11th Week

18 Aralık 2021 Cumartesi 12:16

Linear Transformation

$* \to$ Every linear transformation corresponds to a matrix.

mappings between vector spaces

$V, W \to$ vector space

$$L : \underset{(V,\oplus,0)}{V} \longrightarrow \underset{(W,\boxplus,\boxdot)}{W}$$

1) $\forall v_1, v_2 \in V$ $\quad \underbrace{L(\underbrace{v_1}_{\in V} \oplus \underbrace{v_2}_{\in V})}_{LHS} \stackrel{?}{=} \underbrace{\underbrace{L(v_1)}_{\in W} \boxplus \underbrace{L(v_2)}_{\in W}}_{RHS}$ ✓

2) $\forall \alpha \in \mathbb{R}, \forall v \in V$ $\quad \underbrace{L(\alpha \odot v)}_{LHS} = \underbrace{\alpha \boxdot L(v)}_{RHS}$ ✓

$\Rightarrow L$ is a l.t.

Ex/ $L : \mathbb{R}^2 \to \mathbb{R}^3$

$(x,y) \longmapsto (x+y, x-y, x \cdot y)$

Is L a linear transformation?

1) $L\big(\underbrace{(x_1,y_1)}_{\in \mathbb{R}^2} + \underbrace{(x_2,y_2)}_{\in \mathbb{R}^2}\big) \stackrel{?}{=} \underbrace{L((x_1,y_1))}_{\in \mathbb{R}^3} + \underbrace{L((x_2,y_2))}_{\in \mathbb{R}^3}$

LHS

$L\big((x_1,y_1)+(x_2,y_2)\big) = L\big((x_1+x_2, y_1+y_2)\big)$

$= \big(x_1+x_2+y_1+y_2,\ x_1+x_2-y_1-y_2,\ (x_1+x_2).(y_1+y_2)\big)$

RHS

$= (x_1+y_1, x_1-y_1, x_1.y_1) + (x_2+y_2, x_2-y_2, x_2.y_2)$

$= (x_1+y_1+x_2+y_2,\ x_1-y_1+x_2-y_2,\ x_1y_1+x_2y_2)$

$\stackrel{?}{=} \quad \neq \quad \Rightarrow$ 1st property doesn't hold.

$\Rightarrow L$ is not a linear transformation.

Ex/ $L : \mathbb{R}^2 \to \mathbb{R}^3$

$(x,y) \longmapsto (y, x, x+y)$ $\quad \to L((x,y)) = (y, x, x+y)$

Is L a linear transformation?

1) $L\big((x_1,y_1) + (x_2,y_2)\big) \stackrel{?}{=} L((x_1,y_1)) + L((x_2,y_2))$

$= L\big((x_1+x_2, y_1+y_2)\big)$

$= (y_1+y_2, x_1+x_2, x_1+x_2+y_1+y_2)$

LHS

$= (y_1, x_1, x_1+y_1) + (y_2, x_2, x_2+y_2)$

$= (y_1+y_2, x_1+x_2, x_1+y_1+x_2+y_2)$

RHS

$= ✓$

2) $L\big(\alpha(x_1,y_1)\big) \stackrel{?}{=} \alpha . L(x_1,y_1) \quad \forall \alpha \in \mathbb{R}$

$L\big((\alpha x_1, \alpha y_1)\big)$ $\qquad = \alpha . (y_1, x_1, x_1+y_1)$

2) $L\left(\alpha(x_1,y_1)\right) \quad = \quad \alpha \cdot L(x_1,y_1)$ check

$= L\left((\alpha x_1, \alpha y_1)\right) \qquad = \alpha \cdot (y_1, x_1, x_1+y_1)$

$= (\alpha y_1, \alpha x_1, \alpha x_1 + \alpha y_1) \qquad = (\alpha y_1, \alpha x_1, \alpha(x_1+y_1))$

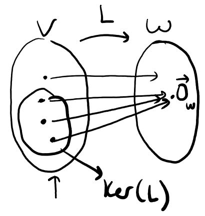
$\checkmark \quad \checkmark \quad \checkmark$

$= \checkmark$

Since 1 and 2 both hold, L is a linear transformation.

Kernel and Image of a Linear Transformation

Let $L: V \to W$ be a linear transformation.



Kernel of L: $\quad Ker(L) = \{ \forall \vec{v} \in V : L(\vec{v}) = \vec{0}_w \}$

* At least, $\vec{0}_V \in Ker(L) \Rightarrow \boxed{Ker(L) \neq \emptyset}$

* $Ker(L) \leq V$ — is a subspace of V

Soru: $T((x,y)) = (x, y, x+y+3)$ → is not a linear transformation.

$T: \mathbb{R}^2 \to \mathbb{R}^3$ Gekirdek = ?

$T(x,y) = (0,0,0)$ $\quad x=0$, $y=0$, $x+y+3 \neq 0$ $\quad$ Gekirdek $= \emptyset$, $Ker(T) = \emptyset$

Let's check. $\quad T: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad$ Is T a linear transformation?

$(x,y) \mapsto (x, y, x+y+3)$

1) $T\left((x_1,y_1)+(x_2,y_2)\right) \overset{?}{=} T\left((x_1,y_1)\right) + T\left((x_2,y_2)\right)$

$= T\left((x_1+x_2, y_1+y_2)\right) \qquad = (x_1, y_1, x_1+y_1+3) + (x_2, y_2, x_2+y_2+3)$

$= (x_1+x_2, y_1+y_2, x_1+x_2+y_1+y_2+3) \neq (x_1+x_2, y_1+y_2, x_1+y_1+3+x_2+y_2+3)$

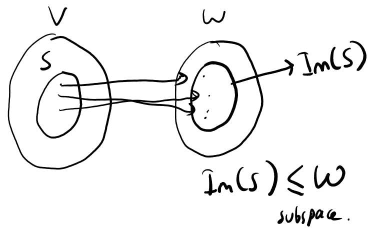
$\checkmark \quad \checkmark \quad \times \; (+3) \qquad \checkmark \quad \checkmark \quad \times \; (+6)$

$\Rightarrow$ T is NOT a lin. trans.

Image of L = $\quad$ Let $S \leq V$.

$L: V \to W$

$$\boxed{Im(S) = \{ L(s) : \forall s \in S \}}$$



$Im(S) \leq W$ subspace.

$$\boxed{Im(V) = \text{Range of } L = Range(L)}$$