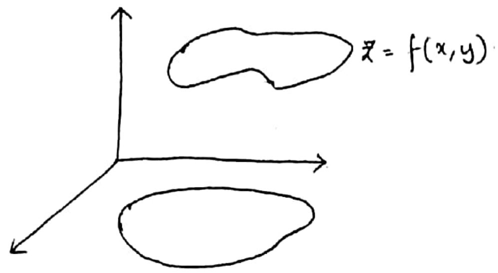
# Several Variable Calculus :-

Let $z = f(x, y)$ be a function defined on $D \subseteq \mathbb{R}^2$.




Let $(a, b) \in D$.

The a $\delta$ neighbourhood of

$$N_\delta(a,b) = \{ x, y : \sqrt{(x-a)^2 + (y-b)^2} < \delta \}$$

$$\text{or } = \{ x, y : |x-a| < \delta,\ |y-b| < \delta \}.$$

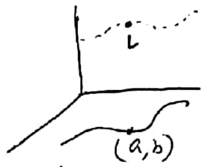
**Limit** :- A function $f(x,y)$ tends to a limit '$l$' $\in \mathbb{R}$, when $(x,y)$ tends to $(a,b)$, if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x,y) - l| < \epsilon \quad \text{, whenever } (x,y) \in N_\delta(a,b)$$

~~Theorem~~ Remark :- Let $y = \phi(x)$ such that $y = \phi(x) \to b$ as $x \to a$.

Then $y = \phi(x)$ is a path of approaching to $(a,b)$.

**Theorem** :- $\lim\limits_{(x,y)\to(a,b)} f(x,y) = l$, iff $\lim\limits_{x\to a} f(x, \phi(x)) = l$, for every $\phi(x) \to b$ as $x \to a$.




**Corrollary** :- If $y = \phi_1(x)$ & $y = \phi_2(x)$ be two curves approaching to $(a,b)$ as $x \to a$. (paths)

and $\lim\limits_{x\to a} f(x, \phi_1(x)) = l_1 \neq l_2 = \lim\limits_{x\to a} f(x, \phi_2(x))$.

Then we say that $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exists.

Ex. i) using defn, s.t. a) $\lim\limits_{(x,y)\to(a,b)} xy \cdot \dfrac{x^2-y^2}{x^2+y^2} = 0$.

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x \cdot y^2}{x^2+y^2} = 0$. c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0$.

Soln: a) Let $\epsilon > 0$ be an arbitrary quantity.,

then $|f(x,y) - l| = \left| xy \cdot \left(\dfrac{x^2-y^2}{x^2+y^2}\right)\right| \leq |x||y| < \epsilon$.

whenever $|x - 0| < \sqrt{\epsilon} = \delta$ $\quad |y - 0| < \sqrt{\epsilon} = \delta$.

Hence by defn., $\lim\limits_{(x,y)\to(0,0)} xy \cdot \dfrac{x^2-y^2}{x^2+y^2} = 0$

$\mathcal{C}$) Let $\epsilon > 0$ be an arbitrary no.

Then $\left|f(x,y) - l\right| = \left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| = \left|\dfrac{r^2 \sin\theta \cdot \cos\theta}{r}\right| = \dfrac{r}{2}|\sin 2\theta|$

$$\leq \frac{r}{2} = \frac{\sqrt{x^2+y^2}}{2} < \epsilon.$$

whenever $(x-0)^2 + (y-0)^2 < 4\epsilon^2 - \delta$

By defn, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0.$

Ex (2) : S.T the limit $(x,y) \to (0,0)$ does not exist in each case.

i) $\lim \dfrac{2xy}{x^2+y^2}$ ii) $\lim \dfrac{xy^3}{x^2+y^6}$ iii) $\lim \dfrac{x^3+y^3}{x-y}$.

$[x-y = mx^3]$.

Soln. i) Here $y = mx \to 0$ as $x \to 0$.

$y = mx$ is a ~~a~~ path approaching to $(0,0)$.

Then $\lim\limits_{x\to 0} f(x, mx) = \lim\limits_{x\to 0} \dfrac{2x \cdot mx}{x^2 + m^2x^2} = \lim\limits_{x\to 0} \dfrac{2m}{1+m^2}$ which depends on $m$.

Hence the lt does not exist as they are path-dependent.

ii) Here $y^3 = mx \to 0$ as $x \to 0$.

$y^3 = mx$ is a path approaching to $(0,0)$.

Then $\lim\limits_{x\to 0} f(x, (mx)^{1/3}) = \lim\limits_{x\to 0} \dfrac{x \cdot mx}{x^2 + m^2x^2} = \dfrac{m}{1+m^2}$ which depends on $m$.

Hence the lt. does not exist as they are dependent on path.

iii) $x - y = mx^3$. $\lim\limits_{x\to 0} \dfrac{x^3 + (x - mx^3)^3}{mx^3} = \lim\limits_{x\to 0} \dfrac{1 + (1-mx^2)^3}{m} = \dfrac{2}{m}$ which depends on $m$.

Hence the lt. does not exist as they are path-dependent.

Ex 3 Show that a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2} = 0$.

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{1+x^2y^2} - 1}{x^2+y^2} = 0$. c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y^3}{x^2+y^2} = 0$.

Soln: a) Let $\epsilon > 0$ be an arbitrary quantity

Then $\left|f(x,y) - l\right| = \left|\dfrac{xy^2}{x^2+y^2}\right| \leq |x| < \epsilon \quad \left[\because \left|\dfrac{y^2}{x^2+y^2}\right| \leq 1\right]$

whenever $|x-0| < \delta = \epsilon$, $|y-0| < \delta = \epsilon$

$(x,y)\to(0,0) \quad x^2+y^2$

c/ Let $\epsilon > 0$ be an arbitary quantity.

Then $\left|f(x,y) - l\right| = \left|\dfrac{r^6 \sin^3\theta \cdot \cos^3\theta}{r^2}\right| = \left|\dfrac{r^4}{8}\right| \left|\sin^3 2\theta\right| \leq \dfrac{r^4}{8}$

$= \dfrac{(x^2+y^2)^2}{8} < \epsilon$

whenever $(x-0)^2 + (y-0)^2 < \delta = 2\sqrt{2\epsilon}$.

b/ $\dfrac{\sqrt{1+x^2y^2}-1}{x^2+y^2} = \dfrac{(1+x^2y^2)^{1/2}-1}{x^2+y^2} \simeq \dfrac{\frac{1}{2}x^2y^2}{x^2+y^2}$ [for $x \simeq 0, y \simeq 0$].

$= f(x,y) \simeq \dfrac{1}{2} r^4 \dfrac{\sin^2\theta \cdot \cos^2\theta}{r^2} = \dfrac{r^2 \sin^2 2\theta}{8} \leq \dfrac{r^2}{8} = \dfrac{x^2+y^2}{8}$

$\left|f(x,y) - 0\right| \leq \dfrac{x^2+y^2}{8} < \epsilon$

whenever $x^2+y^2 < 8\epsilon = \delta$.

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{1+x^2y^2}-1}{x^2+y^2} = 0.$$

JAM - 2008.

Let $f(x,y) = \begin{cases} \dfrac{x(x^2-y^2)}{x^2+y^2} & (x,y) \neq 0 \\ 0 & (x,y) = (0,0) \end{cases}$

Examine the continuity at $(0,0)$.

Soln: $\left|f(x,y) - l\right| = \left|x \dfrac{x^2-y^2}{x^2+y^2}\right| \leq |x| < \epsilon$ $\left[\because \left|\dfrac{x^2-y^2}{x^2+y^2}\right| \leq 1\right]$

whenever $|x-0| < \delta = \epsilon$, $|y-0| < \delta = \epsilon$.

Hence by defn., $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

JAM 2012. Find $\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} \sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right)$.

Hint:- $\left|f(x,y) - 0\right| \leq \sqrt{x^2+y^2} < \epsilon$. whenever $(x-0)^2 + (y-0)^2 < \epsilon^2 = \delta$.

① ② $\lim_{y\to 0}\left\{\lim_{x\to 0}\left(y\sin 1/x + \frac{xy}{x^2+y^2}\right)\right\}$ does not exist. $\left[\because \lim_{x\to 0}\sin 1/x \text{ doesn't exist}\right]$

For $y = mx \to 0$ as $x \to 0$.

$$\lim_{x\to 0} f(x, mx) = \lim_{x\to 0}\left[mx\sin 1/x + \frac{mx^2}{x^2+m^2x^2}\right]$$

$$= m\cdot 0 + \frac{m}{1+m^2} = \frac{m}{1+m^2} \text{ depends on } m$$

$[\because |x\sin 1/x| \le |x| < \epsilon$ whenever $|x-0| < \epsilon = \delta]$

Ex 4. 1. S.T. the repeated limits exist but the double limit doesn't when $(x,y)\to(0,0)$

$$f_1(x,y) = \frac{x^2y^2}{x^4+y^4} - x^2y^2$$

$$f_2(x,y) = \frac{x^3+y^3}{x-y} \quad \text{when } x \ne y$$

$$= 0 \quad \text{when } x = 0.$$

Soln. $\lim_{x\to 0}\left(\lim_{y\to 0} f_1(x,y)\right) = 0-0 = 0.$ $\lim_{y\to 0}\left(\lim_{x\to 0} f_1(x,y)\right) = 0-0 = 0.$

For $y^2 = mx^2$, $y = \pm\sqrt{m}\,x \to 0$ as $x\to 0$.

$$\therefore \lim_{x\to 0} f_1(x, \sqrt{m}\,x) = \lim_{x\to 0}\left[\frac{mx^4}{x^4+m^2x^4} - mx^4\right] = \frac{m}{1+m^2} \text{ depends on } m.$$

Thus the double limit doesn't exist.

