MATHEMATICAL PROGRAMMING

Sophie Demassey (CMA, Mines ParisTech/PSL)
sophie.demassey@mines-paristech.fr
`http://sofdem.github.io`

Overview

introduction

modeling LPs

geometry and algebra

the simplex methods

duality

sensitive analysis

INTRODUCTION

INTRODUCTION

DECISION IS OPTIMIZATION

select the **best** of all **possible** alternatives – the solutions – regarding a quantitative criterion – the objective.

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Examples:

- minimize travel **time**, project duration
- maximize storage **space** capacity, minimize travel distance
- minimize system design **cost**, maximize system operation profit
- maximize **good** production, minimize energy consumption
- mechanical **equilibrium**: minimize potential energy

MODELING FOR SOLVING

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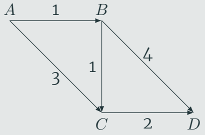
ex: models to minimize the route duration

- the sequences of locations: $(ABCD4, ABD5, ACD5)$

- graph + restrictions:



- logic+arithmetic relations:

$\min d_1 + d_2 + d_3,$

$x_1x_2d_1 \in \{AB1, AC3\} \wedge$

$x_2x_3d_2 \in \{BC1, BD4, CD2\} \wedge$

$x_3x_4d_3 \in \{CD2, DD0\}$

MODELING FOR SOLVING

A model is an abstract representation of the problem solutions, implicitly as **relationships** between **unknowns** (rather than explicitly as a list of tuples)

A mathematical optimization model, or mathematical program, employs **real-values functions** of **real-valued variables**:

$$\min\{f(x) | g(x) \geq 0, x \in \mathbb{R}^n\}$$

with $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the objective, and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ the constraints.

SOLUTIONS: THEORY VS PRACTICE

feasibility

- approximate models (e.g., abstract routes)
- uncertain data (e.g., erratic duration)
- floating-point numerical errors

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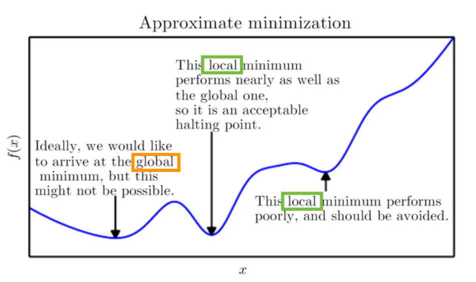
optimality

- finite time complexity $\neq$ reachable
- provable within a gap tolerance
- provable locally:



SOLVING TECHNIQUES

analytical methods come from a **provable theory**, e.g.:

- $min\ x^2 - 4x + 3, x \in [0, 5]$ (*Fermat, derivative*)
- shortest path in a graph (*Dijkstra, Bellman*)

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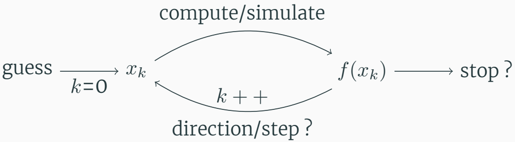
numerical methods **evaluate** assignments **iteratively**:



driven by f (derivative-free) or f' (1st-order) or f'' (2nd-order methods) depending on the structure of f

DIFFERENT SOLVING TECHNIQUES FOR DIFFERENT MODELS

- **fixed** / probabilistic data
- **single** / **multiple** objectives
- **with** / without constraints
- **analytic** / **logic** / **graphic** models
- **linear** / convex / nonconvex functions
- **differentiable** / non-differentiable functions
- **continuous** / **discrete** decisions

APPLICATIONS

operational research : system operation and design in human activities: transport, scheduling, packing, cutting, rostering, allocation,...

optimal control : command (u_t) to optimize trajectory (x_t) s.t. $x'(t) = g(x_t, u_t)$

machine learning : find a best match (e.g. a linear fit)

artificial intelligence : for robot decision

game theory : find the best respective strategies of multiple players

INTRODUCTION

mathematical optimization

OSE: demandez le programme

linear programs

standard form

linear algebra (review)

MATHEMATICAL OPTIMIZATION IN OSE

lectures & practice

(Integer) Linear Programming	9	SD (8), JPM (gurobi)	oct
Machine Learning	10	VR, VS	oct-nov
Integer Nonlinear Programming	1	SD	nov
Nonlinear Programming	3	WO	nov
Stochastic Programming	9	WO	feb
Prospective Modelling	8	NM, EA, SS	feb

application & project

ILP: power plant provision	6	SD	oct
INLP: water network operation	4	SD	nov

Edi Assoumou (EA), Sophie Demassey (SD), Welington de Oliveira (WO), Nadia Maïzi (NM), Jean-Paul Marmorat (JPM), Valérie Roy (VR), Valentina Sessa (VS), Sandrine Selosse (SS).

agenda

1.introduction	30/09		
2.models	1/10		[Chap 1]*
3.gurobipy	2/10	(JPM)	
3.geometry	2/10	TP models	[Chap 2]
4.simplex	6/10		[Chap 3]
5.duality	7/10	TP duality	[Chap 4]
6.sensitivity analysis	8/10	TP analysis	[Chap 5]
7.discrete models	12/10	TP integer	[Chap 10]
8.discrete solutions	15/10		[Chap 11]

*of [Bertsimas-Tsitsiklis]: Bertsimas, Dimitris, and John N. Tsitsiklis. Introduction to linear optimization. Vol. 6, Athena Scientific, 1997.

INTRODUCTION

MATHEMATICAL PROGRAMMING

programming = **planning** (military/industrial) operations

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$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } g(x) \geq 0 \\ &\qquad\qquad\quad x \in \mathbb{R}^n \end{aligned}$$

- x: the decision variables
- $f : \mathbb{R}^n \to \mathbb{R}$: the objective function (maximize $-f \equiv$ minimize f)
- $g : \mathbb{R}^n \to \mathbb{R}^m$: the constraints

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solution/assignment	$X \in \mathbb{R}^n$
feasible solution	$X \in g^{-1}(\mathbb{R}^m_+)$
optimal solution	$X \in \arg\min\{f(x) : g(x) \geq 0, x \in \mathbb{R}^n\}$

LINEAR PROGRAM

a mathematical program $min \{f(x)|g(x) \geq 0, x \in \mathbb{R}^n\}$
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Example with $n = 3$, $m = 2$:

$$f(x) = x_1 = \begin{pmatrix} 1, & 0, & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c^T x \quad \text{with } c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^{3\times 1}, \text{ transpose } c^T \in \mathbb{R}^{1\times 3}$$

$$g(x) = \begin{pmatrix} 5x_1 + 3x_2 - 2x_3 - 4 \\ x_1 + x_2 + x_3 + 1 \end{pmatrix} = \begin{pmatrix} 5, & 3, & -2 \\ 1, & 1, & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} -4 \\ 1 \end{pmatrix} = Ax + b$$

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$$\begin{aligned} \min\ & x_1 \\ \text{s.t.}\ & 5x_1 + 3x_2 - 2x_3 \geq 4 \\ & x_1 + x_2 + x_3 \geq -1 \\ & x \in \mathbb{R}^3 \end{aligned}$$

TAKING ADVANTAGE OF

- many optimization problems can be modeled as linear programs possibly after approximation or adding discrete variables $x \in \mathbb{Z}^p \times \mathbb{R}^{n-p}$

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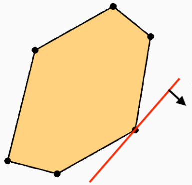
- many optimization problems can be modeled as linear programs possibly after approximation or adding discrete variables $x \in \mathbb{Z}^p \times \mathbb{R}^{n-p}$
- easy in theory: differentiable & convex (& concave) functions
- easy in practice: convenient geometric and algebraic properties



INTRODUCTION

LINEAR PROGRAM IN STANDARD FORM

equality constraints and **nonnegative** variables:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax = b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \min\ & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.}\ & \sum_{j=1}^{n} a_{ij} x_j = b_i, & \forall i = 1, \dots, m \\ & x_j \geq 0 & \forall j = 1, \dots, n \end{aligned}$$

with $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

REDUCTION TO STANDARD FORM

Every linear program

$$\min\{c^T x | Ax \geq b, x \in \mathbb{R}^n\}$$

can be transformed into an equivalent problem in standard form

$$\min\{d^T y | Ey = f, y \in \mathbb{R}^p_+\}$$

$$\begin{aligned} \min\ & x_1 \\ \text{s.t.}\ & 5x_1 + 3x_2 - 2x_3 \geq 4 \\ & x_1 + x_2 + x_3 \geq -1 \\ & x \in \mathbb{R}^3 \end{aligned}$$

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$$\begin{aligned} \min\ & (x_1^+ - x_1^-) \\ \text{s.t.}\ & 5(x_1^+ - x_1^-) + 3(x_2^+ - x_2^-) - 2(x_3^+ - x_3^-) - z_1 = 4 \\ & (x_1^+ - x_1^-) + (x_2^+ - x_2^-) + (x_3^+ - x_3^-) - z_2 = -1 \\ & x^+ \in \mathbb{R}^3_+, x^- \in \mathbb{R}^3_+, z \in \mathbb{R}^2_+ \end{aligned}$$

REDUCTION TO STANDARD FORM (RECIPE)

replace by		
negative variable	$x \leq 0$	$x = -z, z \geq 0$
free variable	y free	$y = y^+ - y^-, y^+, y^- \geq 0$
slack constraint	$Ax \geq b$	$Ax - s = b$, $s \geq 0$
slack constraint	$Ey \leq f$	$Ey + u = f$, $u \geq 0$
maximization	$max\ cx$	$-min(-c)x$

$$\begin{aligned} \max\ & c^T x + d^T y \\ \text{s.t.}\ & Ax \geq b \\ & Ey \leq f \\ & x \leq 0, y\ free \end{aligned}$$

$$\begin{aligned} \min\ & (-c)^T(-z) + (-d)^T(y^+ - y^-) \\ \text{s.t.}\ & A(-z) - s = b \\ & E(y^+ - y^-) + u = f \\ & z, y^+, y^-, s, u \geq 0 \end{aligned}$$

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LINEAR ALGEBRA REVIEW AND NOTATION (1)

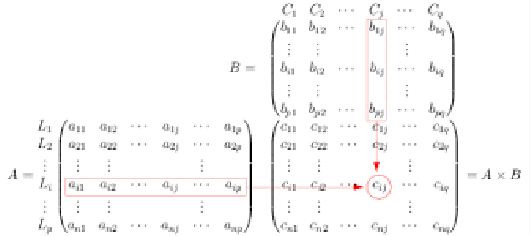
matrix $A \in \mathbb{R}^{m \times n}$ with entry a_{ij} in row $1 \le i \le m$, column $1 \le j \le n$

transpose $A^T \in \mathbb{R}^{n \times m}$ with $a^T_{ji} = a_{ij}$

(column) vector $a \in \mathbb{R}^n \equiv \mathbb{R}^{n \times 1}$

scalar product $a, b \in \mathbb{R}^n$, $\langle a, b \rangle = a^T b = b^T a = \sum_{j=1}^n a_j b_j$

matrix product $A \in \mathbb{R}^{m \times p}$, $B \in \mathbb{R}^{p \times n}$, $C = AB \in \mathbb{R}^{m \times n}$ with $c_{ij} = \sum_{k=1}^p a_{ik} b_{kj}$.
matrix product is associative $(AB)C = A(BC)$ and $(AB)^T = B^T A^T$



