14th Week

08 Ocak 2022 Cumartesi 12:35

Some Operations on Vectors

* Dot Product

inputs are two vector; output is a scalar

$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3 \in \mathbb{R}$

Ex/ $\left.\begin{array}{l}\vec{u} = (u_1, u_2, u_3)\\ \vec{v} = (v_1, v_2, v_3)\end{array}\right\} \in \mathbb{R}^3$

$u, v \in V$ → same vector space

Ex/ $\vec{u} = (3, -1, 2, 0)$

$\vec{v} = (1, -2, 0, 1)$

$\vec{u} \cdot \vec{v} = 3.1 + (-1).(-2) + 2.0 + 0.1$

$3 + 2 + 0 + 0 = 5$

$\|\vec{u}\| = \sqrt{9+1+4+0} = \sqrt{14}$

$\|\vec{v}\| = \sqrt{1+4+0+1} = \sqrt{6}$

* Norm of a Vector → $\|\vec{u}\| \in \mathbb{R}$

↳ length

$\vec{u} = (u_1, u_2, u_3) \in \mathbb{R}^3 \Rightarrow \|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2} \in \mathbb{R}$

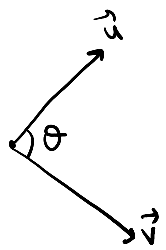
! $u \cdot u = u_1.u_1 + u_2.u_2 + u_3.u_3 = \|\vec{u}\|^2$

Normed Vector $\vec{u}_0$: $\frac{1}{\|u\|} \cdot \vec{u}$ → has norm $= 1$

$\vec{u}_0 = \frac{1}{\sqrt{14}}(3, -1, 2, 0) = \left(\frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, 0\right)$

$\|\vec{u}_0\| = \frac{9}{14} + \frac{1}{14} + \frac{2}{14} = 1$

!



$$\boxed{\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}}$$

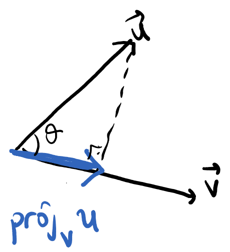
dot

$\left.\begin{array}{l}\theta = 90^\circ\\ \vec{u} \perp \vec{v}\end{array}\right\} \Leftrightarrow \cos 90^\circ = 0 \Leftrightarrow \vec{u} \cdot \vec{v} = 0$

! If $\vec{u}$ and $\vec{v}$ is orthogonal (dik) $\vec{u} \cdot \vec{v} = 0$

Projection :

izdüşüm



(norm) length of $proj_v u = \|\vec{u}\| . \cos\theta$ → scalar component

direction of $\vec{v}$: $\frac{\vec{v}}{\|\vec{v}\|}$ ↳ normed vector

$$\boxed{proj_v u = \underbrace{\|\vec{u}\| . \cos\theta}_{\text{scalar}} \underbrace{\frac{\vec{v}}{\|\vec{v}\|}}_{\text{vector}}}$$

$\cos\theta = \frac{u \cdot v}{\|u\| \, \|v\|}$

$= \underbrace{\frac{u \cdot v}{\|\vec{v}\|^2}}_{\text{scalar}} \cdot \underbrace{\vec{v}}_{\text{vector}}$

Orthogonal Spaces

dik ↑ $\boxed{u \cdot v = 0}$

" Subspaces

$S \leq \mathbb{R}^n$, $T \leq \mathbb{R}^n$ S and T are orthogonal subspaces

$S \leq \mathbb{R}^n \quad , \quad T \leq \mathbb{R}^n$ S and T are orthogonal subspaces

$$\forall \vec{s} \in S, \ \forall \vec{t} \in T \quad \vec{s}\cdot\vec{t} = 0 \quad \Rightarrow \quad S \perp T$$

$\mathbb{R}^3$ $\quad \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} = S \qquad \text{Span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = T$

$u = \begin{bmatrix}3\\5\\0\end{bmatrix} \in S \qquad v = \begin{bmatrix}0\\0\\-7\end{bmatrix} \in T$

$u = (3,5,0)$
$v = (0,0,-7)$ $\qquad u \cdot v = 3.0 + 5.0 + 0.-7 = 0$

$\forall s \in S$ and $\forall t \in T$

$s \cdot t = 0$

Orthogonal Complement

$$S = \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} \leq \mathbb{R}^3$$

$$S = \{ v \in \mathbb{R}^3 : v \cdot s = 0, \ \forall s \in S \}$$

Ex/

$$S = \text{Span}\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\} \leq \mathbb{R}^3$$