Again,

$\lim_{x\to 0}\left(\lim_{y\to 0} f_2(x,y)\right) = \lim_{x\to 0} x^2 = 0.$ $\lim_{y\to 0}\left(\lim_{x\to 0} f_2(x,y)\right) = \lim_{y\to 0} -y^2 = 0$

Let $x - y = mx^3$ $\therefore y = x \mp mx^3 \to 0$ as $x\to 0$.

$$\therefore \lim_{x\to 0} f_2(x, x-mx^3) = \lim_{x\to 0}\frac{x^3+(x\mp mx^3)^3}{mx^3} = \lim_{x\to 0}\frac{1+(1\mp mx^2)^3}{m}$$

$$= \frac{2}{m} \text{ depends on } m.$$

Thus the double limit doesn't exist.

Let $f(x,y)$ be a continuous function on $D$. Then for $(a,b) \in D$

$$f(a+h, b+k) = f(a,b) + \{h f_x(a,b) + k f_y(a,b)\} + \frac{1}{2!}\{h^2 f_{xx}(a^*,b^*) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a^*,b^*)\}$$

~~where $a^*, b^*$~~

Infact $f(a+h, b+k) = e^{\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)} f(a,b)$

Differentiability of Functions of two variables :-

Let $f(x,y)$ be a function of two variables $(x,y)$ and $f(x,y)$ be defined on $D \subseteq \mathbb{R}^2$.

Now, $f(x,y)$ is said to be differentiable at $(a,b) \in$, if

$$f(a+h, b+k) = f(a,b) + \{h f_x(a,b) + k f_y(a,b)\} + \frac{1}{2!}\{h^2 f_{xx}(a^*,b^*) + 2hk f_{xy}(a^*,b^*) + k^2 f_{yy}(a^*,b^*)\};$$

$$f(a+h, b+k) - f(a,b) = h\cdot A + k\cdot B + h\,\phi(h,k) + k\,\psi(h,k).$$

where $A = f_x(a,b)$ $B = f_y(a,b)$ & $\phi(h,k) \to 0$, $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$

$$h\cdot\left\{\frac{h}{2!} f_{xx} + k f_{xy}\right\}$$

**Theorem** :- Every differentiable function must be continuous.

Hint :- $\lim_{(h,k)\to(0,0)} \{f(a+h, b+k) - f(a,b)\} = 0$.

$\therefore \lim_{(h,k)\to(0,0)} f(a+h, b+k) = f(a,b)$

Ex/ Examine the differentiability of the following function at $(0,0)$ :-

i) $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

ii) $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Soln: i) $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = 0$ ; $f_y(0,0) = 0$.

Now, $f(0+h, 0+k) - f(0,0)$.

$$= \frac{hk}{\sqrt{h^2+k^2}}$$

$$= h\cdot 0 + k\cdot 0 + h\cdot\frac{k}{2\sqrt{h^2+k^2}} + k\cdot\frac{h}{2\sqrt{h^2+k^2}}$$

$$= h\cdot f_x(0,0) + h\cdot f_y(0,0) + h\cdot\phi(h,k) + k\,\psi(h,k).$$

$$\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \frac{k}{2\sqrt{h^2+k^2}}$$

and

$$\lim_{(h,k)\to(0,0)} \psi(h,k) = \lim_{(h,k)\to(0,0)} \frac{h}{2\sqrt{h^2+k^2}}.$$

[For $k = mh$, $\frac{k}{2\sqrt{h^2+k^2}} = \frac{m}{2\sqrt{1+m^2}}$, depends on m].

Hence $f(x,y)$ is not differentiable at $(0,0)$.

(ii)
$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 1$$
$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = -1.$$

Now, $f(0+h, 0+k) - f(0,0) = \frac{h^3-k^3}{h^2+k^2} = \frac{(h-k)(h^2+k^2+hk)}{(h^2+k^2)}$ $\left(1+\frac{hk}{h^2+k^2}\right)$

$$= h - k + h\cdot\frac{hk}{h^2+k^2} - k\cdot\frac{hk}{h^2+k^2}.$$

$$= h\cdot 1 + k\cdot(-1) + h\cdot\left(\frac{hk}{h^2+k^2}\right) + k\left(\frac{-hk}{h^2+k^2}\right)$$

$$= h f_x(0,0) + k f_y(0,0) + h\,\phi(h,k) + k\,\psi(h,k).$$

$$\lim_{h,k\to(0,0)} \phi(h,k) = \lim_{h,k\to(0,0)} \frac{hk}{h^2+k^2} \left[= \frac{m}{1+m^2} \text{ if } k = mh, \text{ depends on } m\right].$$

$= h f_x(0,0) + h f_y(0,0) + h\cdot\phi(h,k) + k\,\psi(h,k)$.

$$\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \frac{-k^2}{h^2+k^2} = -\frac{1}{1+m}. \text{ where } h^2 = mk^2$$
$$\lim_{(h,k)\to(0,0)} \psi(h,k) = \lim_{(h,k)\to(0,0)} -\frac{hk}{h^2+k^2} = \frac{-m}{1+m^2} \text{ where } h = mk$$
$\Big\}$ depends on m.

Hence $f(x,y)$ is not differentiable at $(0,0)$ but it is continuous [previous].

Ex3 Let $f(x,y) = \begin{cases} x\cdot\frac{x^2-y^2}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

i) Find the 1st order partial deriavative at $(0,0)$ if they exist

ii) Check the continuity & differentiability of $f$ at $(0,0)$.

Soln
$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0.$$

$$f(0+h, 0+k) = h\cdot\frac{h^2-k^2}{h^2+k^2} = h\left\{\frac{(h^2+k^2)-2k^2}{h^2+k^2}\right\} = h - \frac{2hk^2}{h^2+k^2}$$

$$= h\cdot f_x(0,0) + k f_y(0,0) + h\left(\frac{-k^2}{h^2+k^2}\right) + k\left(\frac{-hk}{h^2+k^2}\right).$$

Let $z = f(x,y)$ be a func. of two variable defined on $D \subseteq \mathbb{R}^2$

Absolute Maximum/ Minimum :—

$f(x,y)$ has an absolute max | min. at $(a,b) \in D$ if $f(x,y) \leq f(a,b)$ or $f(x,y) \geq f(a,b)$, for all $(x,y) \in D$.

local maximum| minimum :—

$f(x,y)$ has a local maximum or minimum at $(a,b) \in D$. if $f(x,y) \leq f(a,b)$ or $f(x,y) \geq f(a,b)$, for all $(x,y) \in N_\delta(a,b)$

$= \{(x,y) : (x-a)^2 + (y-b)^2 < \delta^2\}$ for some $\delta > 0$.

A necessary condition for Extremum :—

For a partial differentiable function $f(x,y)$, a necessary condition for extremum at $(a,b)$ is that $f_x(a,b) = 0 = f_y(a,b)$.

Note that $f(x,y) = |x| + |y|$ has a minimum at $(0,0)$ but $f_x(0,0)$ & $f_y(0,0)$ donot exist.

A sufficient condition for Extremum :—

Let $f(x,y)$ be a function of two variables defined on $D$ and let the 2nd order partial derivative of $f$ be continuous on $D$.

Let $(a,b) \in D$ and $f_x(a,b) = f_y(a,b) = 0$.

Then (i) $f(a,b)$ is a local maximum if $f_{xx}(a,b) < 0$ and $\{f_{xx}(a,b) f_{yy}(a,b) - f_{xy}^2(a,b)\} > 0$.

ii) $f(a,b)$ is a local minimum if $f_{xx}(a,b) > 0$ and $\{f_{xx}(a,b) f_{yy}(a,b) - f_{xy}^2(a,b)\} > 0$.

iii) $f(a,b)$ is neither a maximum nor a minimum if $\{f_{xx}(a,b) f_{yy}(a,b) - f_{xy}^2(a,b)\} < 0$.

iv) The test fails to give a conclusion if $f_{xx}(a,b) f_{yy}(a,b) - f_{xy}^2(a,b) = 0$. ~~a $f_{xx}(a,b) = 0$~~.

Proof: Bivariate Taylor's theorem:

$$f(a+h, b+k) = f(a,b) + \{h f_x(a,b) + k f_y(a,b)\} + \frac{1}{2!}\left[h^2 f_{xx}(a^*,b^*) + 2hk f_{xy}(a^*,b^*) + k^2 f_{yy}(a^*,b^*)\right]$$

Here $f_x(a,b) = 0 = f_y(a,b)$.

$$f(a+h, b+k) - f(a,b) = \frac{1}{2!}\left[ f_{xx}(a^*,b^*)\left\{h + k\cdot\frac{f_{xy}(a^*,b^*)}{f_{xx}(a^*,b^*)}\right\}^2 + \left\{ f_{yy}(a^*,b^*) - \frac{f_{xy}^2(a^*b^*)}{f_{xx}(a^*b^*)}\right\}k^2\right]$$

$$\left[ = \frac{1}{2!}\left\{\frac{f_{xx}(a^*,b^*)f_{yy}(a^*,b^*) - f_{xy}^2(a^*,b^*)}{f_{xx}(a^*,b^*)}\right\}\cdot k^2 \quad \text{set } h = -k\frac{f_{xy}(a^*,b^*)}{f_{xx}(a^*,b^*)}\right]$$

$$\begin{cases} \leq 0 & \text{if } f_{xx}(a,b) < 0 \text{ and } \left(f_{xx}f_{yy} - f_{xy}^2\right)_{(a,b)} > 0 \\ \geq 0 & \text{if } f_{xx}(a,b) > 0 \text{ and } \left(f_{xx}f_{yy} - f_{xy}^2\right)_{(a,b)} > 0 \end{cases}$$

1/ Find the local rel. max. & min. of the func. $f(x,y) = 2x^2 - xy + 2y^2 - 20x$

Soln. $f(x,y) = 2x^2 - xy + 2y^2 - 20x$.

For stationary points,

$$\left.\begin{aligned} 0 = f_x &= 4x - y - 20 \\ 0 = f_y &= -x + 4y \end{aligned}\right\} \Rightarrow x = 4y, \quad y = \frac{20}{15} = 4/3, \quad x = 16/3.$$

$f_{xx} = 4 \quad f_{xy} = -1 \quad f_{yy} = 4$.

At $(x,y) = (16/3, 4/3)$ $\quad f_{xx} = 4 > 0 \quad f_{xx}f_{yy} - f_{xy}^2 = 16 - 1 = 15 > 0$.

