Malik Arora Page 80

$$\frac{n+1}{(n!)^{1/n}} \to e$$

$$\frac{(n!)^{1/n}}{n+1} \to \frac{1}{e}$$

Proof

$$\frac{n+1}{(n!)^{1/n}} = \left\{\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2 \cdots \left(\frac{(n+1)}{n}\right)^n\right\}^{1/n}$$

$$= \left\{\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^k\right\}^{1/n}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

~~Then~~ Now, $\lim_{n\to\infty}\left(\prod_{i=1}^{n} a_i\right)^{1/n} = \lim_{n\to\infty} a_n$

Thus, $\frac{n+1}{(n!)^{1/n}} \to e$

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$$\lim \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n = 0, \quad |x|<1$$

$$\lim \binom{m}{n} x^n = 0, \quad |x|<1$$

Proof

$$a_n = \frac{m(m-1)\cdots(m-n+1)}{n!} x^n$$

$$\lim \frac{a_{n+1}}{a_n} = \lim\left(\frac{m-n}{n+1}\right)x = \lim \frac{\frac{m}{n}-1}{1+\frac{1}{n}} x = -x$$

Now, $|-x| = |x| < 1$

Thus $\lim \binom{m}{n} x^n = 0, \quad |x|<1$

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\# If $\{S_n\}$ be a sequence of positive real numbers such that $S_n = \frac{1}{2}(S_{n-1} + S_{n-2})$, $\forall$ $n > 2$ then $\{S_n\}$ converges and $\lim_{n \to \infty} S_n = \frac{1}{3}(S_1 + 2S_2)$.

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\# The sequence $\{a_n\}$ defined by $a_{n+1} = \frac{1}{2}\left(a_n + \frac{9}{a_n}\right), n \geq 1$ and $a_1 > 0$ converges to 3.

\# The sequence $\{S_n\}$ defined by $S_n = \frac{S}{1 + S_{n-1}}$ where $S > 0$, $S_1 > 0$, $n \geq 2$, converges to positive root of the equation $x^2 + x - S = 0$.

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Two sequences $\{x_n\}$ and $\{y_n\}$ are defined inductively by $x_1 = \frac{1}{2}$ and $y_1 = 1$

and $x_n = \sqrt{x_{n-1} y_{n-1}}$, $n = 2, 3, 4, \ldots$

$$\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-1}}\right), \quad n = 2, 3, 4, \ldots$$

Then $x_{n-1} < x_n < y_n < y_{n-1}$, $n = 2, 3, \ldots$

$\{x_n\}$ and $\{y_n\}$ converge to the same limit $l \ni \frac{1}{2} < l < 1$

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Given that $\{a_n\}$ is a sequence such that

$$a_2 \leq a_4 \leq a_6 \leq \cdots \leq a_5 \leq a_3 \leq a_1$$

and a sequence $\{b_n\}$, where $b_n = a_{2n-1} - a_{2n}$, converges to zero, then $\{a_n\}$ is convergent.

[Intuitively]

Infinite Series

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If $u_n > 0$ and Σu_n is convergent, with the sum S, then ~~prove that~~ $\frac{u_n}{\sum_{i=1}^{n} u_i} < \frac{2u_n}{S}$ when n is sufficiently large. Also ~~prove that~~ $\Sigma \frac{u_n}{u_1 + u_2 + \cdots + u_n}$ is convergent.

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For large n, $e^{an} >> n^b >> (\log n)^c$ where a, b, c are positive numbers

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The series $\Sigma \{(n^3+1)^{1/3} - n\}$ converges.

Soln Proof

$$u_n = \{(n^3+1)^{1/3} - n\}$$
$$= n\left\{\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right\}$$
$$= n\left\{\frac{1}{3n^3} + \cdots\right\}$$
$$= \frac{1}{3n^2} + \cdots$$

Let $v_n = \frac{1}{n^2}$

$$\lim \frac{u_n}{v_n} = \frac{1}{3}$$

Thus by Comparison Test the result follows.

The series $\Sigma \frac{1}{n^{1+1/n}}$ is divergent.