Find a basis for $S^\perp = ?$

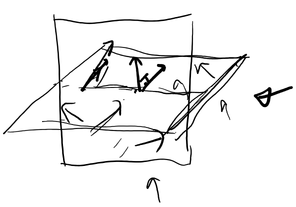
$$S^\perp = \{ \vec{v} \in \mathbb{R}^3 : \vec{v} \cdot \vec{s} = 0, \ \forall s \in S \}$$

$(v_1, v_2, v_3) \cdot (1, -1, 1) = 0$

$v_1.1 - v_2.1 + v_3.1 = 0$

$v_1 - v_2 + v_3 = 0$

$v_3 = a \in \mathbb{R}$
$v_2 = b \in \mathbb{R}$
$v_1 = b - a$



$$\begin{bmatrix}b-a\\b\\a\end{bmatrix} = a\begin{bmatrix}-1\\0\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\begin{bmatrix} b-a \\ b \\ a \end{bmatrix} = a \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$S^{\perp} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

2

$S^{\perp}$

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} < \mathbb{R}^3$$

2 3

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$S^{\perp} = \left\{ (v_1, v_2, v_3) : (v_1,v_2,v_3) \cdot (1,0,0) = 0 \text{ and } (v_1,v_2,v_3) \cdot (0,1,0) = 0 \right\}$$

$v_1 . 1 + v_2 . 0 + v_3 . 0 = 0$

$v_1 . 0 + v_2 . 1 + v_3 . 0 = 0$

$v_3 = r \in \mathbb{R}$

$v_2 = 0$

$v_1 = 0$

$$\text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = S^{\perp}$$

1

Ex/ $S = \text{Span}\left\{ \begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix} \right\} \leq \mathbb{R}^3$

1

Find a basis for $S^{\perp}$

$$S^{\perp} = \left\{ (v_1, v_2, v_3) : (v_1,v_2,v_3) \cdot (3,5,-7) = 0 \right\}$$

$3v_1 + 5v_2 - 7v_3 = 0$

$v_3 = r \in \mathbb{R}$

$v_2 = s \in \mathbb{R}$

$v_1 = \dfrac{7r - 5s}{3}$

$$\begin{bmatrix} \frac{7r-5s}{3} \\ s \\ r \end{bmatrix} = r \begin{bmatrix} 7/3 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -5/3 \\ 1 \\ 0 \end{bmatrix}$$

$$S^{\perp} = \text{Span}\left\{ \begin{bmatrix} 7/3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -5/3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

2

Ex/ $S = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\} \leq \mathbb{R}^2$

2 $\quad 2 + 0 = 2$

$$\rightarrow S^{\perp} = \left\{ (v_1, v_2) : (v_1,v_2) \cdot (1,3) = 0 \text{ and } (v_1,v_2) \cdot (-1,2) = 0 \right\} = \{\vec{0}\}$$

0

$v_1 + 3v_2 = 0$

$-v_1 + 2v_2 = 0$

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ -1 & 2 & 0 \end{array}\right] \quad \begin{array}{ccc} 1 & 3 & 0 \\ 0 & 5 & 0 \end{array}$$

$v_2 = 0$

$v_1 = 0$

4. Let S be the subspace of $\mathbb{R}^4$ spanned by $\mathbf{x}_1 =$

$$S = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \leq \mathbb{R}^4$$

4. Let S be the subspace of $\mathbb{R}^4$ spanned by $\mathbf{x}_1 = (1, 0, -2, 1)^T$ and $\mathbf{x}_2 = (0, 1, 3, -2)^T$. Find a basis for $S^{\perp}$.

$$S = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \end{bmatrix} \right\} \leqslant \mathbb{R}^4$$

$$\overset{2}{\dim(S)} + \overset{2}{\dim(S^{\perp})} = \underbrace{\dim(V)}_{4}$$

$$S^{\perp} = \{ (v_1, v_2, v_3, v_4) : (v_1, v_2, v_3, v_4) \cdot (1, 0, -2, 1) = 0 \text{ and } (v_1, v_2, v_3, v_4) \cdot (0, 1, 3, -2) = 0$$

$$v_1 - 2v_3 + v_4 = 0$$
$$v_2 + 3v_3 - 2v_4 = 0$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{array}\right]$$

$v_4 = r \in \mathbb{R}$

$v_3 = s \in \mathbb{R}$

$v_2 = 2r - 3s$

$v_1 = -r + 2s$

$$\begin{bmatrix} -r+2s \\ 2r-3s \\ s \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

$$S^{\perp} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$$