8th Week Monday

12 Nisan 2021 Pazartesi 09:35

* Vector Spaces

✓1) $\underbrace{\vec{v}_1 \oplus \vec{v}_2}_{LHS} \overset{?}{=} \underbrace{\vec{v}_2 \oplus \vec{v}_1}_{RHS}$

2) closed under vector addition $v_1, v_2 \in V \quad v_1 \oplus v_2 \in V$; closed under scalar multiplication $\alpha \in \mathbb{R} \quad v \in V \quad \alpha v \in V$

✓2) $(\vec{v}_1 \oplus \vec{v}_2) \oplus \vec{v}_3 = \vec{v}_1 \oplus (\vec{v}_2 \oplus \vec{v}_3)$

✓3) The identity element of vector addition $\in V$

✓4) Inverse element of vector addition $\in V$

✓5) $\alpha \in \mathbb{R}$ (scalar) $\quad \alpha \circ (\vec{v}_1 \oplus \vec{v}_2) = \alpha \circ \vec{v}_1 \oplus \alpha \circ \vec{v}_2$

✓6) $r, s \in \mathbb{R}$ (scalars) $\vec{v} \in V \quad (r+s) \circ \vec{v} = r \circ \vec{v} \oplus s \circ \vec{v}$

✓7) $r, s \in \mathbb{R}$ (scalars) $\vec{v} \in V \quad (rs) \circ \vec{v} = r \circ (s \circ \vec{v})$

✓8) $1 \in \mathbb{R}$ $\vec{v} \in V \quad 1 \circ \vec{v} = \vec{v}$

v_1, v_2, x, y vectors are positive real numbers

12. Let R^+ denote the set of positive real numbers. Define the operation of scalar multiplication, denoted $\circ$, by scalars are all real numbers. $\alpha, r, s \in \mathbb{R}$

scalar multiplic. $\rightarrow$ $\boxed{\alpha \circ x = x^\alpha}$

for each $x \in R^+$ and for any real number α. Define the operation of addition, denoted $\oplus$, by

$\boxed{x \oplus y = x \cdot y}$ * vector addition for all $x, y \in R^+$

Thus, for this system, the scalar product of -3 times $\frac{1}{2}$ is given by

$$-3 \circ \frac{1}{2} = \left(\frac{1}{2}\right)^{-3} = 8$$

and the sum of 2 and 5 is given by

1) $\rightarrow 2 \oplus 5 = 2 \cdot 5 = 10$

$\rightarrow 5 \oplus 2 = 5 \cdot 2 = 10$

Is R^+ a vector space with these operations? Prove your answer.

3) $e \in R^+$, $\forall v \in R^+$ $\quad v \oplus e = v \cdot e = v$ $\quad e = 1$ ✓

4) $v \oplus ? = 1$ $\quad \forall v \in R^+$ $\quad ? = 1/v \in R^+$ ✓

2) $(v_1 \oplus v_2) \oplus v_3 = (v_1 \cdot v_2) \oplus v_3 = v_1 \cdot v_2 \cdot v_3$

$v_1 \oplus (v_2 \oplus v_3) = v_1 \oplus (v_2 \cdot v_3) = v_1 \cdot v_2 \cdot v_3$ $\Big\} = $ ✓

5) $\alpha \circ (\vec{v}_1 \oplus \vec{v}_2) = \alpha \circ (v_1 \cdot v_2) = (v_1 \cdot v_2)^\alpha$

$(\alpha \circ \vec{v}_1) \oplus (\alpha \circ \vec{v}_2) = v_1^\alpha \oplus v_2^\alpha = v_1^\alpha \cdot v_2^\alpha$ $=$ ✓

6) $(r+s) \circ \vec{v} = v^{r+s}$

$r \circ \vec{v} \oplus s \circ \vec{v} = v^r \oplus v^s = v^r \cdot v^s$ $=$ ✓

7) $(r.s) \circ \vec{v} = v^{rs}$

$r \circ (s \circ \vec{v}) = r \circ (v^s) = (v^s)^r$ $=$ ✓

8) $1 \circ \vec{v} = v^1 = v$ ✓

* Supspaces $\quad S \subset (V, \oplus, \circ)$

1) $\vec{0}_V \in S$ ✓

2) $s_1 \oplus s_2 \in S$ ✓

3) $\forall \alpha \in \mathbb{R} \quad \alpha s_1 \in S$ ✓

$\Big\} \Rightarrow S < V$

* Linear Combinations & Span

Q: Do $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ span $\vec{V}$? $\rightarrow$ write a typical vector of V

$\vec{V} = (a_1, a_2, \dots, a_n)$

$\rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = \vec{V}$

Can you find $\alpha_1, \alpha_2, \dots \alpha_n$ in terms of $a_1, a_2, \dots, a_n$

$$\alpha_1 \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} + \alpha_2 \begin{bmatrix} | \\ v_2 \\ | \end{bmatrix} + \dots + \alpha_n \begin{bmatrix} | \\ v_n \\ | \end{bmatrix} = \boxed{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{\vec{V}}}$$

Try to solve this linear system.

no solution $\Rightarrow$ ✗

unique soln., inf. many solutions $\Rightarrow$ ✓

Linear Independence

Q: Are $v_1, v_2, \dots, v_n$ linearly independent?