Proof

$u_n = \frac{1}{n^{1+1/n}}$, $v_n = \frac{1}{n}$

$$\lim \frac{u_n}{v_n} = \lim \frac{1}{n^{1/n}} = 1$$

Hence the result

$\sum \cos\frac{1}{n}$ diverges as, $\lim \cos\frac{1}{n} = 1 \neq 0$.

$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots$ converges.

Hint: $n^n > 2^n \quad \forall\, n > 2$

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$\sum \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ converges.

$a_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

$a_n^{1/n} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$

$\lim a_n^{1/n} = \frac{1}{e} < 1$

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$\frac{\alpha}{\beta} + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)} + \cdots$ converges for $\beta > \alpha$

diverges " $\beta \leq \alpha+1$

Proof

$$a_n = \frac{(1+\alpha)(2+\alpha) \times \cdots \times (n-1+\alpha)}{(1+\beta)(2+\beta)\cdots(n-1+\beta)}$$

$$\cancel{\frac{a_{n+1}}{a_n}} \quad \frac{a_n}{a_{n+1}} = \frac{n+\beta}{n+\alpha}$$

$$\lim n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim n\left(\frac{\beta-\alpha}{n+\alpha}\right) = \beta - \alpha$$

So, by Raabe's Test,

it converges for $\beta - \alpha > 1$

" diverges " $\beta - \alpha < 1$

For $\beta = \alpha+1$, $\frac{\alpha}{\alpha+1} + \frac{(1+\alpha)}{\alpha+2} + \frac{1+\alpha}{\alpha+3} + \cdots = \frac{\alpha}{\alpha+1} + \sum \frac{1+\alpha}{n+\alpha}$

which diverges by Comparison with $\sum \frac{1}{n}$.

The series $\sum_2^\infty \frac{1}{n(\log n)^p}$, $p>0$ converges for $p>1$ and diverges for $p \leq 1$.

Sol^n $u_n = \frac{1}{n(\log n)^p}$

Now $\sum_2^\infty \frac{1}{n(\ln n)^p}$ and $\int_2^\infty u_n \, dn$ converges or diverges simultaneously.

$$\int_2^x \frac{1}{n(\ln n)^p} dn = \int_{\ln 2}^{\ln x} \frac{dz}{z^p} = \begin{cases} \left[\frac{z^{1-p}}{1-p}\right]_{\ln 2}^{\ln x} & \text{when } p \neq 1 \\ [\ln z]_{\ln 2}^{\ln x} & \text{" } p=1 \end{cases}$$

Let $\ln n = z$ when $p \neq 1 \Rightarrow \frac{1}{n} dn = dz$

$$= \begin{cases} \frac{(\ln x)^{1-p} - (\ln 2)^{1-p}}{1-p} & \text{when } p \neq 1 \\ \ln \ln x - \ln \ln 2 & \text{" } p=1 \end{cases}$$

Thus $\sum_2^\infty \frac{1}{n(\ln n)^p}$ converges for $p>1$
diverges " $p \leq 1$

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$1 + \frac{\alpha}{\beta} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} + \cdots$, where α and β are positive, converges if $\beta > \alpha+1$, diverges if $\beta \leq \alpha+1$.

Proof $a_n = \frac{(\alpha+n-1)(\alpha+n-2)\cdots(\alpha+1)\alpha}{(\beta+n-1)(\beta+n-2)\cdots(\beta+1)\beta}$ $n = 1, 2, \ldots$

$$\frac{a_{n+1}}{a_n} = \frac{\alpha+n}{\beta+n}$$

$$\frac{a_n}{a_{n+1}} - 1 = \frac{\beta-\alpha}{\alpha+n}$$

$$\lim n\left(\frac{a_n}{a_{n+1}} - 1\right) = \beta - \alpha$$

wh $\beta > \alpha+1$ converges
" $\beta < \alpha+1$ diverges.

