23/7/18

References

C.R. Rao : Linear Statistical Inference.

Rao Bhimasankaran.

Bapat : Linear Algebra & Linear Models (3rd edition)

~~v : vectors~~

A is $m \times n$. Show that $A = 0 \Leftrightarrow A'A = 0$
$\Leftrightarrow \text{trace}(A'A) = 0$

Proof $\Rightarrow$ obvious.

$\Leftarrow$ $A'A = 0 \Rightarrow tr(A'A) = 0$

$\Rightarrow \sum\sum a_{ij}^2 = 0$

$\Rightarrow a_{ij} = 0 \quad \forall (i,j)$

$\Rightarrow A = 0.$

$A = \begin{pmatrix} 1 \\ i \end{pmatrix} \Rightarrow A'A = 1 + i^2 = 0$

now suppose. $A^* = \bar{A}'$

Then, if A is $m \times n$ a complex matrix, then,

$A = 0 \Leftrightarrow A^*A = 0$ (for real matrix $A^* = A'$)

~~v be~~ Examples of vector spaces

1. $\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_1, \ldots, x_n \in \mathbb{R} \right\}$

2. All 2×3 matrices.

It is isomorphic to $\mathbb{R}^6$.

we can construct a one-to-one function,

$$f\left(\begin{bmatrix} a_{11} & \cdots & \\ & & a_{23} \end{bmatrix}\right) = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \\ a_{13} \\ a_{23} \end{pmatrix}$$

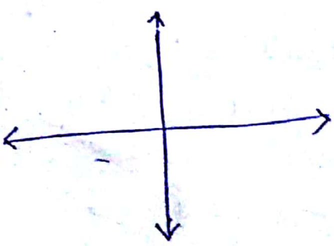
such that it preserves addition and scalar multiplication of the original vector space.

Subspaces

$V \subset W$ vector spaces

then V is a subspace of W.

e.g $\mathbb{R}^2$



what are the subspaces of $\mathbb{R}^2$?

$\mathbb{R}^2$, $\{0\}$, any straight line through origin.

For $\mathbb{R}^3$, $\{0\}$, $\mathbb{R}^3$, any st. line through origin and any plane through origin are the subspaces.

V : vector space.

S : $\{\alpha_1, \dots \alpha_k\}$ finite set of vectors. (not a vector space.)

$\{c_1\alpha_1 + \dots + c_k\alpha_k \mid c_1, \dots, c_k \in \mathbb{R}\}$ = linear span of S.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

linear span of column vectors of A

$= \{y \mid y = Ax : x \in \mathbb{R}^n\}$

= column space of A / Range of A.

A : $k \times n$ matrix

$\{x \in \mathbb{R}^n \mid Ax = 0\}$ is a subspace of $\mathbb{R}^n$

called null space of A / kernel of A.

1. The null vector by itself is a dependent set
2. Any set containing the null vectors is a dependent set.
3. $u_1, \dots, u_k$ is dependent when and only when a member in the set is linear combination of its predecessors.

<u>Def^n</u>: (Basis) $u_1, \dots, u_k \in V$ is called a basis if

(i) $u_1, \dots, u_k$ is lin indep

(ii) span$\{u_1, \dots, u_k\} = V$.

$V = \mathbb{R}^2$ $\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}$ is a basis.

All polynomials in x $\{a_0 + a_1x + \dots + a_nx^n \mid n \in \mathbb{N}\}$ is an infinite dimensional vector space.

$\mathbb{R}^\infty$

set of all functions ~~of~~ $f: \mathbb{R} \to \mathbb{R}$. / set of all continuous functions $f: \mathbb{R} \to \mathbb{R}$.

It is difficult to construct basis.

Let V be a vector space of dim. k.

$u_1, \dots, u_k$ is a basis of V iff $u_1, \dots, u_k$ are lin indep.

alternatively,

$u_1, \dots, u_k$ is a basis of V iff $span\{u_1, \dots, u_k\} = V$.

Ex()

Let $V =$ all functions $f: \mathbb{R} \to \mathbb{R}$.

Show that ~~∄~~ V has no finite basis.

Ans: set of all polynomials in x is a subspace of V and previous is not finite dimensional.

Let V be a vector space of dim k. $u_1, \dots, u_k$ is a basis of V. Then $u \in V$ can be written as lin. comb of $u_1, \dots, u_k$ uniquely.

Theorem

Any vector space of dim k is isomorphic to $\mathbb{R}^k$.

left null space of A : $\{x \mid x^T A = 0\}$.

Theorem

row space and column space of A have the same dim.

Note: Row space and column space are actually isomorphic.

Def^n: (Isomorphism)

V and W are ~~secu~~ two vector spaces are said to be isomorphic if f is one to one f^n, $f: V \to W$ such that $f(x+y) = f(x) + f(y)$

$f(cx) = c f(x)$.

NB the two underlying field must ~~be~~ same.

1
$1+x$
$1+x+x^2$
$\vdots$
$1+x+\cdots+x^k$

V: set of all f^n $f: \mathbb{R} \to \mathbb{R}$.
let V is finite dimensional.
let $f_1, .., f_k$ is a basis of V.
these are $\in V$ and can be written as lin comb of $f_1, .., f_k$.
but there are $k+1$ vectors, so must be lin dependent.

Hence contradiction.

So we can say that V is not finite dimensional.

Note: dimension is for finite dimensional vector spaces, because that's how dim is defined.

Ex 1)
Let A be $n \times n$ matrix. Show that $A = A'$ iff $A^2 = AA'$.

(necessary part)
let $B = (A - A')$ and given that $A^2 = AA'$.

then $BB' = (A-A')(A'-A)$
$= \cancel{AA'} - \cancel{A^2} - A'^2 + A'A = A'A - A'^2$
$= A'A - AA'$

$tr(BB') = \cancel{tr(A^2 - A'^2) = 0}$
$tr(A'A - AA') = 0$

[as $tr(A-B) = tr(A) - tr(B)$
and $\cancel{tr(A) = tr(A')}$
$tr(AB) = tr(BA)$]

hence $tr(BB') = \sum_i \sum_j a_{ij}^2 = 0 \Rightarrow a_{ij} = 0 \;\forall (i,j)$.
let $B = ((a_{ij}))$

$\Rightarrow B = 0$
$\Rightarrow A = A'$.

suff part is trivial.

Linear Equations

26/7/18

Let $\alpha_1, .., \alpha_m$ m fixed vectors in an arbitrary vector space V_F and consider the linear eq^n in the set scalars $x_1, .., x_m \in F$.

$$\sum x_i \alpha_i = 0$$

$A^{m\times n}$ matrix $= \begin{bmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \end{bmatrix}$ $\quad \underline{x} = (x_1, \dots, x_m)$

$\{\underline{x} : (x_1, .., x_m)A = 0\}$ = left null space of A.

isomorphic to ~~N(A)~~ null space of A'

$V = \{(x_1, x_2) : (x_1, x_2)\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = (0,0,0)\}$

$W = \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\}$

$V \neq W$ but V and W are isomorphic.

2 subspace V and W are isomorphic iff they are of same dim.

sol^n space of a sym of lin eq^n forms a subspace.

$\alpha_1, .., \alpha_m$ are fixed vectors.

$S = \{(x_1, .., x_m) \mid x_1\alpha_1 + \dots + x_m\alpha_m = 0, \ x_1, .., x_m \in \mathbb{F}\}$

$M = \{y \mid y = \beta_1\alpha_1 + \dots + \beta_m\alpha_m, \ \beta_1, .., \beta_m \in \mathbb{F}\}$

$d(S) + d(M) = m.$ (rank plus nullity Th.)

$A = m\times n$ matrix. $= \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}$

dim (left null space of A) = m − dim (row space of A)

dim (null space of A) = n − dim (col space of A)

rank of a matrix + nullity of matrix = no. of columns

" " " + " " A' = " " rows.

non homogeneous case. $Ax = b$.

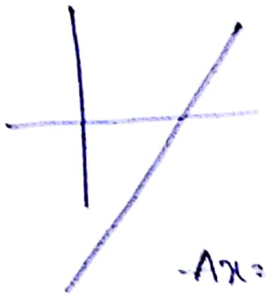
let x_0 be a sol^n. then a class of all solutions is given by $\{x_0 + y \mid Ay = 0\}$.

$A(x_0 + y) = Ax_0 + Ay = b$.

conversely. let z be a sol^n of $Ax = b$.

$z = x_0 + (z - x_0)$

- sol^n of non-homogeneous eq^n does not forms a vector space.



$y = mx + c$

$y - mx = c$

$(1 \ -m)\begin{pmatrix} y \\ x \end{pmatrix} = c$

- $Ax = b$ has sol^n iff $b \in \mathcal{C}(A)$.

the sol^n will be unique if $Ay = 0 \Leftrightarrow y = 0$
then x_0 is the only sol^n.

dim rowspace = dim col space.

$A^{n \times m} = \begin{bmatrix} - \beta_1 - \\ - \beta_2 - \\ \vdots \\ - \beta_n - \end{bmatrix}$ let dim $R(A) = r$.

Suppose, $\beta_1, \dots, \beta_r$ form a basis for the row space.

$B = \begin{bmatrix} - \beta_1 - \\ \vdots \\ - \beta_r - \end{bmatrix}$

$\dim(\text{span}(\alpha_1, \dots, \alpha_m)) = m - \dim(\mathcal{N}(A))$

$\dim(\text{span}(\beta_1, \dots, \beta_n)) = n - \dim(\text{left null space of } A)$

Imp: If $S \overset{\text{subspace}}{\subset} T$
then $\dim S \leq \dim T$

Further if $\dim S = \dim T$
then $S = T$.

Consider a non-homogeneous eq^n $Ax=b$. The following are equivalent.

(1) $b \in \mathcal{C}(A)$

(2) rank $[A]$ = rank $[A \ \ b]$.

(3) $c'A=0 \Rightarrow c'b=0$.

$\Big[\quad \Big]_{m \times n} = \Big[\ \Big]$ If rank $A = m$ then $Ax=b$ is consistent for any $b \in \mathbb{R}^m$.

Inner product

standard inner product, $V = \mathbb{R}^n$ $\langle x,y \rangle = \sum_1^n x_i y_i$

$V = \mathbb{R}^2$ $\langle x,y \rangle = 2x_1y_1 + 2x_2y_2 + x_1y_2 + x_2y_1$.

$(x_1 \ x_2)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ this is an inner product iff A is pd

$\|x\| = \sqrt{\langle x,x \rangle}$

The two vectors are said to be orthogonal if the inner product is 0.

A set of non-null vectors $\alpha_1, \dots, \alpha_k$ orthogonal in pairs is necessarily independent.

$c_1\alpha_1 + \dots + c_k\alpha_k = 0$.

$\langle \sum c_i\alpha_i, \alpha_j \rangle = 0$.

$\Rightarrow c_j = 0$ as $\langle \alpha_j, \alpha_j \rangle > 0$ $\forall j$. $\langle \alpha_i, \alpha_j \rangle = 0$, $i \neq j$

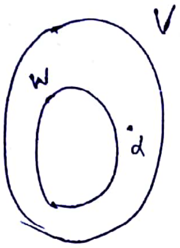
Gram Schmidt Orthogonalization

basis: $\{\alpha_1, \alpha_2, \ldots \alpha_k\}$

orthogonal basis: $\{\beta_1, \beta_2, \ldots, \beta_k\}$

β_i are lin. comb. of $\alpha_1, \ldots, \alpha_{i-1}$.

Projection



V: finite dim vec sp

$W \subseteq V$ $\quad \alpha \in V, \alpha \notin W$.

Then there is a unique pair of vectors γ, β such that $\alpha = \beta + \gamma$, where $\beta \in W$, $\gamma \perp W$.

β is called the orthogonal projection of α on W, γ is the $\perp^r$ from α to W.

Property: $\|\gamma\| = \inf \|\alpha - x\|$, $x \in W$.

Therefore, the shortest distance b/w α and any vector in W is the projection (orthogonal) β, of α on W.

27/7/18

Home Work

1. (a) $2x_1 + x_2 + x_3 = 1$
$2y_1 + y_2 + y_3 = 1$

$\underset{\sim}{x}, \underset{\sim}{y} \in S$. $\alpha \in \mathbb{R}$.

$\underset{\sim}{x} + \alpha \underset{\sim}{y} \notin S$. $\forall \alpha \neq 1$.

So S is not a subspace.

(b). take $\alpha < 0, \in \mathbb{R}$ then $\alpha \underset{\sim}{x} \notin S$ if $\underset{\sim}{x} \in S$.

~~Similarly, if $\alpha < 0$ then~~

let $\underset{\sim}{x}, \underset{\sim}{y} \in T$. then $(\underset{\sim}{x} + \alpha \underset{\sim}{y}) \Rightarrow$

$(x_1 + \alpha y_1)(x_2 + \alpha y_2)$
$= x_1 x_2 + \alpha^2 y_1 y_2$
$+ \alpha(x_2 y_1 + x_1 y_2)$

take $\underset{\sim}{x} = (-10, \cancel{10})$, $\alpha = \frac{1}{2}$
$\hat{y} = (1, 1)$

then $(x_1 + \alpha y_1)(x_2 + \alpha y_2) < 0 \Rightarrow \underset{\sim}{x} + \alpha \underset{\sim}{y} \notin T$.

T is not a subspace.

(e) let $p_1, p_2 \in V$.

take $p_1 = 2x^2 + 3x + 5 + 6$ then $p_1 + p_2 = 3x+5+6 \notin S$.

$p_2 = -2x^2$ S is not a sub space.

2. take $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$A_1, A_2 \in$ set all all 2×2 ns matrix. but $A_1 + A_2 = 0 \notin \{\dots\}$

so, it is not a vector space.

now take $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $B_1 + B_2$ is ns.

hence ~~contradi~~ set of all $2\times n$ sin. mat. is not a vect sp.

3. (a) $(0, b_2, \dots)$ vector space.

(b) $(1,1) + (1,2) = (2,3) \notin \{\underset{\sim}{b} : b_1 = 1\}$

(c) $\{\underset{\sim}{b} : b_1 b_2 = 0\} \Rightarrow \underset{\sim}{a}, \underset{\sim}{c}$. $\underset{\sim}{a} + \alpha \underset{\sim}{c}$. $\alpha \in \mathbb{R}$.

$\Rightarrow (a_1 + \alpha c_1)(a_2 + \alpha c_2)$

$= a_1 a_2 + \alpha^2 c_1 c_2 + \alpha(a_2 c_1 + a_1 c_2)$

$= \alpha(a_1 c_2 + a_2 c_1)$.

~~as either~~ a_1

counter example,

take, $(1,0)$ and $(0,2)$ ~~then~~ $(3,0)$

then $(1,2) \notin \{\underset{\sim}{b} : b_1 b_2 = 0\}$.

(d). span$\{(1,1,0),(2,0,1)\} = \{l_1(1,1,0) + l_2(2,0,1),\ l_1, l_2 \in \mathbb{R}\}$

$a(1,1,0) + b(2,0,1) + \lambda[c(1,1,0) + d(2,0,1)]$

$= (a+\lambda c)(1,1,0) + (b+\lambda d)(2,0,1)$. $\in$ span$\{(\,),(\,)\}$.

so vector ~~space~~ subspace.

(e) $\{(b_1,b_2,b_3) : b_3 - b_2 + 3b_1\} = N\left\{\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}^T\right\} = S$

$\underset{\sim}{x}, \underset{\sim}{y} \in S$ now $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}^T (\underset{\sim}{x} + \alpha \underset{\sim}{y}) = 0 \Rightarrow \underset{\sim}{x} + \alpha \underset{\sim}{y} \in S$.

$\alpha \in \mathbb{R}\}$

4. If ~~no~~ sufficient part: either $S \subseteq T$ / $T \subseteq S$. $\Rightarrow$ ~~cit~~ $S \cup T = S/T$. is a subspace of V.

necessary part: ~~$S \cup T$ is a subspace of V.~~

now suppose, $S \not\subseteq T$ and $T \not\subseteq S$.

then $(S-T)$ and $(T-S)$ is non empty.

$\underset{\sim}{x} \in (S-T)$ and $\underset{\sim}{y} \in (T-S)$. $\Rightarrow$ $\underset{\sim}{x}+\underset{\sim}{y} \in S \cup T$.

~~now~~ as $(\underset{\sim}{x}+\underset{\sim}{y}) \in (S \cup T)$

$\Rightarrow (\underset{\sim}{x}+\underset{\sim}{y}) \in S$ or $(\underset{\sim}{x}+\underset{\sim}{y}) \in T$.

but neither can be true as if

$(\underset{\sim}{x}+\underset{\sim}{y}) \in S$ $\Rightarrow$ $\underset{\sim}{y} \in S$ [due to property of subspace]

contradiction as $\underset{\sim}{y} \in (T-S)$.

hence. $S \cup T$ is a subspace $\Rightarrow$ $S \subseteq T$ / $T \subseteq S$.

5. (a) $\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & 3 & -4 & -1 \end{pmatrix} \underset{\sim}{R_{31}(-2)} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -2 & -1 \end{pmatrix}$. lin. dep.

(b) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 2 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$ lin Indep.

6) let $x, y \in \mathbb{Q}$ s.t. $x + y\sqrt{2} = 0 \Rightarrow$ $x = -y\sqrt{2}$, $y = -x/\sqrt{2}$ ~~contradiction~~

So $x, y \in \mathbb{R}\setminus\mathbb{Q}$

So only possible sol^n $x = 0 = y$. $\Rightarrow$ lin indep.

Similar argument for $\{\sqrt{2}, \sqrt{3}, \sqrt{6}\}$.

$\sqrt{12} = 2\sqrt{3}$ $\Rightarrow$ So $\{\sqrt{12}, \sqrt{3}, \sqrt{2}\}$ is lin dependent.

7). consider $c_1, c_2, c_3 \in \mathbb{R}$. such that,

$c_1(x+y) + c_2(y+z) + c_3(z+x) = 0$

$(c_1+c_3)x + (c_1+c_2)y + (c_2+c_3)z = 0$.

x,y,z are lin indep $\Rightarrow$ $\left.\begin{array}{l} c_1+c_2=0 \\ c_1+c_3=0 \\ c_2+c_3=0 \end{array}\right\}$ provided $1+1\neq 0$. $c_1=c_3=c_2=0$.

suppose $1+1=0$

then $1.(x+y)+1.(y+z)+1.(z+x)$ for $c_1=c_2=c_3=1\neq 0$. $\cancel{1,0,0}$

$= (1+1).x + (1+1)y + (1+1)z = 0$

So they becomes linearly dependent.

counter example: $(1,-1,-1), (1,1,1), \cancel{(0,0,1)}$. $(0,-1,1)$.

