# Modelling in Linear Programming

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## 1 Exercises

### 1.1 PV panels

How to equip two roofs with PV panels, respectively 4m and 6m long, to maximize the total power with an installation budget limited to 18ke, given the following cost/power of one linear meter of PV installed: 3k€/150Wp on roof 1, 2k€/250Wp on roof 2

$$\begin{aligned} \max\ & 3x_1 + 5x_2 \\ \text{s.t.}\ & x_1 \leq 4 \\ & x_2 \leq 6 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

### 1.2 Nuclear waste management

A company eliminates nuclear wastes of 2 types A and B, by applying a sequence of 3 processes I, II and III in any order. The processes I, II, III, have limited availability, respectively: 450h, 350h, and 200h per month. The unit processing times depend on the process and waste type: One unit of waste A is processed in 1 hour with process I, in 2 hours with II, in 1 hour with III. One unit of waste B is processed in 3 hours with process I, in 1 hour with II, in 1 hour with III. The profit for the company is 4000 euros to eliminate one unit of waste A and 8000 euros to eliminate one unit of waste B. Its objective is to maximize the profit.

$$\begin{aligned} \max\ & 4x_A + 8x_B \\ \text{s.t.}\ & x_A + 3x_B \leq 450 \\ & 2x_A + x_B \leq 350 \\ & x_A + x_B \leq 200 \\ & x_A, x_B \geq 0 \end{aligned}$$

## 1.3 The two crude petroleum problem [Ralphs]

A petroleum company distills crude imported from Kuwait (9000 barrels available at 20€ each) and from Venezuela (6000 barrels available at 15€ each), to produce gasoline (2000 barrels), jet fuel (1500 barrels), and lubricant (500 barrels). The topping process first separates the crude into cuts, then the final products result from conversion, treating, and mixing cuts. The crude oil is present in the products in the following proportions (e.g.: *30% of a barrel of crude from Kuwait and 40% from Venezuela are used to produce one barrel of gasoline*):

| | gasoline | jet fuel | lubricant |
|---|---|---|---|
| Kuwait | 0.3 | 0.4 | 0.2 |
| Venezuela | 0.4 | 0.2 | 0.3 |

Objective: minimize the production cost.

$$\begin{aligned}
\min\ & 20x_K + 15x_V \\
\text{s.t.}\ & 0.3x_K + 0.4x_V \geq 2 \\
& 0.4x_K + 0.2x_V \geq 1.5 \\
& 0.2x_K + 0.3x_V \geq 0.5 \\
& 0 \leq x_K \leq 9 \\
& 0 \leq x_V \leq 6
\end{aligned}$$

## 1.4 The steel factory

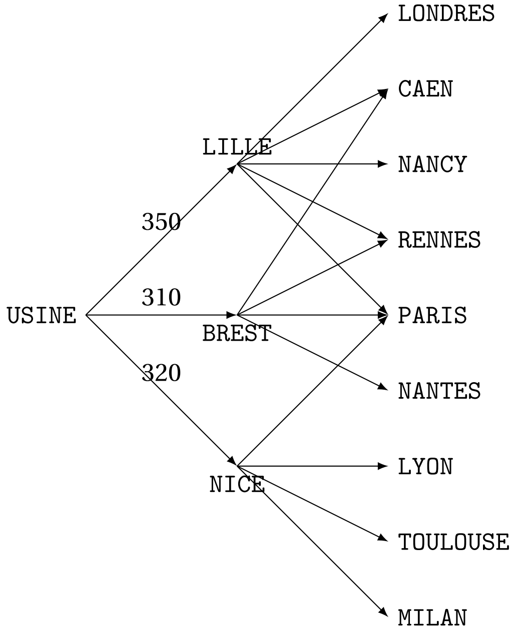
A factory produces steel in coils/tapes/sheets up to 6000/4000/3500 tons a week, sold at 25/30/2 euros per ton of product, respectively. The heating mill is available up to 35 hours a week and can process 200 tons of each product each hour. The rolling mill is available 40 hours and processes hourly either 200/140/160 tons of coils/tapes/sheets. Objective: maximize profit.

$$\begin{aligned}
\max\ & 25x_C + 30x_T + 2x_S \\
\text{s.t.}\ & \frac{x_C}{200} + \frac{x_T}{200} + \frac{x_S}{200} \leq 35 && (heating) \\
& \frac{x_C}{200} + \frac{x_T}{140} + \frac{x_S}{160} \leq 40 && (rolling) \\
& 0 \leq x_C \leq 6000 && (coils) \\
& 0 \leq x_T \leq 4000 && (tapes) \\
& 0 \leq x_S \leq 3500 && (sheets)
\end{aligned}$$