Hence, $f(16/3, 4/3)$ is the unique local minimum.

$\Rightarrow f(16/3, 4/3)$ is the least value.

2/ Show that $f(x,y) = (y-x)^4 + (x-2)^4$ has a minimum at $(2,2)$.

Soln, $f(x,y) = (y-x)^4 + (x-2)^4 \geq 0$. $\quad f(x,y) = 0$ iff $(y-x)^4 = 0 = (x-2)^4$. iff $(x,y) = (2,2)$

Hence $f(x,y) \geq 0 = f(2,2) \; \forall (x,y)$

3/ Show that $f(x,y) = y^2 + x^2y + x^4$ has a minimum at $(0,0)$. whereas $f_{xx}f_{yy} - f_{xy}^2 = 0$.

Soln, $f_x = 2xy + 4x^3$. $\quad f_y = 2y + x^2$. $\quad f_{xx} = 2y + 12x^2$

$f_{yy} = 2 \quad f_{xy} = 2x$.

Ex 4/

Soln.

Hint

Hint

Ex 5/

Ex 6/

Ex 7/

Soln 7

and $f_{xx} f_{yy} - f_{xy}^2 = 0 \cdot 2 - 0^2 = 0.$

The test fails to give a conclusion.

Now,

$$f(x,y) - f(0,0) = y^2 + x^2 y + x^4 = \left(y + \frac{x^2}{2}\right)^2 + \frac{3x^4}{4} \geq 0 \;\forall (x,y).$$

'=' holds iff $\left(y + \frac{x^2}{2}\right)^2 = 0 = \frac{3x^4}{4}$ iff $(x,y) = (0,0)$

Hence $f(0,0)$ is a minimum.

Ex. 4. Show that $f(x,y) = 2x^4 - 3x^2 y + y^2$ has neither a maximum nor a minimum at $(0,0)$ whereas $f_{xx} f_{yy} - f_{xy}^2 = 0$.

Soln.

$$f_x = 8x^3 - 6xy \qquad f_y = -3x^2 + 2y. \qquad f_{xy} = -6x.$$

$$f_{xx} = 24x^2 - 6y \qquad f_{yy} = 2.$$

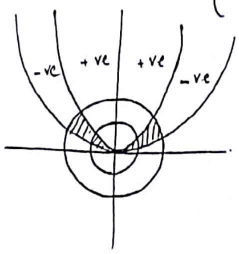
Hint

$$f(x,y) - f(0,0) = 2x^4 - 3x^2 y + y^2.$$

$$= (y - x^2)(y - 2x^2)$$

$$= \begin{cases} < 0, & x^2 < y < 2x^2 \\ > 0, & y < x^2 \text{ or } y > 2x^2. \end{cases}$$

~~Hint~~




Clearly, the difference can take -ve values as well as +ve values in every neighbourhood of $(0,0)$.
Hence $f(0,0)$ is neither a maximum or a minimum.

Ex. 5 Find all the maxima & minima of the func. given by ~~f(x,y)~~

$$f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$$

Ex. 6 P.T the function f where $f(x,y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^5$ has neither minimum nor a maximum at the origin.

Ex. 7 Given $n$ points $(x_i, y_i)$ $i = 1(1)n$ where all the $x_i$'s are not equal, find $a$ & $b$ such that $\sum_{i=1}^{n} \{(y_i - a) - bx_i\}^2$ is minimum.

Soln 7

$$f(a,b) = \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

Note that $f_a = \sum_{i=1}^{n} 2(y_i - a - bx_i)(-1)$

$$f_b = \sum_{i=1}^{n} 2(y_i - a - bx_i)(-x_i)$$

For stationary points $f_a = 0$ $f_b = 0$.

$$\begin{cases} b = \dfrac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \hat{b} \\ a = \bar{y} - \hat{b}\bar{x} = \hat{a} \end{cases}$$

$f_{aa} = 2n$ $\quad f_{bb} = 2\sum_{i=1}^{n} x_i^2$ $\quad f_{ab} = 2\sum_{i=1}^{n} x_i$

Here $f_{bb}$ or $f_{aa} > 0$ & $f_{aa} f_{bb} - f_{ab}^2 = 4n\sum x_i^2 - 4(\sum x_i)^2$

$$= 4n\{\sum x_i^2 - n\bar{x}^2\}$$

$$= 4n\sum (x_i - \bar{x})^2 > 0.$$

Hence $f(\hat{a}, \hat{b})$ is the least value.

Ex (JAM 2014).

The func. $f(x,y) = 3(x^2+y^2) - 2(x^3-y^3) + 6xy$ has.

a) a pt of max. b) pt of min. c) a saddle point. (neither max nor min).

d) no saddle point.

A stationary pt where $f$ has a maximum nor a minimum is called a <u>saddle point</u>.

Ex (ISI)

The minimum value of

$f(x,y,z) = 4x^2 + 9y^2 + z^2 - 12x - 12y + 14$

a) 1 b) 3 c) 14 d) None of these.

Soln.

$f(x,y,z) = (2x-3)^2 + (3y-2)^2 + z^2 + 1 \geq 1.$

'=' holds iff $(x,y,z) = (3/2, 2/3, 0)$.

Soln (JAM 2014) $f(x,y) = 3(x^2+y^2) - 2(x^3-y^3) + 6xy$.

$\therefore$ $f_x = 6x - 6x^2 + 6y = 0$ —(*) $\quad f_y = 6y + 6y^2 + 6x$. —(**)

$f_{xx} = 6 - 12x$ $\quad f_{yy} = 6 + 12y$.

$f_{xy} = 6$.

(**)−(*) we get, $x^2 + y^2 = 0 \Rightarrow x = 0, y = 0$.

$\therefore f_{xx} = 6$ $\quad f_{yy} = 6$ $\quad f_{xy} = 6$ $\quad \therefore f_{xx} f_{yy} - f_{xy}^2 = 0$.

$\therefore$ It has no saddle pts.

Supp. In th $(x,y)$ $z = f($

Assum visual $f(x,$ $f(x,$

At

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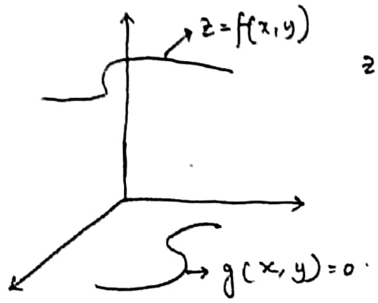
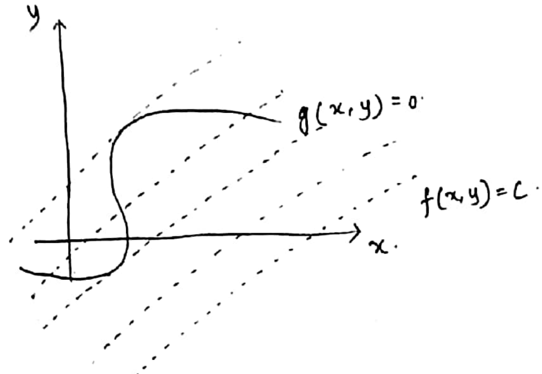
x

[Maximization ~~of f~~ or minimization of functions of several variables subject to some restriction].




$z = f(x,y)$ subject to $g(x,y)=0$

Suppose we wish to maximize/minimize $z = f(x,y)$ subject to $g(x,y)=0$
In the previous part, we have maximized/minimized $z = f(x,y)$ where $(x,y)$ is unrestricted. But here we have to maximize or minimize $z = f(x,y)$ subject to a restriction $g(x,y)=0$.




Assume that $f$ and $g$ have 1st order partial derivative. We can visualize $g(x,y)=0$ as a curve along with the level curves:
$f(x,y)=c$, increasing $c$

Intuitively, the local maxima/minima occur at the points where $f(x,y)=c$ touches $g(x,y)=0$.

At the touching pt, from $g(x,y)=0$, $\frac{dy}{dx} = -\frac{g_x}{g_y}$

from $f(x,y)=c$, $\frac{dy}{dx} = -\frac{f_x}{f_y}$.

$0 = \Delta g = \frac{\Delta g}{\Delta x}\cdot\Delta x + \frac{\Delta g}{\Delta y}\cdot\Delta y$

Taking limit

$0 = \frac{\partial g}{\partial x}\cdot dx + \frac{\partial g}{\partial y} dy$

At the stationary point,

$$-\frac{f_x}{f_y} = \frac{dy}{dx} = -\frac{g_x}{g_y} \Rightarrow \frac{f_x}{g_x} = \frac{f_y}{g_y} = -\lambda \text{ (say)}.$$

Then $(f_x, f_y) = -\lambda(g_x, g_y)$.

$$\Rightarrow (f_x + \lambda g_x,\ f_y + \lambda g_x) = (0,0).$$

Define $F(x,y) := f(x,y) + \lambda g(x,y)$.

Then the stationary points are the solutions of.

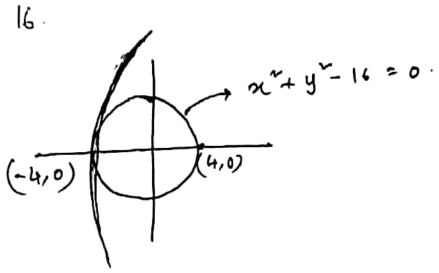
$$* \begin{cases} \frac{\partial F}{\partial x} = f_x + \lambda g_x = 0 \; ; \; \frac{\partial F}{\partial y} = f_y + \lambda g_y = 0 \\ \text{and} \quad g(x,y) = 0 \end{cases}$$

The local maxima / minima are obtained at the solutions of the (*) The unknown multiplier is known as Lagrange's multiplier.

If $g(x,y) = 0$ is a closed and bounded curve, then the absolute maximum / minimum are obtained at the solutions.

Ex 1. Find the maximum / minima of $z = (x-4)^2 + y^2$ subject to the restriction $x^2+y^2 = 16$.

Soln. Method I.




Clearly the level curves : $(x-4)^2+y^2 = c$ touches the $g(x,y) = x^2+y^2-4^2 = 0$ at $(\pm 4, 0)$.

