Determinants

Elementary Matrix of type I $r_i \leftrightarrow r_j$ → $\det(E) = -1$
" " " type II $kr_i \to r_i$ → $\det(E) = k$
" " " type III $kr_j + r_i \to r_i$ → $\det(E) = 1$

Ex/ A, $\boxed{\det(A) = 5}$ Apply $r_1 \leftrightarrow r_3$ and then $3r_3 + r_1 \to r_1$

what happens to $\det(A)$?

$$A \xrightarrow{r_1 \leftrightarrow r_3} E_1A \xrightarrow{3r_3 + r_1 \to r_1} A' = E_2E_1A$$

E_1 of type-I, E_2 of type-III

$\det(A') = ?$

$\det(AB) = \det(A).\det(B)$

$A' = E_2 E_1 A$

$$\det(A') = \det(E_2 E_1 A) = \underbrace{\det(E_2)}_{1}\underbrace{\det(E_1)}_{-1}\underbrace{\det(A)}_{5} = -5$$

Ex/ $A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ $\det(A) = 2.(-3).4 = -24$

$$A \xrightarrow{\frac{1}{4}r_3 \to r_3} \begin{bmatrix} 2 & 1 & -2 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{2r_3 + r_1 \to r_1} \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A'}$$

E_1 type-2, E_2 type-3, $A' = E_2E_1A$

diagonal, upper, lower: $\det(A) = \prod_{i=1}^{n} a_{ii}$

$\det(A') = 2.-3.1 = -6$

$$\det(A') = \underbrace{\det(E_2)}_{1}\underbrace{\det(E_1)}_{\frac{1}{4}}\underbrace{\det(A)}_{-24} = -6 \checkmark$$

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$\dots E_2E_1A = I_n$

→ $\dots \det(E_2)\det(E_1)\det(A) = 1$

$A_{n\times n}$ square matrix $\xrightarrow{RREF}$ $= I_n$ ⟷ $\det(I_n) = 1$ ⟺ $\boxed{\det(A) \neq 0}$ $\boxed{A \text{ is invertible.}}$

$\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$E_1, E_2, \dots$

$\xrightarrow{RREF}$ $\neq I_n$ → $\det(\neq I_n) = 0$ ⟺ $\boxed{\det(A) = 0}$ $\boxed{A \text{ is not invertible.}}$

→ $\dots \underset{\neq 0}{\det(E_2)}\underset{\neq 0}{\det(E_1)}\underset{\neq 0}{\det(A)} = 0$

RREF $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & & & \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Adjoint Matrix

adj(A)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

square

cofactors of 1st row, 2nd row

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{21} & & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ & & & \\ A_{1n} & A_{2n} & & A_{nn} \end{bmatrix}$$

$A_{ij} = (-1)^{i+j}|M_{ij}|$

$$A.\,\text{adj}(A) = \begin{bmatrix} \overbrace{a_{11}A_{11} + a_{12}A_{12} + \dots + a_{1n}A_{1n}}^{\det(A)} & 0 & 0 & 0 \\ 0 & \overbrace{a_{21}A_{21} + \dots + a_{2n}A_{2n}}^{\det(A)} & 0 & \\ & & \ddots & \end{bmatrix} = \begin{bmatrix} \det(A) & & 0 \\ & \det(A) & \\ 0 & & \ddots \end{bmatrix} = \det(A).\,I_n$$

$$A \cdot \operatorname{adj}(A) = \begin{bmatrix} \underbrace{a_{11}A_{11} + a_{12}A_{12} + \dots + a_{1n}A_{1n}}_{} & & & \\ 0 & \underbrace{a_{21}A_{21} + \dots + a_{2n}A_{2n}}_{\det(A)} & & 0 \\ 0 & & \ddots & \\ 0 & 0 & & \underbrace{a_{n1}A_{n1} + a_{n2}A_{n2} + \dots + a_{nn}A_{nn}}_{\det(A)} \end{bmatrix} = \begin{bmatrix} \det(A) & & 0 \\ & \det(A) & \\ 0 & \ddots & \det(A) \end{bmatrix} = \det(A) \cdot I_n$$

$$\boxed{A \cdot \operatorname{adj}(A) = \det(A) \cdot I_n} \rightarrow A \cdot \underbrace{\operatorname{adj}(A) \cdot \frac{1}{\det(A)}}_{A^{-1}} = I_n$$

$$\boxed{A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)}$$

$\det(A)$ → number, scalar; $\operatorname{adj}(A)$ → matrix

Ex

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$\operatorname{adj}(A) = ?$ $\quad \det(A) = ?$ $\quad A^{-1} = ?$ → $\frac{1}{\det(A)} \cdot \operatorname{adj} A =$

$$A_{ij} = (-1)^{i+j} |M_{ij}|$$

$$\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix} \longrightarrow A^{-1} = \begin{bmatrix} 2/5 & 1/5 & -2/5 \\ -7/5 & 4/5 & 2/5 \\ 4/5 & -3/5 & 1/5 \end{bmatrix}$$

$$A_{11} = (-1)^{1+1}|M_{11}| = 1 \cdot \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} = 6-4 = 2 \qquad A_{21} = (-1)^{2+1}|M_{21}| = -1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 1 \qquad A_{31} = (-1)^{3+1}|M_{31}| = 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = -2$$

$$A_{12} = (-1)^{1+2}|M_{12}| = -1 \cdot \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = -(9-2) = -7 \qquad A_{22} = (-1)^{2+2}|M_{22}| = 1 \cdot \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4 \qquad A_{32} = (-1)^{3+2}|M_{32}| = -1 \cdot \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = 2$$

$$A_{13} = (-1)^{1+3}|M_{13}| = 1 \cdot \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 6-2 = 4 \qquad A_{23} = (-1)^{2+3}|M_{23}| = -1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3 \qquad A_{33} = (-1)^{3+3}|M_{33}| = 1 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1$$

1st row

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 2 \cdot 2 + 1 \cdot -7 + 2 \cdot 4 = 4 - 7 + 8 = 5$$