when $\beta = \alpha+1$, then,

$$a_n = \frac{\alpha}{\alpha+n}$$

$$\frac{a_{n+1}}{a_n} = \frac{\alpha+n+1}{\alpha+n} = 1 + \frac{1}{\alpha+n}$$

$1 + \sum a_n = 1 + \sum_1^\infty \frac{\alpha}{\alpha+n}$ diverges [By comparing $\sum_1^\infty \frac{\alpha}{\alpha+n}$ with $\sum_1^\infty \frac{1}{n}$]

Result: ① $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \forall x$

② $\lim_{n\to\infty} \binom{m}{n-1} x^n = 0 \quad \forall |x|<1$

$\lim \binom{m}{n} x^n = 0 \quad \forall |x|<1$

③ If $\sum a_n^2$ and $\sum b_n^2$ are convergent infinite series, then $\sum a_n b_n$ is an absolutely convergent series.

(AM-GM ineq)

④ If $\sum a_n$ is absolutely convergent then $\sum \frac{n+1}{n} a_n$ is also absolutely convergent.

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If b_n is a positive, monotonic decreasing function and if A_n is bounded, then the series $\sum A_n(b_n - b_{n+1})$ is absolutely convergent.

Proof $|A_n| < k \ \forall n$

$$\sum_{n=1}^{k} |A_n(b_n - b_{n+1})| = \sum_{n=1}^{k} |A_n|(b_n - b_{n+1})$$
$$< K \sum (b_1 - b_{k+1})$$
$$< k b_1$$

Thus, seq^n of partial sum of $\sum |A_n(b_n - b_{n+1})|$ is bounded above and hence is convergent.

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~~Abel's T~~

If b_n is a +ve monotonic decreasing function and if Σu_n is a convergent series, then the series $\Sigma u_n b_n$ is convergent.

Page 138 (Abel's Test)

\# A convergent series Σu_n (not necessarily absolutely) remains convergent if its terms are multiplied by a factor a_n, provided that the seq^n $\{a_n\}$ is bounded and monotonic.

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\# (Dirichlet's Test)

If b_n is a positive monotonic decreasing function with limit zero, and if, for the series Σu_n, the sequence $\{S_n\}$ of partial sums is bounded, then the series $\Sigma u_n b_n$ is convergent.

Note:

$v_n = u_n b_n$, $S_n = \sum_{n=1}^{n} u_n$, $\overline{V_n} = \sum_{1}^{n} v_n$

$$\overline{V_n} = \sum_{i=1}^{n} u_i b_i$$

$$= S_1 b_1 + (S_2 - S_1) b_2 + \cdots + (S_n - S_{n-1}) b_n$$

$$= \sum_{n=1}^{n-1} S_n (b_n - b_{n+1}) + S_n b_n$$

The series $0 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{3} + \frac{2}{3^2} - \frac{1}{4} + \frac{3}{4^2} - \cdots$ conv

Proof Given series is obtained by multiplying the terms of the series

$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \cdots$

$\frac{1}{2} - \frac{1}{2} + \frac{3}{4} - \frac{1}{4} + \cdots$

by the terms of the following seq^n

$0, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \frac{3}{4}, \cdots$

Hence from Abel's Test the result follows

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Riemann's Theorem

By appropriate rearrangement of terms a conditionally convergent series $\sum U_n$ can be made ① to converge to any number σ or

② " diverge " $+\infty$ "

③ " " " $-\infty$ "

④ " oscillate finitely "

⑤ " " infinitely.

① The Dirichlet's function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is irrational} \\ -1, & \text{" } x \text{ " " } \end{cases}$$

is discontinuous at every point.

② The function $f(x)$ defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{when } x \text{ is irrational} \\ -x, & \text{" } x \text{ " rational} \end{cases}$$

is continuous only at $x = 0$.

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If a function is continuous in a closed interval, then its bounded therein.

Page 168

If a function f is continuous at an interior point c of an interval $[a,b]$ and $f(c) \neq 0$. then $\exists$ a neighbourhood N of c wherein $f(x)$ has the same sign as $f(c)$ for all $x \in N$.

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Intermediate Value Theorem

If a function f is continuous on $[a,b]$ and $f(a) \neq f(b)$, then it assumes every value between $f(a)$ and $f(b)$

\# A function f, which is continuous on a closed interval $[a,b]$, assumes every value between its bounds.