{ As $g(x,y) = x^2+y^2-4^2 = 0$ is a closed & bounded curve, absolute maxima/minimum are among the soln.

Note that $z_{(-4,0)} = 64$ (absolute maxima) $\quad z_{(4,0)} = 0$ (absolute minima). }

Method II.

Define $F(x) = f(x,y) + \lambda g(x,y) = (x-4)^2 + y^2 + \lambda(x^2+y^2-16)$

For stationary points,

$$\frac{\partial F}{\partial x} = 2(x-4) + 2\lambda x = 0 \qquad (1+\lambda)x = 4.$$

$$\frac{\partial F}{\partial y} = 2y + 2\lambda y = 0 \qquad (1+\lambda)y = 0.$$

$$g(x,y) = x^2+y^2 = 16$$

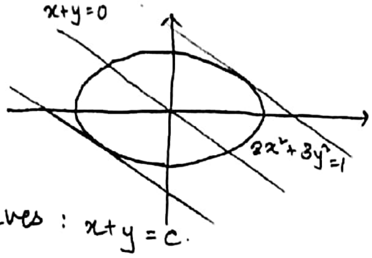
Here $x = 4/1+\lambda$, hence $1+\lambda \neq 0$ and then $y = 0$

$\therefore 16 = x^2 + 0^2 \Rightarrow x = \pm 4$.

Hence $(x,y) = (\pm 4, 0)$ are the pts of maxima & minima.

{ }

Soln:




To find the maximum of $z = x+y$ subject to $2x^2+3y^2 \le 1$

The pt. of max/min are the pts where the level curves touches : $x+y=c$ touches $2x^2+3y^2=1$.

level curves : $x+y=c$.

$$4x + 6y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x}{3y}$$

By problem,

$$\frac{dy}{dx} = -1. \quad -\frac{2x}{3y} = -1 \Rightarrow \frac{x}{3} = \frac{y}{2} = \lambda. \Rightarrow \begin{array}{l} x = 3\lambda \\ y = 2\lambda \end{array}$$

$$\therefore 18\lambda^2 + 12\lambda^2 = 1 \Rightarrow 30\lambda^2 = 1 \Rightarrow \lambda = \pm 1/\sqrt{30}.$$

$$\therefore (x,y) = \pm 1/\sqrt{30}\,(3,2)$$

$$\therefore \text{Max}(x+y) = 5/\sqrt{30} \qquad \text{Min}(x+y) = -5/\sqrt{30}$$

Alt:

$$F(x,y) = x+y+\lambda(2x^2+3y^2-1).$$

For pts of extremum,

$$0 = \frac{\partial F}{\partial x} = 1 + 4\lambda x \qquad 0 = \frac{\partial F}{\partial y} = 1 + 6\lambda y$$

and $2x^2+3y^2 = 1$.

$$\Rightarrow (x,y) = \left(-1/4\lambda, -1/6\lambda\right)$$

$$\therefore 1 = 2\cdot\frac{1}{16\lambda^2} + 3\cdot\frac{1}{36\lambda^2} = \left(1/8 + 1/12\right)1/\lambda^2 \Rightarrow 1/\lambda^2 = \frac{24}{5}.$$

$$\Rightarrow 1/\lambda = \pm\frac{2\sqrt{6}}{\sqrt{5}} = \pm\frac{12}{\sqrt{30}}.$$

$$\therefore (x,y) = \pm\frac{12}{\sqrt{30}}\left(1/4, 1/6\right) = \pm\left(3/\sqrt{10}, 2/\sqrt{10}\right).$$

$$\therefore \text{Max}(x+y) = 5/\sqrt{30}.$$

Ex. 3 If $ax^2+by^2 = ab$ S.T. the maximum & minimum value of $(x^2+y^2+xy)$ will be the values of $u$, given by the eqn. $4(u-a)(u-b) = ab$.

Soln:

$$F(x,y) = x^2+y^2+xy+\lambda(ax^2+by^2-ab).$$

For pts of extremum;

$$0 = \frac{\partial F}{\partial x} = 2x + y + 2\lambda a x \text{ — (i)} \; ; \; 0 = \frac{\partial F}{\partial y} = 2y + x + 2\lambda b y. \text{ — (ii)}$$

and $ax^2+by^2 = ab$. — (iii)

(1) $\Rightarrow (1+\lambda a)2x + y = 0$  (2) $\Rightarrow (1+\lambda b)2y + x = 0$.

and $ax^2+by^2 = ab$.

Note that, $(0,0)$ cannot be a pt. of extremum.

For non-trivial soln of ①, ⓘⓘ, ⓘⓘⓘ ;

$$\begin{vmatrix} 2(1+\lambda a) & 1 \\ 1 & 2(1+\lambda b) \end{vmatrix} = 0 \quad \Rightarrow 4(1+\lambda a)(1+\lambda b) - 1 = 0 \longrightarrow (*)$$

$① \times x + ⓘⓘ\, y \Rightarrow$

$$2(x^2+xy+y^2) + 2\lambda(ax^2+by^2) = 0$$

[The values of $(x^2+y^2+xy)$ at solns. of ① & ⓘⓘ are the max/min value of $u$].

$$\Rightarrow \quad 2u + 2\lambda ab = 0$$

$$\Rightarrow \quad \lambda = -\frac{u}{ab}.$$

Putting $\lambda$ in $(*)$.

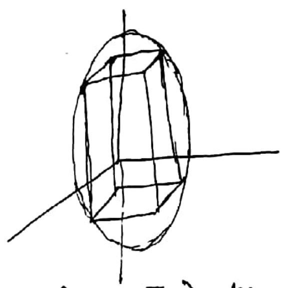
$$4\left(1-\frac{u}{b}\right)\left(1-\frac{u}{a}\right) = 1.$$

$$\Rightarrow 4(u-a)(u-b) = ab.$$

Ex 4/ Prove that the vol. of the greatest ~~reg~~ rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ is $\frac{8abc}{3\sqrt{3}}$.

Ex. 5/ A rectangular box. opened at the top is to have a vol. of 32 cubic feet. what must be the dimension so that the total surface area is min.?

Soln. 4.




Let $P(x, y, z)$ be a point on the ellipsoid. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

Then vol. of the parallelopiped. is $V = 2x \cdot 2y \cdot 2z = 8xyz$.

To maximize $V = 8xyz$ subject to $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$.

Define $F(x,y,z) = 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} - 1\right)$.

$$0 = \frac{\partial F}{\partial x} = 8yz + \frac{2\lambda x}{a^2} \quad — ⓘ \qquad 0 = \frac{\partial F}{\partial y} = 8xz + \frac{2\lambda y}{b^2} \quad — ⓘⓘ$$

$$0 = \frac{\partial F}{\partial z} = 8xy + \frac{2\lambda z}{c^2} \quad — ⓘⓘⓘ \quad \& \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1.$$

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$$\text{(i)} \Rightarrow \frac{V}{x} + \frac{2\lambda x}{a^2} = 0 \qquad \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = -\frac{V}{2\lambda}$$

$$\text{(ii)} \Rightarrow \frac{V}{y} + \frac{2\lambda y}{b^2} = 0 \qquad \text{and } 1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3\left(-\frac{V}{2\lambda}\right).$$

$$\text{(iii)} \Rightarrow \frac{V}{z} + \frac{2\lambda z}{c^2} = 0. \qquad \Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = 1/3 \quad \Rightarrow (x,y,z) = \left(\frac{a}{\sqrt3}, \frac{b}{\sqrt3}, \frac{c}{\sqrt3}\right)$$

$$\therefore V\left(a/\sqrt3, b/\sqrt3, c/\sqrt3\right) = \frac{8abc}{3\sqrt3}.$$

Note that AM $\geq$ GM.

$$\Rightarrow \frac{1}{3} = \frac{x^2/a^2 + y^2/b^2 + z^2/c^2}{3} \geq \sqrt[3]{\frac{x^2y^2z^2}{a^2b^2c^2}} = \left(\frac{xyz}{abc}\right)^{2/3}.$$

$$\therefore V = 8xyz \leq \frac{8\,abc}{3\sqrt3} = V\left(a/\sqrt3, b/\sqrt3, c/\sqrt3\right).$$

<u>Soln 5.</u> Let $a, b, c \to$ length, breadth & height.

$$V = abc = 32.$$

$$S = ab + 2bc + 2ac$$

AM $\geq$ GM

$$\frac{ab + 2bc + 2ac}{3} \geq \left[4a^2b^2c^2\right]^{1/3} = \left[8\cdot4\cdot8\cdot4\cdot4\right]^{1/3} = 4\cdot4 = 16.$$

$$ab + 2bc + 2ac \geq 48$$

<u>Ex. 6</u> Find the maximum & minimum value of $\sum_{i=1}^{n} c_i x_i$ subject to $\sum_{i=1}^{n} x_i^2 = 1$.

<u>Soln.</u>
$$F(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i + \lambda\left(\sum_{i=1}^{n} x_i^2 - 1\right).$$

For stationary points

$$0 = \frac{\partial F}{\partial x_i} = c_i + 2\lambda x_i, \quad i = 1(1)n \quad \& \quad \sum_{i=1}^{n} x_i^2 = 1.$$

$$\Rightarrow x_i = -\frac{c_i}{2\lambda} \quad i = 1(1)n. \quad \& \quad 1 = \sum_{i=1}^{n} x_i^2 = \frac{1}{4\lambda^2}\sum_{i=1}^{n} c_i^2$$

$$\therefore \frac{1}{2\lambda} = \pm\frac{1}{\sqrt{\sum_{i=1}^{n} c_i^2}} \quad \Rightarrow x_i = \frac{c_i}{\sqrt{\sum_{i=1}^{n} c_i^2}} \text{ or } -\frac{c_i}{\sqrt{\sum_{i=1}^{n} c_i^2}}, \quad i = 1(1)n.$$

$$\text{At } (x_1, \ldots, x_n) = \frac{1}{\sqrt{\sum c_i^2}}(c_1, \ldots, c_n)$$

$$, \sum_{i=1}^{n} c_i x_i = \sqrt{\sum c_i^2} \quad (\max).$$

$$\text{similarly, } \sum_{i=1}^{n} c_i x_i = -\sqrt{\sum_{i=1}^{n} c_i^2} \quad (\min)$$

<u>Remark.</u> C-S inequality

$$\left(\sum_{i=1}^{n} c_i x_i\right)^2 \leq \left(\sum_{i=1}^{n} c_i^2\right)\left(\sum_{i=1}^{n} x_i^2\right) = \left(\sum_{i=1}^{n} c_i^2\right).$$

$$\Rightarrow -\sqrt{\sum c_i^2} \leq \sum c_i x_i \leq +\sqrt{\sum c_i^2}$$

'=' holds iff $x_i = \lambda c_i$ & $\sum x_i^2 = 1$, iff $x_i = \pm \frac{c_i}{\sqrt{\sum c_i^2}}$.