LINEAR ALGEBRA REVIEW AND NOTATION (2)

linear combination $\sum_{i=1}^{p} \lambda_i x^i \in \mathbb{R}^n$
of vectors $x^1, \dots, x^p \in \mathbb{R}^n$ with scalars $\lambda_1, \dots, \lambda_p \in \mathbb{R}$

linearly independence $\sum_{i=1}^{p} \lambda_i x^i = 0 \;\Rightarrow\; \lambda_1 = \cdots = \lambda_p = 0$

vector-space span $V = \{\sum_{i=1}^{p} \lambda_i x^i \mid \lambda_1, \dots, \lambda_p \in \mathbb{R}\} \subseteq \mathbb{R}^n$

dimension $dim(V) = p$ if $x^1, \dots, x^p$ are linearly independent, i.e. form a basis for V

row space of $A \in \mathbb{R}^{m \times n}$ span of the rows $rs_A = \{\lambda^T A, \lambda \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$

column space of $A \in \mathbb{R}^{m \times n}$ span of the columns $cs_A = \{A\lambda, \lambda \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$

rank of $A \in \mathbb{R}^{m \times n}$: $rk_A = dim(rs_A) = dim(cs_A) \leq \min(m, n)$

Reading:

to go further:

read [Bertsimas-Tsitsiklis]:
Section 1.1

for the next class:

read [Bertsimas-Tsitsiklis]:
Section 1.5: Linear algebra background

MODELING LPs

HOWTO

1. what are the decision variables ?
2. what are the constraints and the objective ?
3. are they linear ?

MODELING LPs

exercise *verre*

exercise *petroleum*

exercise *steel*

exercise *waste*

exercise *network*

exercise *planning*

exercise *distance*

exercise *data fitting*

EX 1: VERRE

Verre

Une ligne de production de portes et fenêtres est composée de 3 postes. Les postes 1 (fabrication de portes), 2 (fabrication de fenêtres) et 3 (finitions) ouvrent respectivement 4 heures, 12 heures et 18 heures par semaine et ne peuvent traiter chacun, indépendamment, qu'un élément à la fois. La fabrication d'une porte occupe le poste 1 pendant 1h puis le poste 3 pendant 3 heures. La fabrication d'une fenêtre occupe le poste 2 pendant 2h puis le poste 3 pendant 2h. Avec un prix de vente de 3000 euros par porte et 5000 euros par fenêtre, comment l'usine peut-elle maximiser ses revenus ?

ex 1: modèle PL

- variables de décision ?
  - x_P, x_F le nombre (fractionnaire) de portes et de fenêtres produites par jour
- contraintes ?
  - occupation des postes

$$
\begin{aligned}
\max\ & 3000x_P + 5000x_F \\
\text{s.t. } & x_P \leq 4 \\
& 2x_F \leq 12 \\
& 3x_P + 2x_F \leq 18 \\
& x_P, x_F \geq 0
\end{aligned}
$$

MODELING LPs

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EX 2: PRODUCTION PLANNING

The two crude petroleum problem [Ralphs]

A petroleum company distills crude imported from Kuwait (9000 barrels available at 20€ each) and from Venezuela (6000 barrels available at 15€ each), to produce gasoline (2000 barrels), jet fuel (1500 barrels), and lubricant (500 barrels) in the following proportions:

	gasoline	jet fuel	lubricant
Kuwait	0.3	0.4	0.2
Venezuela	0.4	0.2	0.3

(first entry reads *producing 1 unit of gasoline requires 0.3 units of crude from Kuwait*)

Objective: minimize the production cost.

ex 2: LP model

- decision variables ?
  - x_K, x_V the quantity (in thousands of barrel) to import from Kuwait or from Venezuela
- constraints ?
  - production and availability

$$
\begin{aligned}
\min\ & 20x_K + 15x_V \\
\text{s.t.}\ & 0.3x_K + 0.4x_V \geq 2 \\
& 0.4x_K + 0.2x_V \geq 1.5 \\
& 0.2x_K + 0.3x_V \geq 0.5 \\
& 0 \leq x_K \leq 9 \\
& 0 \leq x_V \leq 6
\end{aligned}
$$

MODELING LPs

EX 3: STEEL FACTORY

steel factory

A factory can produce steel in coils (*bobines*), tapes (*rubans*), and sheets (*tôles*) every week up to 6000 tons, 4000 tons and 3500 tons, respectively. The selling prices are 25, 30, and 2 euros, respectively, per ton of product. Production involves two stages, heating (*réchauffe*) and rolling (*laminage*). These two mills are available up to 35 hours and 40 hours a week, respectively. The following table gives the number of tons of products that each mill can process in 1 hour:

	heating	rolling
coils	200	200
tapes	200	140
sheets	200	160

The factory wants to maximize its profit.

EX 3: LP MODEL

- decision variables ?
  - x_C, x_T, x_S the quantity (in tons) of weekly produced coils, tapes and sheets
- constraints ?
  - mill occupation
  - maximum production

$$\max\ 25x_C + 30x_T + 2x_S$$

$$\text{s.t.}$$

$$\frac{x_C}{200} + \frac{x_T}{200} + \frac{x_S}{200} \leq 35$$

$$\frac{x_C}{200} + \frac{x_T}{140} + \frac{x_S}{160} \leq 40$$

$$0 \leq x_C \leq 6000$$

$$0 \leq x_T \leq 4000$$

$$0 \leq x_S \leq 3500$$

MODELING LPs

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EX 4: WASTE MANAGEMENT

waste management

A company eliminates nuclear wastes of 2 types A and B, by applying a sequence of 3 processes I, II and III in any order. The processes I, II, III, have limited availability, respectively: 450h, 350h, and 200h per month. Times to process one unit of waste are:

process	I	II	III
waste A	1h	2h	1h
waste B	3h	1h	1h

The unit profit for the company is 4000 euros for waste A and 8000 euros for waste B.

Objective: maximize the profit.

ex 4: LP model

- decision variables ?
  - x_A, x_B the number of units of waste of type A or B to process each month
- constraints ?
  - availability and operation

$$
\begin{aligned}
\max\ & 4000x_A + 8000x_B \\
\text{s.t.}\ & x_A + 3x_B \leq 450 \\
& 2x_A + x_B \leq 350 \\
& x_A + x_B \leq 200 \\
& x_A, x_B \geq 0
\end{aligned}
$$

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EX 5: NETWORK FLOW

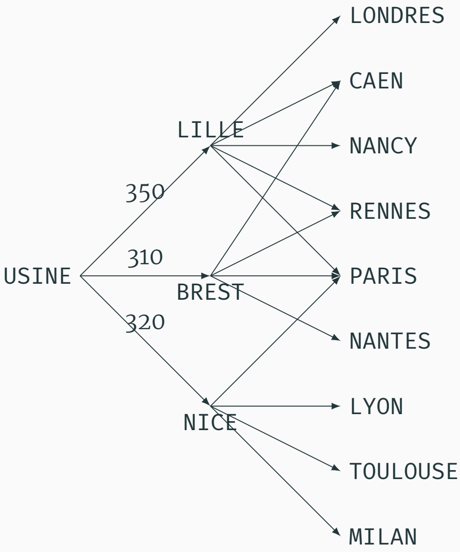
network flow

A company delivers retail stores in 9 cities in Europe from its unique factory *USINE*.
How to manage production and transportation in order to:

- meet the demand of each store,
- not exceed the production limit,
- not exceed the line capacities,
- minimize the transportation costs ?

```
demand = {
    'PARIS': 110,
    'CAEN': 90,
    'RENNES': 60,
    'NANCY': 90,
    'LYON': 80,
    'TOULOUSE': 50,
    'NANTES': 50,
    'LONDRES': 70,
    'MILAN': 70
}
LINES, unitary_cost, capacity = multidict({
    ('USINE','LILLE'): [2.9, 350],
    ('USINE','NICE') : [3.5, 320],
    ('USINE','BREST') : [3.1, 310],
    ('LILLE', 'PARIS') : [1.1, 150],
    ('LILLE','CAEN'): [0.7, 150],
    ('LILLE','RENNES'): [1.0, 150],
    ('LILLE','NANCY'): [1.3, 150],
    ('LILLE','LONDRES'): [1.3, 150],
    ('NICE','LYON'): [0.8, 200],
    ('NICE','TOULOUSE'): [0.2, 110],
    ('NICE','PARIS'): [1.3, 100],
    ('NICE','MILAN'): [1.3, 150],
    ('BREST','NANTES'): [0.9, 150],
    ('BREST','CAEN'):[0.8, 200],
    ('BREST','RENNES'): [0.8, 150],
    ('BREST','PARIS'): [0.9, 100]
})
MAX_PRODUCTION = 900
```

EX 5: GRAPH MODEL



- find a flow on a capacitated directed graph
- flow conservation at each node: IN=OUT

EX 5: LP MODEL

- x_l the quantity of products transported on line $l = (i,j) \in \texttt{LINES}$
- $\texttt{TRANSITS} = \{\texttt{LILLE}, \texttt{NICE}, \texttt{BREST}\}$

$$
\begin{aligned}
\min \quad & \sum_{l \in \texttt{LINES}} \texttt{COST}_l x_l \\
\text{s.t.} \quad & \sum_{i \in \texttt{TRANSIT}} x_{(\texttt{USINE},i)} \le \texttt{MAXPROD} \\
& \sum_{i \in \texttt{TRANSITS}} x_{(i,j)} \ge \texttt{DEMAND}_j, && \forall j \in \texttt{STORES} \\
& x_{(\texttt{USINE},i)} = \sum_{j \in \texttt{STORES}} x_{(i,j)}, && \forall i \in \texttt{TRANSITS} \\
& 0 \le x_l \le \texttt{CAPACITY}_l, && \forall l \in \texttt{LINES}.
\end{aligned}
$$

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EX 6: CAPACITY PLANNING

capacity planning [Bertsimas-Tsitsiklis]

find a least cost electric power capacity expansion plan:

- planning horizon: the next $T \in \mathbb{N}$ years
- forecast demand (in MW): $d_t \geq 0$ for each year $t = 1, \ldots, T$
- existing capacity (oil-fired plants, in MW): $e_t \geq 0$ available for each year t
- options for expanding capacities: (1) coal-fired plant and (2) nuclear plant
  - lifetime (in years): $l_j \in \mathbb{N}$, for each option $j = 1, 2$
  - capital cost (in euros/MW): c_{jt} to install capacity j operable from year t
  - political/safety measure: share of nuclear should never exceed 20% of installed capacity

ex 6: LP model

- decision variables ?
  - x_{jt}: installed capacity (in MW) of type $j = 1, 2$ starting at year $t = 1, \ldots, T$
- constraints ?
  - each year: total capacity meets the demand + nuclear share
- available capacity $j = 1, 2$ for year t: $y_{jt} = \sum_{s=\max\{1,t-l_j+1\}}^{t} x_{js}$

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$$
\begin{aligned}
\min \; & \sum_{t=1}^{T} \sum_{j=1}^{2} c_{jt} x_{jt} \\
\text{s.t. } & y_{jt} = \sum_{s=\max\{1,t-l_j+1\}}^{t} x_{js}, && \forall j = 1, 2, t = 1, \dots, T \\
& e_t + y_{1t} + y_{2t} \geq d_t, && \forall t = 1, \dots, T \\
& 0.8 y_{2t} \leq 0.2 e_t + 0.2 y_{1t}, && \forall t = 1, \dots, T \\
& x_{jt} \geq 0, y_{jt} \geq 0, && \forall j = 1, 2, t = 1, \dots, T
\end{aligned}
$$

EX 6: IN STANDARD FORM

$$
\begin{aligned}
\min \ & \sum_{t=1}^{T} \sum_{j=1}^{2} c_{jt} x_{jt} \\
\text{s.t.} \ & y_{jt} - \sum_{s=\max\{1, t-l_j+1\}}^{t} x_{js} && = 0, && \forall j = 1, 2, t = 1, \dots, T \\
& y_{1t} + y_{2t} - u_t && = d_t - e_t, && \forall t = 1, \dots, T \\
& 0.8 y_{2t} - 0.2 y_{1t} + v_t && = 0.2 e_t, && \forall t = 1, \dots, T \\
& x_{jt} \geq 0, y_{jt} \geq 0, u_t \geq 0, v_t \geq 0 && && \forall j = 1, 2, t = 1, \dots, T
\end{aligned}
$$

$n = 6T$ variables, $m = 4T$ constraints.

