Modelling in
Linear Programming

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1 Exercises

1.1 Doors and windows

$$
\begin{aligned}
\max\ & 3x_1 + 5x_2 \\
\text{s.t.}\ & x_1 \leq 4 \\
& x_2 \leq 6 \\
& 3x_1 + 2x_2 \leq 18 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

1.2 Nuclear waste management

A company eliminates nuclear wastes of 2 types A and B, by applying a sequence of 3 processes I, II and III in any order. The processes I, II, III, have limited availability, respectively: 450h, 350h, and 200h per month. The unit processing times depend on the process and waste type, as reported in the following table:

process	I	II	III
waste A	1h	2h	1h
waste B	3h	1h	1h

(first entry reads *one unit of A-type waste is processed in 1 hour with process I*) The profit for the company is 4000 euros to eliminate one unit of waste A and 8000 euros to eliminate one unit of waste B.

Objective: maximize the profit.

$$
\begin{aligned}
\max\ & 4x_A + 8x_B \\
\text{s.t.}\ & x_A + 3x_B \leq 450 \\
& 2x_A + x_B \leq 350 \\
& x_A + x_B \leq 200 \\
& x_A, x_B \geq 0
\end{aligned}
$$

1.3 The two crude petroleum problem [Ralphs]

A petroleum company distills crude imported from Kuwait (9000 barrels available at 20€ each) and from Venezuela (6000 barrels available at 15€ each), to produce gasoline (2000 barrels), jet fuel (1500 barrels), and lubricant (500 barrels) in the following proportions:

	gasoline	jet fuel	lubricant
Kuwait	0.3	0.4	0.2
Venezuela	0.4	0.2	0.3

(first entry reads: *producing 1 unit of gasoline requires 0.3 units of crude from Kuwait*)
Objective: minimize the production cost.

$$\begin{aligned}
\min\ & 20x_K + 15x_V \\
\text{s.t. }\ & 0.3x_K + 0.4x_V \geq 2 \\
& 0.4x_K + 0.2x_V \geq 1.5 \\
& 0.2x_K + 0.3x_V \geq 0.5 \\
& 0 \leq x_K \leq 9 \\
& 0 \leq x_V \leq 6
\end{aligned}$$

1.4 The steel factory

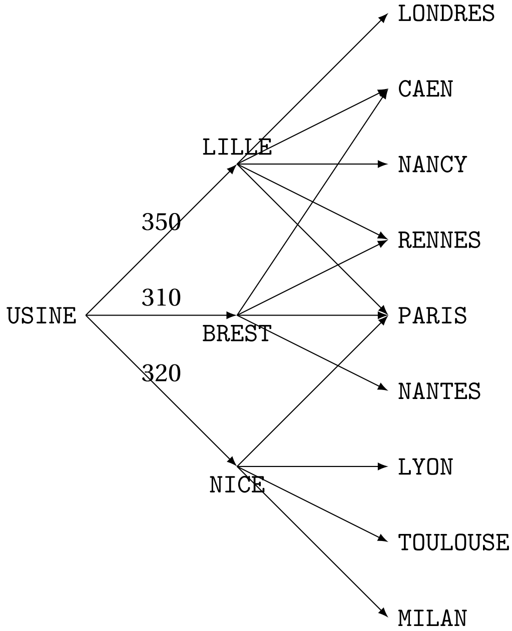
A factory produces steel in coils/tapes/sheets up to 6000/4000/3500 tons a week, sold at 25/30/2 euros per ton of product, respectively. The heating mill is available up to 35 hours a week and can process 200 tons of each product each hour. The rolling mill is available 40 hours and processes hourly either 200/140/160 tons of coils/tapes/sheets. Objective: maximize profit.

$$\begin{aligned}
\max\ & 25x_C + 30x_T + 2x_S \\
\text{s.t. }\ & \frac{x_C}{200} + \frac{x_T}{200} + \frac{x_S}{200} \leq 35 && (heating) \\
& \frac{x_C}{200} + \frac{x_T}{140} + \frac{x_S}{160} \leq 40 && (rolling) \\
& 0 \leq x_C \leq 6000 && (coils) \\
& 0 \leq x_T \leq 4000 && (tapes) \\
& 0 \leq x_S \leq 3500 && (sheets)
\end{aligned}$$