8. $c_1(0,1,\alpha) + c_2(\alpha,1,0) + c_3(1,\alpha,1) = 0$

$\Rightarrow$ $\left.\begin{array}{l} c_3+\alpha c_2 = 0 \\ c_1+c_2 \cancel{=0} + \alpha c_3 = 0. \\ c_1\alpha + c_3 = 0 \end{array}\right\}$ for $\alpha\neq 0$ $\begin{array}{l} c_2 = -\frac{1}{\alpha}c_3 \\ c_1 = -\frac{1}{\alpha}c_3. \end{array}$

$(-\frac{2}{\alpha}+\alpha)c_3 = 0 \Rightarrow c_3 = 0$.

$\forall \alpha\neq 0$. three vectors are linearly dependent.

9. $\left.\begin{array}{l} (a,b,b,b)\, c_1 \\ (b,a,b,b)\, c_2 \\ (b,b,b,a)\, c_3 \end{array}\right\} \Rightarrow$ $\left.\begin{array}{l} \cancel{ac_1+bc_2+bc_3=0} \\ ac_1+bc_2+bc_3=0 \\ bc_1+ac_2+bc_3=0 \\ bc_1+bc_2+bc_3=0 \\ bc_1+bc_2+ac_3=0 \end{array}\right\}$ $\begin{pmatrix} a & b & b \\ b & a & b \\ b & b & b \\ b & b & a \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underset{\frown}{0}$

$\Downarrow$ lin indep

if $\underset{\frown}{c} = \underset{\frown}{0}$. now $\underset{\frown}{c} = \underset{\frown}{0}$ be the only sol^n if the matrix A in $A\underset{\frown}{c} = \underset{\frown}{0}$ is left invertible

but

SOURCE: GITHUB.COM/SOURAVSTAT

11. (Steinitz Extention theorem)

~~$x_1, x_2, (1,0,-1,0,0)$~~

u_1 ~~u_2, u_3, u_4~~

u_2 $(1,0,0,2,-3) = 3(1,0,0,0,-1) - 2(1,0,0,-1,0)$

u_3

u_4

any one can be dropped.

$(1,0,0,2,-3), (1,-1,0,0,0), (1,0,-1,0,0), (1,0,0,-1,0)$

$\Downarrow$

$(1,1,0,4,-6) = 2(1,0,0,2,-3) - (1,-1,0,0,0)$.

$(1,0,-1,0,0), (1,0,0,-1,0)$.

So, new basis. $x_1, x_2, (1,0,-1,0,0), (1,0,0,-1,0)$

9. $$\begin{pmatrix} 1 & 0 & 1 & 2 & -1 \\ 2 & 1 & -1 & 3 & 0 \\ 0 & -1 & 3 & 1 & -2 \\ 3 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 3 & 0 \end{pmatrix} \begin{matrix} (\times 2) \\ -(\times 1) \end{matrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 3 & 0 \\ 0 & -1 & 3 & 1 & -2 \\ 3 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 & 4 & -2 \\ 0 & -1 & 3 & 1 & -2 \\ 3 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

last four can form a basis.

12. $\mathcal{P}_4$ is isomorphic to $\mathbb{R}^4$.

$x_1 = (2,3,0,0)$
$x_2 = (3,5,0,0)$
$x_3 = (5,0,-8,1)$
$x_4 = (0,4,-1,0)$

linearly independent

dim $\mathcal{P}_4 = 4$

they form a basis.

13. (a) $$\underset{\sim}{x} = \begin{pmatrix} 2x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(b) $$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \end{pmatrix}$$

$x_1 + x_2 - x_3 = 0$

$x_2 + 2x_3 - x_4 = 0$.

$\Rightarrow$ $x_2 = -2x_3 + x_4$

$x_1 = 3x_3 - x_4$

~~$\underset{\sim}{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} x_3 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} x_4$~~

$$\underset{\sim}{x} = \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} x_4 .$$

14 $S+T$ and $S\cup T$ is equal when ~~S=~~ $S\cap T = \{\phi\}$.

15. ~~$(S\cap T)$~~
$$\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 0 & -1 & 2 \\ 5 & 2 & 3 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \quad \cancel{\underset{\sim}{x} = \underset{\sim}{0}}$$

17. $W = \{ A^{3\times 3} : A \text{ is skew symmetric}\}$.

$A, B \in W$. $c \in \mathbb{R}$. $((A+cB))_{ij} = a_{ij} + cb_{ij}$
$= -(a_{ji} + cb_{ji}) \quad \forall\ i=j$.

So, W is a subspace.

if $A \in W \Rightarrow A \in \{A^{3\times 3}\} = V$

$\therefore$ W is a subspace of V.

$$sp(W) = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

18. trivial

19. $N(A) = \{ \underset{\sim}{x} \in \mathbb{R}^4 : A\underset{\sim}{x} = \underset{\sim}{0} \}$.

(a)
$$\begin{pmatrix} 0 & 2 & -6 & 2 \\ 0 & 2 & -6 & 2 \\ 1 & 1 & 6 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 1 \\ 1 & 0 & 3 & 4 \end{pmatrix}$$

So
$$\left. \begin{aligned} x_2 - 3x_3 + x_4 &= 0 \\ x_1 + 3x_3 + 4x_4 &= 0 \end{aligned} \right\} . \qquad \underset{\sim}{x} = \begin{pmatrix} -3x_3 - 4x_4 \\ 3x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

~~basis~~ $N(A) = span\left\{ \begin{pmatrix} -3 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

(b) $R(A) = span\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 6 \\ 3 \end{pmatrix} \right\}$. $\rightarrow$ last 2 rows.

(c) $C(A)$
$$\begin{pmatrix} 2 & 4 & 1 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 1 & 8 \\ 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 3 & 1 & 5 \\ 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\Downarrow$

$sp\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix} \right\}$.

20) if $a \neq 0$ then W_a is not a vector space.

suppose $g, f \in W_a$ and then $g+f \notin W_a$

as $(g+f)(0) = g(0)+f(0) = 2a \neq a$

So, a must be 0 to be a vector space if $a \neq 0$.

$\therefore W_0 = \{ f \in C(\mathbb{R}) : f(0) = 0 \}$ is a vector space.

now if $f \in W_0$ then $f \in C(\mathbb{R})$ trivial.

So it is also a vector subspace

15)

Assume rank plu

To prove dim R

A : $n \times m$ matrix

$\dim R(A) = r$

B : $r \times m$ wh

$\dim N(A) = m -$ d

$\dim N(B) = m -$ d

$N(A) = N(B)$.

$\dim C(A) = \dim$

Similarly, we

The following are e

(i) $Ax = b$ is con

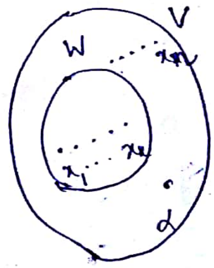
(ii) $c'A = 0 \Rightarrow c'b = 0$

(i) $\Rightarrow$ (ii) $Ax_0 = b$

(ii) $\Rightarrow$ (i) after as

~~Att~~ $\alpha = \beta + \gamma$

β is called th



di

β

α

30/7/18

Assume rank plus nullity theorem

To prove $\dim R(A) = \dim \mathcal{C}(A)$.

$A : n \times m$ matrix

$\dim R(A) = r$

$B : r \times m$ where the rows are from A and lin indep

$\dim \mathcal{N}(A) = m - \dim \mathcal{C}(A)$

$\dim \mathcal{N}(B) = m - \dim \mathcal{C}(B)$

$\mathcal{N}(A) = \mathcal{N}(B)$. $\left\{ \text{write } A = \begin{bmatrix} B \\ C \end{bmatrix} \quad C = DB. \text{ Then } \mathcal{N}\begin{bmatrix} B \\ C \end{bmatrix} = N(B) \right\}$

$\dim \mathcal{C}(A) = \dim \mathcal{C}(B) \leq r$ [as $\mathcal{C}(B) \subseteq \mathbb{R}^r$].

$= \dim R(A)$

Similarly, we can show

$\dim R(A) \leq \dim \mathcal{C}(A)$.

The following are equivalent.

(i) $Ax = b$ is consistent

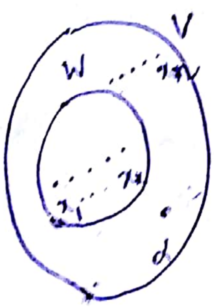
(ii) $c'A = 0 \Rightarrow c'b = 0$.

(i) $\Rightarrow$ (ii) $\quad Ax_0 = b \quad c'Ax_0 = c'b \Rightarrow$ if $c'A = 0 \Rightarrow c'b = 0$.

(ii) $\Rightarrow$ (i) $\quad$ after orthogonality.

~~def~~ $\alpha = \beta + \gamma \quad ; \beta \in W, \gamma \in W^\perp$

β is called the orthogonal projection of α on W.

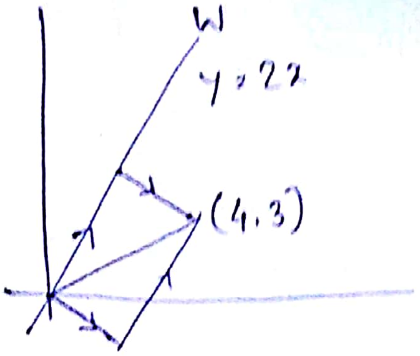


$\dim W = k, \quad \dim V = n$

$\beta = \langle \alpha, x_1 \rangle x_1 + \cdots + \langle \alpha, x_k \rangle x_k$.

$\alpha = \beta + \gamma \quad \beta \in W, \gamma \in W^\perp$

β is called the orthogonal projection of α on W.



Find orthogonal projection of (4, 3) on $y = 2x$.

$\{(x, y): y = 2x\} \xrightarrow{\text{basis}} \{(1, 2)^T\}$. basis for W.

Extend it to a basis of $V = \mathbb{R}^2$ by adding $\begin{pmatrix}0\\1\end{pmatrix}$.

do gram schmidt on $\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$.

take, ~~$u_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $u_2 = \begin{pmatrix}0\\1\end{pmatrix} - c\begin{pmatrix}1\\2\end{pmatrix}$~~

~~s.t. $u_1' u_1 = 0$. $\Rightarrow 1 - 2c = 0$.~~

~~$\Rightarrow c = \frac{1}{2}$.~~

~~$\Rightarrow \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1/2\\0\end{pmatrix}$~~ $u_1 = \begin{pmatrix}1\\2\end{pmatrix}$

~~$u_2 = \begin{pmatrix}1\\2\end{pmatrix} - c\begin{pmatrix}0\\1\end{pmatrix}$~~ ~~s.t. $u_1' u_1 = 0$~~

$1 + 2(2-c) = 0$.

~~suppose $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is the orthonormal basis.~~

on basis for W: $\begin{pmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{pmatrix}, \begin{pmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{pmatrix}$

$$\beta = \left\langle \begin{pmatrix}4\\3\end{pmatrix}, \begin{pmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{pmatrix} \right\rangle \begin{pmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$$

$$\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix} + \begin{pmatrix}2\\-1\end{pmatrix}$$

Best approximation to $\begin{pmatrix}4\\3\end{pmatrix}$ in the W is the vector $x \in W$ that attains $\inf_{x \in W} \|\alpha - x\|$

let $\alpha = \beta + \gamma$, $\beta \in W$, $\gamma \perp W$.

then $\langle x - \beta, \gamma \rangle = 0 \quad \forall x \in W$.

Hence, $\langle x-\alpha, x-\alpha\rangle = \langle x-\beta-\gamma, x-\beta-\gamma\rangle$

$= \langle x-\beta, x-\beta\rangle + \langle\gamma,\gamma\rangle \geq \|\gamma\|^2$

equal iff $x=\beta$.

Therefore, the shortest distance b/w α and an element of W is the length of the $\perp^r$ from α to W.

Bessel's Inequality. (Proof ??)

Let $\alpha_1, \ldots, \alpha_m$ be an orthonormal set of vectors in an inner product space and β be any other vector.

Then, $|\langle\alpha_1,\beta\rangle|^2 + \cdots + |\langle\alpha_m,\beta\rangle|^2 \leq \|\beta\|^2$

with equality iff $\beta \in sp\{\alpha_1, \ldots, \alpha_m\}$.

Proof Let $\delta = \sum \langle\alpha_i,\beta\rangle\alpha_i$, projection of β on $sp\{\alpha_1, \ldots, \alpha_m\}$

and by proposition, $\beta = \delta + \pi$ where $\langle\pi,\delta\rangle = 0$

hence $\|\beta\|^2 = \|\delta\|^2 + \|\pi\|^2 \geq \|\delta\|^2 = \langle\delta,\delta\rangle$

$= \sum |\langle\alpha_i,\beta\rangle|^2$

Def^n: ($S^\perp$)

V: vector space. $S \subset V$ (not necessarily a subspace)

$S^\perp = \{y \in V \mid \langle y, x\rangle = 0 \ \forall\ x \in S\}$.

Suppose ~~$\mathcal{S}$~~ S is not a subspace.

$M(S) = span\{S\}$ = smallest subspace containing S.

Important Result: $\dim[M(S)] + \dim(S^\perp) = \dim(V)$.

Suppose, $V = \mathbb{R}^n$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.

$A = [\alpha_1, \ldots, \alpha_n]$ ~~$C(A)$~~.

$S = M(\alpha_1, \ldots, \alpha_n) = C(A)$.

$S^\perp$ = left null space of A.

$\dim C(A) + \dim(\text{left null space of } A)$

$= n = \dim V$.

S, T are subspaces of V.

$S+T = \{x+y \mid x \in S, y \in T\}$ It is a subspace.

$S \cup T$ is usually not a subspace.

$S \cap T$ is a subspace.

We say that V is a direct sum of S and T and write $V = S \oplus T$ if any $x \in V$ has a unique decomposition $x = y + z$ $y \in S, z \in T$.

suppose $x \in S \cap T$, $x \neq 0$

then $\left.\begin{array}{l} x = x + 0 \\ x = 0 + x \end{array}\right\}$ not unique decomposition.

So, $\cancel{x}$ $S \cap T = \{0\}$.

$V = S+T$ $\quad S \cap T = \{0\} \Rightarrow V = S \oplus T$.

$x \in V$, $\cancel{\;\;}$ $x = y + z$, $y \in S$, $z \in T$.

$x = \tilde{y} + \tilde{z}$, $\tilde{y} \in S$, $\tilde{z} \in T$.

then $(y - \tilde{y}) + (z - \tilde{z}) = 0$

$(y - \tilde{y}) \in S$ $\quad$ as $(y - \tilde{y}) = (\tilde{z} - z)$

$(z - \tilde{z}) \in T$ $\quad \in S \cap T$

$\Rightarrow y - \tilde{y} = 0$ and $z - \tilde{z} = 0$

$\Rightarrow y = \tilde{y}$ and $z = \tilde{z}$.

$V = S \oplus T \Rightarrow \dim V = \dim S + \dim T$.

CR Rao Problem 5, 6, 7, 8, 9, 10, 11, 18, 19.

Q2 Modular law $\dim(S+T) = \dim(S) + \dim(T) - \dim(S \cap T)$

(Problem 11). S, A.

$\mathcal{C}(A) = \{y \mid y = Ax \text{ for some } x\}$

$\mathcal{C}_S(A) = \{y \mid y = Ax$ for some x and $y \in S\}$.

(Problem 18) Steinitz Replacement Result

V vector space. $\dim V = n$.

$\{x_1, \ldots, x_n\}$: basis for V.

$y_1, \ldots, y_r \in V$ linearly independent. $r < n$.

Then there exists $n-r$ vectors $x_{i_1}, \ldots, x_{i_{n-r}}$ such that, $\{y_1, \ldots, y_r, x_{i_1}, \ldots, x_{i_{n-r}}\}$ is a basis for V.

31/7/18

Example 1.6.4 (Rao Bhima)

S, T.

$S+T = \{x+y \mid x \in S, y \in T\}$

$S = \{(\xi_1, \ldots, \xi_4) : \xi_1 + \xi_2 = 0 \text{ \& } \xi_3 + \xi_4 = 0\}$

$T = \{(\xi_1, \ldots, \xi_4) : \xi_1 + \xi_3 = 0, \xi_2 + \xi_4 = 0\}$.

Show that $S+T = \{(\xi_1, \ldots, \xi_4) : \xi_1 + \cdots + \xi_4 = 0\}$.

(u_1, u_2, u_3, u_4) such that $u_1 + u_2 + u_3 + u_4 = 0$

$$(u_1, u_2, u_3, u_4) = \underbrace{(0, 0, u_1+u_3, -u_1-u_3)}_{\in S} + \underbrace{(u_1, u_2, -u_1, -u_2)}_{\in T}.$$

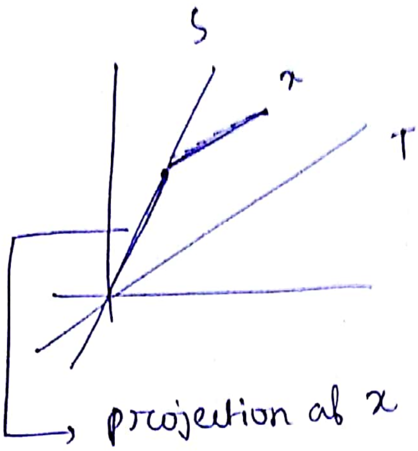
Theorem 1.7.2 (with proof)

$V = S \oplus T$ T is complement of S
S is " " T.

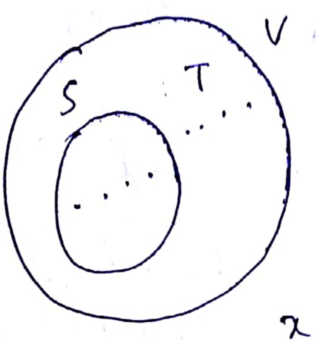
S and T are said to be complement. $(S, T \subseteq V)$
if $S \oplus T = V$.

Theorem 1.7.4

Def^n 1.7.5



→ projection of x on S along T.

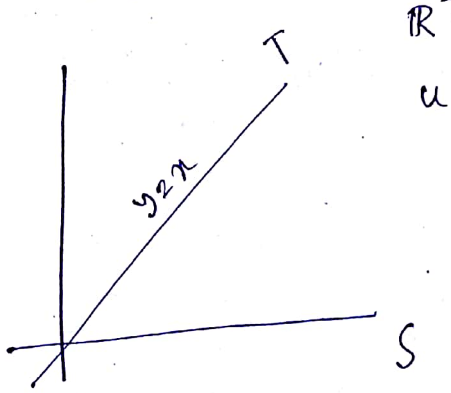


$V = S \oplus T$.
$\exists$ unique $x \in S$, $y \in T$
$u \in V$
$\ni u = x + y$.

x is called projection of u on S along T.

y is called the projection of u on T along S.

Orthogonal projection of x on S is projection of x on S along $S^\perp$.