MODELING LPs

exercise *verre*

exercise *petroleum*

exercise *steel*

exercise *waste*

exercise *network*

exercise *planning*

exercise *distance*

exercise *data fitting*

EX 7: MINIMUM DISTANCE

minimize $\|\|_1$ and $\|\|_\infty$

Find a solution $x \in \mathbb{R}^n$ of the system of equation $Ax = b$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ of minimum

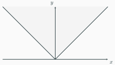
- 1-norm:

$$\|x\|_1 = \sum_{j=1,\dots,n} |x_j|$$

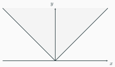
- ∞-norm:

$$\|x\|_\infty = \max_{j=1,\dots,n} |x_j|$$

EX 7: LP MODELS $\min \|x\|_1 = \min \sum_j |x_j|$

how to model $|x|$, $x \in \mathbb{R}$? 

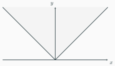
EX 7: LP MODELS $\min \|x\|_1 = \min \sum_j |x_j|$

how to model $|x|$, $x \in \mathbb{R}$? 

$$|x| = \min\{y + z \mid x = y - z, y \geq 0, z \geq 0\}$$

$$\begin{aligned} |x| &= \max\{x, -x\} \\ &= \min\{y \geq 0 \mid y \geq x, y \geq -x\} \end{aligned}$$

EX 7: LP MODELS $\min \|x\|_1 = \min \sum_j |x_j|$

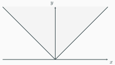
how to model $|x|$, $x \in \mathbb{R}$? 

$$|x| = \min\{y + z \mid x = y - z, y \geq 0, z \geq 0\}$$

$$\begin{aligned} \min \sum_{j=1}^{n} & (y_j + z_j) \\ \text{s.t.} \quad & Ax = b, \\ & x_j = y_j - z_j, \qquad \forall j \\ & y_j, z_j \geq 0, \qquad \forall j \end{aligned}$$

$$\begin{aligned} |x| &= \max\{x, -x\} \\ &= \min\{y \geq 0 \mid y \geq x, y \geq -x\} \end{aligned}$$

EX 7: LP MODELS $\min \|x\|_1 = \min \sum_j |x_j|$

how to model $|x|$, $x \in \mathbb{R}$? 

$$|x| = \min\{y + z \mid x = y - z, y \geq 0, z \geq 0\}$$

$$\begin{aligned} \min \sum_{j=1}^{n} (y_j + z_j) & \\ \text{s.t. } Ax = b, & \\ x_j = y_j - z_j, & \qquad \forall j \\ y_j, z_j \geq 0, & \qquad \forall j \end{aligned}$$

$$\begin{aligned} |x| &= \max\{x, -x\} \\ &= \min\{y \geq 0 \mid y \geq x, y \geq -x\} \end{aligned}$$

$$\begin{aligned} \min \sum_{j=1}^{n} y_j & \\ \text{s.t. } Ax = b, & \\ y_j \geq x_j, & \qquad \forall j \\ y_j \geq -x_j, & \qquad \forall j \\ y_j \geq 0, & \qquad \forall j \end{aligned}$$

EX 7: LP MODEL $\min \|x\|_\infty = \min \max_j |x_j|$

- $|x_j| = \min\{y_j \geq 0 \mid y_j \geq x_j, y_j \geq -x_j\}$
- $\max_j |x_j| = \min\{y \geq 0 \mid y \geq y_j, y_j \geq x_j, y_j \geq -x_j \; \forall j\}$

$$
\begin{aligned}
\min \; & y & \\
\text{s.t.} \; & y \geq x_j, & \forall j \\
& y \geq -x_j, & \forall j \\
& Ax = b, & \\
& y \geq 0, & \forall j
\end{aligned}
$$

EX 7: NORMS AND DISTANCES

- min$\{y \geq 0 \mid y \geq x$ AND $y \geq -x\}$ is a linear program
  but NOT max$\{x, -x\}$ = max$\{y \geq 0 \mid y = x$ OR $y = -x\}$
- we will see how to formulate disjunctions using binary (0/1) variables
  e.g. to formulate $\max_{x \mid Ax=b} \|x\|_1$ and $\max_{x \mid Ax=b} \|x\|_\infty$ as I(nteger)LPs
- modeling $\|x\|_p = (\sum_j |x_j|^p)^{1/p}$ for $p \geq 2$ exactly requires nonlinear functions

modeling LPs

exercise *verre*

exercise *petroleum*

exercise *steel*

exercise *waste*

exercise *network*

exercise *planning*

exercise *distance*

exercise *data fitting*

EX 7: DATA FITTING

data fitting [Bertsimas–Tsitsiklis]

Given m observations – data points $a_i \in \mathbb{R}^n$ and associate values $b_i \in \mathbb{R}$, $i = 1..m$ – predict the value of any point $a \in \mathbb{R}^n$ according to a linear regression model ?

EX 7: DATA FITTING

data fitting [Bertsimas-Tsitsiklis]

Given m observations – data points $a_i \in \mathbb{R}^n$ and associate values $b_i \in \mathbb{R}$, $i = 1..m$ – predict the value of any point $a \in \mathbb{R}^n$ according to a linear regression model ?

a linear fit:

$$b(a) = a^T x + y, \ x \in \mathbb{R}^n, y \in \mathbb{R}$$

the prediction error is evaluated by the residuals $|b(a_i) - b_i| \ \forall i = 1..m$.

EX 7: DATA FITTING

data fitting [Bertsimas-Tsitsiklis]

Given m observations – data points $a_i \in \mathbb{R}^n$ and associate values $b_i \in \mathbb{R}$, $i = 1..m$ – predict the value of any point $a \in \mathbb{R}^n$ according to a linear regression model ?

a linear fit:

$$b(a) = a^T x + y, \; x \in \mathbb{R}^n, y \in \mathbb{R}$$

the prediction error is evaluated by the residuals $|b(a_i) - b_i| \; \forall i = 1..m$. *best* linear fits:

1. minimize the sum of the residuals $\sum_i |b(a_i) - b_i|$
2. minimize the largest residual $max_i |b(a_i) - b_i|$

EX 7: DATA FITTING

sum of residuals $sum_i|b(a_i) - b_i|$

EX 7: DATA FITTING

sum of residuals $sum_i|b(a_i) - b_i|$

$$
\begin{aligned}
& min \sum_i z_i \\
\text{s.t.}\quad & z_i \geq a_i x + y - b_i, && \forall i \\
& z_i \geq b_i - a_i x - y, && \forall i \\
& z_i \geq 0, && \forall i \\
& x \in \mathbb{R}^n, y \in \mathbb{R}
\end{aligned}
$$

largest residual $max_i|b(a_i) - b_i|$

EX 7: DATA FITTING

sum of residuals $sum_i|b(a_i) - b_i|$

$$\begin{aligned} & min \sum_i z_i \\ \text{s.t.} \quad & z_i \geq a_i x + y - b_i, && \forall i \\ & z_i \geq b_i - a_i x - y, && \forall i \\ & z_i \geq 0, && \forall i \\ & x \in \mathbb{R}^n, y \in \mathbb{R} \end{aligned}$$

largest residual $max_i|b(a_i) - b_i|$

$$\begin{aligned} & min\ z \\ \text{s.t.} \quad & z \geq a_i x + y - b_i, && \forall i \\ & z \geq b_i - a_i x - y, && \forall i \\ & z \geq 0, && \forall i \\ & x \in \mathbb{R}^n, y \in \mathbb{R} \end{aligned}$$

Reading:

to go further:

read [Bertsimas-Tsitsiklis]:
Sections 1.2, 1.3, 1.4

for the next class:

read [Bertsimas-Tsitsiklis]:
Section 2.1: Polyhedra and convex sets

GEOMETRY AND ALGEBRA

GEOMETRY AND ALGEBRA

graphical representation

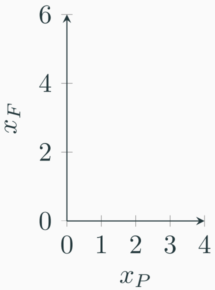
polyhedra and extreme points

bases and degeneracy

adjacent bases

graphical representation (ex: LP verre)

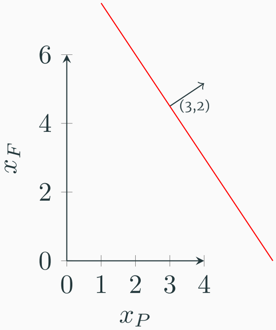
$$
\begin{aligned}
\max\ & 3000x_P + 5000x_F \\
\text{s.t.}\ & x_P \leq 4 \\
& 2x_F \leq 12 \\
& 3x_P + 2x_F \leq 18 \\
& x_P, x_F \geq 0
\end{aligned}
$$



- solution space $\mathbb{R}^2$

graphical representation (ex: LP verre)

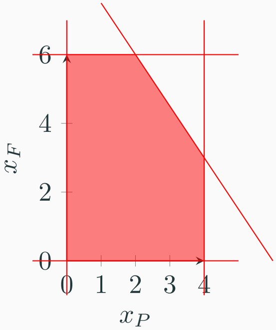
$$\begin{aligned} \max\ & 3000x_P + 5000x_F \\ \text{s.t. } & x_P \leq 4 \\ & 2x_F \leq 12 \\ & 3x_P + 2x_F \leq 18 \\ & x_P, x_F \geq 0 \end{aligned}$$



- solution space $\mathbb{R}^2$
- linear constraint $\equiv$ halfspace, ex: $\{x \in \mathbb{R}^2 \mid 3x_P + 2x_F \leq 18\}$

