2nd Week Monday

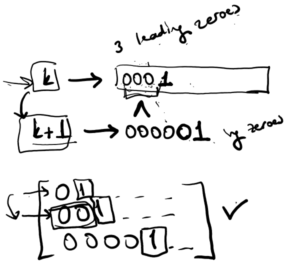
01 Mart 2021 Pazartesi 09:35

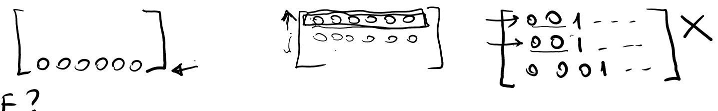
REF

leading element of the row

! A matrix is said to be in **row echelon form** if

- ✓ (i) The first nonzero entry in each nonzero row is $\boxed{1}$.
- (ii) If row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k.
- (iii) If there are rows whose entries are all zero, they are below the rows having nonzero entries.





Which ones are in REF?

✓ (a) $\begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & \boxed{1} & 2 \end{bmatrix}$

✗ (b) $\begin{bmatrix} 1 & 0 & 0 \\ \boxed{0 \;\; 0 \;\; 0} \\ 0 & 0 & 1 \end{bmatrix}$

✓ (c) $\begin{bmatrix} \boxed{1} & 3 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$

✓ (d) $\begin{bmatrix} 0 & \boxed{1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ REF ✓ ✓

✗ (e) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \cancel{3} \end{bmatrix}$

✗ (f) $\begin{array}{c} 0 \\ 2 \\ 1 \end{array}\begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

✓ (g) $\begin{array}{c} 0 \\ 1 \\ 2 \end{array}\begin{bmatrix} \boxed{1} & 0 & 0 & 1 & 2 \\ 0 & \boxed{1} & 0 & 2 & 4 \\ 0 & 0 & \boxed{1} & 3 & 6 \end{bmatrix}$

✓ (h) $\begin{array}{c} 1 \\ 2 \\ 4 \end{array}\begin{bmatrix} 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & \boxed{1} & 3 \\ \boxed{0 \;\; 0 \;\; 0 \;\; 0} \end{bmatrix}$ ✓

✓ ✓ 1) leading nonzero = 1

✓ ~~X~~ → 2) # leading zeros of k^{th} row < # leading zeros of $k+1^{th}$ row

X → 3) if all zero row → bottom

(it may be more than one)

Apply Elementary Row Operations

1. $r_i \leftrightarrow r_j$
2. $c \cdot r_i \rightarrow \boxed{r_i}$
3. $c \cdot \boxed{r_j} + r_i \rightarrow \boxed{r_i}$!

$$\begin{bmatrix} 0 & -1 & -1 & 1 & 0 \\ \boxed{1} & 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} \boxed{1} & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{bmatrix} \xrightarrow[-3r_1+r_4 \rightarrow r_4]{-2r_1+r_3 \rightarrow r_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 0 & \boxed{-1} & -1 & 1 & 0 \\ 0 & 2 & -1 & -4 & -13 \\ 0 & -2 & -5 & -1 & -15 \end{bmatrix}$$

$$\begin{array}{rrrrrl} -2 & -2 & -2 & -2 & -12 & \\ + \;\; 2 & 4 & 1 & -2 & -1 & \\ \hline 0 & 2 & -1 & -4 & -13 & \rightarrow r_3 \end{array}$$

$$\begin{array}{rrrrrl} -3 & -3 & -3 & -3 & -18 & -3r_1 \\ + \;\; 3 & 1 & -2 & 2 & 3 & \leftarrow r_4 \\ \hline 0 & -2 & -5 & -1 & -15 & \rightarrow r_4^{new} \end{array}$$

$$\xrightarrow{-1r_2 \rightarrow r_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 0 & \boxed{1} & 1 & -1 & 0 \\ 0 & 2 & -1 & -4 & -13 \\ 0 & -2 & -5 & -1 & -15 \end{bmatrix} \xrightarrow[2r_2+r_4 \rightarrow r_4]{-2r_2+r_3 \rightarrow r_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & \boxed{-3} & -2 & -13 \\ 0 & 0 & -3 & -3 & -15 \end{bmatrix}$$

