Eigenvalues and Eigenvectors

$A_{n\times n}$ → a square matrix

$\lambda \in \mathbb{R}$

$\underbrace{A}_{\text{your matrix}} \vec{x} = \underbrace{\lambda}_{\text{scalar}} \vec{x}$ → EIGENVECTORS corresponding to this λ.

scalar → EIGENVALUES

If a vector $\vec{x}$ satisfying $A\vec{x} = \lambda\vec{x}$ can be found,

⇒ λ is an eigenvalue of A.

$$A\vec{x} = \lambda\vec{x} \Rightarrow A\vec{x} - \lambda\vec{x} = 0$$

$$\Rightarrow \underbrace{(\underset{\text{matrix}}{A} - \underset{\text{scalar}}{\lambda} I)}_{n\times n}\underset{n\times 1}{\vec{x}} = \underset{n\times 1}{\vec{0}}$$

If this system has a nontrivial solution for $\vec{x}$ ⇒ λ is an eigenvalue.

⇒ When does such a homogeneous system has only the trivial solution?

$$A-\lambda I \,|\, 0 \quad \left[\begin{array}{c|c} & 0 \\ & 0 \\ & \vdots \\ & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & & & 0 & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ 0 & & & 1 & 0 \end{array}\right] \rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ \vdots \\ x_n = 0 \end{array}$$

the trivial soln.

→ $A-\lambda I$ → only if $\det(A-\lambda I) \neq 0$ → I_n

! WE DON'T WANT THIS!

⇒ We should have the case where $\det(A-\lambda I) = 0$ ←

λ → unknown

⇒ $p(\lambda) = \det(A - \lambda I) = 0$ → The characteristic polynomial of A.

⇒ The roots of this polynomial → eigenvalues

⇒ Once you find an eigenvalue $(A - \lambda I)\vec{x} = 0$ ⇒ Find all solutions for $\vec{x}$.

λ → this is known

⇒ Basis of the eigenspace ⇒ eigenvectors corresponding to that λ.

⇒ eigenspace corresponding to that λ.

⇒ Do the last 2 steps for each λ.

Ex/ $A = \begin{bmatrix} 3 & 2 \\ \cdot & \cdot \end{bmatrix}$ Find the characteristic polynomial.

Ex/ $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$ Find the characteristic polynomial.
Find the eigenvalues and corresponding eigenvectors

$$\det(A-\lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = \underbrace{(3-\lambda)(-2-\lambda)} - 6 = 0$$

$$\Rightarrow -6 + 2\lambda - 3\lambda + \lambda^2 - 6 = 0$$

$$\Rightarrow \lambda^2 - \lambda - \underset{-4,3}{12} = 0$$

$$\Rightarrow (\lambda-4)(\lambda+3) = 0 \rightarrow \boxed{\lambda_1 = 4}, \boxed{\lambda_2 = -3}$$

eigenvalues of A.

The characteristic polynomial of A $\Rightarrow$ $\boxed{\lambda^2 - \lambda - 12 = 0}$

For $\lambda_1 = 4$: $(A - \lambda I)\vec{x} = 0$

$$A - 4I = \begin{bmatrix} 3-4 & 2 \\ 3 & -2-4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}}_{\det = 0} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \text{Find nontrivial solutions.}$$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc} 1 & -2 & 0 \\ 3 & -6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - 2x_2 = 0$

$x_2 = r \in \mathbb{R}$

$x_1 = 2r$

Eigenspace for $\lambda = 4$: span $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ $\xrightarrow{\text{a typical element}}$ $\begin{bmatrix} 2r \\ r \end{bmatrix} = r \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\hookrightarrow$ the eigenvector corresponding to $\lambda = 4$.

For $\lambda_2 = -3$:

$$A - (-3)I = \begin{bmatrix} 3+3 & 2 \\ 3 & -2+3 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$$

$(A+3I)\vec{x} = \vec{0}$ $\left[\begin{array}{cc|c} 6 & 2 & 0 \end{array}\right.$ $\left[\begin{array}{cc|c} 3 & 1 & 0 \end{array}\right]$ $3x_1 + x_2 = 0$

$$(A+3I)\vec{x}=\vec{0} \Rightarrow \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$3x_1+x_2=0$
$x_1=r\in\mathbb{R}$
$x_2=-3r$

Eigenspace of $\lambda=-3$: $\text{Span}\left\{\begin{bmatrix}1\\-3\end{bmatrix}\right\}$ $\xrightarrow{\text{a typical element}} \begin{bmatrix}r\\-3r\end{bmatrix}=r\begin{bmatrix}1\\-3\end{bmatrix} \rightarrow$ eigenvector corresponding to $\lambda=-3$

Ex

$$A=\begin{bmatrix}3 & -1 & -2\\ 2 & 0 & -2\\ 2 & -1 & -1\end{bmatrix}$$

characteristic polynomial of A? eigenvalues of A? corresponding eigenvectors?