GRAPHICAL REPRESENTATION (EX: LP VERRE)

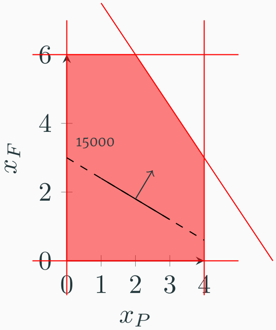
$$\begin{aligned} \max\ & 3000x_P + 5000x_F \\ \text{s.t.}\ & x_P \leq 4 \\ & 2x_F \leq 12 \\ & 3x_P + 2x_F \leq 18 \\ & x_P, x_F \geq 0 \end{aligned}$$



- solution space $\mathbb{R}^2$
- linear constraint $\equiv$ halfspace, ex: $\{x \in \mathbb{R}^2 \mid 3x_P + 2x_F \leq 18\}$
- feasible region $\equiv$ intersection of a finite number of halfspaces $\triangleq$ polyhedron

graphical representation (ex: LP verre)

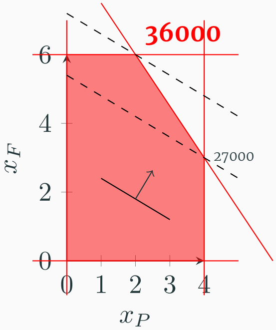
$$\begin{aligned} \max\ & 3000x_P + 5000x_F \\ \text{s.t.}\ & x_P \leq 4 \\ & 2x_F \leq 12 \\ & 3x_P + 2x_F \leq 18 \\ & x_P, x_F \geq 0 \end{aligned}$$



- solution space $\mathbb{R}^2$
- linear constraint $\equiv$ halfspace, ex: $\{x \in \mathbb{R}^2 \mid 3x_P + 2x_F \leq 18\}$
- feasible region $\equiv$ intersection of a finite number of halfspaces $\triangleq$ polyhedron
- objective: $z = 3000x_P + 5000x_F$

graphical representation (ex: LP verre)

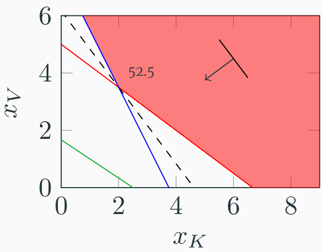
$$\begin{aligned} \max\ & 3000x_P + 5000x_F \\ \text{s.t.}\ & x_P \leq 4 \\ & 2x_F \leq 12 \\ & 3x_P + 2x_F \leq 18 \\ & x_P, x_F \geq 0 \end{aligned}$$



- solution space $\mathbb{R}^2$
- linear constraint $\equiv$ halfspace, ex: $\{x \in \mathbb{R}^2 \mid 3x_P + 2x_F \leq 18\}$
- feasible region $\equiv$ intersection of a finite number of halfspaces $\triangleq$ polyhedron
- objective: $z = 3000x_P + 5000x_F$
- optimum: move the line up as long as feasible

GRAPHICAL REPRESENTATION (EX: PETROLEUM)

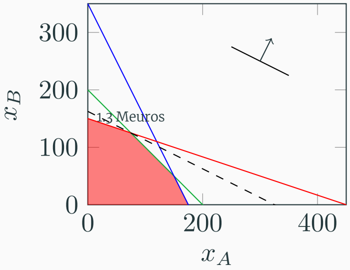
$$
\begin{aligned}
\min\ & 20x_K + 15x_V \\
\text{s.t.}\ & 0.3x_K + 0.4x_V \geq 2 \\
& 0.4x_K + 0.2x_V \geq 1.5 \\
& 0.2x_K + 0.3x_V \geq 0.5 \\
& 0 \leq x_K \leq 9 \\
& 0 \leq x_V \leq 6
\end{aligned}
$$



- constraint $0.2x_K + 0.3x_V \geq 0.5$ is redundant
- constraints $0.3x_K + 0.4x_V \geq 2$ and $0.4x_K + 0.2x_V \geq 1.5$ are active/binding at the optimum $(2, 3.5)$ but not constraints $x_k \geq 0$ or $x_V \leq 6$

GRAPHICAL REPRESENTATION (EX: WASTE MANAGEMENT)

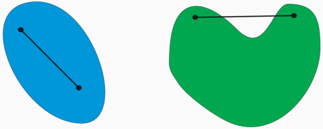
$$\begin{aligned} \max\ & 4000x_A + 8000x_B \\ \text{s.t.}\ & x_A + 3x_B \leq 450 \\ & 2x_A + x_B \leq 350 \\ & x_A + x_B \leq 200 \\ & x_A, x_B \geq 0 \end{aligned}$$



GEOMETRY AND ALGEBRA

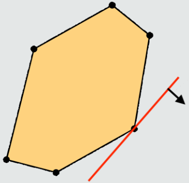
GEOMETRY OF LINEAR PROGRAMMING

- the feasible region is defined as a polyhedron
- thus it is convex (intersection of convex regions)



where are the optimal solutions ?

intuition: the optimum of a linear function cx on a polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n | Ax \geq b\}$ is reached at a "*corner point*" (under conditions of existence)



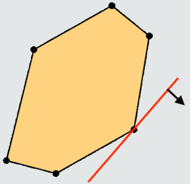
idea: solving an LP = evaluate corner points

CHARACTERIZING THE CORNER POINTS

Theorem [BT 2.3]

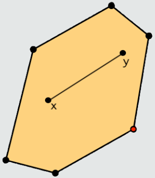
A nonempty polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n | Ax \geq b\}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and a feasible solution $\hat{x} \in \mathcal{P}$, then these are equivalent: $\hat{x}$ is a

vertex



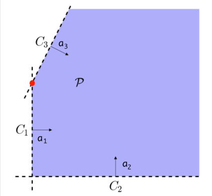
$\exists c, \forall x \in \mathcal{P}, c^T \hat{x} < c^T x$

extreme point



$$\hat{x} = \lambda x + (1 - \lambda) y, \quad x, y \in \mathcal{P} \Rightarrow \lambda = 0$$

basic feasible solution



$\exists n$ linearly independent rows a_i in A s.t. $a_i x = b_i$

vertices and extreme points are independent of the model of $\mathcal{P}$

their number is finite $\leq \binom{m}{n}$ but may be large

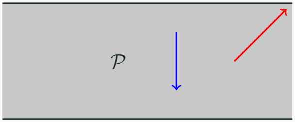
EXISTENCE OF OPTIMA AND EXTREME POINTS

Theorem: existence of an extreme point [BT 2.6]

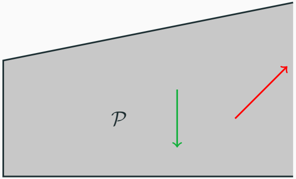
Nonempty $\mathcal{P} = \{x \in \mathbb{R}^n | Ax \geq b\}$, $A \in \mathbb{R}^{m \times n}$ has at least one extreme point
$\iff$ it has no line: $\forall x \in \mathcal{P}, d \in \mathbb{R}^n$, $\{x + \theta d | \theta \in \mathbb{R}\} \not\subseteq \mathcal{P}$
$\iff$ A has n linearly independent rows

Theorem: existence of an optimal solution [BT 2.8]

Minimize cx over $\mathcal{P}$ which has at least one extreme point.
Either optimal cost is $-\infty$ or an extreme point is optimal.



unbounded
∞ optima / 0 vertex
∞ optima including 1 vertex



OPTIMA AND EXTREME POINTS (EXERCISE)

show that:

- $\mathcal{P} = \{(x, y) \in \mathbb{R}^2 \mid x + y = 0\}$ is nonempty and has no extreme point
- $5(x + y)$ has a finite optimum on $\mathcal{P}$
- $\min\{5(x + y) \mid (x, y) \in \mathcal{P}\}$ has an optimal solution which is an extreme point

OPTIMA AND EXTREME POINTS (EXERCISE)

show that:

- $\mathcal{P} = \{(x, y) \in \mathbb{R}^2 \mid x + y = 0\}$ is nonempty and has no extreme point
- $5(x + y)$ has a finite optimum on $\mathcal{P}$
- $\min\{5(x + y) \mid (x, y) \in \mathcal{P}\}$ has an optimal solution which is an extreme point

answer: put in standard form
$\min\{5(x^+ - x^- + y^+ - y^-) \mid x^+ - x^- + y^+ - y^- = 0,\ x^+, x^-, y^+, y^- \geq 0\}$ reaches its optimum in $(0, 0, 0, 0)$

GEOMETRY AND ALGEBRA

CONSTRUCTING A BASIC SOLUTION

Theorem

A nonempty polyhedron **in standard form** $\mathcal{P} = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ with m linear independent rows $A \in \mathbb{R}^{m \times n}$: $x \in \mathbb{R}^n$ is a basic solution iff $Ax = b$ and there exists m linear independent columns A_j, $j \in \beta \subset \{1, \dots, n\}$ s.t. $x_j = 0, \forall j \notin \beta$.

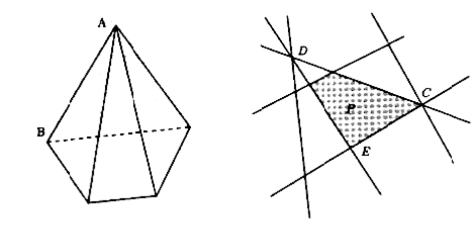
Find a basic (perhaps not feasible) solution:

1. pick m linear independent columns A_j, $j \in \beta \subset \{1, \dots, n\}$
2. fix $x_j = 0, \forall j \notin \beta$
3. solve the system of m equations in $\mathbb{R}^m$: $A_{|\beta} x_{|\beta} = b$

The columns A_j, $j \in \beta$ is a **basis** of $\mathbb{R}^m$ and form an invertible basis matrix $A_{|\beta} \in \mathbb{R}^{m \times m}$; x_j, $j \in \beta$ are the basic variables

DEGENERACY

2 different basic solutions correspond to 2 different bases
but different bases may correspond to the same basic solution, when it is
degenerate $\iff$ more than n active constraints
$\iff$ more that $n - m$ variables to 0 (in standard form).



basic nonfeasible degenerate ?
basic feasible nondegenerate ?
basic feasible degenerate ?