Ex.6 Find the max. value of $(x_1 x_2 \cdots x_n)^2$ subject to $\sum_{i=1}^{n} x_i^2 = 1$.

$F(x_1, \ldots, x_n) = 2\sum_{i=1}^{n} \log x_i + \lambda\left(\sum_{i=1}^{n} x_i^2 - 1\right)$.

$\therefore\ 0 = \frac{\partial F}{\partial x_i} = \frac{2}{x_i} + 2\lambda x_i = 0 \Rightarrow x_i^2 = -1/\lambda$

Again. $\sum x_i^2 = 1 \Rightarrow -n/\lambda = 1 \Rightarrow -1/\lambda = 1/n$  $\therefore\ x_i^2 = 1/n$.

$\therefore$ Max. value of $(x_1 \cdots x_n)^2 = (1/n)^n$.

Alt

$\frac{\sum x_i^2}{n} \geq (\pi x_i)^{2/n} \Rightarrow \left(\frac{1}{n}\right)^n \geq (\pi x_i)^2$

Ex.7 Let $x_1, x_2, \ldots, x_n$ be $n$ independent R.V.s with common mean $\mu$ & $Var(x_i) = \sigma_i^2$; $i = 1(1)n$

Define $T = \sum_{i=1}^{n} a_i x_i$

Find the minimum value of var (T) subject to the restriction $E(T) = \mu$.

$\mu = E(T) = E\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i E(x_i) = \left(\sum_{i=1}^{n} a_i\right)\mu$.

$\Rightarrow \sum_{i=1}^{n} a_i = 1$.

$var(T) = var\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} var(a_i x_i)$ $\left[cov(a_i x_i, a_j x_j) = 0 \text{ as } x_i\text{'s independent}\right]$.

$= \sum_{i=1}^{n} a_i^2 \sigma_i^2$

To minimize $\sum_{i=1}^{n} a_i^2 \sigma_i^2$ subject to $\sum_{i=1}^{n} a_i = 1$

$\sum a_i^2 \sigma_i^2 \geq (\sum \sigma_i)^2$.

Ex.8. Let $0 < x_i < 1$ & $\sum x_i = 1$ find the point of max. of

$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i(1 - x_i)$.

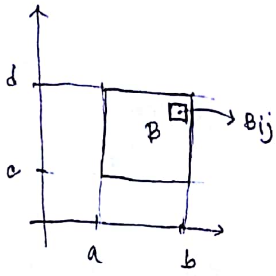
$\Rightarrow F(x_1, x_2, \ldots x_n) = \sum_{i=1}^{n} x_i(1 - x_i) + \lambda\left(\sum x_i - 1\right)$.

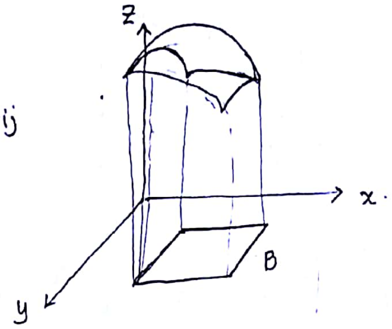
$0 = \frac{\partial F}{\partial x_i} = 1 - 2x_i + \lambda \Rightarrow \lambda = 2x_i - 1 \Rightarrow \frac{\lambda + 1}{2} = x_i$

Again, $\sum x_i = 1 \Rightarrow \sum \frac{\lambda+1}{2} = 1 \Rightarrow \lambda + 1 = \frac{2}{n}$.  $\therefore\ x_i = 1/n$.

$\therefore\ \sum_{i=1}^{n} x_i(1 - x_i) = \sum_{i=1}^{n} 1/n(1 - 1/n) = 1 - 1/n$.

Let $f(x,y)$ be a bounded function defined on $[a,b]\times[c,d]=B$



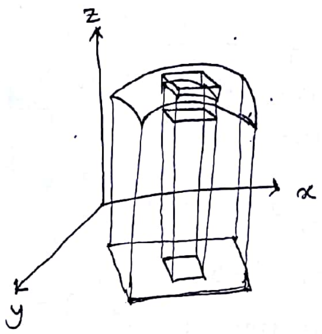

Consider the partition, $a=x_0<x_1<\cdots<x_{m-1}<x_m=b$.
$c=y_0<y_1<\cdots<y_{n-1}<y_n=d$.

Then $B_{ij}=\{(x,y): x_{i-1}<x\leq x_i,\ y_{j-1}<y\leq y_j\}$ $\begin{matrix} i=1(1)m \\ j=1(1)n \end{matrix}$.

Let $M_{ij}=\sup f(x,y)$ $(x,y)\in B_{ij}$
$m_{ij}=\inf f(x,y)$ $(x,y)\in B_{ij}$

Define $U(P,f)=\sum_{i=1}^{m}\sum_{j=1}^{n}M_{ij}\,a(B_{ij})$.

as the upper sum of $f(x,y)$ and
$L(P,f)=\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\,a(B_{ij})$. as the lower sum of $f(x,y)$.




$[L(P,f)\leq V\leq U(P,f)$ for $P]$.

$\Rightarrow \sup_P L(P,f)\leq V\leq \inf_P U(P,f)$

Define $\sup_P L(P,f)=\underline{\iint} f$, as the lower double integral of $f$ over $B$ and $\inf_P U(P,f)=\overline{\iint} f$, as the upper double integral of $f$ over $B$.

If $\underline{\iint} f$ and $\overline{\iint} f$ are equal, we say that $f$ is double integrable over $B$.

<u>Method of evaluation of Double integral</u>:—

<u>Repeated or Iterated integrals</u>:—

$f(x,y)$ is a bounded function on $B=[a,b]\times[c,d]$ For fixed $x\in[a,b]$ $f(x,y)$ can be treated as a function of $y\in[c,d]$ and if $f(x,y)$ is integrable on $[c,d]$, then

$\int_c^d f(x,y)\cdot dy$ gives a unique value for each $x \in [a,b]$.

Clearly $\int_c^d f(x,y)\cdot dy = g(x)$ defines a func. of $x \in [a,b]$

Then $\int_a^b g(x)\, dx = \int_a^b \left\{ \int_c^d f(x,y)\cdot dy \right\}\cdot dx$ is a repeated integral of $f(x,y)$ on $B$.

Similarly the integral $\int_c^d \left\{ \int_a^b f(x,y)\cdot dx \right\} dy$ is another repeated integral of $f(x,y)$ on $B$.

**Fubini's Theorem :—**

If a double integral $\iint_B f(x,y)\cdot dx\, dy$ exist, if the repeated integrals exists then $\iint_B f(x,y)\cdot dx dy = \int_c^d \left\{ \int_a^b f(x,y) dx \right\}\cdot dy = \int_a^b \left\{ \int_c^d f(x,y) dy \right\}\cdot dx$.

Remark :— If a double integral exist, then two repeated integrals cannot exist without being equal. However, if the double integral doesn't exist, nothing can be said about the repeated integrals, they may or may not exist. and one of the repeated integrals may exist or even both may exist & be equal yet the double integral may not exist. If $(x,y)$ is continuous on $B$ then all the integrals in question exists and the double integral can be evaluated by repeated integrals.

Ex.1 Let $f(x,y) = \begin{cases} 1/y^2, & 0<x<y<1 \\ -1/x^2, & 0<y<x<1. \end{cases}$

Evaluate the repeated integrals of $f(x,y)$ on $B = [0,1]\times[0,1]$ and comment on $\iint_B f$.

Soln. $\int_0^1 \left\{ \int_0^1 f(x,y)\cdot dx \right\}\cdot dy = \int_0^1 \left\{ \int_0^y \frac{1}{y^2}\cdot dx + \int_y^1 -\frac{1}{x^2}\cdot dx \right\}\cdot dy$.

$= \int_0^1 \left\{ \frac{1}{y} + \left(1 - 1/y\right) \right\}\cdot dy = 1.$

$\int_0^1 \left\{ \int_0^1 f(x,y)\cdot dy \right\} dx = \int_0^1 \left\{ \int_0^x -\frac{1}{x^2}\cdot dy + \int_x^1 \frac{1}{y^2}\cdot dy \right\}\cdot dx$

$= \int_0^1 \left\{ -1/x + \left(1/x - 1\right) \right\}\cdot dx = -1$

... two repeated integrals ...

Hence $\iint_R f$ cannot exist

Ex 2 Let $f(x,y) = \frac{x-y}{(x+y)^3}$ $x, y \in B = [0,1] \times [0,1]$.

Evaluate the two repeated integrals. Does $\iint_R f$ exist? Justify.

Ex. 3 $f(x,y) = \begin{cases} 1/2, & \text{if } y = \text{rational} \\ x, & \text{if } y = \text{irrational}. \end{cases}$ $(x,y) \in B = [0,1] \times [0,1]$

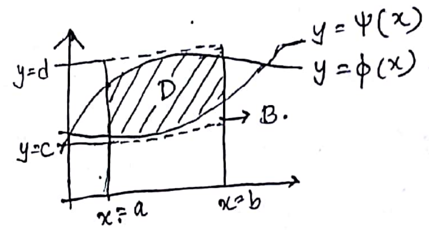
P.T. $\int_0^1 \left\{ \int_0^1 f(x,y) \cdot dx \right\} dy = 1/2$ & $\int_0^1 \left\{ \int_0^1 f(x,y) \cdot dy \right\} \cdot dx$ does not exist.

