7th Week

20 Kasım 2021 Cumartesi 20:54

03 Nisan 2021 Cumartesi 14:36

1) $\vec{v}_1 \oplus \vec{v}_2 \overset{?}{=} \vec{v}_2 \oplus \vec{v}_1$ (LHS, RHS)

(2) closed under vector addition $\vec{v}_1, \vec{v}_2 \in V$, $\vec{v}_1 \oplus \vec{v}_2 \in V$; closed under scalar multiplication $\alpha \in \mathbb{R}$, $\vec{v} \in V$, $\alpha\vec{v} \in V$

2) $(\vec{v}_1 \oplus \vec{v}_2) \oplus \vec{v}_3 = \vec{v}_1 \oplus (\vec{v}_2 \oplus \vec{v}_3)$

3) The identity element of vector addition $\in V$

4) Inverse element of vector addition $\in V$

5) $\alpha \in \mathbb{R}$ (scalar) $\alpha \odot (\vec{v}_1 \oplus \vec{v}_2) = \alpha \odot \vec{v}_1 \oplus \alpha \odot \vec{v}_2$

6) $r, s \in \mathbb{R}$ (scalars), $\vec{v} \in V$: $(r+s) \odot \vec{v} = r\odot\vec{v} \oplus s\odot\vec{v}$

7) $r, s \in \mathbb{R}$ (scalars), $\vec{v} \in V$: $(rs)\odot\vec{v} = r\odot(s\odot\vec{v})$

8) $1 \in \mathbb{R}$, $\vec{v} \in V$: $1 \odot \vec{v} = \vec{v}$

$V, \oplus, \odot$ — $\oplus$: vector addition, $\odot$: scalar multiplication ✓ → vector space

8 + 2 properties

$(\mathbb{R}^2, +, \cdot), \ldots, (\mathbb{R}^n, +, \cdot)$ ✓

Ex/ real number → vectors of $(\mathbb{R}, +, \cdot)$ ✓ (set, vector addition, scalar multiplication)

→ matrices → vectors of $(\mathbb{R}^{m\times n}, \oplus, \odot)$ ✓ (matrix addition, scalar multip. in matrices)

→ polynomials → vector of $(P_n, +, \odot)$

P_n : polynomials with degree less than n. ✓

SUBSPACES

$S \subseteq V$ → vector space

S: a subset of V; $\subseteq$: subset notation

$V, \oplus, \odot$

✓ 1) $\vec{0}_V \in S$

✓ 2) $\forall \vec{s}_1, \vec{s}_2 \in S$ $\vec{s}_1 \oplus \vec{s}_2 \in S$ → (S should be a closed subset of V wrt $\oplus$).

✓ 3) $\forall \lambda \in \mathbb{R}$, $\forall \vec{s} \in S$ $\lambda \odot \vec{s} \in S$ → (S should be a closed subset of V wrt $\odot$).

If S satisfies all these 3 properties, then S is a subspace of V.

$\boxed{S \leqslant V}$ — $\leqslant$: subspace notation.

Ex/ $(\mathbb{R}^2, +, \cdot)$ is a vector space → $(\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}, \oplus, \odot)$

$S = \{(x,y) : \boxed{y = 2x}, x, y \in \mathbb{R}\}$ ... (1,2), (2,4), (4,8) ...

↑ property of S

Is S a subspace of $\mathbb{R}^2$?

1) $\vec{0}_V = (0,0)$ $0 = 2.0$ → $(0,0) \in S$ ✓

$x=0$ $y=0$ satisfies the property of S. ✓

2) Let (x_1, y_1), (x_2, y_2) be elements of S.

→ $y_1 = 2x_1$ $y_2 = 2x_2$

$$\Rightarrow y_1 = 2x_1 \qquad y_2 = 2x_2$$

$$(x_1,y_1) \oplus (x_2,y_2) = (x_1+x_2,\ y_1+y_2) \overset{?}{\in} S \quad \checkmark$$

$$y_1+y_2 \overset{?}{=} 2(x_1+x_2)$$

$$y_1+y_2 = 2x_1+2x_2 = 2(x_1+x_2)$$

3) Let $\lambda \in \mathbb{R}$, $(x_1,y_1) \in S \rightarrow y_1 = 2x_1$

$$\lambda \odot (x_1,y_1) = (\lambda x_1, \lambda y_1) \overset{?}{\in} S \quad \checkmark$$

$$\lambda y_1 \overset{?}{=} 2.(\lambda x_1)$$

$$\lambda y_1 = \lambda.(2x_1) = 2.(\lambda x_1)$$

$\Rightarrow$ 1,2,3 holds. $\Rightarrow$ $S \leqslant \mathbb{R}^2$

Ex/ $(\mathbb{R}^2, \oplus, \odot)$ is a vector space.

$$S = \{ (x,1) : x \in \mathbb{R} \}$$

$(0,1)$, $(2,1)$ $(3,1)$ ...

Is S a subspace of $\mathbb{R}^2$?