DEGENERACY

2 different basic solutions correspond to 2 different bases
but different bases may correspond to the same basic solution, when it is
degenerate $\iff$ more than n active constraints
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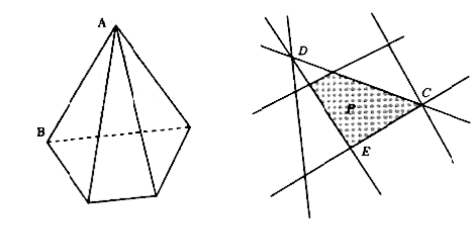


basic nonfeasible degenerate ? D
basic feasible nondegenerate ? B and E
basic feasible degenerate ? A and C

EX: BASIC SOLUTION (CAPACITY PLANNING)

$$
\begin{aligned}
\min \; & \sum_{t=1}^{T} \sum_{j=1}^{2} c_{jt} x_{jt} \\
\text{s.t. } & y_{jt} - \sum_{s=\max\{1, t-l_j+1\}}^{t} x_{js} && = 0, && \forall j = 1, 2,\; t = 1, \ldots, T \\
& y_{1t} + y_{2t} - u_t && = d_t - e_t, && \forall t = 1, \ldots, T \\
& 0.8 y_{2t} - 0.2 y_{1t} + v_t && = 0.2 e_t, && \forall t = 1, \ldots, T \\
& x_{jt} \geq 0,\; y_{jt} \geq 0,\; u_t \geq 0,\; v_t \geq 0 && && \forall j = 1, 2,\; t = 1, \ldots, T
\end{aligned}
$$

$$
\begin{pmatrix}
L & 0 & I & 0 & 0 & 0 \\
0 & L & 0 & I & 0 & 0 \\
0 & 0 & I & I & -I & 0 \\
0 & 0 & -0.2I & 0.8I & 0 & I
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ y_1 \\ y_2 \\ u \\ v
\end{pmatrix}
=
\begin{pmatrix}
0 \\ 0 \\ d - e \\ 0.2e
\end{pmatrix}
$$

$n = 6T$ variables, $m = 4T$ linearly independent rows

EX: BASIC SOLUTION (CAPACITY PLANNING)

$$
\begin{aligned}
\min \ & \sum_{t=1}^{T} \sum_{j=1}^{2} c_{jt} x_{jt} \\
\text{s.t. } & y_{jt} - \sum_{s=\max\{1,t-l_j+1\}}^{t} x_{js} && = 0, && \forall j = 1, 2, t = 1, \ldots, T \\
& y_{1t} + y_{2t} - u_t && = d_t - e_t, && \forall t = 1, \ldots, T \\
& 0.8 y_{2t} - 0.2 y_{1t} + v_t && = 0.2 e_t, && \forall t = 1, \ldots, T \\
& x_{jt} \geq 0, y_{jt} \geq 0, u_t \geq 0, v_t \geq 0 && && \forall j = 1, 2, t = 1, \ldots, T
\end{aligned}
$$

$$
\begin{pmatrix}
L & 0 & I & 0 & 0 & 0 \\
0 & L & 0 & I & 0 & 0 \\
0 & 0 & I & I & -I & 0 \\
0 & 0 & -0.2I & 0.8I & 0 & I
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ u \\ v \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ d - e \\ 0.2e \end{pmatrix}
$$

basic solution $(0, 0, 0, 0, e - d, 0.2e)$ is feasible iff $e_t \geq d_t, \forall t$,
degenerate ($4T > n - m$ zeros), other basis e.g (x_1, x_2, u, v)

EX: BASIC SOLUTION (CAPACITY PLANNING)

Exercise: reformulate by dropping the redundant variables y_1 and y_2, find a basic solution, and give conditions of degeneracy

EX: BASIC SOLUTION (CAPACITY PLANNING)

Exercise: reformulate by dropping the redundant variables y_1 and y_2, find a basic solution, and give conditions of degeneracy

$$
\begin{aligned}
\min \; & \sum_{t=1}^{T} \sum_{j=1}^{2} c_{jt} x_{jt} \\
\text{s.t.} \; & \sum_{s=\max\{1,t-l_1+1\}}^{t} x_{1s} + \sum_{s=\max\{1,t-l_2+1\}}^{t} x_{2s} - u_t = d_t - e_t, && \forall t = 1, \dots, T \\
& 0.8 \sum_{s=\max\{1,t-l_2+1\}}^{t} x_{2s} - 0.2 \sum_{s=\max\{1,t-l_1+1\}}^{t} x_{1s} + v_t = 0.2 e_t, && \forall t = 1, \dots, T \\
& x_{jt} \geq 0, u_t \geq 0, v_t \geq 0 && \forall j = 1, 2, t = 1, \dots, T
\end{aligned}
$$

$$
\begin{pmatrix} L & L & -I & 0 \\ -0.2L & 0.8L & 0 & I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ u \\ v \end{pmatrix} = \begin{pmatrix} d - e \\ 0.2e \end{pmatrix}
$$

EX: BASIC SOLUTION (CAPACITY PLANNING)

Exercise: reformulate by dropping the redundant variables y_1 and y_2, find a basic solution, and give conditions of degeneracy

$$
\begin{aligned}
\min \; & \sum_{t=1}^{T} \sum_{j=1}^{2} c_{jt} x_{jt} \\
\text{s.t.} \; & \sum_{s=\max\{1,t-l_1+1\}}^{t} x_{1s} + \sum_{s=\max\{1,t-l_2+1\}}^{t} x_{2s} - u_t = d_t - e_t, && \forall t = 1, \dots, T \\
& 0.8 \sum_{s=\max\{1,t-l_2+1\}}^{t} x_{2s} - 0.2 \sum_{s=\max\{1,t-l_1+1\}}^{t} x_{1s} + v_t = 0.2 e_t, && \forall t = 1, \dots, T \\
& x_{jt} \geq 0,\ u_t \geq 0,\ v_t \geq 0 && \forall j = 1, 2,\ t = 1, \dots, T
\end{aligned}
$$

$$
\begin{pmatrix} L & L & -I & 0 \\ -0.2L & 0.8L & 0 & I \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ u \\ v \end{pmatrix}
=
\begin{pmatrix} d - e \\ 0.2e \end{pmatrix}
$$

basic solution $(0, 0, e - d, 0.2e)$ is feasible iff $e_t \geq d_t, \forall t$, degenerate iff $\exists t$, $e_t = 0$ or $e_t = d_t$

GEOMETRY AND ALGEBRA

graphical representation

polyhedra and extreme points

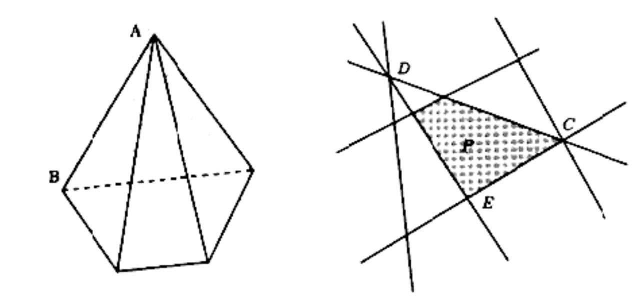
bases and degeneracy

adjacent bases

MOVING TO ANOTHER BASIC SOLUTION

Adjacency

- two basic solutions x and y are adjacent if there exists $n-1$ linearly independent constraints active at x and y
- the line segment between 2 adjacent basic feasible solutions is an edge of $\mathcal{P}$
- (nondegenerate) adjacent basic feasible solutions correspond to adjacent bases (in standard form), i.e. that share $m-1$ columns



SUMMARY

- the feasible set of an LP is a polyhedron $\mathcal{P}$
- if $\mathcal{P}$ is nonempty and bounded, then (i) there exists an optimal solution which is an extreme point
- if $\mathcal{P}$ is unbounded, then either (i), or (ii) there exists an optimal solution but no extreme point (not in standard form), or (iii) the optimal cost is infinite
- if (i) then the LP can be solved in a finite (probably exponential) number of steps by evaluating all extreme points

Instead of complete enumeration: the simplex algorithm moves along the edges of $\mathcal{P}$ while improving the objective

READING:

to go further:

read [Bertsimas-Tsitsiklis]:
Sections 2.2, 2.3, 2.4, 2.5, 2.6

for the next class:

read [Bertsimas-Tsitsiklis]:
Section 1.6: Algorithms and operation count

THE SIMPLEX METHODS

THE SIMPLEX METHODS

basic directions

optimality condition

the simplex method

REVIEW

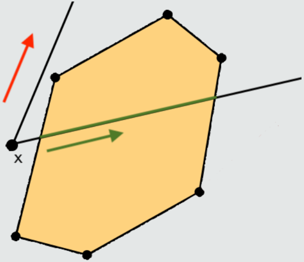
- optimal solutions of $\min_{x \in \mathcal{P}} cx$ with $\mathcal{P} = \{Ax = b, x \geq 0\}$, $A \in \mathbb{R}^{m \times n}$, $rk(A) = m$ are basic feasible solutions
- the solution associated to a basis $\beta \subseteq \{1, \ldots, n\}$ of m linearly independent columns of A is: $x_\beta = A_\beta^{-1} b$, $x_{\neg\beta} = 0$
- adjacent basic solutions share $m - 1$ basic variables: $\beta' = \beta \cup \{j'\} \setminus \{j''\}$
- adjacent basic solutions may coincide if degenerate (if $x_{j'} = 0$)

the simplex method goes from a basic feasible solution to an adjacent one as the cost decreases

FEASIBLE IMPROVING DIRECTION

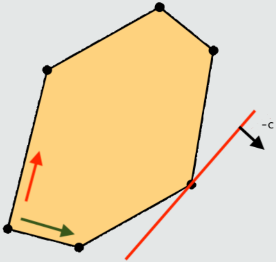
feasible direction from $x \in \mathbb{R}^n$

$d \in \mathbb{R}^n$ such that $\exists \theta > 0,\, x + \theta d \in \mathcal{P}$



improving direction from $x \in \mathbb{R}^n$

$d \in \mathbb{R}^n$ such that $c^T d < 0$



following a feasible improving direction d with a step $\theta > 0$ leads to a feasible solution $x' = x + \theta d \in \mathcal{P}$ of better cost $c^T x' = c^T x + \theta . c^T d < cx$

FEASIBLE IMPROVING BASIC DIRECTION

Let x be a basic feasible solution of basis β, and $j' \notin \beta$:

the j'th basic direction

$d \in \mathbb{R}^n$: $d_{j'} = 1$, $d_j = 0, \forall j \notin \beta \cup \{j'\}$, $d_\beta = -A_\beta^{-1} A_{j'}$

is a feasible direction if x nondegenerate:

- $x_\beta > 0 \Rightarrow \exists \theta > 0, x_\beta + \theta d_\beta \geq 0 \Rightarrow x + \theta d \geq 0$
- $Ad = A_\beta d_\beta + A_{j'} = 0 \Rightarrow \forall \theta > 0,\ A(x + \theta d) = Ax = b$

reduced cost of nonbasic variable $x_{j'}$

$\bar{c}_{j'} = c_{j'} - c_\beta^T A_\beta^{-1} A_{j'}$

- $\bar{c}_{j'} = c^T d = c^T x' - c^T x$ is the cost increase when $\theta = 1$ and $x' = x + d$
- d is an improving direction iff $\bar{c}_{j'} < 0$
- the reduced cost of a basic variable $j \in \beta$ is always 0: $\bar{c}_j = c_j - c_\beta^T A_\beta^{-1} A_j = c_j - c_\beta^T e_j = 0$

EXAMPLE: BASIC IMPROVING DIRECTION

$$
\begin{aligned}
min_{x\geq 0} &\; 2x_1 + x_2 + x_3 + x_4 \\
\text{s.t.} &\; x_1 + x_2 + x_3 + x_4 = 2 \\
&\; 2x_1 + 3x_3 + 4x_4 = 2
\end{aligned}
$$

- $m = 2$, $n = 4$, $rk(A) = 2$
- $\beta = \{1, 2\}$ is a basis

EXAMPLE: BASIC IMPROVING DIRECTION

$$\begin{aligned} min_{x \geq 0} \; & 2x_1 + x_2 + x_3 + x_4 \\ \text{s.t.} \; & x_1 + x_2 + x_3 + x_4 = 2 \\ & 2x_1 + 3x_3 + 4x_4 = 2 \end{aligned}$$

- $m = 2$, $n = 4$, $rk(A) = 2$
- $\beta = \{1, 2\}$ is a basis
- $x = (1, 1, 0, 0)$ feasible nondegenerate ($x_j > 0 \; \forall j \in \beta$)
- basic direction $j = 3$: $d_3 = 1$, $d_4 = 0$, $Ad = 0 \; \Rightarrow \; d_\beta = \binom{-3/2}{1/2}$
- improving direction: $\bar{c} = c^T d = -3/2 < 0$

STEP LENGTH θ

Let x be a nondegenerate basic feasible solution of basis β, and $j' \notin \beta$ of feasible improving direction d, i.e. $\bar{c}_{j'} < 0$

Theorem [BT 3.2]

if $d \geq 0$ then the LP is unbounded, otherwise
if $j'' \in argmin\{-x_j/d_j, j \in \beta, d_j < 0\}$ and $\theta = -x_{j''}/d_{j''}$ then $x' = x + \theta d$ is a basic feasible solution of basis $\beta' = \beta \cup \{j'\} \setminus \{j''\}$: j' enters the basis, j'' exits the basis.