1.5 network flow

A company delivers retail stores in 9 cities in Europe from its unique factory *USINE*. How to manage production and transportation in order to: meet the demand of each store, not exceed the production limit, not exceed the line capacities, minimize the transportation costs ?

```
demand = {
    'PARIS': 110,
    'CAEN': 90,
    'RENNES': 60,
    'NANCY': 90,
    'LYON': 80,
    'TOULOUSE': 50,
    'NANTES': 50,
    'LONDRES': 70,
    'MILAN': 70
}
LINES, unitary_cost, capacity = multidict({
    ('USINE','LILLE'): [2.9, 350],
    ('USINE','NICE') : [3.5, 320],
    ('USINE','BREST') : [3.1, 310],
    ('LILLE', 'PARIS') : [1.1, 150],
    ('LILLE','CAEN'): [0.7, 150],
    ('LILLE','RENNES'): [1.0, 150],
    ('LILLE','NANCY'): [1.3, 150],
    ('LILLE','LONDRES'): [1.3, 150],
    ('NICE','LYON'): [0.8, 200],
    ('NICE','TOULOUSE'): [0.2, 110],
    ('NICE','PARIS'): [1.3, 100],
    ('NICE','MILAN'): [1.3, 150],
    ('BREST','NANTES'): [0.9, 150],
    ('BREST','CAEN'):[0.8, 200],
    ('BREST','RENNES'): [0.8, 150],
    ('BREST','PARIS'): [0.9, 100]
})
MAX_PRODUCTION = 900
```



- x_ℓ the quantity of products (*flow*) transported on line $\ell = (i,j) \in \mathtt{LINES}$
- $\mathtt{TRANSITS} = \{\mathtt{LILLE}, \mathtt{NICE}, \mathtt{BREST}\}$

$$
\begin{aligned}
\min \quad & \sum_{\ell \in \mathtt{LINES}} \mathtt{COST}_\ell x_\ell \\
\text{s.t.} \quad & \sum_{i \in \mathtt{TRANSITS}} x_{(\mathtt{USINE},i)} \leq \mathtt{MAXPROD} \\
& \sum_{i \in \mathtt{TRANSITS}} x_{(i,j)} \geq \mathtt{DEMAND}_j, && \forall j \in \mathtt{STORES} \\
& x_{(\mathtt{USINE},i)} = \sum_{j \in \mathtt{STORES}} x_{(i,j)}, && \forall i \in \mathtt{TRANSITS} \\
& 0 \leq x_\ell \leq \mathtt{CAPACITY}_\ell, && \forall \ell \in \mathtt{LINES}.
\end{aligned}
$$

1.6 minimum distance (1-norm)

Find a solution $x \in \mathbb{R}^n$ of the system of equation $Ax = b$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ of minimum L^1 norm:

$$\|x\|_1 = \sum_{j=1,\ldots,n} |x_j|$$

- variable splitting:

$$|x| = \min\{x^+ + x^- \mid x = x^+ - x^-, x^+, x^- \geq 0\}$$

$$\begin{aligned}
\min \sum_{j=1}^{n} (x_j^+ + x_j^-) & \\
\text{s.t. } Ax = b, & \\
x_j = x_j^+ - x_j^-, & \quad \forall j \\
x_j^+, x_j^- \geq 0, & \quad \forall j
\end{aligned}$$

- supporting plane model:

$$|x| = \max\{x, -x\} = \min\{y \mid y \geq x, y \geq -x\}$$

$$\begin{aligned}
\min \sum_{j=1}^{n} y_j & \\
\text{s.t. } Ax = b, & \\
y_j \geq x_j, & \quad \forall j \\
y_j \geq -x_j, & \quad \forall j
\end{aligned}$$

1.7 minimum distance (infinity-norm)