Comment on $\iint_B f$.

**Evaluation of Double integral over a closed region:—**

To evaluate $\iint_D f(x,y)\, dx\, dy$ where $D$ is a closed region in $\mathbb{R}^2$, bounded by the curves: $y = \phi(x)$, $y = \psi(x)$, $a \le x \le b$.

and $\phi(x) \le \psi(x)$.




Here $D = \{ x,y : a \le x \le b, \phi(x) \le y \le \psi(x) \} \subseteq B = [a,b] \times [c,d]$.

Define $f^*(x,y) = \begin{cases} f(x,y) & (x,y) \in D. \\ 0 & (x,y) \in B - D. \end{cases}$

Now, $\iint_B f^*(x,y) \cdot dx\,dy = \iint_D f(x,y) \cdot dx\,dy + \iint_{B-D} 0 \cdot dx \cdot dy$.

$= \iint_D f(x,y) \cdot dx \cdot dy$.

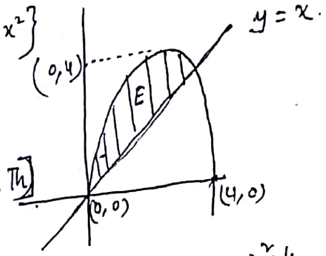
Now $\iint_D f(x,y) \cdot dx\,dy = \iint_B f^*(x,y) \cdot dx\,dy = \int_a^b \left\{ \int_c^d f^*(x,y)\, dy \right\} \cdot dx$.

$= \int_a^b \left\{ \int_c^{\phi(x)} 0 \cdot dy + \int_{\phi(x)}^{\psi(x)} f(x,y) \cdot dy + \int_{\psi(x)}^{d} 0 \cdot dy \right\} \cdot dx$.

$= \int_a^b \left\{ \int_{\phi(x)}^{\psi(x)} f(x,y) \cdot dy \right\} \cdot dx$.

1) Evaluate $\iint_E y \cdot dx\,dy$ over the region $E$ bounded by the curves $y = 4x - x^2$ and $y = x$.

Soln. Here $E = \{ x, y : 0 \le \underline{x \le 4} , \ x \le y \le 4x - x^2 \}$ $(\sqrt{3})$




$y = 4x - x^2 = -(x-2)^2 + 4.$

$\Rightarrow y - 4 = -(x-2)^2$

Now,

$$\iint_E y \cdot dx\,dy = \int_0^4 \left( \int_x^{4x-x^2} y \cdot dy \right) \cdot dx \quad [\text{by Fubini's Th}]$$

$$= \int_0^4 \left[ \frac{y^2}{2} \right]_x^{4x-x^2} \cdot dx.$$

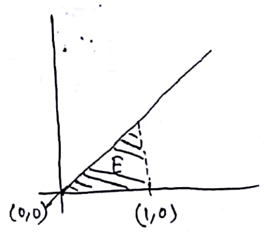
$$= \int_0^4 \frac{(4x-x^2)^2 - x^2}{2} \cdot dx = \frac{1}{2} \int_0^4 (15x^2 - 8x^3 + x^4) \cdot dx.$$

$$= \frac{1}{2} \left[ 15 \cdot \frac{x^3}{3} - 8 \cdot \frac{x^4}{4} + \frac{x^5}{5} \right]_0^4 = \frac{32}{5} \text{ (Ans)}$$

Ex 2 Evaluate $\iint_E e^{y/x} \cdot dx \cdot dy$ where $E$ is the triangle bounded by the lines $y = x$, $y = 0$, $x = 1$.

Soln. $E = \{ (x, y) : 0 \le y \le 1 , \ 0 \le y \le x \}$

$= \{ (x, y) : 0 \le y \le 1 , \ y \le x \le 1 \}.$



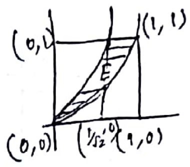

$$\iint_E e^{y/x} \cdot dx \cdot dy = \int_0^1 \left\{ \int_0^x e^{y/x} \cdot dy \right\} \cdot dx$$

$$= \int_0^1 \left[ x e^{y/x} \right]_0^x \cdot dx = \int_0^1 x(e-1) \cdot dx = \frac{e-1}{2}.$$

$\mathbb{R} = [0,1] \times [0,1]$ . $f(x,y) = \begin{cases} x+y & \text{if } \ldots \\ 0 & \text{otherwise} \end{cases}$

Soln.

$$\iint_R f(x,y)\,dx\,dy = \iint_E (x+y)\cdot dx\cdot dy + \iint_{R-E} 0\, dx\cdot dy.$$




$E = \{(x,y) : 0<x<1/\sqrt{2},\ x^2<y<2x^2$ or $1/\sqrt{2} \le x < 1,\ x^2 < y < 1\}$

or $\{(x,y) : 0 \le y \le 1, \sqrt{\frac{y}{2}} < x < \sqrt{y}\}$

$$= \int_0^1 \left\{ \int_{\sqrt{y/2}}^{\sqrt{y}} (x+y)\cdot dx \right\}\cdot dy.$$

$$= \int_0^1 \left[\frac{x^2}{2} + xy\right]_{\sqrt{y/2}}^{\sqrt{y}} \cdot dy$$

$$= \int_0^1 \left[\frac{y}{2} + y\sqrt{y} - \frac{y}{4} - y\sqrt{\frac{y}{2}}\right]\cdot dy$$

Ans. $\frac{21-8\sqrt{2}}{40}$.

$$= \int_0^1 \left\{\frac{y}{4} + y^{3/2}(1 - 1/\sqrt{2})\right\}\cdot dy = \left[\frac{y^2}{8} + \frac{2y^{5/2}}{5}(1-1/\sqrt{2})\right]_0^1 = \frac{1}{8} + \frac{2}{5}(1-1/\sqrt{2}).$$

$$= \frac{21-8\sqrt{2}}{40}.$$

**Change of order of integration :—**

Consider the repeated integral $\int_a^b \left\{ \int_{\phi(x)}^{\psi(x)} f(x,y)\cdot dy \right\}\cdot dx$

If $f(x,y)$ is double integrable over $D$, where $D = \{(x,y) : a \le x \le b, \phi(x) \le y \le \psi(x)\}$

$= \{(x,y) : c \le y \le d,\ \phi^*(y) \le x \le \psi^*(x)\}$

By Fubini's theorem :

$$\int_a^b \left\{ \int_{\phi(x)}^{\psi(x)} f(x,y)\cdot dy \right\}\cdot dx = \iint_D f(x,y)\cdot dx\cdot dy = \int_c^d \left\{ \int_{\phi^*(y)}^{\psi^*(y)} f(x,y)\cdot dx \right\}\cdot dy.$$

Ex1 Evaluate the following repeated integrals :—

i) $\int_0^1 \left\{ \int_{2y}^2 e^{x^2}\cdot dx \right\}\cdot dy$  ii) $\int_0^1 \left\{ \int_x^1 y \sin \pi y^3 \cdot dy \right\} dx$.  iii) $\int_0 \left( \int_x^1 y^{-1} e^{-y}\cdot dy \right)\cdot dx$

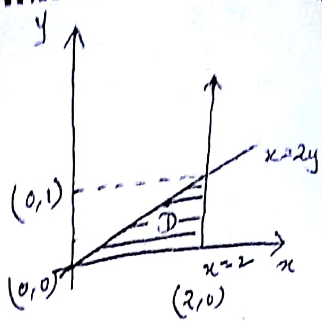
Soln. i) $D = \{(x,y) : 0 < y < 1,\ 2y < x < 2\}$.

$= \{(x,y) : 0 < y < x/2 < 1\}$.

$= \{(x,y) : 0 < x < 2,\ 0 < y < x/2\}$

As $f(x,y)$ is continuous,

$\iint_D f(x,y)\cdot dx\,dy$ is exists.




By Fubini's Theorem :-

$$\int_0^1\left\{\int_{2y}^{2} e^{x^2}\cdot dx\,dy\right. = \int_0^2\left\{\int_0^{x/2} e^{x^2}\cdot dy\right\}\cdot dx$$

$$= \int_0^2 e^{x^2}\cdot\frac{x}{2}\cdot dx = \left[\frac{1}{4}e^{x^2}\right]_0^2 = \frac{1}{4}(e^4-1).$$

ii) $D = \{(x,y): 0<x<1,\ x<y<1\} = \{(x,y): 0<x<y<1\}$.

$= \{(x,y): 0<x<y,\ 0<y<1\}$.

$$\therefore \int_0^1\left\{\int_x^1 y\sin\pi y^3\cdot dy\right\}\cdot dx = \int_0^1\left\{\int_0^y y\sin\pi y^3\cdot dx\right\}\cdot dy$$

$$= \int_0^1 y\sin\pi y^3\cdot[x]_0^y\cdot dy = \int_0^1 y^2\sin\pi y^3\cdot dy = \int_0^\pi \sin z\cdot\frac{dz}{3\pi}\cdot\ [\text{Putting } \pi y^3 = z]$$

$$= \frac{1}{3\pi}\left[-\cos z\right]_0^\pi = \frac{-1}{3\pi}[-1-1] = \frac{2}{3\pi}.\ [\text{Ans}].$$

iii) Similar change of order of integration as before.

$$\therefore \int_0^1\left\{\int_x^1 y^{-1}e^{-y}\cdot dy\right\}\cdot dx = \int_0^1\left\{\int_0^y e^{-y}\cdot y^{-1}dx\right\}\cdot dy$$

$$= \int_0^1 y^{-1}e^{-y}\cdot y\cdot dy = \int_0^1 e^{-y}\cdot dy = -\left[e^{-y}\right]_0^1 = -\left[e^{-1}-1\right] = 1-e^{-1} = \frac{e-1}{e}$$

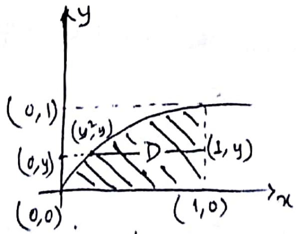
Ex-2 Change the order of integration in a) $\int_0^1\left(\int_0^{\sqrt{x}} f(x,y)\cdot dy\right)dx$.