- $x' \in \mathcal{P}$ iff $x_\beta + \theta d_\beta \geq 0, \forall j \in \beta$
- if $d \geq 0$ then $x + \theta d \in \mathcal{P}$ $\forall \theta > 0$ and $\theta \nearrow \Rightarrow c(x + \theta d) \searrow$
- θ is the largest value s.t. $x_j + \theta d_j \geq 0, \forall j \in \beta$
- $A_\beta^{-1} A_j = e_j, \forall j \in \beta \setminus \{j''\}$, and $A_\beta^{-1} A_{j'} = -d_\beta$ has a nonzero j'' component $\Rightarrow \{A_j, j \in \beta'\}$ are linear independent $Rightarrow$ β' is a basis
- $\theta > 0$ since x nondegenerate ($x_j > 0, \forall j \in \beta$)
- several possible entering columns j''

EXAMPLE: BASIC IMPROVING DIRECTION (CONTINUATION)

$$
\begin{aligned}
min_{x \geq 0} \; & 2x_1 + x_2 + x_3 + x_4 \\
\text{s.t.} \; & x_1 + x_2 + x_3 + x_4 = 2 \\
& 2x_1 + 3x_3 + 4x_4 = 2
\end{aligned}
$$

- $\beta = \{1, 2\}$ is a basis: $x = (1, 1, 0, 0)$ feasible nondegenerate
- basic feasible improving direction $j = 3$: $d = (-3/2, 1/2, 1, 0)$, $\bar{c}_3 = c^T d = -3/2$
- $x'_\beta = x_\beta + \theta d_\beta \geq 0 \Rightarrow x'_1 = 1 - (3/2)\theta \geq 0 \Rightarrow \theta \leq 2/3$
- $x' = (0, 4/3, 2/3, 0)$ basic feasible solution $\beta' = \{2, 3\}$, $cx' = cx + \theta \bar{c}_3 = cx - 1$

basic directions

optimality condition

the simplex method

THE SIMPLEX METHODS

OPTIMALITY CONDITION

Theorem [BT 3.1]

Let x be a basic feasible solution of basis β and $\bar{c} \in \mathbb{R}^n$ the vector of reduced costs.

- if $\bar{c}_j \geq 0 \; \forall j \notin \beta$ then x is optimal
- if x is optimal and nondegenerate then $\bar{c} \geq 0$

($\Rightarrow$) for any $y \in \mathcal{P}$, let $d = y - x$ and $c_{\neg\beta} \geq 0$:
$A_\beta d_\beta + A_{\neg\beta} y_{\neg\beta} = Ad = Ay - Ax = b - b = 0 \Rightarrow d_\beta = -A_\beta^{-1} A_{\neg\beta} y_{\neg\beta} \Rightarrow$
$c^T y - c^T x = c_\beta^T d_\beta + c_{\neg\beta}^T y_{\neg\beta} = (c_{\neg\beta}^T - c_\beta^T A_\beta^{-1} A_{\neg\beta}) y_{\neg\beta} = \bar{c}_{\neg\beta} y_{\neg\beta} \geq 0$
($\Leftarrow$) if x nondegenerate and $\bar{c}_j < 0$, then j is nonbasic and of feasible improving direction, then x nonoptimal

EXAMPLE: BASIC IMPROVING DIRECTION (CONTINUATION)

$$\begin{aligned} min_{x\geq 0}\ & 2x_1 + x_2 + x_3 + x_4 \\ \text{s.t. } & x_1 + x_2 + x_3 + x_4 = 2 \\ & 2x_1 + 3x_3 + 4x_4 = 2 \end{aligned}$$

- note that optimum ≥ 2 since $cx = x_1 + 2$, $\forall x$ feasible
- $\beta = \{2, 3\}$ is a basis with $x = (0, 4/3, 2/3, 0)$ nondegenerate
- basic directions are not improving:
    - $j = 1$: $d = (1, -1/3, -2/3, 0)$ and $\bar{c}_1 = cd = 1 \geq 0$
    - $j = 4$: $d = (0, 1/3, -4/3, 1)$ and $\bar{c}_4 = cd = 0 \geq 0$
- then x is optimal

THE SIMPLEX METHODS

The simplex method

steps	howto:
1. get a basis β	find m linearly independent columns
2. get a basic feasible x	$x_{\neg\beta} = 0$, $x_\beta = A_\beta^{-1} b$ if $x_\beta \geq 0$
halt condition (optimality)	$\bar{c} = c - c_\beta^T A_\beta^{-1} A \geq 0$ if nondegenerate
3. find an improving direction	any $j' \notin \beta$ s.t. $\bar{c}_{j'} < 0$ if nondegenerate
halt condition (unboundness)	$d_\beta = -A_\beta^{-1} A_{j'} \geq 0$
4. find the largest step length	any $j'' \in argmin\{-x_j/d_j \mid j \in \beta, d_j < 0\}$
5. update the basis	j' enters, j'' exits
6. goto 2.	$x := x - (x_{j''}/d_{j''})d$

convergence [BT 3.3]

if $\mathcal{P} \neq \emptyset$ and every basic feasible solution is nondegenerate then the simplex method terminates after a finite number of iteration with an optimal basis β or with some direction $d \geq 0$, $Ad = 0$, $c^T d < 0$, and the optimal cost is $-\infty$

- cx decreases at every iteration, all x are basic feasible solutions, the number of basic feasible solutions is finite

PIVOTING RULES

- choice of the entering column $j' \notin \beta$ s.t. $\bar{c}_{j'} < 0$:
  - largest cost decrease per unit change: $\min \bar{c}_j$
  - largest cost decrease: $\min \theta\bar{c}_j$
  - smallest subscript: $\min j$
- choice of the exiting column $j'' \in argmin\{-x_j/d_j \mid j \in \beta, d_j < 0\}$
- trade-off between computation burden and efficiency, e.g. compute a subset of reduced costs

IN CASE OF DEGENERARY ?

- if x degenerate and $\exists j \in \beta, d_j < 0, x_j = 0$ then $\theta = 0$: the basis changes but not the basic feasible solution
- a sequence of basis changes may lead to a cost reducing feasible direction or may cycle
- cycles can be avoided (and convergence ensured) using the smallest subscript pivoting rules for both entering and exiting columns (see [Bertsimas-Tsitsiklis] Section 3.4 for details)

THE INITIAL BASIC FEASIBLE SOLUTION ?

- if $\mathcal{P} = \{Ax \leq b, x \geq 0\}$, we can form the basis with the slack variables: $\mathcal{P} = \{Ax + Is = b, x \geq 0, s \geq 0\}$
- if the LP $min\{cx, Ax = b, x \geq 0\}$ is already in standard form, then we can first solve the auxiliary LP:

$$min\{1.y, Ax + Iy = b, x \geq 0, y \geq 0\}$$

if optimum is 0 we get a feasible solution, otherwise the original LP is unfeasible (see [Bertsimas-Tsitsiklis] Section 3.5 for details)

IMPLEMENTATIONS

- each iteration involves costly arithmetic operations:
  - computing $p^T = c_\beta^T A_\beta^{-1}$ or $A_\beta^{-1} A_j$ takes $O(m^3)$ operations
  - computing $\bar{c}_j = c_j - p^T A_j$ for all $j \notin \beta$ takes $O(mn)$ operations
- revised simplex: update matrix $A^{-1}_{\beta \cup \{j'\} \setminus \{j\}}$ from A_β^{-1} in $O(mn)$
- full tableau: maintain and update the $m \times (n+1)$ matrix $A_{\beta^{-1}}(b|A)$
- the complexity in the worst case remains exponential: the LP may have 2^n extreme points and the simplex method visits them all
- good implementations of the simplex method perform usually very well

(see [Bertsimas-Tsitsiklis]Section 3.3 for details)

READING:

to go further:

read [Bertsimas-Tsitsiklis]:
Sections 3.1, 3.2, 3.3

for the next class:

read [Bertsimas-Tsitsiklis]:
Section 1.6: Algorithms and operation count

DUALITY

DUALITY: MOTIVATION

$$P : z = min\, x^2 + y^2$$
$$\text{s.t. } x + y = 1$$

- penalization: $P_u = z_u = min\, x^2 + y^2 + u(1 - x - y)$
  the constraint is relaxed but violations are penalized with price $u \in \mathbb{R}$
- $z_u \leq z$:
  $\forall(x, y)$ feasible for $P \Rightarrow$ feasible for P_u and $z_u \leq x^2 + y^2 + u(1 - x - y) = x^2 + y^2$
- the optimal solution of P_u is $(u/2, u/2)$ (zero of the partial derivative)
- when $u = 1$, $(1/2, 1/2)$ is optimal for P_u and feasible for P, thus it is optimal for P: $1/2 = z_1 \leq z \leq (1/2)^2 + (1/2)^2 = 1/2$

LAGRANGIAN MULTIPLIERS

$$
\begin{aligned}
P : z = min\ & c^T x \\
\text{s.t.}\ & Ax = b \\
& x \geq 0
\end{aligned}
$$

$$
\begin{aligned}
P_u : z_u = min\ & c^T x + u^T (b - Ax) \\
\text{s.t.}\ & x \geq 0
\end{aligned}
$$

with multipliers $u \in \mathbb{R}^m$

- lagrangian problems P_u, $u \in \mathbb{R}^m$ provide lower bounds $z_u \leq z$
- dual problem $D : d = max_{u \in \mathbb{R}^m} z_u$ is the tightest lower bound
- if x is optimal for some P_u and satisfies $Ax = b$ then x is optimal for P and $d = z$

for LPs: $z = d$ (strong duality) and D is a linear program, the dual of which being P

dual linear program

optimality condition

feasibility condition

DUALITY

DUAL LINEAR PROGRAM

Theorem

- the dual of $P = min\{c^T x | Ax = b, x \geq 0\}$ is a linear program:

$$(D) : max\ u^T b$$
$$\text{s.t.}\ \ u^T A \leq c^T$$

- the dual of D is the primal P
- equivalent forms of P give equivalent forms of D

