Representation Matrix for a Linear Transformation

We can represent any linear transformation with a matrix.

$$L : \mathbb{R}^{n} \to \mathbb{R}^{m} \qquad L(\vec{x}) = A\vec{x}$$

A → matrix representation for L.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \longmapsto \underset{m\times n}{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} | \\ | \\ | \end{bmatrix}_{m\times 1}$$

Case 1 : The formula of L are given wrt the standard bases.

Ex $L : \mathbb{R}^2 \to \mathbb{R}^3$ $\quad A_{3\times 2}$ $\quad$ Find the representation matrix for L.

$$(x,y) \longmapsto (x, x+y, x-y)$$

$$x\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{e_1} + y\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{e_2} \qquad x\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{e_1} + (x+y)\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{e_2} + (x-y)\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{e_3}$$

$$A = \begin{bmatrix} | & | \\ L(e_1) & L(e_2) \\ | & | \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}_{3\times 2}$$

$\uparrow L(e_1)$ $\quad \uparrow L(e_2)$

$\Rightarrow$ The j^{th} column of $A = L(e_j)$

$L(e_1) = L((1,0)) = (1, 1+0, 1-0) = (1,1,1)$ → wrt the standard basis → 1st column of A

$L(e_2) = L((0,1)) = (0, 0+1, 0-1) = (0,1,-1)$ → " → 2nd column of A.

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x+y \\ x-y \end{bmatrix} = L((x,y))$$

$$\underset{\text{mapping}}{L} \longleftrightarrow \underset{\text{matrix}}{A}$$

they do the same operation

Case 2 : The formula of L is given with some other bases, and the matrix is asked wrt those bases.

Ex $L : \mathbb{R}^3 \to \mathbb{R}^2$ $\qquad \vec{b}_1 = \begin{bmatrix}1\\1\end{bmatrix} \quad \vec{b}_2 = \begin{bmatrix}-1\\1\end{bmatrix}$

$$(x_1, x_2, x_3) \longmapsto x_1\vec{b}_1 + (x_2+x_3)\vec{b}_2$$

standard basis

$B = \{\vec{b}_1, \vec{b}_2\}$ → another basis for $\mathbb{R}^2$

$A_{E,B} = ?$

E: the basis for the LHS

B: the basis for the RHS

standard basis $\quad B=\{b_1,b_2\}$ another for $\mathbb{R}^2$

$$L(e_1) = L((1,0,0)) = 1.\vec{b_1} + (0+0)\vec{b_2} = 1\vec{b_1} + 0\vec{b_2} \qquad \begin{bmatrix}1\\0\end{bmatrix} \to \text{1st column of } A_{E,B}.$$

$$L(e_2) = L((0,1,0)) = 0.\vec{b_1} + (1+0)\vec{b_2} = 0\vec{b_1} + 1\vec{b_2} \qquad \begin{bmatrix}0\\1\end{bmatrix} \to \text{2nd column}$$

$$L(e_3) = L((0,0,1)) = 0.\vec{b_1} + (0+1)\vec{b_2} = 0\vec{b_1} + 1\vec{b_2} \qquad \begin{bmatrix}0\\1\end{bmatrix} \to \text{3rd column}$$

$$\Rightarrow A_{E,B} = \begin{bmatrix}1&0&0\\0&1&1\end{bmatrix}_{2\times3}$$

! Ex

$$L: \mathbb{R}^2 \to \mathbb{R}^2 \qquad \mathcal{U} = \left\{ u_1 = \begin{bmatrix}1\\1\end{bmatrix}, u_2 = \begin{bmatrix}-1\\1\end{bmatrix} \right\} \to \text{another basis for } \mathbb{R}^2.$$

$$\to \underbrace{x_1\vec{u_1} + x_2\vec{u_2}}_{\substack{\text{wrt the basis } \mathcal{U} \\ \{u_1,u_2\}}} \longmapsto \underbrace{(x_1+x_2)\vec{u_1} + 2x_2\vec{u_2}}_{\substack{\text{wrt the basis } \mathcal{U}. \\ \{u_1,u_2\}}}$$

Find a representation matrix for L wrt the basis $\mathcal{U}$.