Soln: The domain of integration

$D = \{(x,y): 0<x<1,\ 0<y<\sqrt{x}\}$

$= \{(x,y): 0<y<\sqrt{x}<1\}$

$= \{(x,y): 0<y<1,\ y^2<x<1\}$




Interchanging the order of integration, we have $\int_0^1\left\{\int_{y^2}^1 f(x,y)\cdot dx\right\}\cdot dy$.

$\int_{-2} \{ \int_{0} f(x,y)\cdot dy \}$ ... $\int_{0} \{ \int_{0}$ ...

Soln: b) $D = \{(x,y) : -2 < x < 1 , \; 0 < y < \sqrt{1-x^2}\}$.

c) $D_1 = \{(x,y) : 0<x<1 , \; 0<y<x\} = \{(x,y) : 0<y<x<1\}$

$= \{(x,y) : 0<y<1 , \; y<x<1\}$.

$D_2 = \{(x,y) : 1 < x < 2 , \; 0<y<2-x\}$. | $0<y<2-x$, $-2<y-2<-x$, $+2>2-y>x$

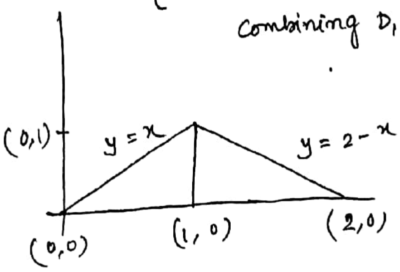
$= \{(x,y) : 1< x < 2-y < 2\}$.

$= \{(x,y) ; 1<2-y<2 , \; 1<x<2-y\}$.

$= \{(x,y) : 0<y<1 , \; 1<x<2-y\}$

Combining $D_1$ & $D_2$ $\rightarrow$ $\{(x,y) : 0<y<1 ; \; y<x<2-y\}$

$D = \{(x,y) : 0<y<1 , \; y<x<2-y\}$




$\therefore$ Change in the order of integration

$$I = \int_0^1 \left\{ \int_y^{2-y} f(x,y)\cdot dx \right\} \cdot dy .$$

<u>Transformation of Variables</u> :—

[ $\int_a^b f(x)\cdot dx$. Let $u = \phi(x)$ be a continuous & one to one transformation.

$x = \phi^{-1}(u)$.

| $x$ | $a$ | $b$ |
|---|---|---|
| $u$ | $\phi(a)$ | $\phi(b)$ |

, assuming $\phi(\cdot)$ is increasing.

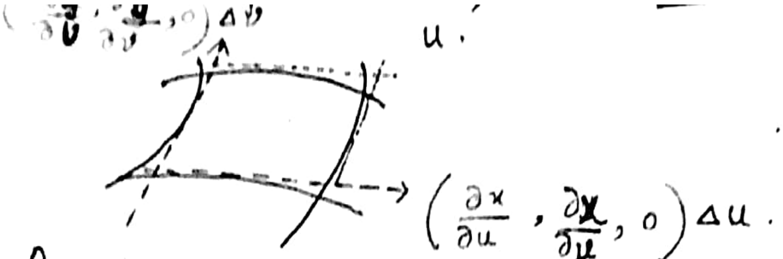
$$\int_a^b f(x)\cdot dx = \int_{\phi(a)}^{\phi(b)} f[\phi^{-1}(u)] \cdot \underbrace{\frac{d\phi^{-1}(u)}{du}}_{\text{magnification factor}} \cdot du . \quad ]$$

□ Consider the double integral; $\iint_D f(x,y)\cdot dx\, dy$.

Let $u = \phi(x;y)$ & $v = \psi(x,y)$ be a one-to-one tranf$\underline{n}$ from D. onto E. Then $x = \phi^{-1}(u,v)$, $y = \psi^{-1}(u,v)$

Assume that $\phi^{-1}$ & $\psi^{-1}$ have continuous partial derivatives w.r.t $u$ & $v$

Then the quantity $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is known as <u>local magnification factor</u> or <u>Jacobian</u>.



Area of $D_{ij}$ = $\left| \left( \frac{\partial x}{\partial u}, \frac{dx}{\partial v}, 0 \right) \Delta u \times \left( \frac{\partial y}{\partial u}, \frac{dy}{dv}, 0 \right) \Delta v \right|$

$$= \left\| \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{dx}{\partial v} & 0 \\ \frac{\partial y}{du} & \frac{dy}{\partial v} & 0 \end{vmatrix} \right\| \Delta u \Delta v = \text{mod} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{dy}{dv} \end{vmatrix} \Delta u \Delta v$$

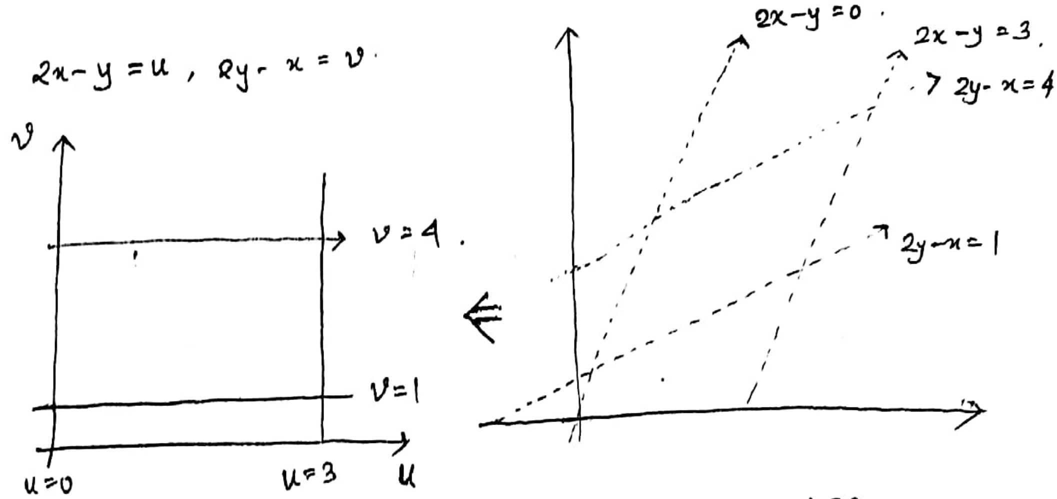
$$= |J| \text{ area of } E_{ij}$$

Ex1. Evaluate $\iint_D (x+y)\,dx\,dy$ where D is the region bounded by the st. lines $2x-y=0$, $2y-x=1$, $2x-y=3$ & $2y-x=4$.

Ex2. Evaluate $\iint (y-x)\,dx\,dy$ over the region E in the x-y plane bounded by the st. lines $y=x-3$, $y=x+1$, $3y+x=5$, $3y+x=7$.

Ex3. Integrate $x^3y^3$ over the area bounded by the parabolas $y^2=ax$, $y^2=bx$, $x^2=py$, $x^2=qy$. where $0<a<b$, $0<p<q$.

Soln 1. Let $2x-y=u$, $2y-x=v$.




Then $x+y = u+v$ $\Rightarrow$ $x = \frac{2u+v}{3}$, $y = \frac{u+2v}{3}$.

Here

and

Clearly

E =

J =

$= \{(u,v) : 0 < u < 3,\ 1 < v < 4\}.$

Jacobian is $J = \begin{vmatrix} \frac{\partial x}{du} & \frac{\partial x}{\partial v} \\ \frac{dy}{du} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{vmatrix} = 1/3.$

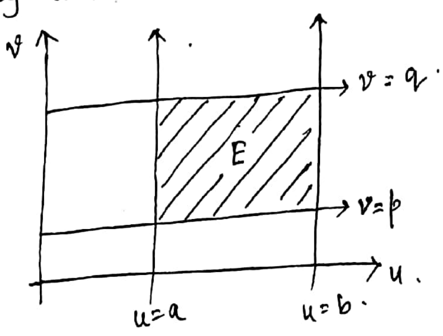
Hence $\iint_D (x+y)\cdot dx \cdot dy = \iint_E (u+v)\cdot |J| \cdot du\, dv$

$= \frac{1}{3}\int_1^4 \left\{ \int_0^3 (u+v)\cdot du \right\} \cdot dv = \frac{1}{3}\int_1^4 \left[\frac{u^2}{2} + uv\right]_0^3 \cdot dv = \frac{1}{3}\int_1^4 \left(\frac{9}{2} + 3v\right) \cdot dv$

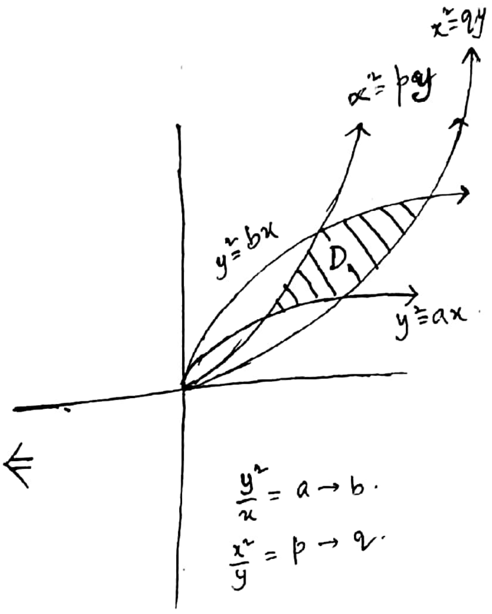
$= \int_1^4 \frac{3}{2} \cdot dv + \int_1^4 v\, dv = \frac{3}{2}(4-1) + \frac{1}{2}(16-1) = \frac{9}{2} + \frac{15}{2} = 12.$

Soln. 3. Let $u = \frac{y^2}{x}$, $v = \frac{x^2}{y}$.

Clearly $a < u < b$, $p < v < q$.