The range of a continuous function whose domain is a closed interval is as well ~~as~~ a closed interval.

The image of a closed interval under a continuous function is a closed interval.

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① Fixed Point Theorem

If f is continuous on $[a,b]$ and $f(x) \in [a,b]$ for every $x \in [a,b]$, then f has a fixed point, i.e. there exists a point, $c \in [a,b]$ $\ni$ $f(c) = c$.

② A function f defined on $[a,b]$ is said to satisfy the Intermediate-Value Property on $[a,b]$ if for every $x_1, x_2 \in [a,b]$ with $x_1 < x_2$ and for every λ between $f(x_1)$ and $f(x_2)$ there is a $c \in (x_1, x_2)$ with $f(c) = \lambda$.

A function which satisfies the intermediate-value property on $[a,b]$ need not be continuous on $[a,b]$. e.g.

$f(x) = \sin\frac{1}{x}$ with $f(0) = 0$ defined on $[-\frac{2}{\pi}, \frac{2}{\pi}]$ satisfies the property.

If f satisfies the intermediate value property on $[a,b]$, then ~~prove that~~ f has no discontinuity (removable and of first kind).

If f is one-to-one and satisfies IVP on $[a,b]$, then f is continuous on $[a,b]$.

If f is continuous on $[a,b]$ and $f(a)=f(b)$ then show that there exist $x, y \in (a,b) \ni$ $f(x)=f(y)$.

Monotone functions have no discontinuities of the second kind.

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A function f defined on an interval I is said to be uniformly ~~convergent~~ continuous on I if ~~xx~~ to each $\epsilon>0$ there exists a $\delta>0$ $\ni$ $|f(x_2)-f(x_1)|<\epsilon$, for arbitrary points x_1, x_2 of I for which $|x_1-x_2|<\delta$.

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A function which is continuous everywhere but not derivable anywhere is

$$f(x) = \sum_{1}^{\infty} \frac{1}{2^n} \cos(3^n x) \quad \forall x \in \mathbb{R}$$

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The function $f(x) = x|x|$ is derivable at $x=0$.

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If ~~f(x)~~ f and g are defined on $[0,\infty[$ by $f(x) = \lim_{n\to\infty} \frac{x^n - 1}{x^n + 1}$ and $g(x) = \int_0^x f(t)\,dt$. Then g is continuous but not derivable at $x=1$.

Some inequalities ~~etc~~

1. $$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)} \quad \forall x > 0$$
2. $$\frac{x}{1+x} < \log(1+x) < x \quad \forall x > 0$$
3. $$\frac{x}{1+x^2} < \tan^{-1} x < x \quad \forall x > 0$$
4. $$\frac{2}{\pi} < \frac{\sin x}{x} < 1 \quad \forall\ 0 < |x| \leq \frac{\pi}{2}$$
5. $$\tan x > x \quad \forall\ 0 < x < \frac{\pi}{2}$$
6. $$2x < \log\frac{1+x}{1-x} < 2x\left(1 + \frac{x^2}{3(1-x^2)}\right), \quad 0 < x < 1$$
7. $$\frac{2}{2x+1} < \log\left(1 + \frac{1}{x}\right) < \frac{1}{\sqrt{x(x+1)}}, \quad \forall x > 0$$

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Darbow Th

If a function f is derivable on a closed interval $[a, b]$ and $f'(a), f'(b)$ are of opposite sign then there exits atleast one point c between a and b $\ni f'(c) = 0$.

IVT of derivatives

If a function f is derivable on a closed interval $[a, b]$ and $f'(a) \neq f'(b)$ and k is a number lying between $f'(a)$ and $f'(b)$ then $\exists$ atleast one point $c \in (a, b) \ni f'(c) = k$.

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Between two consecutive zeros of $f'(x)$ there lies at the most one zero of $f(x)$.

Page 190

If $f(x)$ satisfies all the conditions of LMVT and $f'(x) = 0 \ \forall \ x \in (a, b)$, then $f(x)$ is constant on $[a, b]$.