0 0 2 −2 2 −5 −2 −1 0 −15

$$\xrightarrow{-\frac{1}{3}r_3 \rightarrow r_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & \boxed{1} & 2/3 & 13/3 \\ 0 & 0 & -3 & -3 & -15 \end{bmatrix} \xrightarrow{3r_3+r_4 \rightarrow r_4} \begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2/3 & 13/3 \\ 0 & 0 & 0 & \boxed{-1} & -2 \end{bmatrix}$$

0 0 0 0 3 −3 2 −3 13 −15

$$\xrightarrow{-1r_4 \rightarrow r_4} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & 6 \\ 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2/3 & 13/3 \\ 0 & 0 & 0 & 1 & 2 \end{array} \rightarrow \text{REF}$$

RREF

$x_1 + \overset{-1}{x_2} + \overset{3}{x_3} + \overset{2}{x_4} = 6$

$x_2 + x_3 - x_4 = 0$

$x_3 + \frac{2}{3}x_4 = \frac{13}{3}$

$\boxed{x_4 = 2}$

$x_3 + \frac{4}{3} = \frac{13}{3} \Rightarrow \boxed{x_3 = 3}$

$x_2 + 3 - 2 = 0$

$\boxed{x_2 = -1}$

$\boxed{x_1 = 2}$

$(x_1, x_2, x_3, x_4) = (2, -1, 3, 2)$ → unique solution.

2. The augmented matrices that follow are in row echelon form. For each case, indicate whether the corresponding linear system is consistent. If the system has a unique solution, find it.

(a) $\begin{array}{ccc} x_1 & x_2 & b \\ \hline 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}$

(b) $\begin{array}{c|cc|c} & x_1 & x_2 & \\ 1 & 1 & 3 & 1 \\ 2 & 0 & 1 & -1 \\ 3 & \cancel{0} & \cancel{0} & \cancel{0} \end{array} \rightarrow 0=0$ ✓

2 eqns left
2 unknowns
unique soln.

(c) $\begin{array}{c|ccc|c} & x_1 & x_2 & x_3 & \\ 1 & 1 & -2 & 4 & 1 \\ 2 & 0 & 0 & 1 & 3 \\ & 0 & 0 & 0 & 0 \end{array} \rightarrow 0=0$ ✓

RREF
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ & & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -2 & 2 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow$ unique soln

(e) $\begin{bmatrix} 1 & 3 & 2 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow 0=1$
impossible.
NO SOLUTION.

EX

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix} \rightarrow 0=1$!

2 eqns
2 unknowns

no soln !

a) $0x_1 + 0x_2 = 1 \rightarrow$ bottom line

$0 = 1 \rightarrow$ impossible !

$\Rightarrow$ The system has NO SOLUTION !

b) $x_1 + 3\overset{-1}{x_2} = 1$ $\qquad x_1 - 3 = 1$

$\boxed{x_2 = -1}$ $\qquad \boxed{x_1 = 4}$

$(4,-1) \rightarrow$ is a unique SOLN !

c) $x_1 - 2x_2 + 4\overset{3}{x_3} = 1$ $\qquad \boxed{\overset{s}{x_1} - 2x_2 = -11}$

$\boxed{x_3 = 3}$

$x_2 = r \in \mathbb{R} \rightarrow$ free variable

$\boxed{x_1 = -11 + 2r}$

$(x_1, x_2, x_3) = (-11+2r, ⓡ, 3)$, $r \in \mathbb{R} \rightarrow$ infinitely many solutions

you may also give $x_1 = s \in \mathbb{R}$

$x_2 = \dfrac{s+11}{2}$

$(x_1, x_2, x_3) = (ⓢ, \dfrac{s+11}{2}, 3)$, $s \in \mathbb{R} \rightarrow$ represents the same solution.