3rd Week Monday

08 Mart 2021 Pazartesi 09:22

→ Gaussian Elimination Method

Augmented Matrix ⟶ REF (to solve the system)

→ Gauss-Jordan Elimination Method

Augmented Matrix ⟶ RREF (to solve the system)

Ex (HW)

$$\begin{aligned} \to\ x_1 + x_2 + 3x_3 + x_4 \quad &= 3 \\ 2x_3 + x_4 \quad &= 7 \\ \to -x_1 - x_2 \quad - 2x_5 &= 4 \end{aligned}$$

$$\longrightarrow \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b \end{array} \\ \left[\begin{array}{ccccc|c} \boxed{1} & 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 2 & 1 & 0 & 7 \\ -1 & -1 & 0 & 0 & -2 & 4 \end{array}\right]_{3\times 6} \end{array} \quad r_j = r_1$$

$$r_1 + r_3 \to r_3 \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & \boxed{2} & 1 & 0 & 7 \\ 0 & 0 & 3 & 1 & -2 & 7 \end{array}\right] \xrightarrow{\frac{1}{2}r_2 \to r_2} \left[\begin{array}{cccccc} 1 & 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & \boxed{1} & 1/2 & 0 & 7/2 \\ 0 & 0 & \boxed{3} & 1 & -2 & 7 \end{array}\right] \leftarrow r_j = r_2$$

$$\begin{array}{cccccc} 0 & 0 & -3 & -3/2 & 0 & -21/2 \\ 0 & 0 & 3 & 1 & -2 & 7 \end{array}$$

$$-3r_2 + r_3 \to r_3 \left[\begin{array}{cccccc} 1 & 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1/2 & 0 & 7/2 \\ 0 & 0 & 0 & \boxed{-1/2} & -2 & -7/2 \end{array}\right] ✓$$

$$-2r_3 \to r_3 \left[\begin{array}{cccccc} \boxed{1} & 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & \boxed{1} & 1/2 & 0 & 7/2 \\ 0 & 0 & 0 & \boxed{1} & 4 & 7 \end{array}\right] \to r_3 \to r_j \quad ✓ \to \boxed{\text{REF}}$$

$-7/2$

$$\begin{array}{l} \to -1r_3 + r_1 \to r_1 \\ -\frac{1}{2}r_3 + r_2 \to r_2 \end{array} \left[\begin{array}{cccccc} 1 & 1 & \boxed{3} & 0 & -4 & -4 \\ 0 & 0 & \boxed{1} & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 7 \end{array}\right] \to r_2 \to r_j$$

$$-3r_2 + r_1 \to r_1 \left[\begin{array}{ccccc|c} \boxed{1} & 1 & 0 & 0 & 2 & -4 \\ 0 & 0 & \boxed{1} & 0 & -2 & 0 \\ 0 & 0 & 0 & \boxed{1} & 4 & 7 \end{array}\right] \to \text{RREF!} ✓$$

Back to the system:

$$\begin{aligned} x_1 + x_2 + 2x_5 &= -4 \\ x_3 - 2x_5 &= 0 \\ x_4 + 4x_5 &= 7 \end{aligned}$$

[x_5 = free variable $\quad x_5 = r \in \mathbb{R}$

$\Rightarrow x_3 = 2x_5 \Rightarrow \quad x_3 = 2r$

$\Rightarrow x_4 = 7 - 4x_5 \Rightarrow \quad x_4 = 7 - 4r$

$\to x_1 + x_2 + 2r = -4 \quad$ [$x_1 = s \in \mathbb{R}$ → free variable

$\to s + x_2 + 2r = -4 \Rightarrow \quad x_2 = -4 - s - 2r$

$(x_1, x_2, x_3, x_4, x_5)$

$= (s, -4-s-2r, 2r, 7-4r, r) : r, s \in \mathbb{R}$

The system has this Infinitely many solutions.

Ex

$$\left[\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right] \begin{array}{l} ?r_1 + r_2 \to r_2 \\ ?r_1 + r_3 \to r_3 \\ ?r_1 + r_4 \to r_4 \\ ?r_1 + r_5 \to r_5 \end{array} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right]$$

$$\begin{array}{l} ?r_2 + r_3 \to r_3 \\ ?r_2 + r_4 \to r_4 \\ ?r_2 + r_5 \to r_5 \end{array} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} ?r_3 + r_4 \to r_4 \\ ?r_3 + r_5 \to r_5 \end{array} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & \boxed{-4} \\ 0 & 0 & 0 & 0 & 0 & \boxed{-3} \end{array}\right] \to \text{REF}$$

result

→ 1, → 1 } we don't need to make these final elements 1 which are in results column.

$0 = -4$ Impossible!
$0 = -3$ ″ } ⇒ NO SOLUTION!