If two functions have equal derivatives at all points of (a, b), then they differ only by a constant.

If f' exists and is bounded on some interval I, then f is uniformly continuous on I.

Can be proved by Lagrange's MVT,

$$\frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2}, \text{ if } 0<u<v$$

Using it it can be proved.

$$\frac{\pi}{4}+\frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4}+\frac{1}{6}.$$

A twice differentiable function f is $\ni$ $f(a)=f(b)=0$ and $f(c)>0$ $\forall$ $c\in(a,b)$ then there exist atleast one value $\xi\in(a,b)$ $\ni f''(\xi)<0$

#① $\frac{\tan x}{x} > \frac{x}{\sin x}$ $\forall$ $x\in(0,\frac{\pi}{2})$

② $\cos x < \left(\frac{\sin x}{x}\right)^3$ $\forall$ $x\in(0,\frac{\pi}{2})$

If f, ϕ and ψ are continuous on $[a,b]$ and derivable on (a,b) then ~~show~~ there is a value $c\in(a,b)$ $\ni$

$$\begin{vmatrix} f(a) & f(b) & f'(c) \\ \phi(a) & \phi(b) & \phi'(c) \\ \psi(a) & \psi(b) & \psi'(c) \end{vmatrix} = 0$$

If $f'(x)$ and $g'(x)$ exists for all $x\in[a,b]$, and if $g'(x)$ doesn't vanish anywhere on (a,b) Then $\exists$ some $c\in(a,b)$ $\ni$

$$\frac{f(c)-f(a)}{g(b)-g(c)} = \frac{f'(c)}{g'(c)}$$

[Can be proved using Rolle's Th. on $f(x)g(x) - f(a)g(x) - f(b)$

If $|f(x)-f(y)| \leq (x-y)^2$, for all real numbers x and y then f is a constant function.

If $g(x)=0$ has two equal roots, then $g'(x)=0$ has one root equal to either.

If $\lim_{(x,y)\to(a,b)} f(x,y) = l$ and if $y=\phi(x)$ is any function $\ni \phi(x)\to b$, when $x\to a$, then $\lim_{x\to a} f(x,\phi(x))$ must exist and should be equal to l.

If we can find two functions $\phi_1(x)$ and $\phi_2(x) \ni$ the limit of $f(x,\phi_1(x))$ and $f(x,\phi_2(x))$ are different, then the simultaneous limit in question doesn't exist.

\# Let $f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & , \text{ if } x^4+y^2 \neq 0 \\ 0 & , \text{ '' } x+y=0. \end{cases}$

Then $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist.

Suppose we approach $(0,0)$ along the line $y=mx^2$

Thus, $f(x,y) = f(x,mx^2) = \dfrac{m}{1+m^2}$ which $\to \dfrac{m}{1+m^2}$ as $x\to 0$

Thus $\dfrac{m}{1+m^2}$ is different for different values of m thus the result follows.

\# $\lim_{(x,y)\to(0,0)} xy\dfrac{x^2-y^2}{x^2+y^2} = 0$

Let $x = r\cos\theta$, $y = r\sin\theta$

Then $\left| xy\dfrac{x^2-y^2}{x^2+y^2}\right| = \left| r^2\sin\theta\cos\theta\cos 2\theta\right|$

$= \left|\dfrac{r^2}{4}\sin 4\theta\right| \leq \dfrac{r^2}{4} = \dfrac{x^2+y^2}{4} < \epsilon$

whenever $x^2+y^2 < \delta^2 = 4\epsilon$

Thus, $\lim_{(x,y)\to(0,0)} xy\dfrac{x^2-y^2}{x^2+y^2} = 0$

$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = 0$

Use $\frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} \simeq \frac{1}{2}\frac{x^2y^2}{x^2+y^2}$

$\lim_{(x,y)\to(0,0)} \left(\frac{1}{|x|}+\frac{1}{|y|}\right) = \infty$?

as $\lim_{(x,y)\to(0,0)} |x| = 0$

$\lim_{(x,y)\to(0,0)} |y| = 0$

$\lim_{(x,y)\to(0,0)} \frac{1}{xy} \sin(x^2y+xy^2) = 0$

Since ~~$(x,y)\to(0,0)$~~ x and y are sufficiently small, $\sin(x^2y+xy^2) \simeq x^2y+xy^2$

$\lim_{(x,y)\to(0,0)} \frac{1}{xy}\sin(x^2y+xy^2) = \lim_{(x,y)\to(0,0)} (x+y) = 0$

$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y}$ doesn't exist.