Find a solution $x \in \mathbb{R}^n$ of the system of equation $Ax = b$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ of minimum L^∞ norm:

$$\|x\|_\infty = \max_{j=1,\ldots,n} |x_j|$$

- $y \geq |x_j| \iff y \geq x_j \wedge y \geq -x_j$
- $y \geq \max_j |x_j| \iff y \geq x_j \wedge y \geq -x_j \ (\forall j)$

$$\begin{aligned}
\min y & \\
\text{s.t. } Ax = b, & \\
y \geq x_j, & \quad \forall j \\
y \geq -x_j, & \quad \forall j
\end{aligned}$$

1.8 data fitting (LAD regression)

Given m observations – data points $a_i \in \mathbb{R}^n$ and associate values $b_i \in \mathbb{R}$, $i = 1..m$ – predict the value of any point $a \in \mathbb{R}^n$ according to a linear regression model ?

A best **linear fit** is a function : $b(a) = a^T x + y$, for chosen $x \in \mathbb{R}^n, y \in \mathbb{R}$ minimizing the **residual/prediction error** $|b(a_i) - b_i|$, globally over the dataset $i = 1..m$, e.g:

Least Absolute Deviation or L_1-regression: $\min \sum_i |b(a_i) - b_i|$

supporting planes

$$\begin{aligned} & min \sum_i d_i \\ \text{s.t. } & d_i \geq \sum_j a_{ij} x_j + y - b_i, && \forall i \\ & d_i \geq -(\sum_j a_{ij} x_j + y - b_i), && \forall i \\ & d \in \mathbb{R}^m, x \in \mathbb{R}^n, y \in \mathbb{R} \end{aligned}$$

sparse supporting planes

$$\begin{aligned} & min \sum_i d_i \\ \text{s.t. } & r_i = \sum_j a_{ij} x_j + y - b_i, && \forall i \\ & d_i \geq r_i, && \forall i \\ & d_i \geq -r_i, && \forall i \\ & r, d \in \mathbb{R}^m, x \in \mathbb{R}^n, y \in \mathbb{R} \end{aligned}$$

variable splitting

$$\begin{aligned} & min \sum_i d_i^+ + d_i^- \\ \text{s.t. } & d_i^+ - d_i^- = \sum_j a_{ij} x_j + y - b_i, && \forall i \\ & d_i^+, d_i^- \geq 0, && \forall i \\ & x \in \mathbb{R}^n, y \in \mathbb{R} \end{aligned}$$

dual model

$$\begin{aligned} & max \sum_i b_i z_i \\ \text{s.t. } & \sum_i a_{ij} z_i = 0, && \forall j \\ & \sum_i z_i = 0, \\ & z_i \in [-1, 1], && \forall i \end{aligned}$$

1.9 capacity planning

find a least cost electric power capacity expansion plan over an horizon of $T \in \mathbb{N}$ years, given:

- forecast demand (in MW): $d_t \geq 0$ for each year $t = 1, \ldots, T$
- existing capacity (oil-fired plants, in MW): $e_t \geq 0$ available for each year t
- options for expanding capacities: (1) coal-fired plant and (2) nuclear plant
  - lifetime (in years): $l_j \in \mathbb{N}$, for each option $j = 1, 2$
  - capital cost (in euros/MW): c_{jt} to install capacity j operable from year t
  - political/safety measure: share of nuclear should never exceed 20% of available capacity

$$\begin{aligned} \min \ & \sum_{t=1}^{T} \sum_{j=1}^{2} c_{jt} x_{jt} \\ \text{s.t. } & y_{jt} = \sum_{s=\max\{1, t-l_j+1\}}^{t} x_{js}, && \forall j = 1, 2, t = 1, \ldots, T \\ & e_t + y_{1t} + y_{2t} \geq d_t, && \forall t = 1, \ldots, T \\ & 8 y_{2t} \leq 2 e_t + 2 y_{1t}, && \forall t = 1, \ldots, T \\ & x_{jt} \geq 0, y_{jt} \geq 0, && \forall j = 1, 2, t = 1, \ldots, T \end{aligned}$$

- with decision variables, x_{jt}: installed capacity (in MW) of type $j = 1, 2$ starting at year $t = 1, \ldots, T$
- and implied variables, y_{jt}: available capacity (in MW) $j = 1, 2$ for year t