Ex

Gauss Elimination

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 & 4 & 4 \end{array}\right] \Rightarrow \Rightarrow \left[\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{REF}$$

$\boxed{x_1 = s \in \mathbb{R}}$ → free variable

$\boxed{x_3 = r} \in \mathbb{R}$ → free variable

Gauss Elimination

$$\left[\begin{array}{cccccc|c} -2 & -2 & 0 & 0 & 3 & & 1 \\ 0 & 0 & 1 & 1 & 3 & & 3 \\ 1 & 1 & 2 & 2 & 4 & & 4 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$ REF

$\boxed{x_1 = s \in \mathbb{R}} \to$ free variable

$x_1 + x_2 + x_3 + x_4 + x_5 = 1$
$x_3 + x_4 + 2x_5 = 0$
$\boxed{x_5 = 3}$

$\boxed{x_3 = r \in \mathbb{R}} \to$ free variable

$\Rightarrow x_3 + x_4 + 6 = 0$
$\boxed{x_4 = -6 - r}$

$\Rightarrow s + x_2 + \cancel{r} - 6 - \cancel{r} + 3 = 1$
$\boxed{x_2 = 4 - s}$

$x_1 + x_3 + x_5 = 1 - x_2 - x_4$
$x_3 + 2x_5 = -x_4$
$x_5 = 3$

Soln:

$(s, 4-s, r, -6-r, 3) : r, s \in \mathbb{R}$

infinitely many solutions.

Let m = # of equations n = # of variables.

Over determined System

$\to m > n$

* you should look at at least REF

Under Determined Systems

$m < n$

$\Rightarrow$ can not have a unique soln!

for other cases; * you should look at REF.

Examples of Overdetermined systems

System (a): $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right] \to$ no soln. $0 = 1$

System (b): $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 2 & -1 & 3 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \to$ unique soln.

System (c): $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to$ inf. many solns.

Examples of Underdetermined Systems

System (a): $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \to$ no soln. $0 = 1$

free free

System (b): $\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \to x_4, x_5$

$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$

$\Rightarrow$ $x_1 = 1 - x_2 - x_3$, $x_4 = 2$, $x_5 = -1$ $\Rightarrow$ infinitely many soln.

$m = n$, $Ax = 0$ Linear Systems

Matrices

Matrix Operations

Addition - Subtraction:

* they should have the same type
* the result has the same type
* component-wise add / subtract

Ex/

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3} - \begin{bmatrix} -1 & 0 & 2 \\ -2 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix}_{3\times3} = \begin{bmatrix} 2 & 2 & 1 \\ 6 & 4 & 7 \\ 7 & 10 & 6 \end{bmatrix}_{3\times3}$$

$\to [a_{ij}]_{3\times3}$ $[b_{ij}]_{3\times3}$ $[a_{ij} \pm b_{ij}]_{3\times3}$

Scalar Multiplication:

a real number

* $\overset{\in\mathbb{R}}{c} \cdot [a_{ij}]_{m\times n} = [c \cdot a_{ij}]_{m\times n}$

number . matrix = matrix

$$\overset{\in\mathbb{R}}{-2} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{m\times n} = \begin{bmatrix} -2 & -4 & -6 \\ -8 & -10 & -12 \\ -14 & -16 & -18 \end{bmatrix}_{m\times n}$$

$[a_{ij}]_{m\times n} \longrightarrow [c \cdot a_{ij}]_{m\times n}$

Matrix Multiplication:

$\to$ * $[a_{ij}]_{m\times \boxed{n}} \cdot [b_{ij}]_{\boxed{n}\times k} = [c_{ij}]_{\boxed{m\times k}} \to$ How?

! You can not multiply any two matrix.

! Order is important. $A_{m\times n} \, B_{n\times k} = C_{m\times k}$ ✓

$B_{n\times \boxed{k}} \, A_{\boxed{m}\times n} \to$ can even not multiply

$A_{m\times m} \cdot B_{m\times m} = C_{m\times m}$

$B_{m\times m} \cdot A_{m\times m} = \boxed{D_{m\times m}}$

Ex/

$$\overset{A}{\begin{bmatrix} 1 & 3 & 4 \\ 2 & -2 & -3 \end{bmatrix}}_{2\times3} \cdot \overset{B}{\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & 1 & 2 \\ 3 & -1 & 2 & 3 \end{bmatrix}}_{3\times4} = \overset{C}{\begin{bmatrix} 19 & -2 & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{bmatrix}}_{2\times4}$$

B.A

1st row of A • 1st column of B $1+6+12$ $\longrightarrow C_{11}$

$(1, 3, 4) \cdot (1, 2, 3) = 1.1 + 3.2 + 4.3 = 19$

2nd column of B $\longrightarrow C_{12}$

$(1, 3, 4) \cdot (2, 0, -1) = 1.2 + 3.0 + 4.(-1) = -2$

3rd column of B $\longrightarrow C_{13}$

4^{th} column of B $\longrightarrow C_{14}$

2nd row of A 1st column of B $\longrightarrow C_{21}$

2^{nd} column of B $\longrightarrow C_{22}$

3^{rd} colum of B $\longrightarrow C_{23}$

4^{th} colum of B $\longrightarrow C_{24}$