$\mathbb{R}^2 = S \oplus T$.
$u = (-1, 2)$
$= \underset{\in S}{(-3, 0)} + \underset{\in T}{(2, 2)}$.

$(-3, 0)$ is the projection of u on S along T.

$(2, 2)$ is the projection of u on T along S.

(Theorem 1.7.7)

Construct a 3x3 matrix A s.t A, A^2 are non-zero but $A^3 = 0$.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$A^{n\times r}$, $r<n$ $B: r\times n$. Show that $AB \neq I$

$$\begin{bmatrix} A & \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} \end{bmatrix}^{n\times n} \begin{bmatrix} B \\ \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} \end{bmatrix}^{n\times n} = I$$

take determinant both side and gets contradiction.

Bapt Ex(1.4) (Problem-2) take modulo 2.} to the matrix.

[Laplace Expansion of Determinant] ??

$$\begin{vmatrix} 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -3 & 0 & 0 & 0 \\ -2 & -1 & -1 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} \begin{vmatrix} -1 & -1 & -1 \\ -1 & -1 & -3 \\ -2 & -1 & -1 \end{vmatrix} (-1)^{1+2+3+4+5+6}$$

6/8/18

Linear transformation vs Matrices

$\phi: V_1 \longrightarrow V_2$ is linear if, (it may not be a bijection)

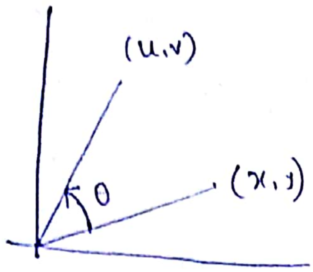
(i) $\phi(x+y) = \phi(x) + \phi(y)$

(ii) $\phi(\alpha x) = \alpha\, \phi(x)$

$\phi: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$\phi(x,y) = (2x-3y,\ x-y,\ x+y)$

$$\phi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$



$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

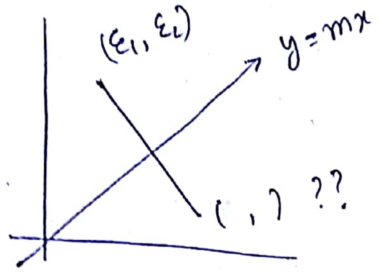
anticlock wise rotation.

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\phi : \mathbb{R}^n \to \mathbb{R}^n$

$x \longmapsto Ax$

Exercise



Consider, V_1, V_2. The set of all linear transformation from V_1 to V_2 form a vector space.

Suppose. $\{x_1, x_2\}$ basis of $\mathbb{R}^2$

$\{y_1, y_2, y_3\}$ " " $\mathbb{R}^3$

Remark If we know $\phi(x_1)$ and $\phi(x_2)$ then we know ϕ completely.

$x \in \mathbb{R}^2$ $\quad x = c_1 x_1 + c_2 x_2$

$\phi(x) = c_1\phi(x_1) + c_2\phi(x_2)$.

$\phi(x_1) = a_{11}y_1 + a_{12}y_2 + a_{13}y_3$

$\phi(x_2) = a_{21}y_1 + a_{22}y_2 + a_{23}y_3$

$\cancel{\phi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}}$ $\quad \phi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\cancel{\phi : V_1 \Rightarrow V_2}$

V basis $x_1, \ldots, x_n$

$x \in V$, $x = c_1 x_1 + \cdots + c_n x_n$.

if we change the basis the matrix changes.

A and B commute if $AB = BA$.

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\,|D - CA^{-1}B|$$

T/F $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|$ → False. Give a counter example.

T/F $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|$ if A and C commute

$|A|\,|D - CA^{-1}B| = |AD - ACA^{-1}B|$

$= |AD - CAA^{-1}B|$

$= |AD - CB|$ True.

Set of all $m \times n$ real matrices form a vector space. (dim mn)

$n \times n$ symmetric matrix. (dim $\frac{n(n+1)}{2}$)

$tr(ABC) = tr(CAB)$ True

$tr(ABC) = tr(ACB)$ False (counter example?)

Ex. $A^{n \times n}$; $tr(AB) = 0$ $\forall$ B. then $A = 0$;

~~the~~ take $B = U_{ij}$ $\forall (i,j)$.

Idea of Sub matrix, Principal sub matrix.

and leading principal submatrix (first k rows and first k columns)

Kronecker Product

(Ex- 2.7, 10.7, 12)

$A^{m\times n}$ $B^{p\times q}$

$A \otimes B = ((a_{ij} B))^{mp\times nq}$

$$= \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & & & a_{mn}B \\ a_{m1}B & a_{m2}B & \cdots & \end{bmatrix}^{mp\times nq}$$

$A \otimes (B \otimes C) = (A \otimes B) \otimes C.$

$A \otimes (B + C) = (A \otimes B) + (A \otimes C).$

$(B+C) \otimes A = (B \otimes A) + (C \otimes A)$ and $I_p \otimes I_n = I_{np}$

$I \otimes A = \text{diag}(A_1, \ldots, A_n)$

$0 \otimes A = 0 = A \otimes 0$

$(AB) \otimes (CD) = (A \times C)(B \otimes D)$ ⇒ Important.

$(A \otimes B)^T = A^T \otimes B^T.$

suppose. $A^{m\times m}$ and $B^{n\times n}$ are n.s square matrix not necessarily of same size.

Show that $A \otimes B$ is n.s

$(A^{-1} \otimes B^{-1})(A \otimes B) = (AA^{-1} \otimes BB^{-1}) = I \otimes I = I.$

Similarly, if A and B are (~~dig~~ diagonal / orthogonal) then $A \otimes B$ is (diagonal / orthogonal) resp.

~~tet~~

let A, B $n\times n$ fixed matrix.
Define $f(C) = ACB$ on the set of $n\times n$ matrices.

$C = [C_{*1}, C_{*2}, \ldots, C_{*n}]$

define $\operatorname{vec} C = \begin{bmatrix} C_{*1} \\ C_{*2} \\ \vdots \\ C_{*n} \end{bmatrix}$

$\operatorname{vec} ACB = \underline{\underline{(A \otimes B^T)}}(\operatorname{vec} C)$

$\Downarrow$
matrix of linear transformation.

13/8/18

Chapter 3

$A^{m\times n}$: column rank $= r$

let $[B_1 \ldots B_r]$ be a basis of $\mathcal{C}(A)$.

there exists $C^{r\times n}$ such that $A = BC$.

- $R(A) \subset R(C)$ $\quad \dim(R(A)) \leq \dim(R(C))$.

row rank of $A \leq$ row rank of C.

row rank of $C \leq r$.

there are r rows of C $\Rightarrow$

row rank of $A \leq$ ~~row~~ column rank of A.

now reversing the argument, we can show the reverse ineq.

~~null matrix is the~~

$A^{m\times n}$ ~~m~~ of rank r. $\qquad \rho(B) \leq \rho(A)$.

B be a submatrix

WLG,

$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$ $\qquad \mathcal{C}\begin{bmatrix} B \\ D \end{bmatrix} \subseteq \mathcal{C}(A) \Rightarrow \rho\begin{pmatrix} B \\ D \end{pmatrix} \leq \rho(A)$.

$R(B) \subseteq R\begin{pmatrix} B \\ D \end{pmatrix}$

$\Rightarrow \rho(B) \leq \rho\begin{pmatrix} B \\ D \end{pmatrix} \leq \rho(A)$.

If $BA = I$ then B is called a left inverse of A.

Eq necessary and suff condition for existence of left inv. is full column rank.

B is not unique unless A is square and ns.

right inv. exists iff full row rank.

Theorem 33.7 (can be extended)

column (rows) are Lin. indep.

strictly diagonally dominated matrix.

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad , i = 1, \ldots, n.$$

has an inverse.

$Ax = 0 \qquad \sum_j a_{ij} x_j = 0 \qquad i = 1, \ldots, n.$

Suppose, $|x_i|$ is maximum among $|x_j|$ $j = 1, \ldots n.$

Suppose $|x_i| > 0$.

$$a_{ii} x_i = -\sum_{j \neq i} a_{ij} x_j.$$

$$|a_{ii}||x_i| \leq \sum_{j \neq i} |a_{ij}||x_j| \leq \left(\sum_{j \neq i} a_{ij}\right)|x_i| < |a_{ii}||x_i|$$

contradiction. $\Rightarrow$ $|x_i| = 0$ $\Rightarrow$ $x = \underset{\sim}{0}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

interchanging column $\Leftrightarrow$ post multiplying by permutation matrix.

$B = AP$

$B^{-1} = P^{-1}A^{-1} = P^{T}A^{-1}$.

(1) $\mathcal{C}(B) \subset \mathcal{C}(A) \iff$ (2) $B = AC$ for some C.

A has full column rank.

To show $\rho(AB) = \rho(B)$

A has left inverse X. $\Rightarrow$ $B = XAB$

$\rho(B) = \rho(XAB) \leq \rho(AB) \leq \rho(B)$.

$\Rightarrow$ $\rho(AB) = \rho(B)$.

Special Case of Rank Cancellation law.

if $AA^TB = AA^TC$

$\Rightarrow$ $A^TB = A^TC$. as in addition, we know $\rho(A) = \rho(AA^T)$.

Theorem 3.5.10

$B^{n \times p}$; $T \subseteq \mathbb{R}^p$ $W = \mathcal{N}(B)$

$S = \{Bx : x \in T\}$

restricted col. space.

Then, $\dim(S) = \dim(T) - \dim(T \cap W)$.

Remark if $T = \mathbb{R}^p$, this is Rank plus nullity theorem

$\rho(AB) = \rho(B) - \dim(\mathcal{C}(B) \cap \mathcal{N}(A))$.

$\rho(ABC) \geq \rho(AB) + \rho(BC) - \rho(B)$.

Rank Factorization

$A : m \times n$ rank r.

$A = PQ$ $\left.\begin{matrix} P^{m \times r} \\ Q^{r \times n} \end{matrix}\right\}$ both of rank r.

If A is idempotent $A^2 = A$.

then rank $A = tr(A)$.

$A = PQ$. $= A^2 = PQPQ$.
P has left inv.

$\Rightarrow$ $QP = I_r$
Q " right " .

$tr(QP) = r = tr(PQ)$

$\Rightarrow$ $r = tr(A) = r(A)$.

14/8/18

Section 3.7

$\rho(A+B) \leq \rho(A) + \rho(B)$.

Proof $(A+B)x = Ax + Bx$.

$\mathcal{C}(A+B) \subseteq \mathcal{C}(A) + \mathcal{C}(B)$.

$\rho(A+B) \leq \dim \mathcal{C}(A) + \dim \mathcal{C}(B) - \dim(\mathcal{C}(A) \cap \mathcal{C}(B))$

$\leq \rho(A) + \rho(B)$

$\because$ two vectors spaces are virtually disjoint $\Leftrightarrow$ only vector in intersection is null vector.

$\rho(A+B) = \rho(A) + \rho(B) \not\Rightarrow \mathcal{C}(A) \cap \mathcal{C}(B) = \{0\}$.

To prove: $\mathcal{C}(A) \cap \mathcal{C}(B) = \{0\}$ $\Rightarrow$ $\rho(A+B) = \rho(A) + \rho(B)$

$R(A) \cap R(B) = \{0\}$

Let $\rho(A) = r$, $\rho(B) = s$.

$A = P_1 Q_1$ rank factorization

$B = P_2 Q_2$ " "

$A+B = P_1 Q_1 + P_2 Q_2 = \begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$

Let $P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}$ $Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$

$\mathcal{C}(A) = \mathcal{C}(P_1)$ $\{\mathcal{C}(P_1) \cap \mathcal{C}(P_2)\} = \{0\}$.

$\mathcal{C}(B) = \mathcal{C}(P_2)$ So $\rho(P) = \rho(P_1) + \rho(P_2)$.

[In addition $\mathcal{C}(P) \subseteq \mathcal{C}(P_1) + \mathcal{C}(P_2)$

SOURCE: GITHUB.COM/SOURAVSTAT

$\rho([P_1 \ \ P_2]) = r+s$. Similarly $\rho(\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}) = r+s$.

So (P, Q) is a rank factorization of $A+B$.

P has full col. rank
Q " " row " $\Big\}$ $\Rightarrow \rho(A+B) = r+s$.

So here, to get this $\Uparrow$ we need both $\mathcal{C}(A) \cap \mathcal{C}(B) = \{0\}$, $R(A) \cap R(B) = \{0\}$.
dropping one of them fails to hold the additivity.

$\mathcal{C}(A) \cap \mathcal{C}(B) = \{0\} \Rightarrow \mathcal{C}(A) + \mathcal{C}(B)$ is direct sum.

<u>Lemma</u> let A be $n \times n$. Then $A^2 = A \Leftrightarrow \rho(A) + \rho(I-A) = n$

necessary part $\rho(A) + \rho(I-A) = tr(A) + tr(I-A)$
$= n$,

suff part $\mathcal{N}(A) = \mathcal{C}(I-A)$
$\Rightarrow$ nullity $A = \rho(I-A)$.
$\Rightarrow n = \rho(A) + \rho(I-A)$ (rank - nullity)

$S \longrightarrow T$
A is a projection. $x \longrightarrow Ax$
Ax is the projection of x on S along T.

$A^2 = A$; To prove $\mathcal{C}(I-A) = \mathcal{N}(A)$.

$x \in \mathcal{C}(I-A) \Rightarrow x = (I-A)y$ for some y
$\Rightarrow Ax = 0 \Rightarrow x \in \mathcal{N}(A)$. $\Rightarrow \mathcal{C}(I-A) \subseteq \mathcal{N}(A)$

$x \in \mathcal{N}(A) \Rightarrow A\underline{x} = 0 \Rightarrow (I-A)x = x$.
$\Rightarrow x \in \mathcal{C}(I-A)$. $\Rightarrow \mathcal{N}(A) \subseteq \mathcal{C}(I-A)$.

any x can be written as. $x = \underline{Ax} + (I-A)x$.
projection of x into on $\mathcal{C}(A)$ along $\mathcal{C}(I-A)$.

$A = PQ$ rank factorization.

then $QP = I$ iff $A^2 = A$.

Suppose, $A = A_1 + \cdots + A_k$.

(a) A is idempotent

(b) A_i " " $\forall i$.

(c) $A_i A_j = 0$ $\forall i \neq j$.

(d) $\rho(A) = \sum_1^k \rho(A_i)$

$\Big\}$ see Th. 3.7.6

(a), (d) $\Rightarrow$ $QP = I$ $\Rightarrow$ $\begin{bmatrix} Q_1 \\ \vdots \\ Q_k \end{bmatrix} \begin{bmatrix} P_1 \cdots P_k \end{bmatrix} = I$

$\Rightarrow$ $Q_i P_i = I$ $\Rightarrow$ $P_i Q_i = I \Rightarrow A_i^2 = A_i$

$Q_i P_j = 0$ $\qquad Q_i P_j \Rightarrow A_i A_j = 0$.

To prove, $\rho(A) + \rho(I - A) = n \Rightarrow A^2 = A$.

$A = P_1 Q_1$ r.f

$I - A = P_2 Q_2$ r.f.

$P_1 Q_1 + P_2 Q_2 = I = [P_1\ P_2] \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$

Square matrix

So we can reverse the order $[P_1\ P_2]$ $\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} [P_1\ P_2] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$Q_1 P_1 = I$ $\qquad Q_1 P_2 = 0 = Q_2 P_1$.

$Q_2 P_2 = I$ $\qquad \Rightarrow A^2 = A$.

(c*): $A_i = P_i Q_i$

$A_i A_j = P_i Q_i P_j Q_j = 0 \Rightarrow Q_i P_j = 0$ $\forall i \neq j$.

Ex () Let A, B be $n \times n$ idempotent.

then $A + B$ is idempotent iff $AB = 0$, $BA = 0$.

if part $(A+B)^2 = A^2+B^2+AB+BA$
$= A+B+AB+BA.$

if $AB=BA$
then $(A+B)$ is idempotent

now if $(A+B)$ is idempotent.

$\Rightarrow AB+BA=0$ ~~BAB~~

$A^2B+ABA=0$ $ABA+BA^2=0$

$AB+ABA=0$ $ABA+BA=0$

$AB=BA$

$\Rightarrow AB=BA=0.$

$A_1+\cdots+A_K = I$
Then following are equivalent
(i) $A_i^2 = A_i \quad \forall i$
(ii) $A_iA_j=0 \quad i\neq j$
(iii) $\rho(A_1)+\cdots+\rho(A_n) = n.$

$X\sim N(0,I)$
$X'X = X'IX$
$= X'A_1X+\cdots+X'A_kX$
A_i's are idempotent matrix.

16/8/18

changing one element, rank varies $+1/0/-1$.

$A = B+(a_{ij}-a_{ij}')U_{ij}$

~~ρ(B) ρ(A) ρ(B)+~~ $\rho(A)\leq \rho(B)+1.$

Section 3.8

$M = \begin{bmatrix} A & B \\ C & D\end{bmatrix}$ suppose A is square and ns
$F = D-CA^{-1}B$ Schur complement of A.

$M^{-1} = \begin{bmatrix} - & - \\ - & -\end{bmatrix}$ inverse of this block is Schur complement of A.

$A^{-1}+A^{-1}BF^{-1}CA^{-1} = (A-BD^{-1}C)^{-1}.$

M is ns iff T is ns.

rank M = rank A + rank F.

Section 3.9

$A=0 \iff A^TA=0 \iff \text{trace}(A^TA)=0$

$\mathcal{N}(A)$ is a complement of $\mathcal{C}(A^T)$.

$x \in \mathcal{N}(A) \cap \mathcal{C}(A^T)$

$x = A^Ty$ $\quad$ $Ax = AA^Ty$

$AA^Ty = 0$

$\Rightarrow y^TAA^Ty = 0$ $\quad$ $\Rightarrow A^Ty = 0$

$\Rightarrow x = 0$.

Check $\mathcal{N}(A^TA) = \mathcal{N}(A)$.

$x \in \mathcal{N}(A^TA)$. $\Rightarrow A^TAx = 0$

$\Rightarrow x^TA^TAx = 0$ $\quad$ $\Rightarrow Ax = 0$

$\Rightarrow x \in \mathcal{N}(A)$.

$\Rightarrow \rho(A) = \rho(A^TA)$. $\quad$ $\rho(A) = \rho(A^T)$

Similarly $\rho(A^T) = \rho(AA^T)$ $\quad$ all are equal.

$\mathcal{C}(A) = \mathcal{C}(AA^T)$. $\quad$ $\rho(A) = \rho(AA^T)$

$\mathcal{C}(AA^T) \subseteq \mathcal{C}(A)$

$\Rightarrow \mathcal{C}(AA^T) = \mathcal{C}(A)$.

A has full col. rank $\Rightarrow AA^T$ is ns.

$(A^TA)^{-1}A^T$ is a left inverse of A.