Suppose we approach $(0,0)$ along $y = x - mx^3$

$\frac{x^3+y^3}{x-y} = \frac{x^3+(x-mx^3)^3}{mx^3} = \frac{1+(1-mx^2)^3}{m} \to \frac{1}{m}$ as $x\to 0$

~~Thus~~ It varies with diff. values of m.

$\lim_{(x,y)\to(2,1)} \frac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)} = \lim_{t\to 0} \frac{\sin^{-1} t}{\tan^{-1} 3t} = \lim_{t\to 0} \frac{\frac{1}{\sqrt{1-t^2}}}{\frac{3}{1+9t^2}} = \frac{1}{3}$.

If the simultaneous limit exists, these two repeated limits if they exist are necessarily equal but the converse is not true.

If the repeated limits aren't equal, the simultaneous limit can't exist.

For $f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x} & , xy \neq 0 \\ 0 & , xy = 0 \end{cases}$

Simultaneous limit exists at (0,0) but repeated limits don't.

$$|f(x,y) - 0| = \left|x\sin\tfrac{1}{y} + y\sin\tfrac{1}{x}\right| \leq |x| + |y| < \epsilon$$

when $|x-0| < \delta = \frac{\epsilon}{2}$, $|y-0| < \delta = \frac{\epsilon}{2}$

So, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

$\lim_{x\to 0} f(x,y)$ and $\lim_{y\to 0} f(x,y)$ doesn't exist. So does the repeated limits.

For $f(x,y) = \begin{cases} 1 & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0 \end{cases}$

The simultaneous limit doesn't exist. Though the repeated limits exist and are equal.

$\exists$ (x,y) in the nbd of $(0,0)$ $\ni$ $f(x,y) = 1$

Thus $|f(x,y) - f(0,0)| \neq \epsilon$ $\forall$ points in the nbd of $(0,0)$.

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If a function f of two variables x, y is continuous at (a, b) then f(x, b) is a continuous function of x at x = a and f(a, y) that of y at y = b. The converse however isn't true.

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Partial derivatives may exist at a point at which the function isn't even continuous

Page 408

A sufficient condition that a function f be continuous at (a, b) is that one of the partial derivatives exists and is bounded in the nbd of (a, b) and that the other exists at (a, b)

A sufficient condition that a function be continuous in a closed region is that both the partial derivatives exist and are bounded throughout the region.

A function which is differentiable at a point possesses the first order partial derivatives thereat.

If we replace

~~Th A function is differentiable at a~~

The function f, where

$$f(x,y) = \begin{cases} x\sin\frac{1}{x} + y\sin\frac{1}{y}, & xy \neq 0 \\ x\sin\frac{1}{x}, & x \neq 0, y = 0 \\ y\sin\frac{1}{y}, & x = 0, y \neq 0 \\ 0, & x = 0 = y \end{cases}$$

is continuous but not differentiable at the origin.

If (a, b) be a point of the domain of definition of a function f $\ni$

① f_x is continuous at (a,b)

② f_y exists at (a,b)

Then f is differentiable at (a,b)

(It's a sufficient condition. It's not necessary)