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0} \Rightarrow \text{ if } c_1 = c_2 = \dots = c_n = 0$$

$\Rightarrow v_1, v_2, \dots, v_n$ are lin. indep. ✓

if not " are not lin. indep ✗

$$c_1 \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} + c_2 \begin{bmatrix} | \\ v_2 \\ | \end{bmatrix} + \dots + c_n \begin{bmatrix} | \\ v_n \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \boxed{\underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}}_{\text{solve this homogeneous system.}} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$$ *

→ trivial solution $\Rightarrow c_1 = c_2 = \dots = c_n = 0$ ✓
$\Rightarrow v_1, v_2, \dots, v_n$ are lin. indep.

→ inf. many solns. ✗

* if we can write any vector in the set, $\{v_1, v_2, \dots, v_n\}$ as a linear combination of some other vectors in the set, $\Rightarrow$ The set is not lin. indep.

* if you have the zero vector in the set $\Rightarrow$ The set is not lin. indep.

***Theorem** Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in \mathbb{R}^n$ (n vectors taken from n-tuples)

$$\underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}}_{X}{}_{n\times n} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow *$$

[X is singular $\Leftrightarrow$ $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are NOT linearly independent.]

* If X is invertible $\Leftrightarrow$ $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent..

$$\underbrace{\left[\begin{array}{cccc|c} | & | & & | & 0 \\ v_1 & v_2 & \cdots & v_n & 0 \\ | & | & & | & 0 \\ & & & & \vdots \\ & & & & 0 \end{array}\right]}_{X}{}_{n\times n}$$

X (square) $\xrightarrow{\text{rref}}$ you may either get I_n or not I_n

get I_n:

$$\left[\begin{array}{cccc|c} 1 & & 0 & & 0 \\ & 1 & & & 0 \\ 0 & & 1 & & \vdots \\ & & & \ddots 1 & 0 \end{array}\right] \Rightarrow \left.\begin{matrix} c_1 = 0 \\ c_2 = 0 \\ \vdots \\ c_n = 0 \end{matrix}\right\} \Rightarrow \text{lin. indep.}$$

$\Rightarrow \det(X) \neq 0 \Rightarrow$ X is invertible

not I_n:

$$\left[\begin{array}{cccc|c} 1 & & & & 0 \\ & 1 & & & 0 \\ 0 & - & - & - & \vdots \\ 0 & - & - & 0 & 0 \end{array}\right] \rightarrow \left.\text{inf. many solns.}\right\} \Rightarrow \text{NOT lin. indep.}$$

$\Rightarrow \det(X) = 0 \Rightarrow$ X is not invertible

$$\Rightarrow \det(X)=0 \Rightarrow X \text{ is not invertible } (X \text{ is singular}).$$

Ex $\vec{v}_1 = (4,2,3)^T$, $\vec{v}_2 = (2,3,1)^T$, $v_3 = (2,-5,3)^T$

book notation. $\rightsquigarrow \langle 4,2,3 \rangle \to$ webwork notation for vectors.

$\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$ Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly independent?

Don't need to solve

$$\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \left[\begin{array}{ccc|c} 4 & 2 & 2 & 0 \\ 2 & 3 & -5 & 0 \\ 3 & 1 & 3 & 0 \end{array}\right] \end{array} \dashrightarrow \cancel{[\,I_n \mid 0\,]} \quad c_1 = c_2 = \dots 0$$

$\to$ Just check the $\det(X)$

$$\det(X) = 4\begin{vmatrix} 3 & -5 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & -5 \\ 3 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}$$

$9+5 \qquad 6+15 = 21 \qquad 2-9$

$$= 56 - 42 - 14$$

$$= 0$$

$\det(X) = 0 \Rightarrow X$ is singular $\Rightarrow \{v_1, v_2, v_3\}$ are NOT linearly independent!

* If X is not square, you can not use the above thm.

Ex $\vec{v}_1 = (1,-1,2,3)^T$ $\vec{v}_2 = (-2,3,1,-2)^T$ $\vec{v}_3 = (1,0,7,7)^T$

Are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly independent?

$$X = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & 0 \\ 2 & 1 & 7 \\ 3 & -2 & 7 \end{bmatrix}_{4\times 3}$$ is not square matrix!