A has full row rank $\Rightarrow A^TA$ is ns.

~~$A(A^TA)$~~ $A^T(AA^T)^{-1}$ is right inv of A.

(3.9) no listed problem.

(3.10, 3.11) X

$A^{m\times n}$ $\quad B^{p\times m}$ left invertible.

$\rho(BA) = \rho(A) \Leftarrow$ To show.

$\rho(BA) \le \rho(A)$. $\quad A = B_L^{-1} B A$

$\rho(A) \le \rho(BA)$.

$\therefore \rho(A) = \rho(BA)$.

(116) $\underset{r}{\text{10.(a)}}$. lin indep

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \} r \text{ lin indep}$$

$\text{rank}(A) = \text{rank}\begin{bmatrix} B & C \end{bmatrix}$

$= \text{rank}[B]$.

$$A^{m\times n} = P^{m\times r} Q^{r\times n}$$
$$= \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} P_1Q_1 & P_1Q_2 \\ P_2Q_1 & P_2Q_2 \end{bmatrix}$$

$B = P_1 Q_1$ $\quad P_1, Q_1$ is ns.

$\therefore P_1Q_1$ is ns.

For symmetric matrix we choose principal submatrix as if one vector in is in col. basis then it is also in row basis.

$\cancel{A_{i*}R = 0}$ $\quad \cancel{\Phi}\ \underset{\sim}{e_i}$ $\quad i = 1(1)m$.

$m\ ee^T$

n column

4. $A^{m\times n}$. $\quad AR = I$ $\quad AR_{*j} = e_j$

$m = n$. ~~[illegible]~~ .

multiplying by elementary matrix

$\Leftrightarrow$ elementary row operation.

elementary row operation does not change row sp
" col. " " " " col. "

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad \rho(M) = \rho(A) + \rho(D - CA^{-1}B).$$

we can perform row & col operation blockwise.

$A^{m \times n}$ of rank r.

Then $\exists$ n.s P, Q s.t. $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

$A = BC$ r.f. $\quad \exists P$ n.s $\because PB = \begin{bmatrix} I_r \\ 0 \end{bmatrix}$

$\exists Q$ n.s $CQ = \cancel{0}[I_r \;\; 0]$

$\Rightarrow \quad PBCQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & \cancel{0} \end{bmatrix} \qquad A = P^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$

$$= \underbrace{P^{-1} \begin{bmatrix} I_r \\ 0 \end{bmatrix}}_{B} \underbrace{[I_r \;\; 0]\, Q^{-1}}_{C}.$$

Chapter 5

Section 5.3

$Ax = b$ is consistent iff $A^T u = 0 \Rightarrow b^T u = 0$.

$A^T u = 0 \Rightarrow u$ is orthogonal to every row of A^T
$\Rightarrow$ " " " col of A.

$\Rightarrow b^T u = 0 \Rightarrow u$ " " "

$$\{C(A)\}^{\perp} \subset \{u\}^{\perp}$$

$$\Rightarrow C(A) \supseteq \{u\}$$

exactly one of $Ax = b$ and $\begin{bmatrix} A' \\ b' \end{bmatrix} y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

is consistent.

Generalised Inverse of a Matrix

$A^{m\times n}$ Then G of order $n \times m$ is called a g-inverse of A if $AGA = A$

If A is $n\times n$ nonsingular then A^{-1} is the only inverse.

~~$B = BGB$. $B = BGB$~~
~~$A = AGA$~~ ~~$B - A = BGB - AGA$~~

$AA^{-1}A = A \Rightarrow A^{-1}$ is a g. inverse.

now $AGA = A$ G is another g inverse.

$\Rightarrow GA = I$, G is square $\Rightarrow G = A^{-1}$.

A left inverse or ~~gight~~ right inverse is a g inverse.

$BB_L^{-1}B = BI = B$; $BB_R^{-1}B = IB = B$

then there are infinitely many g inverse.

suppose B has full col rank. $\begin{bmatrix} E \\ F \end{bmatrix} [B \mid C]$ ns

$EB = I$ now E is left inv.

there are inf many $E \Rightarrow$ inf many g-inverse.

$A = BC$ r.f.   Then $AGA = BCC_R^{-1}B_L^{-1}BC$
$= BC = A$

let $G = C_R^{-1}B_L^{-1}$

So every matrix has a g-inverse.

Alt Def^n of g-inverse

Suppose $Ax = b$ is consistent

Then $x_0 = Gb$ is a sol^n for any g-inv G of A.

~~Proof~~ Proof. $\exists$ y s.t. $Ay = b$.

$Ax_0 = AGb = AGAy = Ay = b$.

$\Rightarrow$ x_0 is a sol^n.

equivalence with def^n 5.4.1.

$A^{m\times n}$. $G^{n\times m}$.

$A = [A_{*1} \dots A_{*n}]$

now $Ax = A_{*i}$ is consistent $\forall i$.

Want to show $AGA = A$.

enough to show $AGA_{*i} = A_{*i}$.

now GA_{*i} is a sol^n ??

$AGA = A \Rightarrow AG$ is idempotent and $\rho(AG) = \rho(A)$.

$A = AGA$. $\rho(AG) \le \rho(A) = \rho(AGA) \le \rho(AG)$

~~AG is idempotent, ρ~~

AG is idempotent and $\rho(AG) = \rho(A)$. ~~$R(A) = R(AG)$~~ $C(A) = C(AG)$

$\Downarrow$

$AGAG = AG$. $A = AGD$ for some D.

$\Rightarrow$ $AGAGD = AGD$

~~$\Rightarrow$ $AGA = A$~~. $AGA = A$.

Corollary: G is a g-inverse of A then AG is the projection into $C(A)$ along $N(AG)$.

a general sol^n of $Ax = 0$ is $(I - GA)\underset{\sim}{z}$ where $\underset{\sim}{z}$ is arbitrary.

$\Rightarrow$ $N(A) = C(I - GA)$

$AGA = A \quad N(A) \subseteq N(GA)$ everytime

$\Downarrow$

$\rho(GA) = \rho(A). \Rightarrow N(A) = N(GA).$

GA is idempotent $\Rightarrow N(GA) = \mathcal{C}(I-GA).$

Chapter 5

192 1, 2, 5(a), (c), 6, 11–15

198 1-16.

p 137 (20) A, B are projectors.

(a) $A-B$ is projector

(b) $AB = BA = B$

(c) $\rho(A-B) = \rho(A) - \rho(B)$.

(d) $\mathcal{C}(B) \subset \mathcal{C}(A)$, $R(B) \subset R(A)$.

(a) $\Leftrightarrow$ $(A-B)$ is idempotent

$\Rightarrow (A-B)^2 = A-B \Rightarrow AB+BA = 2B$ $\quad$ $a \Rightarrow b$

$\Rightarrow ABA + BA = 2BA \qquad AB + ABA = 2AB$

$\Rightarrow ABA = BA. \qquad \Rightarrow ABA = AB.$

$\Rightarrow AB = BA = B$. (b). $\qquad ABA - AB^2$, $AB(A-B) = 0$.

~~(b) $AB = BA = B$~~ $\qquad BA = B$

~~$\Rightarrow AB = B \Rightarrow AB - B^2 = 0$~~ $\qquad \Rightarrow BA - B^2 = 0$

~~$\Rightarrow (A-B)B = 0$~~ $\qquad \Rightarrow B(A-B) = 0.$

$a \Rightarrow c$ as $\rho(A-B) = tr(A-B) = tr(A) - tr(B)$

$= \rho(A) - \rho(B).$

$c \Rightarrow d \quad \rho(A) = \rho(A-B) + \rho(B)$

$\mathcal{C}(B) \subset \mathcal{C}(A)$

$R(B) \subset R(A).$

$d \Rightarrow b \quad \mathcal{C}(B) \subset \mathcal{C}(A) \Rightarrow AB = AX$ for some X.

$AB = A^2X = AX = B.$

23/8/18

$\cancel{A^T(AA^T)^{-1}}$

$A^T(AA^T)^-$ is a g-inverse of A.

~~Proof~~ If $R(A) \subseteq R(B)$ and $C(C) \subseteq C(B)$ then, AB^-C is invariant under diff choices of B^-.

suppose $\rho(A) \leq \cancel{\rho(G)}\, s \leq \min(m,n)$. $\cancel{\forall s}$.

then $\exists$ a g-inverse $\rho(G) = s$

$\cancel{\sum a_{ik}a_{kj} = \sum a_{ik}a_{kj}}$

$\cancel{\rho(A) = \rho(AA')}$

$\cancel{A = ADA'}$

$\cancel{B = A - A'}$

$\cancel{BB' = A^2 - AA' - A'A}$

A is an $m \times n$ of rank r.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \text{ is } r \times r \text{ ns.}$$

Then, $G = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ is a g-inv of A.

$$AGA = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

rank A = rank A_{11} + rank $(A_{22} - A_{21}A_{11}^{-1}A_{12})$.

now if rank A = rank A_{11} = r

then $r(A_{22} - A_{21}A_{11}^{-1}A_{12}) = 0$

$\Rightarrow A_{22} = A_{21}A_{11}^{-1}A_{12}$.

$(X'X)^- X'$, $(X'X)^-$

$$A = \begin{bmatrix} A_{11} & A_{11}X \\ A_{21} & A_{21}X \end{bmatrix}$$

$A_{21}X - A_{21}A_{11}^{-1}A_{11}X$

$= A_{21}X - A_{21}X = 0$.

A $m \times n$ matrix.

1) $AGA = A$ — 1. g_1-inverse / 1-inverse

2) $GAG = G$ — 1,2. reflexive g inverse

3) $(AG)' = AG$ — 2 outer inverse.

4) $(GA)' = GA$ — 1,3 least squares inverse

1,4 minimum norm g-inverse

1,2,3,4 Moore-Penrose inverse.

$A = AGA$ let $H = GAG$

$AHA = AGAGA = A$.

$HAH = H$

let $A = AGA$ then G is reflexive iff $r(A) = r(G)$

G is reflexive $\Rightarrow AGA = A$, $GAG = G$.

$\rho(A) = \rho(AGA) \leq \rho(G) = \rho(GAG) \leq \rho(A)$.

converse,

$\mathcal{C}(GA) \subset \mathcal{C}(G)$ $\rho(G) = \rho(GA)$

$\Rightarrow \mathcal{C}(GA) = \mathcal{C}(G) \Rightarrow G = GAX$. for some X.

$GAG = GAGAX = GAX = G$.

$Ax = y$, $y \in \mathcal{C}(A)$ Gy is a sol^n

min norm g-inv is that Gy is a sol^n with min. norm

To show $y'G'(I - GA) = 0$

$y = Au \Rightarrow u'A'G'(I - GA) = u'(GA)'(I - GA)$

$= u'(GA)' - u'(GA)'GA$.

$= u'(GA)' - u'GAGA$

$= u'GA - u'GA = 0$.

Exercise: A $m \times n$

Show that G is a min norm g-inv iff $GAA' = A'$.

suppose G is a min norm g-inv

$\Rightarrow AGA = A$ $(GA)' = GA$.

$A = A(GA)' \Rightarrow A' = G'A'A'$.

$= AA'G'$.

if $A' = GAA'$

$\Rightarrow A = AA'G'$.

$\Rightarrow GA = GA(GA)'$

$(GA)' = GA(GA)'$

$\Rightarrow GA = (GA)'$

$AA' = AGAA'$.

$A = AGA$ as $\rho(A') = \rho(AA')$.

27/8/18

1) $AGA = A$
2) $GAG = G$
3) $(AG)' = AG$
4) $(GA)' = GA$

$\dagger$

$Ax = b$ may not be consistent.

Suppose, $\min_x \|Ax - b\|$ is attained at x_0. then x_0 is called a least square sol^n of $Ax = b$.

least square g-inv.

then for any x, y $\|AGy - y\| \le \|Ax - y\|$.

$x_0 = Gb$ is a ls sol^n for any ls. g-inv.

Moore Penrose ~~G~~ g-inv

this is unique. denote it by A^+.

If B is full column rank. then,

claim $B^+ = (B'B)^{-1}B'$. (verify that)

~~BB^+~~ if C has full row rank then $A \rightsquigarrow (B, C)$

claim $C^+ = C'(CC')^{-1}$

then $A^+ = C^+ B^+$. (verify).

$BB^+B = B(B'B)^{-1}B'B = B$.

$B^+BB^+ = (B'B)^{-1}B'B(B'B)^{-1}B' = B^+$.

$BB^+ = B(B'B)^{-1}B'$ symm

$B^+B = I$ symm.

if G is a g-inv of A then $G + xy'$ is another g-inv for $x \in N(A)$, y arbitrary.

Set of all g-inverses of A is given by,

$\underline{\underline{G + (I - GA)U + V(I - AG)}}$; U, V arbitrary.

this is a g-inv (can show easily).

$G + (I - GA)(H A G) + (H - G)(I - AG)$

$= H$

7) if A has rank 1 then $A^+ = A'$.

$A = xy'$ $\quad x^+ = (x'x)^{-1}x'$

$y^+ = \cancel{x'(x'x)^{-1}}$ $\cancel{y'(y'y)^{-1}}$ $y(y'y)^{-1}$

$A^+ = y(y'y)^{-1}(x'x)^{-1}x'$ $\quad AA' = xy'yx'$

$= \dfrac{yx'}{(y'y)(x'x)} = \dfrac{A'}{tr(AA')}$. $\quad tr(AA') = (y'y)(x'x)$

8) $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ Moore Penrose inv will work.

Class of all g-inv $\{x^+ + (I - x^+x)U + V(I - xx^+)\}$

$I = x^+x$ as x^+ is a left inv of x.

$= x^+ + V(I - xx^+)$

$\|x^+ + V(I - xx^+)\| = \|x^+\|^2 + \|V(I - xx^+)\|^2 + 2x^+(V(I - xx^+))'$

9) $S \oplus T = V$

proj of x on S along T.

if $T = S^\perp$ then x is the orthogonal projection on S along $S^\perp$.

$X^{m\times n}$: $y \in \mathbb{R}^n$. Orthogonal projection of y on $\mathcal{C}(X)$ is given by $X(X'X)^{-1}X'y$.

$y = X(X'X)^{-1}X'y + (I - X(X'X)^{-1}X)y$.
$\in \mathcal{C}(X)$. $\quad \in \mathcal{C}(X^\perp)$.

$X'X = X'X(X'X)^-(X'X)$.

$X' = X'X(X'X)^-X'$. $\quad$ So $X'(I - X(X'X)^-X) = 0$.

So $X^\perp = I - X(X'X)^-X$.

$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ $\quad A_{11}$ – ns.

then $r(A) = r(A_{11}) + r(A_{22} - A_{21}A_{11}^{-1}A_{12})$

if $\mathcal{C}(A_{12}) \subseteq \mathcal{C}(A_{11})$, $\mathcal{R}(A_{21}) \subset \mathcal{R}(A_{11})$

$$\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$$

$$\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} A \begin{pmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$$

then $\quad r(A) = r(A_{11}) + r(A_{22} - A_{21}A_{11}^{-1}A_{12})$

$\det A = |A_{11}|\,|A_{22} - A_{21}A_{11}^{-1}A_{12}|$

$A^{n\times n}$ idempotent.

$B = \begin{pmatrix} A & A \\ A & I \end{pmatrix}$

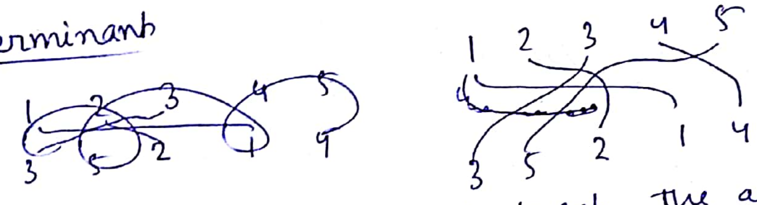
$r(B) = r(A) + r(I - AA^{-}A)$

$= r(A) + r(I - A)$.

$r(B) = r(I) + r(A - A^2)$.

$\Rightarrow r(A) + r(I-A) = n + r(A - A^2)$.

Determinants



no. of transposition req^d to get the ordered arrangement.

3 5 2 1 4

no of inversions

look at each pair count

count 1 if req^d inversion

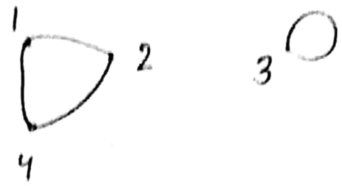
" 0 if not req^d.

Scanned by CamScanner

SOURCE: GITHUB.COM/SOURAVSTAT

1 2 3 4

2 4 3 1

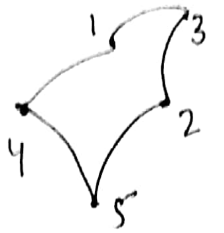


n - no. of cycles

$4-2$

1 2 3 4 5

3 5 2 1 4



$5-1$

if π and σ are two permutations
then $\varepsilon(\pi \circ \sigma) = \varepsilon(\pi)\,\varepsilon(\sigma)$.

$1 = \varepsilon(id) = \varepsilon(\lambda \circ \lambda^{-1}) = \varepsilon(\lambda) \cdot \varepsilon(\lambda^{-1})$

$$|A| = \sum_{\sigma} \varepsilon(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

$$= a_{11} a_{22} \cdots a_{nn} \quad \text{(only term survives)}.$$

$$\begin{bmatrix} a_{11} & 0 \cdots & 0 \\ a_{21} & a_{22} \cdots & 0 \\ \vdots & & \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix}$$

$|A^T| = |A|$ $\quad$ $\cancel{|A^T| = \sum_{\sigma}}$

$a_{1\sigma(1)} \cdots a_{n\sigma(n)} = \prod_i a_{i\sigma(i)} = \prod_i a_{\sigma^{-1}(i)i}$ (?) (exercise!!)

$\left(\prod_i \sigma^{-1}(\sigma(i))\,\sigma(i) = \prod_i \sigma^{-1}(\tau(i))\,\tau(i) \right.$

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & & a_{nn} \end{vmatrix} = \begin{vmatrix} b_{11}+c_{11} & \cdots & b_{1n}+c_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \end{vmatrix}$$

$$= \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ a_{n1} & & a_{nn} \end{vmatrix} + \begin{vmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

$$A = B + C$$

$$\begin{vmatrix} b_{11}+c_{11} & \cdots & b_{1n}+c_{1n} \\ \vdots & & \\ b_{n1}+c_{n1} & \cdots & b_{nn}+c_{nn} \end{vmatrix}$$

$\sum (2^n$ terms $)$

take one term

$r(B) = r$

$r(C) = s$

take $r+s+1$ cols.

they are linearly dependent

$\therefore r(B) + r(C) \geq r(A)$

rank : largest size non-vanishing determinant

Z^6 (modular class of 6)

vander monde matrix.

determinant is a polynomial.

$$\det A = \prod_{i<j} (a_i - a_j)$$

laplace expansion

$$|A^{4\times 4}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} + \cdots$$

$$\left[\begin{array}{cc|cccc} + & + & 0 & 0 & 0 & 0 \\ + & + & 0 & 0 & 0 & 0 \\ + & + & 0 & 0 & 0 & 0 \\ \hline + & + & + & + & + & + \\ + & + & + & + & + & + \end{array}\right]$$

use laplace expansion

also this has rank 2.

here det = 0.