Let

$$f(x,y) = \begin{cases} x^2\sin\frac{1}{x} + y^2\sin\frac{1}{y} & \text{if } xy \neq 0 \\ x^2\sin\frac{1}{x} & \text{if } x \neq 0 \text{ and } y = 0 \\ y^2\sin\frac{1}{y} & \text{if } x = 0 \text{ and } y \neq 0 \\ 0 & \text{if } x = y = 0 \end{cases}$$

$$f_x(0,0) = \lim_{h \to 0} \frac{h^2\sin\frac{1}{h}}{h} = 0$$

$$f_y(0,0) = 0$$

$$f_x(x,y) = 2x\sin\frac{1}{x} - \cos\frac{1}{x} \qquad (x \neq 0)$$

$$f_y(x,y) = 2y\sin\frac{1}{y} - \cos\frac{1}{y} \qquad (y \neq 0)$$

Thus none of the f_x & f_y ~~is~~ continuous at $(0,0)$

$f(h,k) - f(0,0) = h f_x(0,0) + k f_y(0,0) + h\left(h \sin\frac{1}{h}\right) + k\left(k \sin\frac{1}{k}\right)$

Thus, $f(x,y)$ is differentiable at $(0,0)$

For $f(x,y) = \sqrt{|xy|}$ is not differentiable at $(0,0)$ but f_x and f_y exist at the origin and have the values 0 but the two partial derivatives are continuous except at $(0,0)$.

If a function isn't differentiable at a point then the partial derivatives can't be continuous thereat.

Page 509

Young's Theorem

If f_x and f_y are both differentiable at (a,b) of the domain of definition of a function f, then, $f_{xy}(a,b) = f_{yx}(a,b)$

Page 510

Schwarz's Theorem

If f_y exists in a certain nbd of a point (a,b) of the domain of def^n of a function f, and f_{yx} is continuous at (a,b), then $f_{xy}(a,b)$ exists and is equal to $f_{yx}(a,b)$.

If f_{xy} and f_{yx} are both continuous at (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$.

Page 515

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$d^n z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^n z$$

Page 516

~~When $z = f(x_1, x_2, x_3)$ and x_1, x_2, x_3 are n.t the independent veriables then.~~

Page 515

If $z = f(x,y)$ is a differentiable function of x, y and $x = g(u,v)$, $y = h(u,v)$ are themselves differentiable functions of the independent veriables u, v, then z is a differentiable function of u, v and

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

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If (i) x, y be differentiable function of a single veriable, and (ii) z is differentiable function of x and y.

Then z posseses continuous derivative w.r.t t, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

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If $z = f(x,y)$ posseses nth order partial derivatives and x & y are linear functions of a single veriable t, i.e. $x = a + ht$, $y = b + kt$.

where a, b, h, k are constants. then.

$$\frac{d^n z}{dt^n} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n z$$

518

If (i) x, y are differentiable functions of two independent variables u and v, and

(ii) z is a differentiable function of x and y

Then z possesses continuous partial derivatives w.r.t. u and v and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

520

If for all values of the parameter λ, and for some constant n, $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ identically where F is assumed differentiable. Then

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = nF$$

538

A necessary condition for $f(x,y)$ to have an extreme value at (a, b) is that $f_x(a,b) = 0$, $f_y(a,b) = 0$ provided these partial derivatives exist.

\# If $f(x,y) = 0$, if $x = 0$ or $y = 0$, and $f(x,y) = 1$ elsewhere, then both the partial derivatives exist (each equal to 0) at $(0,0)$, but $f(0,0)$ isn't an extreme value.

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Since at (a,b)

$$df = h f_x(a,b) + k f_y(a,b)$$

and $d^2f = h^2 f_{xx}(a,b) + 2hk\, f_{xy}(a,b) + k^2 f_{yy}(a,b)$

$$= Ah^2 + 2Bhk + Ck^2$$

so $f(a,b)$ is an extreme value of $f(x,y)$ if at (a,b), $df = 0$ and d^2f keeps the same sign for all values of $(h,k) \neq 0,0$,

563

$F(x,y,z)$ is a function subject to the constrain condition $G(x,y,z) = 0$. Then at the stationary points $F_x G_y - F_y G_x = 0$

If $f(u)$ be a function of u and u be a function of x and y then

$$\frac{\partial f}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{df}{du} \frac{\partial u}{\partial y}$$

$$f_{xy}(a,b) = \frac{\partial^2}{\partial x \partial y} f(x,y) \Big|_{(a,b)}$$

$$= \lim_{h \to 0} \lim_{k \to 0} \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b)}{hk}$$