}_{4\times 4}$$

$$0[\ ] + 0[\ ]_{3\times3} + 0[\ ]_{3\times3} + \underbrace{[\ ]}_{3\times3\ \text{det.}} \rightarrow \text{1 cofactor expan.}$$

M_{ij}

$$\underbrace{[\ ] + [\ ] + [\ ]}$$

$3 \times 4 = 12$ 2×2 det.

4 3×3 det.

$$A_{41} = (-1)^{4+1}\underbrace{\overbrace{\begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix}}^{M_{41}}}_{3\times3} \begin{array}{l} \rightarrow 13 \\ \rightarrow 23 \\ \rightarrow 33 \end{array}$$

$$\rightarrow \det(A) = \cancel{a_{11}A_{11}} + \cancel{a_{21}A_{21}} + \cancel{a_{31}A_{31}} + a_{41}A_{41} = 12$$

$0 \quad 0 \quad 0 \quad 2.6$

$$(-1)^{1+1}|M_{11}| \quad (-1)^{2+1}|M_{21}| \quad (-1)^{3+1}|M_{31}| \quad (-1)^{4+1}|M_{41}|$$

$$\cancel{0.(-1)^{1+3}\begin{vmatrix} 4 & 5 \\ 1 & 0 \end{vmatrix}_{2\times2}} + \cancel{0(-1)^{2+3}\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}_{2\times2}} + 3.(-1)^{3+3}\underbrace{\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}_{2\times2}}_{10-12=-2} = -6 = |M_{41}|$$

$$A_{41} = -(-6) = 6$$

* If A is a diagonal matrix;

$$A = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots\ d_n \end{bmatrix} \Rightarrow \boxed{\det(A) = d_1 d_2 \dots d_n}$$

Ex/

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$\det(A) = 2(-1)^{1+1}\begin{vmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{vmatrix} + 0 + 0 + 0$$

$$(-1).(-1)^{1+1}\underbrace{\begin{vmatrix} 3 & 0 \\ 0 & 6 \end{vmatrix}}_{} + 0 + 0$$

$$3.6 - 0$$

$$= 2.(-1).3.6$$

* If A is an upper/lower triangular matrix; $\det(A) = d_1 d_2 \dots d_n$

Ex/

$$A = \begin{bmatrix} 1 & -1 & 3 & -6 \\ 0 & 2 & 0 & 9 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 4 \end{bmatrix} \ 44$$

$$\det(A) = 0 + 0 + 0 + 4.(-1)^{4+4}\underbrace{\begin{vmatrix} 1 & -1 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}}_{} \ 33$$

$$0 + 0 + 3.(-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}$$

$$2.1 - 0$$

Ex/

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 99 & 3 & 0 & 0 \\ 11 & 22 & 4 & 0 \\ -2 & -9 & -12 & 6 \end{bmatrix}$$

$$\det(A) = 2 \cdot (-1)^{1+1} \underbrace{\begin{vmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ -9 & -12 & 6 \end{vmatrix}}_{3(-1)^{1+1}\underbrace{\begin{vmatrix} 4 & 0 \\ -12 & 6 \end{vmatrix}}_{4.6} + 0 + 0} + 0 + 0 + 0$$

$= 2.3.4.6$

$= 4.3.2.1$

The Properties of Determinants

* $\det(A) = \det(A^T)$

→ * If A has an all zero row/column. → $\det(A) = 0$

$0 + 0 + \dots + 0$

→ * $\det(AB) = \det(A).\det(B)$

→ * E_1 : Type-I elementary matrix $(r_i \leftrightarrow r_j)$ $\quad \mathrm{Det}(E_1) = -1$

diagonal → E_2 : Type-II " " $(kr_i \to r_i)$ $\quad \mathrm{Det}(E_2) = k$

E_3 : Type-III " " $(kr_j + r_i \to r_i)$ $\quad \mathrm{Det}(E_3) = 1$

upper/lower triangular

Let A be any matrix.

→ If you apply a 1st type of row operation to A; $E_1 A$

$$\det(E_1 A) = \underbrace{\det(E_1)}_{-1} \det(A) = -\det(A)$$

→ If you apply a 2nd type row operation to A; $E_2 A$

$$\det(E_2 A) = \underbrace{\det(E_2)}_{k} \det(A) = k \det(A)$$

→ If you apply a 3rd type row operation to A = $E_3 . A$

$$\det(E_3 A) = \underbrace{\det(E_3)}_{1} . \det(A) = \det(A)$$

HW Get a random square matrix A.
Find its determinant.
Apply one row operation to A. See what happens to the determinant.
Apply another type of " " " " " " " " " "