Evaluate
$$\begin{bmatrix} 0 & 1 & 2 & \cdots & n-1 \\ 1 & 0 & 1 & \cdots & n-2 \\ 2 & 1 & \ddots & & \\ \vdots & \vdots & & \ddots & 0 \\ n-1 & n-2 & & & \end{bmatrix}$$

<u>20/9/18</u>

Cauchy Binet $A^{n\times p}$, $B^{p\times n}$. if $n>p$ then $|C|=0$

$C = AB$

if $n \le p$.

$$|C| = \sum_{1\le j_1<\cdots<j_n\le p} |A(1,\dots,n|j_1,\dots,j_n)|\,|B(j_1,\dots,j_n|1,\dots,n)|.$$

$$\left|\begin{pmatrix} 2 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \\ -1 & 0 \end{pmatrix}\right|.$$

$$= \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix}\begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix}$$
$$+ \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix}.$$

$|A^{n\times p} A'^{p\times n}| \geq 0$. by Cauchy Binet form,

each term will be square so ≥ 0.

~~if A is full column rank~~ iff ~~then~~ ~~|AA'| is positive.~~

~~(at least one~~ ~~if~~ $|A'A| > 0$ iff rank $A = n$.

(at least one summands is positive as rank $A = n$)

$A^{4\times4}$

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + a_{14}A_{14}$.

now $a_{i1}A_{11} + a_{i2}A_{12} + a_{i3}A_{13} + a_{i4}A_{14} = 0 \quad \forall\, i \neq 1$.

as this is same as expanding A by replacing 1st row by i^{th} row, now.

then it has 2 equal row $\Rightarrow$ det is 0.

$\text{adj}\,A = ((|A_{ij}|))'$

$A(\text{adj}\,A) = (\text{adj}\,A)\,A = |A|\,I$

~~Cramer's Rule.~~

~~$A^{n\times n}$ of order n.~~

Jacobi

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad A_{11}: \text{n.s.}$$

$$B = A^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \quad B_{22}: \text{n.s.}$$

then $|A_{11}| = \dfrac{|B_{22}|}{|B|}$

Recall. $|B| = |B_{22}|\,|B_{11} - B_{12}B_{22}^{-1}B_{21}|$.

now $A_{11}^{-1} = B_{11} - B_{12}B_{22}^{-1}B_{21}$ ~~(schur~~

$\therefore |B| = \dfrac{|B_{22}|}{|A_{11}|}$.

now suppose

minor in A $= \dfrac{\text{complementary det in B}}{|A^{-1}|}$

det square submatrix.

$$(AB)^{\oplus} = B^{\oplus} A^{\oplus}$$

Proof : First suppose A, B are ns.

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$(AB)^{\oplus} = (AB)^{-1} |AB|$$
$$= B^{-1} A^{-1} |A| |B|$$
$$= B^{\oplus} A^{\oplus}.$$

then for some ε $(A + \varepsilon I)$, $(B + \varepsilon I)$ are ns.

hence we get the result

now by continuity argument as $\varepsilon \to 0$, we get the proof

$$\begin{vmatrix} A & b \\ c^T & \alpha \end{vmatrix} = \alpha |A| - c' A^{\oplus} b$$

if A is ns then, $|A| (\alpha - c' \frac{A^{\oplus}}{|A|} b)$.

$$= \begin{vmatrix} A & b \\ c^T & \alpha \end{vmatrix}$$

now if $|A|$ is not ns then also by continuity argument we can make it singular.

Chapter 6

p-228 1-11

p-235 4-11

p-239 1-5

p-245 2, 3, 4, 7, 8, 10

p-247 1-7

24/9/1[illegible]

$|A - \lambda I| = 0$: characteristic polynomial.

$|\lambda I - A|$ - , monic polynomial

fundamental thm of algebra says polynomial of deg n with coeff in C have n roots in $\mathbb{C}$.

for real matrix may ev may be complex.

$|\lambda I - A| = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$ λ_i's are ev.

$$|\lambda I - A| = \sum_{k=0}^{n} C_{n-k} \lambda^k$$

$C_n = (-1)^n |A|$.

$C_{n-1} = (-1)^{n-1}$ sum of $(n-1) \times (n-1)$ principal minor of A

C_{n-k} = sum of $k \times k$ principal minors of A.

C_{n-1} = tr(A). = sum of evs.

C_n = prod of ev.

sum of $k \times k$ principal minors $= \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$

$k = 1, 2, \ldots, n$.

$\begin{bmatrix} 4 & 2+i \\ 2-i & 6 \end{bmatrix}$. $A = A^*$ hermitian matrix.

Hermitian matrix

$A = A^* = \bar{A}^T$

A matrix is pd if it is ~~sy~~ Symmetric and $x'Ax > 0$ for all $x \neq 0$

(A pd, B pd then AB may not be pd)

A matrix is positive semidefinite if it is symm and $x'Ax \geq 0 \ \forall x$ and A is symm.

If A is pd then A is ns.

$Ax = 0 \quad x^T A x = 0 \Rightarrow x = 0.$

A, B are pd. if $\alpha \geq 0, \beta \geq 0$ with $\alpha + \beta > 0$ then

$\alpha A + \beta B$ is pd.

$$x'(\alpha A + \beta B) x = \alpha (x^T A x) + \beta (x^T B x)$$
$$> 0.$$

A is pd ~~then~~ $\Rightarrow |A| > 0.$

for $0 \leq \alpha \leq 1$ then. $\alpha A + (1-\alpha) I$ is pd

$$f(\alpha) = |\alpha A + (1-\alpha) I|.$$

$f(0) = 1$ and since f is cont

$(\alpha A + (1-\alpha) I)$ pd $\Rightarrow (\alpha A + (1-\alpha) I)$ is ns.

$\Rightarrow f(\alpha) \neq 0 \ \forall \alpha \in [0,1].$

$\Rightarrow f(1) > 0. \Rightarrow |A| > 0.$

any submatrix of pdm is pd.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$x'Ax = x_1' A_{11} x_1 > 0 \Rightarrow A_{11}$ is pd.

A symm and all principal ... pd

$\Rightarrow$ A is pd.

(A is symm $\Rightarrow$ A^2 is psd ~~and pd~~ pd)

$A^2 = A'A$.

$x'A^2x = (Ax)'(Ax) \geq 0$ ~~(strict $\neq 0$).~~ ~~as A is pd.~~

however if A is ns and symm then A^2 is pd

then $Ax = 0 \Rightarrow x = 0$.

eigen space of λ_i = null space of $(A - \lambda_i I)$

dim of eigen space ~~then~~ is geometric multiplicity of that ev.

multiplicity of λ_i as a root of ch. polynomial is algebraic multiplicity.

alg multiplicity of $\lambda \geq$ geom mult of λ.

(for symm matrix they are equal)

eg $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. geom mult = 1

algeb " = 2. (0)

0 is only ev

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ geom mult = 1

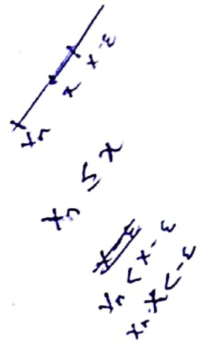
algeb = 3

Q. construct a 10x10 matrix with ev.

2, 4, 6, algeb mult 3, 3, 4 resp,

geom mult 2, 1, 2.

$$\left[\begin{array}{ccc|ccc|cccc} 2 & 1 & & & & & & & & \\ & 2 & 0 & & & & & & & \\ & & 2 & & & & & & & \\ \hline & 0 & & 4 & 1 & & & & & \\ & & & & 4 & 1 & & & & \\ & & & & & 4 & & & & \\ \hline & & & & & & 6 & 1 & & \\ & & & & & & & 6 & 0 & \\ & & & & & & & & 6 & 1 \\ & & & & & & & & & 6 \end{array}\right]$$



$$\begin{bmatrix}2&1&0\\0&2&0\\0&0&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = 2\begin{bmatrix}x\\y\\z\end{bmatrix}$$

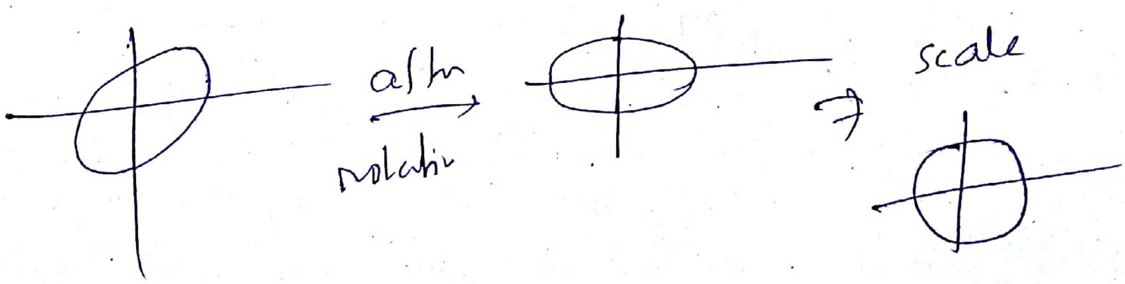
$$\Rightarrow \underline{\underline{\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}}}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$r(A) = 1$ so geo mult $= 2$.

$\dim N(A) = 2$

[SA and $S^{-1}AS$ has same ev (check)]

$\{x \mid x'Ax \leq c\}$ ~~tti~~ if A is diagonal then it is



$x \mapsto Px$ linear transformation of rotation.

$$P'AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad P = [P_1 \; P_2 \; \cdots \; P_n].$$

$$A = P\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}P'$$

P_i = is an e.vector of λ_i

$$AP = P\,\text{diag}(\lambda_1 \ldots \lambda_n)$$

$\lambda_1 \ldots \lambda_n$

$x_1 \ldots x_n$ e.vectors ~~(matrix)~~.

n orthonormal e.vectors

making orthonormal basis.

$$A = \lambda_1 x_1 x_1' + \cdots + \lambda_n x_n x_n'.$$
$$= \lambda_1 E_1 + \cdots + \lambda_n E_n$$

E_i : ~~spectra~~ spectral projection.

they are idempotent.

$E_i E_j = 0 \quad i \neq j$

$r(E_i) = 1$

A is pd $\Rightarrow$ $B'AB$ is psd ~~of~~

however if B is ns. then

$B'AB$ is pd.

For any ~~...~~ A. $A'A$ is psd.
if A is ns then $A'A$ is pd.
+ if A has full col rank ~~...~~ then $A'A$ is pd.

If A is psd
$f: \mathbb{R} \to \mathbb{R}$. $f(A) = P\begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix} P'$.

$\sin A = P\begin{pmatrix} \sin\lambda_1 & & \\ & \ddots & \\ & & \sin\lambda_n \end{pmatrix} P'$.

$A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ $|e^A| = ?$

If λ is ev of A λ^2 is ev of A^2.
if A is sym and all ev. are 0/1 then A is idempotent

If we drop A symm ~~...~~ then A may not be idempotent eg $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

25/9/18

If A symm and principal minors are positive then A is pd

$$|A+\mu I| = \sum_{i=0}^{n} s_i \mu^{n-i} \qquad s_i > 0$$

[start because $s_0 = 1$]

If $\mu \geq 0$ then $|A+\mu I| > 0$

i.e. $|A-(-\mu I)| > 0$ then A can not have negative ev.

~~ev for a~~ non positive

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ det is positive but not pd.

so we need all order principal minors should be positive to be pd.

Show

A is symm and all principle minors are ~~+ved~~ non-negative then A is ~~pnd~~ psd

A is pd, C is ns $\Rightarrow$ $C'AC$ is pd.

$$\begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \underbrace{\begin{bmatrix} B & C \\ C' & D \end{bmatrix}}_{pd} \begin{bmatrix} I & 0X' \\ 0 & I \end{bmatrix} = \begin{bmatrix} \underbrace{B}_{pd} & 0 \\ 0 & \underbrace{D-C'B^{-1}C}_{\text{principle minor . pd}} \end{bmatrix}$$

if ~~A~~ ~~is~~ A has one pd minor and its schur complement also pd then A is pd.

$$A = \begin{bmatrix} B & c \\ c' & d \end{bmatrix}$$

If B is pd and $d - c'B^{-1}c > 0$ then A is pd.

SOURCE: GITHUB.COM/SOURAVSTAT

A leading principal minor is a principal minor formed by rows $1,2,\dots,k$ and columns $1,2,\dots,k$. $k = 1(1)n$.

~~to~~ A to be pd. Only leading principal minor needs to be positive.

Suppose all leading principal minors are positive.

let $A = \begin{pmatrix} B & c \\ c' & d \end{pmatrix}$ suppose true for $(n-1)\times(n-1)$ leading principal minor.

then B is pd. and $|A| = 0$ ← one leading principal minor

now we have. $|A| = \underset{>0}{|B|}\ \underset{>0}{|d - c'B^{-1}c|}$ → schur complement > 0.

hence A is pd.

Remark (not true for nnd) all leading principal minors are nonneg

$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ but this is not nnd.

if A, B are psd and A-B are psd then $A \geq B$.

some times we don need this.

If. A, B are pd. and $A \geq B$ then $B^{-1} \geq A^{-1}$

- If $A \geq B$, $B \geq C$ ⇒ $A \geq C$.
- If $A \geq B$ for any C $C'AC \geq C'BC$.

$A \geq I$ ⇒ $A - I \geq 0$ ⇒ ev ab $(A - I) \geq 1$

(ev ab ~~$(A-I)^{-1} \leq 1$~~ A) ≥ 1

⇒ $I \geq A^{-1}$ ← (ev ab A^{-1}) ≤ 1.

~~If~~ $A \geq B$ ⇒ $B^{-1/2} A B^{-1/2} \geq B^{-1/2} B B^{-1/2} = I$

⇒ $I \geq B^{1/2} A^{-1} B^{1/2}$

⇒ $B^{-1/2}(I) B^{-1/2} \geq B^{-1/2} B^{1/2} A^{-1} B^{1/2} B^{-1/2}$

⇒ $B^{-1} \geq A^{-1}$. (pd ordering)

$A \geq I \Rightarrow$ all ev are ≥ 1

$\Rightarrow |A| \geq 1$.

$A \geq B \Rightarrow I \geq B^{1/2} A^{-1} B^{1/2}$ (by ev argument)

$\Rightarrow 1 \geq |B^{1/2}| |A^{-1}| |B^{1/2}|$

$\Rightarrow |A| \geq |B|$.

$A \geq B$. $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$

$\lambda_1(A) \geq \cdots \geq \lambda_n(B)$.

Fact, $\lambda_i(A) \geq \lambda_i(B)$, $i = 1, 2, \ldots, n$.

(but converse is not true ie if above is true then $A \geq B$ is not necessarily true].

take $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0.5 & 0 \\ 0 & 1.5 \end{bmatrix}$.

$A - B = \begin{bmatrix} 1.5 & 0 \\ 0 & -0.5 \end{bmatrix}$ not pd.

So $A \not\geq B$.

Ex

1. $A^{n \times n}$ symm $A\underset{\sim}{1} = \alpha \underset{\sim}{1}$ $\Rightarrow \alpha$ is an ev.

$\alpha_2, \ldots, \alpha_n$ are remaining ev.

$J = 11'$

ev of $A + \beta J = \{(\alpha + n\beta, \alpha_2, \ldots, \alpha_n)$

A = spectral thm $\Rightarrow A = P' \begin{bmatrix} \alpha & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{bmatrix} P$.

take $P_{*1} = \frac{1}{\sqrt{n}} \underset{\sim}{1}$. $\underset{\sim}{1}$ one e. vector (first column of P').

unit vector

$1'P = \begin{bmatrix} \sqrt{n} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ } remaining are $\perp^{n}$.

$P(A + \beta J)P' = PAP' + \beta PJP'$

$= \begin{bmatrix} \alpha & & \\ & \alpha_2 & \\ & & \alpha_n \end{bmatrix} + \beta \begin{bmatrix} n & & \\ & \ddots & \\ & & 0 \end{bmatrix}$

$$\begin{pmatrix} a & b & \cdots & b \\ \vdots & \ddots & & \\ b & & & a \end{pmatrix} = \begin{pmatrix} a-b & & \\ & \ddots & \\ & & a-b \end{pmatrix} + bJ.$$

one ev is

$a-b+nb$

$= a+(n-1)b.$

$$\left[\begin{array}{ccc|cccc} 5 & -2 & -2 & -1 & -1 & -1 & -1 \\ -2 & 5 & -2 & -1 & -1 & -1 & -1 \\ -2 & -2 & 5 & -1 & -1 & -1 & -1 \\ \hline -1 & -1 & -1 & 3 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 3 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 3 \end{array}\right]$$

all row sums are (-3).

(-3) is an ev.

if I add J then.

$$\left[\begin{array}{ccc|cccc} 6 & -1 & -1 & & & & \\ -1 & 6 & -1 & & & 0 & \\ -1 & -1 & 6 & & & & \\ \hline & & & 3 & -1 & -1 & -1 \\ & 0 & & -1 & 3 & -1 & -1 \\ & & & -1 & -1 & 3 & -1 \\ & & & -1 & -1 & -1 & 3 \end{array}\right]$$

find ev of

they are ev of

Prove !!

A, B are $n \times n$. then AB and BA are same ev counting multiplicity.

$A^{m \times n}$ $B^{n \times m}$

$AB : m \times m$
$BA : n \times n$ } if $m > n$ then every non-zero ev's are same rest are 0.

ev of AB is ev of BA + $(m-n)$ 0's ev's. ~~of A~~

prove that $|I_m - AB| = |I_n - BA|$

now show the 2 ch. poly is same.

Rel^n b/w rank and non-zero ev's.

Rank $\geq$ number of non-zero ev's of A (counting multiplicity).

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow$ rank 1

evs 2 0.

algeb. mult of 0 $\geq$ geom mult of 0.

$\downarrow$ $(n - (\#$ of non-zero ev$))$

$\downarrow$ dim of e. space of 0 $\Updownarrow$ $N(A)$

$\Rightarrow$ ρ rank $A \geq \#$ non-zero ev counting multiplicity.

26/9/18

ev. of xy' : $y'x$ and $\underbrace{0, 0, \ldots, 0}_{n-1 \text{ times}}$.

Cayley Hamilton Theorem

$A = \text{diag}(d_1 \ldots d_n)$. : characteristic eq^n.