## 1.5 network flow

A company delivers retail stores in 9 cities in Europe from its unique factory *USINE*. How to manage production and transportation in order to: meet the demand of each store, not exceed the production limit, not exceed the line capacities, minimize the transportation costs ?

```
demand = {
    'PARIS': 110,
    'CAEN': 90,
    'RENNES': 60,
    'NANCY': 90,
    'LYON': 80,
    'TOULOUSE': 50,
    'NANTES': 50,
    'LONDRES': 70,
    'MILAN': 70
}
LINES, unitary_cost, capacity = multidict({
    ('USINE','LILLE'): [2.9, 350],
    ('USINE','NICE') : [3.5, 320],
    ('USINE','BREST') : [3.1, 310],
    ('LILLE', 'PARIS') : [1.1, 150],
    ('LILLE','CAEN'): [0.7, 150],
    ('LILLE','RENNES'): [1.0, 150],
    ('LILLE','NANCY'): [1.3, 150],
    ('LILLE','LONDRES'): [1.3, 150],
    ('NICE','LYON'): [0.8, 200],
    ('NICE','TOULOUSE'): [0.2, 110],
    ('NICE','PARIS'): [1.3, 100],
    ('NICE','MILAN'): [1.3, 150],
    ('BREST','NANTES'): [0.9, 150],
    ('BREST','CAEN'):[0.8, 200],
    ('BREST','RENNES'): [0.8, 150],
    ('BREST','PARIS'): [0.9, 100]
})
MAX_PRODUCTION = 900
```




- $x_\ell$ the quantity of products (*flow*) transported on line $\ell = (i,j) \in \texttt{LINES}$
- $\texttt{TRANSITS} = \{\texttt{LILLE}, \texttt{NICE}, \texttt{BREST}\}$

$$
\begin{aligned}
\min \quad & \sum_{\ell \in \texttt{LINES}} \texttt{COST}_\ell x_\ell \\
\text{s.t.} \quad & \sum_{i \in \texttt{TRANSITS}} x_{(\texttt{USINE},i)} \le \texttt{MAXPROD} \\
& \sum_{i \in \texttt{TRANSITS}} x_{(i,j)} \ge \texttt{DEMAND}_j, && \forall j \in \texttt{STORES} \\
& x_{(\texttt{USINE},i)} = \sum_{j \in \texttt{STORES}} x_{(i,j)}, && \forall i \in \texttt{TRANSITS} \\
& 0 \le x_\ell \le \texttt{CAPACITY}_\ell, && \forall \ell \in \texttt{LINES}.
\end{aligned}
$$

## 1.6 minimum distance (1-norm)

Find a solution $x \in \mathbb{R}^n$ of the system of equation $Ax = b$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ of minimum $L^1$ norm:

$$\|x\|_1 = \sum_{j=1,\dots,n} |x_j|$$

- variable splitting:

$$|x| = \min\{x^+ + x^- \mid x = x^+ - x^-, x^+, x^- \geq 0\}$$

$$\begin{aligned}
&\min \sum_{j=1}^{n} (x_j^+ + x_j^-) \\
&\text{s.t. } Ax = b, \\
&\quad x_j = x_j^+ - x_j^-, && \forall j \\
&\quad x_j^+, x_j^- \geq 0, && \forall j
\end{aligned}$$

- supporting plane model:

$$|x| = \max\{x, -x\} = \min\{y \mid y \geq x, y \geq -x\}$$

$$\begin{aligned}
&\min \sum_{j=1}^{n} y_j \\
&\text{s.t. } Ax = b, \\
&\quad y_j \geq x_j, && \forall j \\
&\quad y_j \geq -x_j, && \forall j
\end{aligned}$$

## 1.7 minimum distance (infinity-norm)

Find a solution $x \in \mathbb{R}^n$ of the system of equation $Ax = b$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ of minimum $L^\infty$ norm:

$$\|x\|_\infty = \max_{j=1,\dots,n} |x_j|$$

- $y \geq |x_j| \iff y \geq x_j \wedge y \geq -x_j$
- $y \geq \max_j |x_j| \iff y \geq x_j \wedge y \geq -x_j \ (\forall j)$

$$\begin{aligned}
&\min y \\
&\text{s.t. } Ax = b, \\
&\quad y \geq x_j, && \forall j \\
&\quad y \geq -x_j, && \forall j
\end{aligned}$$

## 1.8 data fitting (LAD regression)

Given $m$ observations – data points $a_i \in \mathbb{R}^n$ and associate values $b_i \in \mathbb{R}$, $i = 1..m$ – predict the value of any point $a \in \mathbb{R}^n$ according to a linear regression model ?