$$A-\lambda I=\begin{bmatrix}3 & -1 & -2\\ 2 & 0 & -2\\ 2 & -1 & -1\end{bmatrix}-\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}$$

$$\det(A-\lambda I)=0$$

$$\Rightarrow A-\lambda I=\begin{bmatrix}3-\lambda & -1 & -2\\ 2 & -\lambda & -2\\ 2 & -1 & -1-\lambda\end{bmatrix}$$

$$\det(A-\lambda I)=(3-\lambda)\begin{vmatrix}-\lambda & -2\\ -1 & -1-\lambda\end{vmatrix}-(-1)\begin{vmatrix}2 & -2\\ 2 & -1-\lambda\end{vmatrix}+(-2)\begin{vmatrix}2 & -\lambda\\ 2 & -1\end{vmatrix}$$

$+1$ $\quad 2(-1-\lambda)-(-4$ $\quad -2\,(-2-(-2\lambda))$

$$=(3-\lambda)\big[\underbrace{(\lambda^2+\lambda)}_{\lambda(\lambda+1)}-2\big]+\underline{-2(1+\lambda)+4}+\underline{(-2).[-2+2\lambda]}$$

$-6+2\lambda+4+4-4\lambda$
$-2\lambda+2$

$$=(3-\lambda)\lambda(\lambda+1)-2(3-\lambda)-2(1+\lambda)+4+4(1-\lambda)$$

$(3-\lambda)\lambda-2$
$3\lambda-\lambda^2-2$
$-(\lambda^2-3\lambda+2)$
$-(\lambda-2)(\lambda-1)$

$$=(\lambda+1)\big[(\lambda-2)(1-\lambda)\big]+2(1-\lambda)$$

$$=(1-\lambda)\big[(\lambda+1)(\lambda-2)+2\big]$$

$\lambda^2-2\lambda+\lambda-\cancel{2}+\cancel{2}$

$\lambda^2+\lambda-6$
$(\lambda+3)(\lambda-2)$

$$=(1-\lambda)\big[\lambda^2-\lambda\big]$$

$$=(\lambda-1)\lambda(\lambda-1)$$

a repeated root

$\boxed{\lambda_1=1}$ $\boxed{\lambda_2=0}$

$$=\lambda^3-2\lambda^2+\lambda \rightarrow \text{characteristic poly of } A.$$

For $\lambda_1=1$:

$(A-1I)\vec{x}=0$
The nontrivial solution?

$$A-1I=\begin{bmatrix}3-1 & -1 & -2\\ 2 & -1 & -2\\ 2 & -1 & -1-1\end{bmatrix}=\begin{bmatrix}2 & -1 & -2\\ 2 & -1 & -2\\ 2 & -1 & -2\end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & -2 & 0\\ 2 & -1 & -2 & 0\\ \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & -2 & 0\\ 0 & 0 & 0 & 0\\ & & & 0\end{array}\right]$$

$2x_1-x_2-2x_3=0$
$x_1=r\in\mathbb{R}$

$$\left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 2 & -1 & -2 & 0 \\ 2 & -1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$2x_1 - x_2 - 2x_3 = 0$

$x_1 = r \in \mathbb{R}$

$x_3 = s \in \mathbb{R}$

$x_2 = 2r - 2s$

$$\rightarrow \begin{bmatrix} r \\ 2r-2s \\ s \end{bmatrix} = r \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

eigenvectors corresponding to $\lambda_1 = 1$.

For $\lambda_2 = 0$:

$(A + 0I)\vec{x} = \vec{0}$

$$A - 0I = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

$$3 \begin{vmatrix} 0 & -2 \\ -1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -2 \\ 2 & -1 \end{vmatrix}$$

$-6 + 2 \quad +4 \quad -2+4$

0 ✓

$$\left[\begin{array}{ccc|c} 3 & -1 & -2 & 0 \\ 2 & 0 & -2 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/3 & -2/3 & 0 \\ 2 & 0 & -2 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right]$$

$-2r_1 + r_2 \to r_2$

$-2r_1 + r_3 \to r_3$

$$\left[\begin{array}{ccc|c} 1 & -1/3 & -2/3 & 0 \\ 0 & 2/3 & -2/3 & 0 \\ 0 & -1/3 & 1/3 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1/3 & -2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - \frac{r}{3} - \frac{2r}{3} = 0$

$x_1 - x_2/3 - 2x_3/3 = 0$

$x_2 - x_3 = 0$

$x_3 = r \in \mathbb{R}$

$\Rightarrow x_2 = r$

$\Rightarrow x_1 = r$

$$\begin{bmatrix} r \\ r \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

eigenvector corresponding to $\lambda = 0$.

1. Find the eigenvalues and the corresponding eigenspaces for each of the following matrices:

(a) $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} 3 & -8 \\ 2 & 3 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$ **(f)** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **(h)** $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix}$

(i) $\begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ **(j)** $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

(k) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ **(l)** $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$