$(d_1 - \lambda)(d_2 - \lambda) \ldots (d_n - \lambda)$

$(d_1 I - A)(d_2 I - A) \ldots (d_n I - A) = 0$

A is symmetric.

$A = P' \text{diag}(\lambda_1 \ldots \lambda_n) P$

$|A - \lambda I| = |P'(\text{diag}(\lambda_1 \ldots \lambda_n) - \lambda I) P|$

$= 0$

Result: A is psd and ns then A is pd.

A is psd $\Rightarrow$ ev's are non-neg.

A is ns $\Rightarrow$ all evs are .

(14) A square $(A + A')$ is pd.

if $Ax = 0 \Rightarrow x'A' = 0 \Rightarrow x'(A' + A)x = 0$

$\Rightarrow x = 0$

so $Ax = 0 \Rightarrow x = 0$.

SOURCE: GITHUB.COM/SOURAVSTAT

rank = # of non-zero ev for symm matrix
(can be extended from diagonal matrix).

~~th~~ If A and B symm $\ni$ $AB = BA$. Then $\exists$ orthogonal matrix P $\ni$ $P'AP = \text{diag}(\lambda_1(A) \dots \lambda_n(A))$
$P'BP = \text{diag}(\lambda_1(B) \dots \lambda_n(B))$.

in general P is not same.

7) A is pd, B is symm.

A is pd $\Rightarrow$ $P'AP = (\lambda_1 \dots \lambda_n)$ (2nd part)

take ~~Q~~ $E = P \,\text{diag}\left(\frac{1}{\sqrt{\lambda_1}} \cdots \frac{1}{\sqrt{\lambda_n}}\right) P$, ns.

then ~~$Q'AQ = I$~~ $E'AE = I$

now $\exists$ orthogonal Q $\ni$ $Q'BQ = \text{diag}(\mu_1, \dots \mu_n)$.
(now try the rest)

<u>note</u>:
A symm, B symm. AB may not be symm.
but AB is symm iff $AB = BA$
$AB = BA$..
$(AB)' = B'A' = BA = AB$ if

i) (if A, B are symm. AB need not has real ev. (try a counter example).

But A is symm and <u>B is psd</u> then
AB has real eigen values.

$B = C'C$. now $AB = AC'C$ has same eigen values as of CAC', which is symm.

(as AB and BA has same ev)

if A, B are both psd. then AB has non neg ev.
(similar extension as above).

19) $A^{n\times n}$ symm

A is psd and if B is psd then $tr(AB) \geq 0$.

AB has non-neg ev $\Rightarrow$

conversely, take $B = xx'$.

~~tr(xx'B)~~ $= tr(Axx') \geq 0$

$\Rightarrow tr(x'Ax) \geq 0$.

$\Rightarrow x'Ax \geq 0$. $\Rightarrow$ A is psd.

21. $a_{ij} = \cos(\theta_i - \theta_j)$

$= \cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j$. $\Rightarrow xx' + yy^T$.

Rank = 1 / 2.

$$\cos\theta = \begin{pmatrix} \cos\theta_1 \\ \cos\theta_2 \\ \vdots \\ \cos\theta_n \end{pmatrix}$$

22. $A^{n\times n}$ pd, $n>1$ and $a_{ij} < 0 \;\forall\; i \neq j$.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

diagonal are positive and rest are negative.

any schur complement ≥ 0.

off diag ≥ 0

(A: symm and diagonally dominating then A is ns and all principal minors are positive. so A is pd).

if A is pd with off diagonal ≤ 0. then inv has all elm ≥ 0. } why??

$$\underbrace{a_{kk}}_{\leq 0} - \underbrace{a_{k1}}_{\leq 0}\underbrace{\left(\frac{1}{a_{11}}\right)}_{>0}\underbrace{a_{1k}}_{\leq 0} \leq 0$$

23. A n...

$|A+\mu_1 I|$...

take ...

then $|A$...

$\Rightarrow |A$...

schur p...

24. $A \circ B$

A, B

first supp...

then

$A \circ B =$ (...

= di...

= C...

in general

$B = \mu_1 x$...

$A \circ B =$

=

alternativ...

claim

$A \otimes B$

if ~~$A\otimes$~~

$A \otimes$

23. A not necessarily symm. all principal minors are positive.

$|A+\mu I| = \sum_{k=0}^{n} c_k \mu^{n-k}$ (not necessarily symm).

take $\mu \geq 0$ then rhs is positive. $c_0 \mu^0 > 0$

then $|A+\mu I| > 0$ $\forall \mu \geq 0$.

$\Rightarrow |A-(-\mu)I| > 0$ then no ev can be negative.

schur product.

24. $A \circ B = ((a_{ij} b_{ij}))$. (in particular if we take $B = ((a_{ij}^2))$ then if A is psd then B is psd.)

A, B psd.

first suppose $r(B) = 1$

then $B = xx'$.

$A \circ B = ((a_{ij} \cdot x_i x_j))$

$= \text{diag}(x_1 \ldots x_n)\, A\, \text{diag}(x_1 \ldots x_n)$.

$= CAC' \Rightarrow$ ~~A is psd~~. $A \circ B$ is psd

in general by ~~spec~~ spectral decomposition.

$B = \mu_1 x_1 x_1' + \cdots + \mu_n x_n x_n'$

μ_i are ev

x_i's are orthonormal e vectors.

$A \circ B = \sum_{i=1}^{n} A \circ (\mu_i x_i x_i')$

$= \sum_{i=1}^{n} \mu_i \underbrace{(A \circ x_i x_i')}_{\text{psd}}$ linear comb of psd matrices

hence $A \circ B$ is psd.

<u>alternative</u>:

claim A, B psd $\Rightarrow A \otimes B$ is psd.

$\cdot A \otimes B$ is symm and check that ev's are $\lambda_i(A)\lambda_i(B)$.

if ~~$A \otimes B$~~ $A = XX'$; $B = YY'$

$A \otimes B = (XX') \otimes (YY')$

$= (X \otimes Y)(X' \otimes Y') = CC'$ so it is psd.

Claim : $A \circ B$ is a principal submatrix of $A \otimes B$
(check by example).

then $A \circ B$ is psd.

(alternative)

If X is rv with $Cov(X) = A$.

$Var(u'X) = u'Au \geq 0$ ∴ A is psd.

given A is psd then $A = BB'$

take $Cov(X) = I$ then $Cov(BX) = A$.

now if A is psd then principal submatrix is psd

statistical proof: principal submatrix is cov of marginal distn. So psd.

let X, Y be random vector. indep.

$Cov(X) = A$. $Cov(Y) = B$.

let $Z_i = X_i Y_i$ $i = 1, \dots n$.

$$Cov(Z_i, Z_j) = \cancel{E[X_iY_i - E(X_i)E(Y_i)]}\ \cancel{E[X_jY_j - E(X_j)E(Y_j)]}$$

$$= \cancel{E[Z_iZ_j] - E(Z_i)E(Z_j)}$$

$$= \cancel{E[X_i]}\ cov(X_i, X_j)\ cov(Y_i, Y_j) = a_{ij} b_{ij}$$

$$= (A \circ B)_{ij}.$$

hence $(A \circ B)$ is ~~pd~~ psd.

let $Z_{ij} = X_i Y_j$. $i = 1(1)m$, $j = 1(1)n$.

then $Cov(Z) = A \otimes B$ (check)!

$Cov(Z_{ij}, Z_{kl}) = a_{ij} b_{kl}$ then $A \otimes B$ is psd.

A is pd then any schur complement is pd.
(statistical argument: cov. of condition dist^n is the schur complement.)

25) $x_1 \ldots x_n$ $\quad a_{ij} = \left(\left(\frac{1}{x_i + x_j}\right)\right)$.

$$\frac{1}{x_i + x_j} = \int_0^1 t^{x_i + x_j - 1}\, dt. = \int_0^1 \frac{1}{t}(t^{x_i})(t^{x_j})\, dt.$$

$t > 0$. then $\left(\left(t^{x_i + x_j}\right)\right)$ is psd.

$a_{ij} = t^{x_i + x_j} = t^{x_i} \cdot t^{x_j} \Rightarrow \left(\left(t^{x_i + x_j}\right)\right) = xx'$.

now take $u'Au = \int_0^1 \frac{1}{t}\, u'(t^{x_i + x_j})u\, dt$

$\underline{\text{integral of nonneg values}} \geq 0$.

Results

If $A \geq B$ then $A^2 \ngeq B^2$ (omit 26, 26).
however $A^{1/2} \geq B^{1/2}$

If $A \geq B$, $C \geq D$, show that,
$A \otimes C \geq B \otimes D$.

$$\begin{vmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{vmatrix} \rightarrow \begin{vmatrix} 0 & 1 & 2 & 3 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & -1 \end{vmatrix} \rightarrow \begin{vmatrix} 0 & 1 & 2 & 3 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & -2 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & 0 & -2 \end{vmatrix} \quad \begin{vmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & 0 & -2 \end{vmatrix} \leftarrow \begin{vmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 2 & 1 & -1 & 1 \\ 1 & 1 & 1 & 0 & -2 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & -1 & 1 & 1 \\ 1 & 1 & 0 & -2 & 0 \\ 1 & 1 & 0 & 0 & -2 \end{vmatrix} \to \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & 0 \\ 1 & 1 & 0 & -2 & 0 \\ 1 & 1 & 0 & 0 & -2 \end{vmatrix} \to \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & -2 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & -2 \end{vmatrix}$$

$$(-2)^4 \left| 0 - \underset{\sim}{1}'(-2)^{-4}\underset{\sim}{1} \right|$$

now use schur complement formula

4/10/18

A is symm. $\max\limits_{x \neq 0} \dfrac{x'Ax}{x'x} = \lambda_1$ (largest ev)

max^m attained at any evector corresponding to λ_1,

Similarly,

$$\min_{x \neq 0} \frac{x'Ax}{x'x} = \lambda_n$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{then } \lambda_1(A) \geq \lambda_1(A_{11})$$

Special case $\lambda_1 \geq a_{ii}$, $i = 1, \dots, n$

A ~~B~~ symm, B is pd the AB^{-1} has +ve ev.

$$\max_{x \neq 0} \frac{x'Ax}{x'Bx} \leq \mu. \text{ (largest ev)}.$$

$$A = yy' \qquad \frac{x'Ax}{x'Bx} = \frac{x'yy'x}{x'Bx} = \frac{(x'y)^2}{x'Bx}$$

$AB^{-1} = yy'B^{-1}$ $\quad y'B^{-1}y.$

A $\quad$ B

$A^{1/2}$ $\quad$ $B^{1/2}$

$A \cdot (B^{1/2})^2$

singular values are defined for any matrix.

$\hookrightarrow$ ev of $(AA')^{1/2}$: $\sigma_1(A) \geq \cdots \geq \sigma_n(A)$.

if A is rectangular then some 0 ev comes for $(AA')^{1/2}$.

if we have $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ singular value of $\underset{\sim}{x}$ is $(\sum x_i^2)^{1/2}$

$\rho(A) = \rho(AA')$ AA' symm so $\rho(AA')$ = no. of $\neq 0$ ev.

$\therefore$ # nonzero ev of AA' = # non-zero singular values.

~~(PAQ)'(P~~ <u>Singular value decomposition.</u>

$(PAQ)'(PAQ) = Q'A'P'PAQ = Q'A'AQ$. ~~cst~~ $(AQ_1)'(AQ_1)$, $AQ_1Q_1'A'$, AA'. $A'A$

ev of $AA' = P\underbrace{\text{diag}(\lambda_1 \ldots \lambda_n)}_{D}P'PDP'$

$= PD^2P'$: squares.

so ev of $(AA')^{1/2}$ is $|\alpha_1|, |\alpha_2|, \ldots, |\alpha_n|$.

SVD $\square \; \square \; \square = \begin{pmatrix} \diagdown & 0 \\ & \end{pmatrix}$

~~clo~~ columns of P are the ev of AA'.

$(PAQ)'(PAQ) = \begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{pmatrix}$ | $(PAQ)(PAQ)' = \begin{pmatrix} \sigma_1^2 & \ldots & \sigma_n^2 \end{pmatrix}$

$\Rightarrow PAA'P = \begin{pmatrix} \sigma_1^2 & \ldots & \sigma_n^2 \end{pmatrix}$

$\Rightarrow Q'A'AQ = (\quad)$

columns of Q are the ev of $A'A$.

Thm. $\max\limits_{\|u\|=1, \|v\|=1} |u'Av| = \sigma_1$ (largest SV)

$\max\limits_{u \neq 0, v \neq 0} \dfrac{|u'Av|}{\sqrt{(u'u)(v'v)}} = \sigma_1$

(25)

A $n \times n$ symm.

e.v $\lambda_1 \geq \cdots \geq \lambda_n$ $\{x_1, \ldots, x_n\}$ orthonormal

eivectors $x_1 \cdots x_n$

$$\max_{\|x\|=1} x'Ax = \lambda_1$$

$$\max_{\substack{x \perp x_1 \\ \|x\|=1}} x'Ax = \lambda_2$$ (among x's which perpendicular to x_1 (e.vector of λ_1)).

suppose $A = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ 0 & & \lambda_n \end{pmatrix}$ ev of $\lambda_1 = e_1$.

then, we take vectors perpendicular to e_1.

then $\max^m$ is λ_2.

why s.v decomposition?

$PAQ = \text{diag}(\sigma_1, \ldots, \sigma_n)$

$A = P' \text{diag}(\ldots) Q' = \underline{\underline{\sigma_1 x_1 y_1' + \sigma_2 x_2 y_2' + \cdots + \sigma_n x_n y_n'}}$

SVD

where x_i and y_i are ith columns of P' and Q'

If B is a principal submatrix of A then

$\lambda_1(A) \geq \lambda_1(B)$.

If $A \geq B$ then $\lambda_1(A) \geq \lambda_1(B)$.

$x'(A-B)x \geq 0$ $\forall x$ (as psd)

Proof $x'(A-B)x \geq 0$

$\Rightarrow \max_{\|x\|=1} x'Ax \geq \max_{\|x\|=1} x'Bx$

$\max_{\|x\|=1} x'Ax = \lambda_1(A)$, $\max_{\|x\|=1} x'Bx = \lambda_1(B)$.

$\Rightarrow$ then $\lambda_k(A) \geq \lambda_k(B)$ $\forall k$.

(proof is tough)

but converse is not true. i.e if $\lambda_k(A) \geq \lambda_k(B)$ $\not\Rightarrow A \geq B$.

SOURCE: GITHUB.COM/SOURAVSTAT

Interlacing Principal

$A^{n \times n}$ symm.

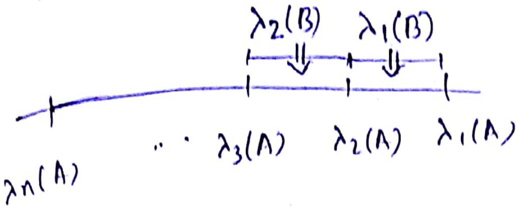
B is principal submatrix of order $n-1$.

$\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ suppose.

$\lambda_1(B) \geq \cdots \geq \lambda_{n-1}(B)$

(we proved that $\lambda_1(A) \geq \lambda_1(B)$).

$\lambda_1(B) \geq \lambda_2(A) \geq \lambda_3(A) \geq \cdots$



however if $\lambda_i(A)$ coincides then also 2 consecutive $\lambda_i(B)$ fall in one interval.

Problems

$A^{n \times n}$ symm and $|A| = 0$

Show that all principal minors of A of order $(n-1)$ are of the same sign.

Chapter 6

$\pi(B) \leq \pi(A) + \pi(B-A)$

$B = A + (B-A)$

$\pi(B) = \pi(A) + \pi(B-A)$

$\Leftrightarrow$ every B^- is a ~~give~~ g-inverse of A.

$A^{n \times n}$ idempotent iff $\pi(A) + \pi(I-A) = n$

if ~~A is part~~

$I = A + (I-A)$

~~then~~ g-inv of I is I so. $AIA = IA$

~~hence Rank additivity follows~~. $\Rightarrow A^2 = A$

A idempotent

$A = UV$ n.f

$I - A = XY$ n.f

$I = A + I - A$

$= UV + XY$

$= [U \; X] \begin{bmatrix} V \\ Y \end{bmatrix}$

$\therefore \begin{bmatrix} V \\ Y \end{bmatrix} [U \; X] = I$

$\Rightarrow VU = I$ $A^2 = UVUV = UV = A$

Proof: (See proof of ...)

$A = XY$ } n.f
$I - A = UV$

$B = [X \; U] \begin{bmatrix} Y \\ V \end{bmatrix}$
full col rank (because $r(A) + r(I-A) = n$)

$\mathcal{C}(A)$, $\mathcal{C}(I-A)$ are virtually disjoint

$\mathcal{C}(A) = \mathcal{C}(X)$ $\mathcal{C}(I-A) = \mathcal{C}(U)$.

So $\begin{bmatrix} Y \\ V \end{bmatrix} B^- [X \; U] = I_n$.

$\Rightarrow YB^- X = I$

$XYB^- XY = XY$

$AB^- A = A$

We will say, ~~A<B (A) for~~ $A < B$ (A is dominated by B)

if $r(B) = r(A) + r(B - A)$.

This rel^n. reflexive, transitive (check)

$A < B$ every g.inv of B is a g.inv of A

$B < C$ " " " C " " " B.

hence every g-inv of C is a g-inv of A.

so $A < C$.

suppose A, B ns.

$r(A-B) = r(A^{-1} - B^{-1})$

$(A^{-1} - B^{-1}) = A^{-1}(B-A)B^{-1}$ so rank is same.

If $A \geq B$ $\Leftrightarrow$ $A-B$ is psd and A, B psd.

$r(A) \geq r(B)$.

If $A \geq B$ then $\mathcal{C}(B) \subseteq \mathcal{C}(A)$

~~$\underset{\sim}{x} \in \mathcal{C}(B) = \underset{\sim}{x} = BY$ $y^T x = y^T B y \geq 0$.~~

~~$y^T A y \geq y^T x$~~

~~$y^T A y - y^T B y \geq 0$.~~

$\underset{\sim}{x} \in N(A)$ $\quad A\underset{\sim}{x} = 0 \quad x^T A x = 0$.

$x^T A x \geq x^T B x \Rightarrow x^T B x = 0$

$\Rightarrow x^T C^T C x = 0 \Rightarrow Cx = 0$

$\Rightarrow C'Cx = 0$

$\Rightarrow Bx = 0$

$\Rightarrow x \in N(B)$

hen $N(A) \subseteq N(B)$

$\Rightarrow \mathcal{C}(B) \subseteq \mathcal{C}(A)$

then $r(A) \geq r(B)$.