- $z_u = min_{x \geq 0} c^T x + u^T (b - Ax) = u^T b + min_{x \geq 0} (c^T - u^T A) x$
- $z_u = \begin{cases} u^T b & \text{if } (c^T - u^T A) \geq 0 \\ -\infty & \text{otherwise} \end{cases}$

HOW TO BUILD THE DUAL ?

primal/dual correspondence

min	max
equality constraint	free variable
inequality constraint:	nonnegative variable:
$min/Ax \geq b$, $max/Ax \leq b$	$u \geq 0$
$min/Ax \leq b$, $max/Ax \geq b$	$u \leq 0$
cost vector c	RHS vector b
matrix A	matrix A^T

$$
\begin{aligned}
P : min\ & c^T x + d^T y \\
\text{s.t.}\ & Ax = b && (u) \\
& Dx + Ey \geq f && (v) \\
& x \geq 0
\end{aligned}
$$

$$
\begin{aligned}
D : max\ & u^T b + v^T f \\
\text{s.t.}\ & u^T A + v^T D \leq c^T && (x) \\
& v^T E = d^T && (y) \\
& v \geq 0
\end{aligned}
$$

EX: DUAL MODEL (STEEL FACTORY)

$$
\begin{aligned}
P : \max\ & 25x_C + 30x_T + 2x_S \\
\text{s.t.}\ & \\
& \frac{x_C}{200} + \frac{x_T}{200} + \frac{x_S}{200} \leq 35 && (heating) \\
& \frac{x_C}{200} + \frac{x_T}{140} + \frac{x_S}{160} \leq 40 && (rolling) \\
& 0 \leq x_C \leq 6000 && (coils) \\
& 0 \leq x_T \leq 4000 && (tapes) \\
& 0 \leq x_S \leq 3500 && (sheets)
\end{aligned}
$$

EX: DUAL MODEL (STEEL FACTORY)

$$
\begin{aligned}
D : \min\ & 35u_H + 40u_R + 6000u_C + 4000u_T + 3500u_S \\
\text{s.t.}\ & \\
& \frac{u_H}{200} + \frac{u_R}{200} + u_C \geq 25 && (coils) \\
& \frac{u_H}{200} + \frac{u_R}{140} + u_T \geq 30 && (tapes) \\
& \frac{u_H}{200} + \frac{u_R}{160} + u_S \geq 2 && (sheets) \\
& u \geq 0
\end{aligned}
$$

WEAK DUALITY

Theorem [BT 4.3]

- if x is feasible for P (min) and u is feasible for D (max) then:

$$u^T b \leq cx$$

- if the optimal cost of P is $-\infty$ then D is unfeasible
- if the optimal cost of D is $+\infty$ then P is unfeasible
- if $u^T b = cx$ then x is optimal for P and u is optimal for D

- if P in standard form: $Ax = b$, $x \geq 0$ and $u^T A \leq c^T$, then $u^T b = u^T Ax \leq cx$.
- in any form: if (x, u) primal-dual feasible then (by construction) $u^T (Ax - b) \geq 0$ and $(c^T - u^T A)x \geq 0$, then $u^T b \leq u^T Ax \leq cx$.

Theorem [BT 4.4]

if a linear programming problem has an optimal solution, so does its dual and their respective optima are equal

- let x an optimal solution of $P = min\{c^T x | Ax = b, x \geq 0\}$ of basis β
- x optimal then the reduced costs are all nonnegative $\bar{c}^T = c^T - c^T_\beta A^{-1}_\beta A \geq 0$
- let $u^T = c^T_\beta A^{-1}_\beta$ then u is feasible for $D = max\{u^T b | u^T A \leq c^T\}$
- $u^T b = c^T_\beta A^{-1}_\beta b = c^T_\beta x_\beta = c^T x$ then u is optimal for D

At optimality: the primal reduced costs $\bar{c}^T$ are the dual slacks $c^T - u^T A$

dual linear program

optimality condition

feasibility condition

DUALITY

COMPLEMENTARY SLACKNESS

Theorem [BT 4.5]

let x feasible for P and u feasible for D then they are optimal iff

$$u_i(a_i^T x - b^i) = 0 \quad \forall i \text{ row of } P$$
$$(c_j - u^T A_j)x_j = 0 \quad \forall j \text{ row of } D.$$

- (x, u) primal-dual feasible then $u^T(Ax - b) \geq 0$ and $(c^T - u^T A)x \geq 0$
- $u^T b - c^T x = \sum_i u_i(a_i^T x - b^i) + \sum_j (c_j - u^T A_j)x_j$ sum of nonnegative terms is zero iff all terms are zero

Either a constraint is binding at the optimum or the dual variable is zero

EXERCISE: OPTIMALITY WITHOUT SIMPLEX

$$
\begin{aligned}
P : min\ & 13x_1 + 10x_2 + 6x_3 \\
\text{s.t. } & 5x_1 + x_2 + 3x_3 = 8 \\
& 3x_1 + x_2 = 3 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

show that the basic solution of P of basis $\beta = \{1, 3\}$ is feasible nondegenerate and optimal using the complementary slackness theorem

EXERCISE: OPTIMALITY WITHOUT SIMPLEX

$$
\begin{aligned}
P : min\ & 13x_1 + 10x_2 + 6x_3 \\
\text{s.t.}\ & 5x_1 + x_2 + 3x_3 = 8 \\
& 3x_1 + x_2 = 3 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
D : max\ & 8u_1 + 3u_2 \\
\text{s.t.}\ & 5u_1 + 3u_2 \leq 13 \\
& u_1 + u_2 \leq 10 \\
& 3u_1 \leq 6
\end{aligned}
$$

- $\beta = \{1, 3\} \Rightarrow x_2 = 0$, $x_1 = 3/3 = 1$, $x_3 = (8-5)/3 = 1$
- $x = (1, 0, 1)$, $x \geq 0 \Rightarrow$ feasible, $x_j > 0, \forall j \in \beta \Rightarrow$ nondegenerate
- P in standard form $\Rightarrow$ first C.S. is always condition satisfied
- second C.S. condition: $5u_1 + 3u_2 = 13$ and $3u_1 = 6 \Rightarrow u = (2, 1)$
- u is feasible for D since $u_1 + u_2 \leq 10$
- C.S. theorem $\Rightarrow$ x and u are optimal with cost 19

OPTIMALITY CONDITIONS

Theorem

x is optimal for $P = min\{c^T x | Ax = b, x \geq 0\}$ if exists $u \in \mathbb{R}^m$ s.t. (x, u) satisfies:

1. primal feasibility: $Ax = b$
2. primal feasibility: $x \geq 0$
3. dual feasibility: $u^T A \leq c$
4. complementary slackness: $x_j > 0 \Rightarrow u^T A_j = c_j$

- The basic feasible solutions of the simplex algorithm always satisfy 1,2 and 4 with $u^T = c_\beta^T A_\beta^{-1}$ ($\bar{c}_\beta = c_\beta^T - u^T A_\beta = 0$). Condition 3 is the halting condition $\bar{c} \geq 0$
- if x is degenerate then solutions u of condition 4 may not be unique

DUAL SIMPLEX

for $P = min\{cx|Ax = b, x \geq 0\}$ and $D = max\{u^T b|u^T A \leq c\}$

- a basis β determines basic solutions for P and D: $x_\beta = A_\beta^{-1} b$ and $u^T = c_\beta^T A_\beta^{-1}$
- if both are feasible, then both are optimal (according to C.S. since $u^T(Ax - b) = 0$ and $(c^T - u^T A)x = (c_\beta^T - u^T A_\beta)x_\beta = 0$)
- simplex algorithm maintains primal feasibility ($x_\beta \geq 0$) while trying to achieve dual feasibility ($\bar{c}^T = c^T - u^T A \geq 0$)
- dual simplex algorithm maintains dual feasibility ($\bar{c} \geq 0$) while trying to achieve primal feasibility ($x_\beta \geq 0$)
- examples of usage: after modifying b or adding a new constraint to P, run the dual simplex starting from the feasible dual solution $c_\beta^T A_\beta^{-1}$

dual linear program

optimality condition

feasibility condition

DUALITY

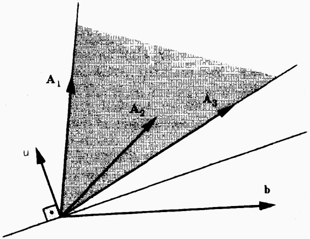
Farka's lemma and unfeasibility

theorem

$A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. Exactly one of the following holds:

1. $\mathcal{P} = \{x \in \mathbb{R}^m, x \geq 0, Ax = b\} \neq \emptyset$
2. $\exists u \in \mathbb{R}^m$, $u^T A \geq 0$ and $u^T b < 0$

($\Leftarrow$) if $x \in \mathcal{P}$ and $u^T A \geq 0$ then $u^T b = u^T Ax \geq 0$
($\Rightarrow$) if $P : max\{0|Ax = b, x \geq 0\}$ is unfeasible then $D : min\ \{u^T b|u^T A \geq 0\}$ is either unbounded or unfeasible. Since $u = 0$ is feasible for D, then (2) holds.



if b is not in the cone $\{Ax, x \geq 0\}$ spanned by the columns of A then a separating hyperplane $\{x \in \mathbb{R}^m | u^T x = 0\}$ exists

READING:

to go further:

read [Bertsimas-Tsitsiklis]:
Sections 4.1, 4.2, 4.5, 4.6, 4.7

for the next class:

read [Bertsimas-Tsitsiklis]:
Section 4.4: Optimal dual variables as marginal costs

SENSITIVE ANALYSIS

GOAL OF SENSITIVE ANALYSIS

models of real-world decision problems are often approximated:

- they rely on forecast/inaccurate data: a model is more reliable if its solutions are less sensitive to changes in the data
- they have incomplete knowledge of the problem: a model is more robust if its solutions are less sensitive to additions of variables/constraints

how to evaluate the sensitivity of an optimal solution of $P : min\{cx \mid Ax = b, x \geq 0\}$ to one local change in A, b or c without having to simulate every possible changes by solving from scratch the LP again and again ?