$\to$ can not use the above thm.

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -1 & 3 & 0 & 0 \\ 2 & 1 & 7 & 0 \\ 3 & -2 & 7 & 0 \end{array}\right] \xrightarrow{REF} \begin{array}{c} c_1, c_2, c_3 \\ \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\to \begin{array}{l} c_1 - 2c_2 + c_3 = 0 \\ c_2 + c_3 = 0 \end{array} \quad \left.\begin{array}{l} c_3 = \text{free} \\ c_1 = \dots \\ c_2 = \dots \end{array}\right\} \Rightarrow \text{inf. many soln.}$$

$\Rightarrow \vec{v}_1, \vec{v}_2, \vec{v}_3$ are NOT lin. independent.

Ex

8. Determine whether the following vectors are linearly independent in P_3: $\to$ poly. with degree < 3

Ex

8. Determine whether the following vectors are linearly independent in P_3: → poly. with degree < 3

→ ~~(a)~~ $1, x^2, x^2 - 2$ ~~(b)~~ $2, x^2, x, 2x+3 = 2v_3 + \frac{3}{2} \cdot v_1$ $1, x, x^2$ → are linearly independent

(c) $x+2, x+1, x^2-1$ (d) $x+2, x^2-1$

→ ex

$\vec{v_1} = 1, \quad \vec{v_2} = x, \quad \vec{v_3} = x^2 \quad \in P_3$ ~~v_2^2~~

$$c_1 \cdot 1 + c_2 \cdot x + c_3 x^2 = 0 \quad \Rightarrow c_1 = c_2 = c_3 = 0$$

$\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent!

→ a) $\vec{v_1} = 1, \quad \vec{v_2} = x^2, \quad \vec{v_3} = x^2 - 2 \qquad (\vec{v_3} = -2 \cdot \vec{v_1} + \vec{v_2})$

$$\to c_1 \cdot 1 + c_2 \cdot x^2 + c_3 (x^2 - 2) = 0$$

$$\Rightarrow c_1 + c_2 x^2 + c_3 x^2 - 2c_3 = 0$$

constant term $= c_1 - 2c_3 = 0$ $\qquad c_3 = r \in \mathbb{R}$ free

coef. of $x^2 = c_2 + c_3 = 0$ $\qquad c_1 = 2r$, $\quad c_2 = -r$

} inf. many solns. ⇒ v_1, v_2, v_3 are NOT linearly independent.

Ex $\vec{v_1} = x - 1, \quad \vec{v_2} = x + 1, \quad \vec{v_3} = x^2 - 1 \quad \in P_3$

$$c_1 \vec{v_1} + c_2 \vec{v_2} + c_3 \vec{v_3} = 0$$

$$c_1 (x-1) + c_2 (x+1) + c_3 (x^2 - 1) = 0$$

$$c_1 x - c_1 + c_2 x + c_2 + c_3 x^2 - c_3 = 0$$

Const. term $= -c_1 + c_2 - c_3 = 0$

coef. of x $= c_1 + c_2 = 0$

coef of x^2 $= c_3 = 0$

} $-c_1 + c_2 = 0$, $+ \; c_1 + c_2 = 0$, $c_2 = 0$, $c_1 = 0$

⇒ $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent.

Ex

4. Determine whether the following vectors are linearly independent in $\mathbb{R}^{2\times 2}$: → vectors here are matrices $\qquad$ the zero vector $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

→ (a) $\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{v_2}$

→ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

→ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

a) $c_1 \vec{v_1} + c_2 \vec{v_2} = \vec{0}$ $\qquad c_1 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\to \begin{bmatrix} c_1 & 0 \\ \ldots & \ldots \end{bmatrix} + \begin{bmatrix} 0 & c_2 \\ \ldots & \ldots \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ \ldots & \ldots \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} c_1 & 0 \\ c_1 & c_1 \end{bmatrix} + \begin{bmatrix} 0 & c_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_1 & c_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$c_1 = 0$, $c_2 = 0$ ✓

$\Rightarrow \vec{v_1}, \vec{v_2}$ are linearly independent!

b-) $$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 & 0 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} 0 & c_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c_3 & 0 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow c_1 = 0$, $c_2 = 0$, $c_3 = 0$

$\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent.

4 ←
5 → depen

$\rightarrow$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + 2c_3 & c_2 + 3c_3 \\ 0 & c_1 + 2c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$c_1 + 2c_3 = 0$ $\quad$ $c_3 =$ free $\Rightarrow$ inf. many solns. $\Rightarrow v_1, v_2, v_3$ are NOT linearly independent.

$c_2 + 3c_3 = 0$