$B = A + (B - A)$

The following are equivalent

(i) rank B = rank A + rank $(B - A)$

ii) $\mathcal{C}(A) \cap \mathcal{C}(B-A) = \{0\}$

$\mathcal{C}(A') \cap \mathcal{C}((B-A)') = \{0\}$.

(iii) any B^- is a g-inv of A.

(iii) $\Rightarrow$ (i), (ii).

For any B^-, $AB^-A = A$. ~~then~~ then AB^-A is invariant under choice of g-inv.

Then $\mathcal{C}(A) \subset \mathcal{C}(B)$, $A = BV$.

$R(A) \subset R(B)$. UBB^-BV

$A = UB$ for some U. ~~UBV~~.

~~UBB~~ then $AB^-B = UBB^-B$

$= UB = A$.

(hence .... look at the book)

<u>Linear Model</u>

$\mathrm{Cov}(Bx, Cy) = B\,\mathrm{Cov}(x, y)C'$

$V(b'x) = b'D(x)b$.

if $\begin{pmatrix} x_1 \\ x_2 \\ x_1+x_2 \end{pmatrix}$ $\Rightarrow$ disp matrix is psd.

if there are no linear ~~&~~ relationship then disp matrix is pd.

$y = X\beta + \varepsilon$ $\quad E(y) = X\beta$

$V(\varepsilon) = \sigma^2 I_p$.

is equivalent as $y = X\beta + \varepsilon$, $E(\varepsilon) = 0$, $V(\varepsilon) = \sigma^2 I_p$.

Estimability

$l'\beta = l_1\beta_1 + \cdots + l_p\beta_p$

if $\exists$ a linear function of $\underset{\sim}{Y}$. $\ni$

$E(c'y) = l'\beta$.

o.w it is inestimable.

$c'E(y) = l'\beta \Rightarrow c'X\beta = l'\beta \Rightarrow$ for all β.

$\Rightarrow c'X = l' \iff l' \in R(X)$.

So, a necessary and suff condition for $l'\beta$ to be estimable if $l' \in R(X)$.

Suppose $Z = X+Y$ and $Z = 10$. But we can not say the value of X, Y. we need additional information.

$X(X'X)^- X'$ is invariant under the choice of g-inv

as $R(X) \subset R(X'X)$
$\mathcal{C}(X') \subset \mathcal{C}(X'X)$.

$u \longmapsto \underline{\underline{X(X'X)^- X'}}u$ projects u orthogonally to the column space of X.

$\Downarrow H$

symm and idempotent.

$u = \underbrace{Hu}_{\in \mathcal{C}(X)} + \underbrace{(I-H)u}_{\in \mathcal{C}(X^\perp)}$.

Theorem (7.1) ⇒ for unbiasedness we don't need least square g-inv.

even if $Ax = b$ is not consistent
we can't take $x = Gb$. G: $\overset{\text{min norm}}{\cancel{ls}}$ g-inv.
to be best sol^n

least square g-inv is not unique but least square estimates are unique.

suppose G_1, G_2 are 2 ls g-inv then.

$l'G_1 y = l'G_2 y$.

$\Rightarrow$ $var(l'Gy) = \underline{\underline{\sigma^2 l'(X'X)^- l}}$

invariant under choice of g-inv.

Given a linear model, we want to find the possible estimable function of parameters.

$S = \{(l_1, l_2, m_1, m_2) \mid l_1 + l_2 = m_1 + m_2\}$ (see the example in p-64)

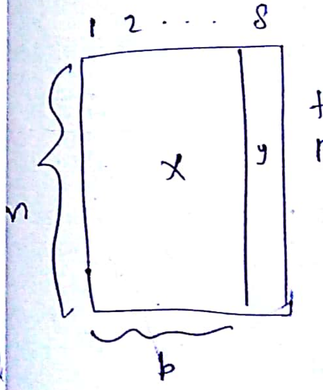
$S = R(X)$.

~~prove~~ show $R(X) \subset S$ and $\dim R(X) = \dim(S)$.

$$\alpha_1 + \beta_1 = (1\ 0\ 1\ 0)\begin{pmatrix}\alpha_1\\ \alpha_2\\ \beta_1\\ \beta_2\end{pmatrix} \qquad (1\,0\,0\,0)(X'X)^- X'y.$$

$$= \underbrace{(1\ 0\ 0\ 0)X}_{1^{st}\text{ row of } X}$$

* Suppose X
Row space
then every
also then
BLUE. $\hat{\beta} =$ (
even if Ax
but $A'Ax$
this is c



then 2^{nd} e
here, we s
$y = X\beta \Rightarrow y$

$y = X\beta$ (X
if we want
project y to
here

* Suppose X has full col rank p.

Row space is $\mathbb{R}^p$.

then every $\ell'\beta$ is estimable as $\ell' \in \mathbb{R}^p \; \forall \; \ell'$

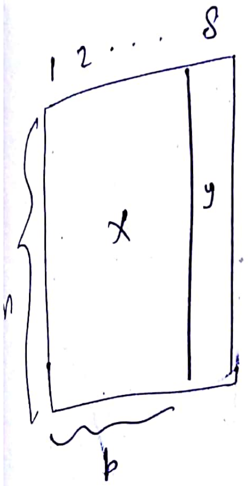
also then $(X'X)^{-1}$ ms.

BLUE. $\hat{\beta} = (X'X)^{-1}X'y$. // Gauss Markov Thm.

even if $Ax = b$ is not consistent

but $A'Ax = A'b$ is always consistent.

this is called normal eq^n.



$y = X\beta$. (may not be consistent)

time points ↓

So many eq^n so few unknowns.

the idea is to reduce to p eq^{n}s.

$ we can take linear combinations of eq^n.

take 1st eq^n as 1st col^{m}s as weight to take linear comb^n of eq^n.

to take linear comb as 2nd col^n as weights.

then 2nd eq^{n}s a

hence, we set $X'y = X'X\beta$.

$y = X\beta \Rightarrow y \in \mathcal{C}(X)$. coordinate free.

$y = X\beta$ (X know, observe y)

if we want to predict y, then should project y to the column space of X

here $\hat{y} = X(X'X)^{-}X'y$.

A p.d. :

$|A| \le a_{11} \cdots a_{nn}$

Suppose $a_{ii} = 1 \ \forall i$ and suppose $\lambda_1 \cdots \lambda_n$ ei...

$|A| = \prod \lambda_i$

by AM–GM, $(\prod \lambda_i)^{1/n} \le \frac{\sum \lambda_i}{n} = \frac{n}{n} = 1$

So $|A| \le 1$.

If $a_{ii} \ne 1$ then make it by $B = D^{-1} A D^{-1}$ for some diagonal matrix. hence get the result.

Gen^n Generalisation

A is p.d. $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

Then $|A| \le |A_{11}||A_{22}|$.

Proof:

Suppose, $A = \begin{bmatrix} a_{11} & x' \\ x & A(1|1) \end{bmatrix}$

then $|A| = |A(1|1)| \, |a_{11} - \underbrace{x' A(1|1)^{-1} x}_{\ge 0 \text{ (as p.d.)}}|$

$\le a_{11} |A(1|1)|$.

now apply induction to prove.

Hadamond Inequality

For any matrix $X^{n \times n}$, $\ni |x_{ij}| \le 1 \ \forall (i,j)$.

Then $|X'X| \le n^n$. (very crude bound)

(Apply Hadamond Inequality) equality holds if $X'X$ is a diagonal matrix.

equality holds for diagonal matrix.

If X is $n \times n$. $x_{ij} = \pm 1 \ \forall (i,j)$

and $X'X$ is diagonal (any two rows / columns are orthogonal).

Then X is called Hadamond. Matrix.

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

for a Hadamond Matrix, $n \times n$. n should divisible by 4.

conjecture: There exists an Hadamond matrix iff ~~$n = 2/4$~~ $n = 2$ or divisible by 4.

left pan	right pan

$v(l'y) = \sigma^2 (X'X)^-$ if we want to make the variance small then we want to make the matrix ~~for~~ $(X'X)$ large.

one can ~~large~~ maximize det / smallest ev / trace.

one can take ~~head~~ Hadamond matrix ~~to~~ as this gives largest det. This is called d-optim.

Problem: $\begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix}$ is pd. Show that,

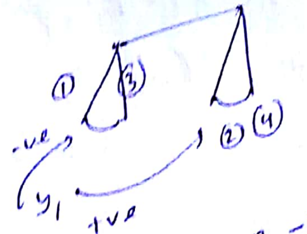
(i) $1 + 2bca + a^2 + b^2 + c^2 > 0$.

(ii) $a^2 + b^2 + c^2 > 2abc$ (iii) $a^4 + b^4 + c^4 > 2a^2b^2c^2$

15/10/18

4 objects.

	wt
1	β_1
2	β_2
3	β_3
4	β_4



y_1 is the wt reqd to balance.

$y_1 = \beta_1 + \beta_2 - \beta_2 - \beta_4 + \varepsilon_1$

$y = X\beta + \varepsilon$

$V(\hat{\beta}) = \sigma^2 (X'X)^{-1}$

Design Matrix.
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

we want $V(\hat{\beta})$ small $\Leftrightarrow$ to make to $|X'X|$ large

by Hadamond ~~matri~~ Ineq:

$|X'X| \le 1 \quad \forall \quad |x_{ij}| \le 1.$

equality iff X is diagonal.

$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ take kronecker's product with itself

$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ ⇒ a hadamond matrix of 4×4.

So, problem reduces to weigh in such a way that $|X'X|$ is maximum.

<u>RSS</u>. to estimate σ^2.

$X'X\beta = X'y$ normal eqn. (always consistent)

$\hat{\beta} = (X'X)^{-} X'y.$

$E(\hat{\beta}) = (X'X)^{-1}(X'X)\beta$

~~if $\ell'\beta$ is~~ estimable. then

~~$E(\ell'\hat{\beta}) =$~~

RSS $= (Y - X\hat{\beta})'(Y - X\hat{\beta})$

unique for any $\hat{\beta}$ as $X\hat{\beta}$ is ~~unique~~ invariant for any g-inv of $(X'X)$.

min $(Y - X\beta)'(Y - X\beta)$ is attained at $\hat{\beta}$.

$\epsilon = Y - X\beta$. $\quad E(\epsilon) = 0. \quad D(\epsilon) = \sigma^2 I$

$E((Y - X\beta)(Y - X\beta)') = \sigma^2 I.$

$E(YY') = X\beta\beta'X' + \sigma^2 I.$

$P = I - X(X'X)^- X'.$

$$E(Y - X\hat{\beta})'(Y - X\hat{\beta}) = E(Y'PY)$$
$$= \operatorname{tr}(P\, E(YY'))$$
$$= \sigma^2 \operatorname{tr}(P).$$

$$\operatorname{tr}(P) = n - \operatorname{tr}(X(X'X)^- X')$$
$$= n - \operatorname{tr}((X'X)^- X'X)$$
$$= n - r(X'X)$$
$$= n - r.$$

$$E\left(\frac{RSS}{n-r}\right) = \sigma^2$$

$$RSS = Y'Y - \hat{\beta}'X'X\hat{\beta}$$
$$= Y'Y - Y'X\hat{\beta}.$$
$$= Y'(Y - X\hat{\beta}).$$
$$= Y'e.$$

one way classification:

$y_{ij} = \alpha_i + \epsilon_{ij} \quad i = 1(1)k, \; j = 1(1)n_i.$

$H_0: l'\beta = c$ (only can be tested if $l'\beta$ is estimable.)

RSS subject to $L\beta = z$. $\quad \mathcal{R}(L) \subset \mathcal{R}(X)$.

$L\beta = z$ is consistent.

there is no dependencies in constraint. ← $r(L)$ is full

$$\tilde{\beta} = \hat{\beta} - (X'X)^{-}L'(L(X'X)^{-1}L')^{-}(L\hat{\beta} - z)$$

RSS: min is attained at $\tilde{\beta}$ subject to $L\beta = z$.

$$\tilde{\beta} = \hat{\beta} - (X'X)^{-}L'(L(X'X)^{-}L')^{-}(L\hat{\beta} - z)$$

$$L\tilde{\beta} = L\hat{\beta} - L(X'X)^{-}L'(L(X'X)^{-}L')^{-}L\hat{\beta} + L(X'X)^{-}L'(L(X'X)^{-}L')^{-}z$$

$$= z.$$

now we want to minimise $(y - X\beta)'(y - X\beta)$ subject to $L\tilde{\beta} = z$.

To show $(\hat{\beta} - \beta)'X'(y - X\hat{\beta}) = 0$

($r(L)$ is full not req^d).

Lemma $r(L) = r(T) = r(L(X'X)^{-}L')$.

A is ~~psd~~ pd $\Rightarrow r(LAL') = r(L)$

(A is nn is not enough)

$r(LAL') = r(LCC'L') = r(LC) = r(L)$

$L_{m\times p}$ $\quad r(L) = m$.

So $r(L(X'X)^{-1}L') = r(L) = m$.

__Example (7.7)__ we might interested in testing the field is square / rectangle.

then additional restriction is θ_i's are equal.

For estimation so far we donot need normality.

General linear model.: $Y = X\beta + \mathcal{E}$.

$E(\mathcal{E}) = 0$ $\quad D(\mathcal{E}) = \sigma^2 V$.

V is pd known.

can be reduced by taking $Z = V^{-1/2} y$.
then it becomes usual linear model.

however if V is psd then the problem becomes complicated.

A pd $\quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ $\quad$ show that. $A_{11} \geq B_{11}^{-1}$.

$A^{-1} = B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$

then by (7).

$$\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} P' & 0 \\ 0 & Q' \end{pmatrix} = \begin{pmatrix} D_1 & * \\ * & D_2 \end{pmatrix}$$

$|A| = \begin{vmatrix} D_1 & * \\ * & D_2 \end{vmatrix} \leq |D_1||D_2|$. (Hadamond ineq)

$= |A_{11}||A_{22}|$

$|A_{11}| = |D_1| \quad |A_{22}| = |D_2|$.

If $S \subset \{1, 2, .. n\}$ $\quad A[S]$ = principal submatrix of A with index S.

if $S = \phi$. $|A[S]| = 1$.

If A is psd and S, T are disjoint subset of $\{1, .., n\}$. Then

$$|A[S \cup T]| \leq |A[S]||A[T]|$$

If $S \cup T = \{1..n\}$ then the prev result.

if we drop $S \cap T = \emptyset$..

then

$$|A[S\cup T]|\,|A[S\cap T]| \le |A[S]|\,|A[T]|.$$

Suppose, $\begin{pmatrix} 1 & a & b \\ b & 1 & c \\ b & c & 1 \end{pmatrix}$ is pd.

$S = \{1,2\}$ $T = \{1,3\}$.

then $1 + 2abc - a^2 - b^2 - c^2 \le (1-a^2)(1-b^2)$

$2abc + a^2 + b^2 + c^2 \geq 0.$

$= (a \;\; b) \underbrace{\begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}}_{\text{pd. (sub matrix of a pd)}} \begin{pmatrix} a \\ b \end{pmatrix} \geq 0.$

Ex. 7.5

1. (i) $E(\sum l_i y_i) = l_1(\beta_1+\beta_2) + l_2(2\beta_1 - \beta_2) + l_3(\beta_1 - \beta_2)$

$= \beta_1(l_1 + 2l_2 + l_3) + \beta_2(l_1 - l_2 - l_3)$

$\left.\begin{array}{l} l_1 + 2l_2 + l_3 = 1 \\ l_1 - l_2 - l_3 = 0 \end{array}\right\}$ $\begin{array}{l} 2l_1 + 3l_2 = 1 \\ 3l_2 + 2l_3 = 1 \end{array}$ $\begin{array}{l} l_2 = -\frac{2}{3}l_3 + \frac{1}{3}. \\ l_1 = \frac{1}{6} + \frac{2}{3}l_3 \end{array}$

$\underset{\sim}{l} = \begin{pmatrix} 1/6 \\ -2/3 \\ 0 \end{pmatrix} + l_3 \begin{pmatrix} 2/3 \\ -2/3 \\ 1 \end{pmatrix}$

take $\frac{1}{3}(y_1 + y_2)$ and $(y_1 + y_3)$

$v\left(\frac{1}{3}(y_1+y_2)\right) = \frac{2\sigma^2}{9}$ $\quad v(y_1+y_3) = 2\sigma^2$

(ii) so, $l_1 + 2l_2 + l_3 = 0$, $l_1 - l_2 - l_3 = 1$ $\Big\}$ $2l_1 + l_2 = 1$, $3l_2 + 2l_3 = -1$

$l_2 = \frac{-1}{3} - \frac{2}{3} l_3 \Rightarrow l_1 = \frac{1}{2} + \frac{1}{3} + \frac{2}{3} l_3$

$= \frac{5}{6} + \frac{2}{3} l_3.$

so $\underset{\sim}{l} = \begin{pmatrix} 5/6 \\ -1/3 \\ 0 \end{pmatrix} + k \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

we also need $\underset{\sim}{l}' \underset{\sim}{m} = 0.$

2 basis vectors of $\begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

orthogonalize them.

$\cancel{\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} - (2\ -2\ 3) \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix}}$

$= \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 70 \\ -28 \\ 0 \end{pmatrix} = \begin{pmatrix} -68 \\ -30 \\ 3 \end{pmatrix}$

(iii) $\theta_1 = \beta_1 + 2\beta_2$, $\theta_2 = \beta_1 - 2\beta_2$ $\Rightarrow \beta_1 = \frac{\theta_1 + \theta_2}{2}$, $\beta_2 = \frac{\theta_1 - \theta_2}{4}.$

write the model in term of β_i.

---

continuity arguments gives weaker result.

1. (i) $V(c'y) = c'D(y)c = c'c\sigma^2 I$

2. $\begin{bmatrix} 2 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & -2 \end{bmatrix}$ dim $N(A) = 1$. (basis $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

then $R(A) = \{ \tilde{x} : \tilde{x}' \tilde{1} = 0 \}$.

$N(A)$ is ortho complement of $R(A)$.

3. $(y_1 - \beta_1 - \beta_2)^2 + (y_2 - \beta_1 + \beta_2)^2 + (y_3 - \beta_1 - 2\beta_2)^2$

min w.r.t β_1, β_2 $\quad \frac{\partial}{\partial \beta_i}(\quad) = 0$

4. $D(\hat{\beta}) = \sigma^2 (X'X)^{-1}$ off diag elm zero.

5. $|A| \geq 1$ Hadamard.

A is pd $\Rightarrow$ $A, ((a_{ij}^2))$ is pd

$\begin{matrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{matrix}$ $\quad \begin{pmatrix} 1 & \alpha & \cdots & \alpha \\ \alpha & 1 & & \\ \vdots & & \ddots & \\ \alpha & & & 1 \end{pmatrix}$ when it is pd/psd.

$(1-\alpha), (1 + \overline{n-1}\,\alpha)$ are ev.

So to be pd, ev have to be pd.