THE CORE IDEA

- let P in standard form $P : min\{cx \mid Ax = b, x \geq 0\}$
- when the simplex method stops with an optimal solution, it returns an optimal basis β and feasible primal and dual solutions x and u such that:

$$
\begin{aligned}
x_\beta &= A_\beta^{-1} b \geq 0, x_{\neg\beta} = 0 \\
u &= c_\beta^T A_\beta^{-1} \\
\bar{c}^T &= c^T - u^T A \geq 0
\end{aligned}
$$

- when the problem changes, check how these conditions are affected

ADDING A NEW VARIABLE/COLUMN

- adding a new variable $x_{n+1} \iff$ assuming $n+1 \notin \beta$ (with $x_{n+1} = 0$)
- β remains a basis and $x_\beta = A_\beta^{-1} b$, $x_{\neg\beta \cup \{n+1\}} = 0$ is primal feasible
- it remains optimal if

$$\bar{c}_{n+1} = c_{n+1} - c_\beta^T A_\beta^{-1} A_{n+1} \geq 0$$

  and the optimal value $c_\beta x_\beta$ does not change
- otherwise the $n+1$-th direction is improving and we must run additional iterations of the **primal** simplex algorithm from β to reach an optimal basis

EXAMPLE: ADDING A VARIABLE

$\beta = \{1,3\}$ optimal basis $x^T = (1,0,1)$, $u^T = (2,1)$ primal-dual feasible, $opt = 19$

$$\begin{aligned} P : min\ & 13x_1 + 10x_2 + 6x_3 + \delta x_4 \\ \text{s.t.}\ & 5x_1 + x_2 + 3x_3 + x_4 = 8 \\ & 3x_1 + x_2 + x_4 = 3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

$$\begin{aligned} D : max\ & 8u_1 + 3u_2 \\ \text{s.t.}\ & 5u_1 + 3u_2 \leq 13 \\ & u_1 + u_2 \leq 10 \\ & 3u_1 \leq 6 \\ & u_1 + u_2 \leq \delta \end{aligned}$$

- β remains a basis, $x^T = (1,0,1,0)$ primal feasible
- $u^T = (2,1)$ remains feasible iff the new constraint is satisfied $u_1 + u_2 = 3 \leq \delta$
- optimal solutions and values do not change while $\delta \geq 3$

CHANGING THE RIGHT HAND SIDE VECTOR

- let $b'_k = b_k + \delta$, i.e. $b = b + \delta e_k$ for some $k = 1, \ldots, m$
- β remains a basis and $u^T = c_\beta^T A_\beta^{-1}$ remains dual feasible
- the primal feasibility condition becomes:

$$A_\beta^{-1}(b + \delta e_k) = x_\beta + \delta h \geq 0$$

  where $h = A_\beta^{-1} e_k$ is the k–th column of A_β^{-1}
- β remains optimal if

$$\max_{i \in \beta | h_i > 0} \frac{-x_i}{h_i} \leq \delta \leq \min_{i \in \beta | h_i < 0} \frac{-x_i}{h_i}$$

  and the optimal cost varies by $u^T(b + \delta e_k) - u^T b = \delta u_k$:
  u_k is the **marginal cost** of changing b_k by one unit
- otherwise we must run additional iterations of the **dual** simplex algorithm from β to reach an optimal basis

EXAMPLE: CHANGING b

$\beta = \{1,3\}$ optimal basis $x^T = (1,0,1)$, $u^T = (2,1)$ primal-dual feasible, $opt = 19$

$$
\begin{aligned}
P : min\ & 13x_1 + 10x_2 + 6x_3 \\
\text{s.t.}\ & 5x_1 + x_2 + 3x_3 = 8 + \delta \\
& 3x_1 + x_2 = 3 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
D : max\ & (8 + \delta)u_1 + 3u_2 \\
\text{s.t.}\ & 5u_1 + 3u_2 \leq 13 \\
& u_1 + u_2 \leq 10 \\
& 3u_1 \leq 6
\end{aligned}
$$

- β remains a basis, u^T remains dual feasible
- $x^T = (1,0,1+\frac{\delta}{3})$ is feasible iff $1 + \frac{\delta}{3} \geq 0$
- $u_1 = 2$ is the marginal cost if $\delta = 1$
- $(1,0,1+\frac{\delta}{3})$ is optimal while $\delta \geq -3$ and the optimum value is $19 + 2\delta$

CHANGING THE COST OF A NON-BASIC VARIABLE

- let $c'_j = c_j + \delta$ for some non-basic $j \notin \beta$
- β remains a basis, and $x_\beta = A_\beta^{-1} b$ remains feasible
- $u^T = c_\beta^T A_\beta^{-1}$ remains feasible iff $\bar{c}'_j = (c_j + \delta) - c_\beta^T A_\beta^{-1} A_j = \bar{c}_j + \delta \geq 0$
  i.e. the basis β remains optimal if

$$\delta \geq -\bar{c}_j$$

  and the optimal value $c_\beta x_\beta$ does not change
- otherwise j is an improving direction and we must run additional iterations of the **primal** simplex algorithm from β to reach an optimal basis

EXAMPLE: CHANGING c (NON-BASIC)

$\beta = \{1, 3\}$ optimal basis $x^T = (1, 0, 1)$, $u^T = (2, 1)$ primal-dual feasible, $opt = 19$

$$
\begin{aligned}
P : min\ & 13x_1 + (10 + \delta)x_2 + 6x_3 \\
\text{s.t.}\ & 5x_1 + x_2 + 3x_3 = 8 \\
& 3x_1 + x_2 = 3 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
D : max\ & 8u_1 + 3u_2 \\
\text{s.t.}\ & 5u_1 + 3u_2 \leq 13 \\
& u_1 + u_2 \leq 10 + \delta \\
& 3u_1 \leq 6
\end{aligned}
$$

- β remains a basis, x^T remains primal feasible
- u^T remains feasible iff $u_1 + u_2 = 3 \leq 10 + \delta$
- optimal solutions and values do not change while $\delta \geq -7 = -rc(x_2)$

CHANGING THE COST OF A BASIC VARIABLE

- let $c'_j = c_j + \delta$ for some basic $j \in \beta$ and j is the l–th element of β
- β remains a basis, and $x_\beta = A_\beta^{-1} b$ remains feasible
- the dual feasibility condition becomes:

$$\bar{c'}^T_{\neg\beta} = c^T_{\neg\beta} - (c_\beta + \delta e_l)^T A_\beta^{-1} A_{\neg\beta} = \bar{c}^T_{\neg\beta} - \delta e_l^T A_\beta^{-1} A_{\neg\beta} \geq 0$$

  i.e. the basis β remains optimal if

$$\delta g \leq -\bar{c}$$

  where g is the l–th row of $A_\beta^{-1} A_{\neg\beta}$ (available in the simplex) and the optimal cost varies by δx_j
- otherwise there exists an improving direction and we must run additional iterations of the **primal** simplex algorithm from β to reach an optimal basis

EXAMPLE: CHANGING c (BASIC)

$\beta = \{1,3\}$ optimal basis $x^T = (1,0,1)$, $u^T = (2,1)$ primal-dual feasible, $opt = 19$

$$
\begin{aligned}
P: min\ & (13+\delta)x_1 + 10x_2 + 6x_3 \\
\text{s.t.}\ & 5x_1 + x_2 + 3x_3 = 8 \\
& 3x_1 + x_2 = 3 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
D: max\ & 8u_1 + 3u_2 \\
\text{s.t.}\ & 5u_1 + 3u_2 \leq 13 + \delta \\
& u_1 + u_2 \leq 10 \\
& 3u_1 \leq 6
\end{aligned}
$$

- β remains a basis, x^T remains primal feasible
- $u^T = (2, 1 + \frac{\delta}{3})$ is feasible iff $u_1 + u_2 = 2 + 1 + \frac{\delta}{3} \leq 10$
- $x_1 = 1$ is the marginal cost if $\delta = 1$
- $(1,0,1)$ is optimal while $\delta \leq 21$ and the optimum value is $19 + \delta$

ADDING A NEW INEQUALITY CONSTRAINT

- adding a new constraint $a_{m+1}^T x \geq b_{m+1}$, by substitution we can assume that $a_{m+1,j} = 0$ for each $j \notin \beta$
- by adding a slack variable x_{n+1}, we get a new basis $\beta' = \beta \cup \{n+1\}$ with

$$A_{\beta'} = \begin{pmatrix} A_\beta & 0 \\ a_{m+1}^T & -1 \end{pmatrix} \quad A_{\beta'}^{-1} = \begin{pmatrix} A_\beta^{-1} & 0 \\ a_{m+1}^T A_\beta^{-1} & -1 \end{pmatrix}$$

$u^T = (c_\beta^T \; 0) A_{\beta'}^{-1} = (c_\beta^T A_\beta^{-1} \; 0)$ is feasible as the reduced costs are unchanged:

$$\bar{c'}^T = (c^T \; 0) - (c_\beta^T \; 0) A_{\beta'}^{-1} A = (\bar{c}^T \; 0)$$

- we must run additional iterations of the **dual** simplex algorithm to recover primal feasibility
- for an equality constraint, we introduce an artificial variable (as in the two-phase method)

CHANGING A NON-BASIC COLUMN

- let $a'_{ij} = a_{ij} + \delta$ for some non-basic $j \notin \beta$
- β remains a basis, and $x_\beta = A_\beta^{-1} b$ remains primal feasible
- $u^T = c_\beta^T A_\beta^{-1}$ remains dual feasible iff $\bar{c}'_j = c_j - c_\beta^T A_\beta^{-1}(A_j + \delta e_i) = \bar{c}_j - \delta u_i \geq 0$
  i.e. the basis β remains optimal if

$$\delta u_i \leq \bar{c}_i$$

  and the optimal value $c_\beta x_\beta$ does not change
- otherwise j is an improving direction and we must run additional iterations of the primal simplex algorithm from β to reach an optimal basis

EXAMPLE: CHANGING A_j (NON-BASIC)

$\beta = \{1,3\}$ optimal basis $x^T = (1,0,1)$, $u^T = (2,1)$ primal-dual feasible, $opt = 19$

$$
\begin{aligned}
P : min\ & 13x_1 + 10x_2 + 6x_3 \\
\text{s.t.}\ & 5x_1 + (1+\delta)x_2 + 3x_3 = 8 \\
& 3x_1 + x_2 = 3 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
D : max\ & 8u_1 + 3u_2 \\
\text{s.t.}\ & 5u_1 + 3u_2 \leq 13 \\
& (1+\delta)u_1 + u_2 \leq 10 \\
& 3u_1 \leq 6
\end{aligned}
$$

- β remains a basis, x^T remains primal feasible
- u^T remains feasible iff $(1+\delta)u_1 + u_2 = 3 + \delta \leq 10$
- optimal solutions and values do not change while $\delta \leq 7$

CHANGING A BASIC COLUMN

- it's complicated...

APPLICATIONS

- parametric simplex method
- (progressive) column generation
- (progressive) constraint generation

EXERCISE (STEEL FACTORY)

- implement the primal and the dual models of steel factory with Gurobipy
- get the dual optimal values: `Constr.pi`
- get the slack values: `Constr.slack`
- get the reduced costs: `Var.rc`
- how to interpret a zero slack value ?
- how to interpret a non-zero reduced cost ? simulate the corresponding change
- how to interpret a non-zero dual value ? simulate the corresponding change
- play also with the attributes VBasis, SAObjLow/Up, SALBLow/Up, SAUBLow/Up of Var and CBasis and SASRHSLow/Up of Constr

ELEMENTS OF ANSWER

- a zero slack value for a mill: the corresponding dual value is the marginal cost of an extra hour of availability of the mill
- a negative reduced cost for a product (that is not in the solution): how much the unit price of the product have to be raised to make it profitable / the marginal cost of producing 1 unit of the product (if feasible)
- note that the model is not in standard form, so be careful with the signs !