$(0,0) \notin S$ ($0 \neq 1$) $\Rightarrow$ it fails the property 1 of being subset

$$(x_1,1) + (x_2,1) = (x_1+x_2, 2) \notin S$$

$$\lambda.(x,1) = (\lambda x, \lambda) \notin S \text{ if } \lambda \neq 1$$

2. Determine whether the following sets form subspaces of $\mathbb{R}^3$: $\mathbb{R}^3 = \{ (x,y,z) : x,y,z \in \mathbb{R} \}$

- ✗ **(a)** $\{(x_1,x_2,x_3)^T \mid x_1 + x_3 = 1\}$ $= S$
- ✓ **(b)** $\{(x_1,x_2,x_3)^T \mid x_1 = x_2 = x_3\}$
- ✓ **(c)** $\{(x_1,x_2,x_3)^T \mid x_3 = x_1 + x_2\}$
- ✗ **(d)** $\{(x_1,x_2,x_3)^T \mid x_3 = x_1$ or $x_3 = x_2\}$

a) $S = \{ (x_1,x_2,x_3) : x_1 + x_3 = 1 \}$

Is S a subspace of $\mathbb{R}^3$?

1) $(0,0,0) \overset{?}{\notin} S$ ($0+0 \neq 1$) $\Rightarrow$ $S \nleqslant \mathbb{R}^3$

b) $S = \{ (x_1,x_2,x_3) : x_1 = x_2 = x_3 \}$

Is S a subspace of $\mathbb{R}^3$?

1) $(0,0,0) \overset{?}{\in} S$ ✓ $\quad 0=0=0$ ✓

2) $(x,x,x) \oplus (y,y,y) = (x+y, x+y, x+y) \in S$ ✓ $\quad x+y = x+y = x+y$ ✓

3) $\lambda \odot (x,x,x) = (\lambda x, \lambda x, \lambda x) \in S$ ✓ $\quad \lambda x = \lambda x = \lambda x$ ✓

$S \leqslant \mathbb{R}^3$ ✓

c) $S = \{ (x_1,x_2,x_3) : x_3 = x_1 + x_2 \}$ Is S a subspace of $\mathbb{R}^3$?

1) $(0,0,0) \in S$ ✓ $0 = 0+0$ ✓

2) $(\underbrace{x_1, x_2, x_1+x_2}_{\in S}) \oplus (\underbrace{y_1, y_2, y_1+y_2}_{\in S}) = (x_1+y_1,\ x_2+y_2,\ x_1+x_2+y_1+y_2) \in S$ ✓

$+ \ =?$ ✓

3) $\forall \lambda \in \mathbb{R}$ $(x_1, x_2, x_1+x_2) \in S$

$\lambda \odot (x_1, x_2, x_1+x_2) = (\lambda x_1, \lambda x_2, \lambda(x_1+x_2)) \in S$ ✓

$+ \ ?$ ✓

✓

$\Rightarrow S \leq \mathbb{R}^3$

2. Determine whether the following sets form subspaces of $\mathbb{R}^3$:

(a) $\{(x_1, x_2, x_3)^T \mid x_1 + x_3 = 1\}$

(b) $\{(x_1, x_2, x_3)^T \mid x_1 = x_2 = x_3\}$

(c) $\{(x_1, x_2, x_3)^T \mid x_3 = x_1 + x_2\}$

(d) $\{(x_1, x_2, x_3)^T \mid x_3 = x_1 \text{ or } x_3 = x_2\}$

d) $S = \{ (x_1, x_2, x_3) : x_3 = x_1 \text{ or } x_3 = x_2 \}$

101, 202, ... 0,1,1 (2,3,3)

1) $(0,0,0) \in S$ ✓

2) $(x_1, x_2, x_3) \oplus (y_1, y_2, y_3)$ 3 0 2

$= (x_1+y_1, x_2+y_2, x_3+y_3)$ may not be S.

$\left.\begin{matrix}(2,0,2) \in S \\ (1,0,0) \in S\end{matrix}\right\}$ $(2,0,2) + (1,0,0) = (3,0,2) \notin S$

$3 \neq 2$ $0 \neq 2$

$\Rightarrow S$ is not a subspace of $\mathbb{R}^3$. $\Rightarrow$ ✗

1. Determine whether the following sets form subspaces of $\mathbb{R}^2$:

(a) $\{(x_1, x_2)^T \mid x_1 + x_2 = 0\}$

(b) $\{(x_1, x_2)^T \mid x_1 x_2 = 0\}$

(c) $\{(x_1, x_2)^T \mid x_1 = 3x_2\}$

(d) $\{(x_1, x_2)^T \mid |x_1| = |x_2|\}$

(e) $\{(x_1, x_2)^T \mid x_1^2 = x_2^2\}$

3. Determine whether the following are subspaces of $\mathbb{R}^{2\times 2}$:

✓**(a)** The set of all 2×2 diagonal matrices

✗**(b)** The set of all 2×2 triangular matrices

✓**(c)** The set of all 2×2 lower triangular matrices

(d) The set of all 2×2 matrices A such that $a_{12} = 1$ ✗

$\begin{bmatrix} x & 1 \\ y & z \end{bmatrix}$

(e) The set of all 2×2 matrices B such that $b_{11} = 0$ ✓

$\begin{bmatrix} 0 & x \\ y & z \end{bmatrix}$

~~**(f)** The set of all symmetric 2×2 matrices~~

~~**(g)** The set of all singular 2×2 matrices~~

$\mathbb{R}^{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}$

$\oplus \to$ matrix addition

$\odot \to$ scalar multiplication

$0_{\vec{v}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

a) $S = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbb{R} \right\}$ $S \leq \mathbb{R}^{2\times 2}$

1) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ ✓

2) $\begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix} + \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 & 0 \\ 0 & y_1+y_2 \end{bmatrix} \in S$ ✓

3) $\lambda \in \mathbb{R}$ $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in S$

$\lambda \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} \lambda x & 0 \\ 0 & \lambda y \end{bmatrix} \in S$ ✓