A best **linear fit** is a function : $b(a) = a^T x + y$, for chosen $x \in \mathbb{R}^n, y \in \mathbb{R}$ minimizing the **residual/prediction error** $|b(a_i) - b_i|$, globally over the dataset $i = 1..m$, e.g:

**Least Absolute Deviation or $L_1$-regression:** $\min \sum_i |b(a_i) - b_i|$

supporting planes

$$
\begin{aligned}
& min \sum_i d_i \\
\text{s.t. } \ & d_i \geq \sum_j a_{ij} x_j + y - b_i, && \forall i \\
& d_i \geq -(\sum_j a_{ij} x_j + y - b_i), && \forall i \\
& d \in \mathbb{R}^m, x \in \mathbb{R}^n, y \in \mathbb{R}
\end{aligned}
$$

sparse supporting planes

$$
\begin{aligned}
& min \sum_i d_i \\
\text{s.t. } \ & r_i = \sum_j a_{ij} x_j + y - b_i, && \forall i \\
& d_i \geq r_i, && \forall i \\
& d_i \geq -r_i, && \forall i \\
& r, d \in \mathbb{R}^m, x \in \mathbb{R}^n, y \in \mathbb{R}
\end{aligned}
$$

variable splitting

$$
\begin{aligned}
& min \sum_i d_i^+ + d_i^- \\
\text{s.t. } \ & d_i^+ - d_i^- = \sum_j a_{ij} x_j + y - b_i, && \forall i \\
& d_i^+, d_i^- \geq 0, && \forall i \\
& x \in \mathbb{R}^n, y \in \mathbb{R}
\end{aligned}
$$

dual model

$$
\begin{aligned}
& max \sum_i b_i z_i \\
\text{s.t. } \ & \sum_i a_{ij} z_i = 0, && \forall j \\
& \sum_i z_i = 0, \\
& z_i \in [-1, 1], && \forall i
\end{aligned}
$$

## 1.9 capacity planning

find a least cost electric power capacity expansion plan over an horizon of $T \in \mathbb{N}$ years, given:

- forecast demand (in MW): $d_t \geq 0$ for each year $t = 1, \ldots, T$
- existing capacity (oil-fired plants, in MW): $e_t \geq 0$ available for each year $t$
- options for expanding capacities: (1) coal-fired plant and (2) nuclear plant
  - lifetime (in years): $l_j \in \mathbb{N}$, for each option $j = 1, 2$
  - capital cost (in euros/MW): $c_{jt}$ to install capacity $j$ operable from year $t$
  - political/safety measure: share of nuclear should never exceed 20% of available capacity

$$
\begin{aligned}
\min \ & \sum_{t=1}^{T} \sum_{j=1}^{2} c_{jt} x_{jt} \\
\text{s.t. } \ & y_{jt} = \sum_{s=\max\{1, t-l_j+1\}}^{t} x_{js}, && \forall j = 1, 2, t = 1, \ldots, T \\
& e_t + y_{1t} + y_{2t} \geq d_t, && \forall t = 1, \ldots, T \\
& 8 y_{2t} \leq 2 e_t + 2 y_{1t}, && \forall t = 1, \ldots, T \\
& x_{jt} \geq 0, y_{jt} \geq 0, && \forall j = 1, 2, t = 1, \ldots, T
\end{aligned}
$$

- with decision variables, $x_{jt}$: installed capacity (in MW) of type $j = 1, 2$ starting at year $t = 1, \ldots, T$
- and implied variables, $y_{jt}$: available capacity (in MW) $j = 1, 2$ for year $t$

## 1.10 traitement de l'eau [Zhou, Sustainability 2019]

La **Demande Biochimique en $O_2$** mesure la pollution de l'eau en masse d'$O_2$ requise pour biodégrader la matière organique présente dans l'eau

- Par jour, deux usines produisent resp. $1200m^3$ (DBO=$850g/m^3$) et $4000m^3$ ($DBO = 400g/m^3$) d'eaux usées. Les systèmes de traitement respectifs ramènent 1 tonne DBO à 100kg et 50kg pour un coût de 400 et 500 euros.
- La part traitée est rejetée dans la rivière dans la limite autorisée de $DBO = 170kg$.
- La part non traitée a un coût d'évacuation de 0.56 et 0.25 euro par $m^3$.

Est-il possible de respecter la limite environnemental dans un budget journalier de 1250 euros ?

$$\begin{aligned}
\min\ & 0.1r_1 + 0.05r_2 \\
\text{s.t.}\ & 400r_1 + 500r_2 + 0.56(1200 - x_1) + 0.25(4000 - x_2) \le 1250 \\
& r_1 = 850 * x_1 * 10^{-3} \\
& r_2 = 400 * x_2 * 10^{-3} \\
& 0 \le x_1 \le 1200 \\
& 0 \le x_2 \le 4000
\end{aligned}$$

- avec $x_1, x_2$ volumes traités par usine ($m^3$)