$A_{\mathcal{U},\mathcal{U}} = ?$

$$L(u_1) = L((1,1)) = L(1.\vec{u_1} + 0.\vec{u_2}) = (1+0)\vec{u_1} + 2.0.\vec{u_2} = 1\vec{u_1} + 0\vec{u_2}$$

$1.\begin{bmatrix}1\\0\end{bmatrix} + 1.\begin{bmatrix}0\\1\end{bmatrix}$

$[u_1]_E \qquad [u_1]_{\mathcal{U}} = ?$

$\to \begin{bmatrix}1\\0\end{bmatrix} \to$ 1st column

$$(1,1) = x_1.\vec{u_1} + x_2.\vec{u_2}$$ ($\vec{u_1}$)

$x_1 = 1 \quad x_2 = 0$

$[u_1]_{\mathcal{U}} =$

$$L(u_2) = L((-1,1)) = L((0.\vec{u_1} + 1.\vec{u_2})) = (0+1)\vec{u_1} + 2.1.\vec{u_2}$$

$$= 1\vec{u_1} + 2\vec{u_2} \to \begin{bmatrix}1\\2\end{bmatrix} \to \text{2nd column}$$

$[u_2]_E = (-1,1) \qquad u_2 = -1\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\end{bmatrix}$

$\to [u_2]_{\mathcal{U}} = (0,1) \leftarrow \qquad u_2 = x_1\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\vec{u_1}} + x_2\underbrace{\begin{bmatrix}-1\\1\end{bmatrix}}_{\vec{u_2}}$

$x_1 = 0 \quad x_2 = 1$

$$\Rightarrow A_{\mathcal{U},\mathcal{U}} = \begin{bmatrix}1&1\\0&2\end{bmatrix}$$

Case 3. L is given in standard bases both sides but A is asked wrt other bases.

Ex

$$L: \mathbb{R}^2 \to \mathbb{R}^3$$

$$\underbrace{(x_1,x_2)}_{\text{coordinates are in standard basis}} \longmapsto \underbrace{(x_2, x_1+x_2, x_1-x_2)}_{\text{coordinates are in standard basis}}$$

Find the representation matrix of L wrt the bases

$$\mathcal{U} = \left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix} \right\} \text{ and } V = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$$

$A_{\mathcal{U},V} = ?$

$$L(\vec{u_1}) = L(\underbrace{(1,2)}_{\text{standard basis}}) = (2, 1+2, 1-2) = \underbrace{(2,3,-1)}_{\text{wrt the standard basis}} \longrightarrow \text{coordinates wrt } V\,?$$

The transition matrix from E to V. $\to V^{-1}E = V^{-1} \Rightarrow V^{-1}\begin{bmatrix}2\\3\\-1\end{bmatrix} = \begin{bmatrix}-1\\4\\-1\end{bmatrix} \to$ 1st column of $A_{\mathcal{U},V}$

matrix from E to V. $\rightarrow V \quad E = V^{-1} \Rightarrow V \quad \begin{bmatrix} \\ 3 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ column of $A_{u,v}$

$$L(\vec{u}_2) = L(\underbrace{(3,1)}_{\text{standard basis}}) = (1, 3+1, 3-1) = \underbrace{(1,4,2)}_{\text{wrt the standard basis}}$$

$$V^{-1}\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix} \rightarrow \text{2nd column}$$

$$\Rightarrow A_{u,v} = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix}$$

Ex

$$L: \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$\underbrace{(x_1, x_2, x_3)}_{\text{wrt the standard basis}} \mapsto \underbrace{(2x_2, -x_1)}_{\text{wrt the standard basis}}$$

Find the representation matrix of L wrt the bases $U = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} \rightarrow \mathbb{R}^3$

$V = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\} \rightarrow \mathbb{R}^2$

$A_{u,v} = ?$

$$V^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

The transition matrix from E to V

$$L(u_1) = L((1,0,-1)) = (2 \cdot 0, -1) = (0,-1) \rightarrow \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \rightarrow \text{1st column}$$

$$L(u_2) = L((1,2,1)) = (2 \cdot 2, -1) = (4,-1) \rightarrow \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \rightarrow \text{2nd column}$$

$$L(u_3) = L((-1,1,1)) = (2 \cdot 1, -(-1)) = (2,1) \rightarrow \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \rightarrow \text{3rd column}$$

$$A_{u,v} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 3 \end{bmatrix}$$

$$L: \qquad \rightarrow A_{u,v} = \begin{bmatrix} | & | & \\ [L(u_1)]_V & [L(u_2)]_V & \cdots \\ | & | & \end{bmatrix}$$

You may need to use change of basis anywhere!