So $\frac{-1}{n-1} \leq \alpha \leq 1$

$$\begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & b & c \\ b & 1 & a \\ c & a & 1 \end{pmatrix}$$ is also p.d.

is p.d. if

$a^2 \leq 1$

$b^2 \leq 1$

$c^2 \leq 1$

and det is +ve.

now take Hadamard product.

$$\begin{pmatrix} 1 & ab & bc \\ ab & 1 & ca \\ bc & ca & 1 \end{pmatrix}$$

$$a^2b^2 + b^2c^2 + c^2a^2 - 2a^2b^2c^2 > 0.$$

6. apply Hadamard on AA'.

8. $A = \begin{bmatrix} a_{11} & x' \\ x & B \end{bmatrix}$

$$|A| = |B|(a_{11} - x'B^{-1}x)$$
$$= a_{11}|B| - (x'B^{-1}x)|B|.$$

$$\frac{x'B^{-1}x}{x'x} \leq \lambda_{max}(B^{-1})$$
$$\geq \lambda_{min}(B^{-1}) = \frac{1}{\lambda_{max}(B)}$$

$$\therefore (x'Bx) \geq \frac{x'x}{\lambda_{max}(B)}.$$

12. use column technique.

13. $\hat{\alpha}_i = \bar{y}_i \quad \forall\ i = 3, \ldots, k.$

$\hat{\alpha}_j = \dfrac{n_1\bar{y}_1 + n_2\bar{y}_2}{n_1+n_2}, \quad j = 1, 2.$ pulled mean of 1st two samples.

If there exist an $n \times n$ Hadamond matrix then $n=2$ or n is divisible by 4.

Hadamond matrix (+1/-1 with any two row/col^n orthog)

if we multiply any row/col^n by (-1) it still remains hadamond

$$\left(\begin{array}{l} 1\ 1\ 1\ 1\ 1\ 1 \cdots 1 \\ 1\ 1\ 1\ 1 \cdots -1\ -1\ -1\ -1 \\ 1 \cdots 1\ -1 \cdots -1\ 1 \cdots 1\ -1 \cdots -1 \\ \ p_1 \quad\ q_1 \quad\ p_2 \quad\ q_2 \end{array}\right.$$

} orthogonal.

same no of +1 and -1.

$p_1 + q_1 = q_1 + q_2$

$p_1 - p_2 = q_1 - q_2$.

Chapter 8

22/10/18

$X^{p\times 1} \sim N_p(\mu, \Sigma)$ Σ p.d.

if $f(x) = \dfrac{1}{(2\pi)^{p/2}\,|\Sigma|^{1/2}} \exp -\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)$.

$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ $u_1, \ldots u_n$ iid $N(0,1)$.

$X_{n\times n}$ matrix , $\mu^{n\times 1}$ vector

$Y = Xu + \mu$ is said to have mult variate normal.

$X \sim MVN$ iff every $l^T X \sim$ univariate Normal.

$E(y) = X E(u) + \mu = \mu$.

$D(y) = X D(u) X^T = XX^T = \Sigma$.

The c.f $\phi_Y(t) = E(e^{it'y})$

$= E(\exp(it'(Xu+\mu)))$

$\phi_u(t) = E(\exp(it'u))$

$= E(e^{i\Sigma t_j u_j})$

$= \prod_{j=1}^{n} E(\exp(it_j u_j))$

$= \prod_j e^{-t_j^2/2}$

$= \exp(\frac{t't}{2})$

$\phi_Y(t) = \exp(it'\mu)\, E(\exp(it'Xu))$

$= \exp(it'\mu)\, \phi_u(t'X)$

$= \exp(it'\mu)\, \exp(-\frac{1}{2} t'XX't)$

$= \exp(it'\mu - \frac{1}{2} t'\Sigma t)$ depends on μ and Σ

c.f determines the distⁿ

so μ, Σ determines MVN.

parameters.

if Σ is ns. we can set density.

$$f(y) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)\right)$$

take $z = \Sigma^{-1/2}(y-\mu)$

$\int \cdots \int_{\mathbb{R}^n} f(y)\, dy = 1$.

$\left\| \frac{\partial z}{\partial y} \right\| = |\Sigma^{-1/2}|$

$= \int \cdots \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} e^{-\frac{1}{2} z'z} |\Sigma|^{1/2}\, dz$.

$= \prod_{j=1}^{n} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_j'} dz_j = 1$

if we can show cf of this density is same as cf of MVN

then this is the density of MVN.

The cf of y is,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)\right) e^{it'y} \, d\underset{\sim}{y}.$$

(use the same transf^n)

$$= e^{it'\mu - \frac{1}{2}t'\Sigma t}.$$

$y \sim N(\mu, \Sigma) \Rightarrow By \sim N(B\mu, B\Sigma B')$

Proof: use cf.

$E(e^{it'BY}) = e^{it'B\mu - \frac{1}{2}t'B\Sigma B't} = \phi_Y(t'B).$

cf of $N(B\mu, B\Sigma B')$.

$y \sim N(\mu, \Sigma)$ $\quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ $\quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$

$y_1 \sim N(\mu_1, \Sigma_{11})$ (use. $B = \begin{pmatrix} I & 0 \end{pmatrix}$)

8.1 y_1 and y_2 are indep iff $\Sigma_{12} = 0$.

$(y-\mu)'\Sigma^{-1}(y-\mu) = ((y_1-\mu_1), (y_2-\mu_2))' \begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix} \begin{pmatrix} y_1-\mu_1 \\ y_2-\mu_2 \end{pmatrix}$

So they are indep.

(Y_1, Y_2) has some bivariate dist^n.

then indep ⇒ uncorrelated

converse not true.

SOURCE: GITHUB.COM/SOURAVSTAT

$y \sim N(\mu, \underset{\sim}{\Sigma})$ with $\Sigma = \sigma^2 I$ $\quad A, B \ni AB' = 0$

then Ay and By are indep.

(A, B can be rectangular)

$$\begin{pmatrix} AY \\ BY \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} Y. \qquad \mathrm{cov}(AY, BY) = A D(Y) B' = \sigma^2 AB' = 0.$$

$\therefore$ AY, BY are indep as ~~AY, BY~~ $\begin{pmatrix} AY \\ BY \end{pmatrix}$ has MVN

Σ n.s. $\quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$

$$\Sigma = \underbrace{\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}}_{S^{-1}} \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{bmatrix} \underbrace{\begin{bmatrix} I & -X' \\ 0 & I \end{bmatrix}}_{S}$$

~~$\tilde{\Sigma}_{22} = \Sigma_{22} - \Sigma_{21}$~~

$$\Sigma^{-1} = S' \begin{bmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \tilde{\Sigma}_{22}^{-1} \end{bmatrix} S.$$

$(Y-\mu)' \Sigma^{-1} (Y-\mu)$

$= (Y_1-\mu_1)' \Sigma_{11}^{-1} (Y_1-\mu_1) + \cancel{(Y_2-\mu_2)' \tilde{\Sigma}_{22}^{-1} (Y_2-\mu_2)}$ $((Y_2-\mu_2)' + (Y_1-\mu_1)'X') \tilde{\Sigma}_{22}^{-1} ((Y_2-\mu_2) + X(Y_1-\mu_1))$

$f(y) = f(y_1) f(y_2|y_1)$

$Y_2|Y_1 \sim N_{n-\alpha}(\quad , \tilde{\Sigma}_{22}^{-1})$

covariance matrix do not depends on Y_1

and mean is linear in Y_1.

$Y_2|Y_1$ is linear in Y_1, cov matrix is independent of Y_1 $\Rightarrow$ does it follow MVN?

(most of the time yes)

even if Σ is singular but Σ_{11} is not then also conditional dist is defined.

$Y \sim N(\mu, \Sigma)$.

(also if Σ_{11} is singular we can define generalised Schur complement to define conditional dist^n).

$y_1, \dots, y_n$ iid $N(0,1)$.

$\sum y_i^2 \sim \chi^2_n$.

$Y \sim N_n(0, I_n)$ $\quad$ A is symm $n \times n$.

(if not take $\left(\frac{A+A^T}{2}\right)$ quad form does not change)

$r(A) = r$.

$Y'AY \sim \chi^2_r$ iff A is idempotent.

($\Leftarrow$) Suppose A is idempotent.

Then $A = P'\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}P$ (ev are 1 / 0).

$$y'Ay = Y'P'\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}PY.$$

$Z = PY \sim N(0, PI)$

P orthogonal.

$$= Z'\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}Z$$

$$= \sum_1^r z_i^2 \sim \chi^2_r$$

($\Rightarrow$) $\lambda_1, \dots, \lambda_n$: ev. (want to show $\lambda_i = 0/1$)

$$\phi(t) = E\left(e^{itY'AY}\right)$$

(by making same transform^n)

$$= E\left(e^{it\sum \lambda_j z_j^2}\right)$$

$$= \prod_j E\left(e^{it\lambda_j z_j^2}\right).$$

$z_j^2 \sim \chi^2_1$.

$= \prod_{j=1} (1 - 2it\lambda_j)^{-1/2}$

Also, $\phi(t) =$ c.f. of χ^2_n.

$= (1-2it)^{-n/2}$.

So, $(1-2it)^{-n/2} = \prod_{j=1}^{n} (1-2it\lambda_j)^{-1/2}$ (by uniqueness of c.f.) $\leftarrow \forall t$

$\Rightarrow (1-2it)^{n} = \prod_{j=1}^{n} (1-2it\lambda_j)$ (square and reciprocal)

poly in t

degree and roots must coincide

roots of LHS. $\frac{1}{2i\lambda_j}$ $j=1(1)n$ roots of RHS $= \frac{1}{2i}$ with multiplicity n.

n of the roots are non-zero

$\Rightarrow \frac{1}{2i\lambda_j} = \frac{1}{4} \Rightarrow \lambda_j = 1.$ for n of them

and rest are 0

So, A is idempotent.

$y \sim N(0, I_n)$ ... A, B $n \times n$ symm idempotent

Then $y'Ay$ and $y'By$ are indep iff $AB = 0$.

$Y'AY$ and $l'Y$ is indep iff $Al = 0$.

$\Leftarrow$ $Y'AY$ and $Y'll'Y$ is indep.

$\Rightarrow Al l' = 0 \Rightarrow Al = 0$ ~~A is pd~~ A symm idempotent if $AB = 0$ ⇒ Ay, By indep.

⇒ $(Ay)'Ay$ and $(By)'By$ indep

as A, B idempotent ⇒ $y'Ay$ and $y'By$ "

If $y'Ay$, $y'By$ are indep then $y'Ay + y'By \sim \chi^2$

⇒ $y'(A+B)y \sim \chi^2$

⇒ $A+B$ idempotent

So A, B, & $(A+B)$ idempotent

⇒ $AB = 0$.

if $y \sim N(0, \Sigma)$ Σ pd

then take $Z = \Sigma^{-1/2} y \sim N(0, I_n)$

A symm

$y'Ay \sim \chi^2(r)$ $A\Sigma A = A$. (generalised idempotent)

generally,

$y \sim N(0, \Sigma)$. A is symm

$y'Ay \sim \chi^2(r)$ ⇔ $\Sigma A \Sigma A \Sigma = \Sigma A \Sigma$ (generalised generalised idempotent)

$y'Ay$ and $y'By$ are indep iff $A \Sigma B = 0$.

25/10/18

A symm, idempotent

$y'Ay$. $y'Ay$ and $l'y$ indep iff $Al = 0$.

$I = A + (I-A)$ $r(A) + r(I-A) = n$. equality holds iff

So $y'Ay$, $y'(I-A)y$ are indep χ^2.

generalisation :

$A_1 + \cdots + A_k = I_n$ then following are equivalent.

(i) $\sum r(A_i) = n$

(ii) $A_i^2 = A_i$ $i = 1(1)k$

(iii) $A_i A_j = 0$ $i \neq j$.

(i) ⇒ (iii) , $\begin{bmatrix} C_1 \\ \vdots \\ C_k \end{bmatrix} [B_1 \cdots B_k]$, $C_i B_j = 0$

$C_i B_i = 0$

⇓

$A_i^2 = A_i$.

also (i) ⇒ rank additivity holds.

⇒ any g-inv of I is g-inv of A_i

∴ $A_i I A_i = A_i$ ⇒ $A_i^2 = A_i$.

(i) each one of then χ^2

(ii) they are indep.

one way ANOVA

$z'z = z'A_1 z + z'A_2 z + z'A_3 z$.

$X_1, \ldots, X_n \overset{iid}{\sim} N(0,1)$.

$\sum (X_i - \bar{X})^2 \sim \chi^2_{n-1}$ becau $\underset{\sim}{x}' \underline{(I - \frac{1}{n} J_n)} \underset{\sim}{x}$

⇓

idempotent

rank $(n-1)$.

So $\bar{X}$ and $\sum (X_i - \bar{X})^2$ is indep

becau $(I - \frac{1}{n} J_n) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 0$. ($x'Ax$ and $l'x$ are indep if $l'A = 0$).

we can not observe z_{ij}'s.

two way classification unequal no ab obsv^n.

$n_{ij} = \frac{n_{i0}\, n_{0j}}{n_{00}}$. -

then F statistic can be derived.

general linear model.

now suppose $Y \sim N(X\beta, \sigma^2 I_n)$

we derived $E(RSS) = (n-r)\sigma^2$.

$\frac{RSS}{\sigma^2} \sim \chi^2_{n-r}$.

(eq^n 8.15)

$Y'PY = (Y-X\beta)'P(Y-X\beta)$

Maximum likelihood estimate

$Y \sim N(X\beta, \sigma^2 I)$ rank $X = p$.

$M(\beta) = \hat{\beta}$ (LSE).

$MLE(\sigma^2) = \frac{1}{n}(Y - X\hat{\beta})'(Y - X\hat{\beta})$.

~~exp~~ exercise 8.6

$\begin{pmatrix} X_1 + X_2 \\ X_1 - X_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & +1 \\ 1 & -1 \end{pmatrix}}_{B} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $N(BX, B\Sigma B')$

$c_1 X_1 + c_2 X_2$ is indep if A $\underline{x}' \cancel{A} \underline{x} = (x_1 - x_2)^2$

12. $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ $\tilde{\Sigma}_{11} = \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}$.

~~if~~ $\cancel{\Sigma}$.

is ped then $\mathcal{C}(\Sigma_{12}) \subset \mathcal{C}(\Sigma_{11})$

then automatically $R(\Sigma_{21}) \subset R(\Sigma_{11})$ by symmetric Σ.

$\Sigma = BB'$ as Σ is psd.

$$= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} B_1' & B_2' \end{bmatrix}$$

$$= \begin{bmatrix} B_1B_1' & B_1B_2' \\ B_2B_1' & B_2B_2' \end{bmatrix}$$

$\mathcal{C}(\Sigma_{12}) = \mathcal{C}(B_1B_2') \subset \mathcal{C}(B_1) = \mathcal{C}(B_1B_1')$
$= \mathcal{C}(\Sigma_{11})$.

then $\tilde{\Sigma}_{11} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-}\Sigma_{12}$ (is invariant under choice of g inv)

$\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, \underset{=}{\Sigma})$, psd.

$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ conditional $X_2 | X_1$.

disp $(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-}\Sigma_{12})$

any g inv.

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} (\Sigma) \begin{pmatrix} \; \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \tilde{\Sigma}_{11} \end{pmatrix}$$

So this hold (replace Σ_{11}^{-1}, by Σ_{11}^{-})

So conditional distⁿ is mvn even if X is singular. (look at 13).

rank additivity $r(\Sigma) = r(\Sigma_{11}) + r(\tilde{\Sigma}_{11})$

gen schur complement.

$r\begin{pmatrix} A & A \\ A & I \end{pmatrix} = r(I) + r(A - A^2)$.

$r\begin{pmatrix} A & A \\ A & I \end{pmatrix} = r(A) + r(I - AA^{-}A)$ for any matrix A.

$= r(A) + r(I - A)$

So, $n + r(A-A^2) = r(A) + r(I-A)$.

A tripotent $A^3 = A$.
get the similar identity.

$y \sim N(0, \Sigma)$. $\Sigma = BB'$.
$x \sim N(0, I)$ so any psd Σ is
$Bx \sim N(0, \Sigma)$. a covariance matrix.

$y \sim N(0, \Sigma)$, Σ p.d.
$y'Ay \sim \chi^2$ iff $A\Sigma$ is idempotent. $\Sigma^{+1/2}A\underline{\Sigma^{+1/2}\Sigma^{+1/2}}A\Sigma^{1/2} = \Sigma^{1/2}A\Sigma^{1/2}$
then $\Sigma^{-1/2}y \sim N(0, I)$. $\Sigma \quad A\Sigma A = A$. $A\Sigma A\Sigma = A\Sigma$

if Σ is psd $y'Ay \sim \chi^2$ iff $\Sigma A \Sigma A \Sigma = \Sigma A \Sigma$.

$\cancel{B'B A \Sigma A = B'AB B'}$
$\cancel{BA}$. $B'BA\Sigma AB'B = B'BAB'B$

$\Leftarrow$ $BA\Sigma AB' = BAB'$ $\Rightarrow$ BAB' is idempotent.
$\Leftarrow$ $(BAB'BAB') = (BAB')$

$X \sim \chi^2$ $X \geq Y$ ($X-Y$ is nng RV).
indep $Y \sim \chi^2$ when $(X-Y) \sim \chi^2$ if $(X-Y)$ is non neg.

A, B: symm idempotent. $A \geq B$.
Show that $(A-B)$ is idempotent
$\left(\begin{array}{l} y'Ay - \chi^2 \\ y'By - \chi^2 \\ y'(A-B)y \sim \chi^2 \end{array} \right)$

$$A = P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} P'$$

$PAP' \geq P'BP = C$.

$$\Rightarrow \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \geq \begin{bmatrix} B_1 & B_2 \\ \cdot & 0 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.$$

Lower

so, $0 \geq c_{22}$ & $c_{22} \geq 0$.

C is psd and $c_{22} = 0$ so. $c_{12} = c_{21} = 0$.

[psd if any diagonal elm is 0 then whole row and colmn is zero].

$$C = \begin{bmatrix} c_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

C is idempotent

& C_{11} is idempotent.

$I_r - C_{11}$ is idempotent.

now $P \begin{bmatrix} I_r - c_{11} & 0 \\ 0 & 0 \end{bmatrix} P' = A - B$.

So. $(A-B)$ is idempotent.

17 (show all ev are 0/1 I think).

19. $Y \sim N(0, I)$. $\bar{Y}$ given Y_1

$\begin{pmatrix} \bar{Y} \\ \Sigma y_i \end{pmatrix}$ get joint distn.

now linear comb of $\begin{pmatrix} Y \\ Y_1 \\ \vdots \\ Y_n \end{pmatrix}$

Omit (24)