$\Leftarrow$




$\frac{y^2}{x} = a \to b.$

$\frac{x^2}{y} = p \to q.$

Here $uv = xy.$

and $u = \frac{y^3}{xy}$ & $v = \frac{x^3}{xy}$. $\Rightarrow x = (uv^2)^{1/3}$, $y = (u^2 v)^{1/3}$.

Clearly $(x,y) \to (u,v)$ is a one to one transformation from $D$ onto

$E = \{(u,v) : a < u < b,\ p < v < q\}.$

$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} u^{-2/3} v^{2/3} & \frac{2}{3} u^{1/3} v^{-1/3} \\ \frac{2}{3} u^{-1/3} v^{1/3} & \frac{1}{3} u^{2/3} v^{-2/3} \end{vmatrix}.$

$= -1/3.$

Hence $\iint_D x^3 y^3 \, dx \, dy = \iint_E u^3 v^3 \left|-\frac{1}{3}\right| du \, dv$

$$= \frac{1}{3}\int_p^q \left(\int_a^b u^3 \, du\right) v^3 \, dv = \frac{1}{3}\left(\int_a^b u^3 \, du\right)\left(\int_p^q v^3 \, dv\right)$$

$$= \frac{1}{3} \cdot \frac{b^4-a^4}{4} \cdot \frac{q^4-p^4}{4}$$

Ex 4 Dirichlet Integral

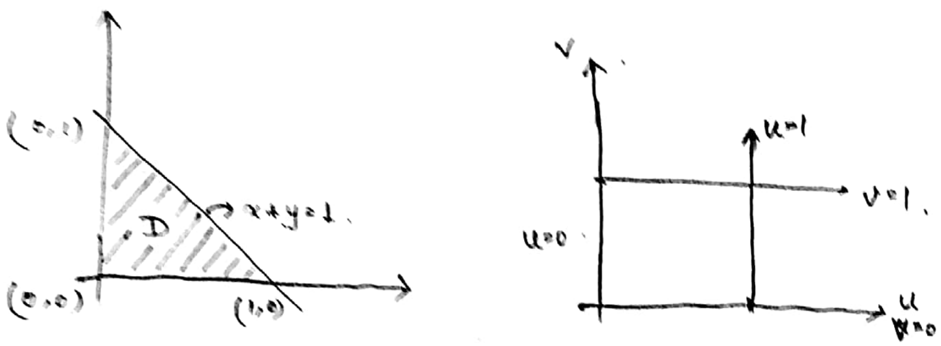
Evaluate $\iint_D x^{l-1} y^{m-1} (1-x-y)^{p-1} \, dx dy$ where $D$ is the region bounded by $x=0$, $y=0$, & $x+y=1$

Ex 5 show that

a) $\iint_E \sqrt{xy} \, dx \, dy = \pi/24$. b) $\iint_E \sqrt{xy(1-x-y)} \, dx dy = 2\pi/105$.

where $E = \{(x,y): 0<x, 0<y, \text{ \& } x+y<1\}$.

Soln 4.




Let $u = x+y$ $\quad v = \frac{x}{x+y}$

Then $x = uv$, $y = u(1-v)$

$x=0 \Rightarrow u=0$ or $v=0$.
$y=0 \Rightarrow u=0$ or $v=1$
$x+y=1 \Rightarrow u=1$

Clearly $(x,y) \to (u,v)$ is a one-to-one transfn from $D$ onto

$E = \{(u,v): 0 \le u \le 1, \ 0 \le v \le 1\}$.

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} v & u \\ (1-v) & (-u) \end{vmatrix} = -u.$$

Hence,

$$\iint_D x^{l-1} y^{m-1} (1-x-y)^{p-1} dx \, dy.$$

$$= \iint_E (uv)^{l-1} (u(1-v))^{m-1} (1-u)^{p-1} |-u| \cdot du \, dv$$

$$\int_0^1 \left\{ \int_0^1 u^{l+m-1}(1-u)^{p-1}\, du \right\} v^{l-1}(1-v)^{m-1}\, dv$$

$$= \left( \int_0^1 u^{l+m-1}(1-u)^{p-1}\, du \right) \left( \int_0^1 v^{l-1}(1-v)^{m-1}\, dv \right)$$

$$= \beta(l+m, p)\cdot \beta(l, m) = \frac{\Gamma(l+m)\,\Gamma p}{\Gamma(l+m+p)} \cdot \frac{\Gamma l\,\Gamma m}{\Gamma(l+m)}$$

$$= \frac{\Gamma l\,\Gamma m\,\Gamma p}{\Gamma(l+m+p)}.$$

**Ex. 6.** Evaluate $\iint_E \sin\left(\frac{x-y}{x+y}\right) dx\,dy$ where $E$ is the region bounded by $x=0$, $y=0$ and $x+y=1$.

**Ex. 7.** Using double integral show that $\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$ if $m>0$, $n>0$.

**Soln.**

$$\Gamma m\,\Gamma n = \left( \int_0^\infty e^{-x} x^{m-1}\, dx \right) \left( \int_0^\infty e^{-y} y^{n-1}\, dy \right).$$

$$= \int_0^\infty \int_0^\infty e^{-(x+y)} x^{m-1} y^{n-1}\, dx\, dy.$$

Let $x+y = u$, $v = \frac{x}{x+y}$ $\therefore$ $x = uv$, $y = u(1-v)$

For $0 < x, y < \infty$, $0 < u < \infty$ & $0 < v < 1$

$$J = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u.$$

Hence,

$$\Gamma m\,\Gamma n = \int_0^1 \int_0^\infty e^{-u} \{uv\}^{m-1} (u(1-v))^{n-1} |-u|\, du\, dv$$

$$= \left( \int_0^\infty e^{-u} u^{m+n-1}\, du \right) \left( \int_0^1 v^{m-1} (1-v)^{n-1}\, dv \right)$$

$$= \Gamma(m+n)\, \beta(m,n).$$

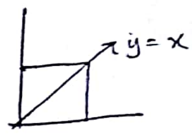
Hence $\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$.

Ex: If $f(x)$ is a non-decreasing integrable function on $[0,1]$. Show that $\left(\int_0^1 f(x)\,dx\right)^2 \leq 2\int_0^1 x f^2(x)\,dx$.

Soln:

$$\left(\int_0^1 f(x)\,dx\right)^2 = \int_0^1 f(x)\,dx \int_0^1 f(y)\cdot dy = \int_0^1\int_0^1 f(x)\cdot f(y)\cdot dx\,dy.$$

$$= \iint_{0<x<y<1} f(x)f(y)\,dx\,dy + \iint_{0<y<x<1} f(x)f(y)\cdot dx\,dy$$

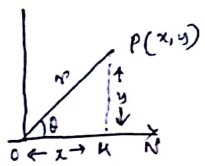



$$= 2\iint_{0<y<x<1} f(x)f(y)\cdot dx\,dy = 2\int_0^1 f(x)\left(\int_0^x f(y)\cdot dy\right)\cdot dx.$$

$$\leq 2\int_0^1 f(x)\left(\int_0^x f(x)\cdot dy\right)\cdot dx = 2\int_0^1 x f^2(x)\cdot dx$$

## Polar Transformation in $\mathbb{R}^2$:—

Let $(x,y)$ be the cartesian co-ordinate point $P(x,y)$




In $\triangle OMP$,

$$\frac{OM}{OP} = \cos\theta, \quad \frac{PM}{OP} = \sin\theta.$$

Here $OP = \sqrt{x^2+y^2} = r$, Also $x = OM = r\cos\theta$, $y = PM = r\sin\theta$.

where $0 < r < \infty$. & $0 < \theta < 2\pi$

Here $(x,y) \to (r,\theta)$ such that $x = r\cos\theta$, $y = r\sin\theta$

$0 < r < \infty$ & $0 < \theta < 2\pi$.

is called polar transformation.

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

1. Evaluate :—

i) $\iint_{x^2+y^2\leq a^2} dx\,dy$. ii) $\iint_{x^2+y^2\leq a^2} x\,dx\,dy$ iii) $\iint_{x^2+y^2\leq a^2} xy\,dx\,dy$

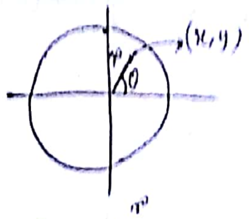
2. Evaluate :—

i) $\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1} dx\,dy$ ii) $\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1} x\,dx\,dy$ iii) $\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1} xy\,dx\,dy$.

consider the polar transformation

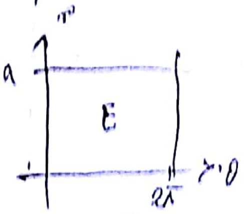
$$x = r\cos\theta \qquad y = r\sin\theta$$

where $0<r<a$, $0<\theta<2\pi$




$|J| = r.$

(i) $\iint_D dx\,dy = \int_0^{2\pi}\int_0^{a} r\cdot dr\,d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{a} r\,dr\right)$




$= \pi a^2$

ii) $\iint_D x\cdot dx\,dy = \int_0^{2\pi}\int_0^{a} r\cos\theta\cdot r\cdot d\theta\cdot dr = \left[\int_0^{2\pi}\cos\theta\cdot d\theta\right]\left[\int_0^{a} r^2\,dr\right]$

$= \left[\sin\theta\right]_0^{2\pi}\left[\frac{r^3}{3}\right]_0^{a}$

$= 0$

Alt. $\iint_{x^2+y^2\le a} x\,dx\,dy = \int_{-a}^{a}\left(\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} x\cdot dx\right)\cdot dy = 0$ [$\because$ odd func]

$\{(x,y) : x^2+y^2\le a^2\} = \{(x,y) : -a\le y\le a, -\sqrt{a^2-y^2}\le x\le\sqrt{a^2-y^2}\}$

Soln. 2. $D = \left\{(x,y) : \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}$

Q.

let $\frac{x}{a} = r\cos\theta$, $\frac{y}{b} = r\sin\theta$ where $0<r<1, 0<\theta<2\pi$.

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = abr \text{ etc.}$$

Ex. 3/ Evaluate i) $\iint_D dx\,dy$ ii) $\iint_D x\,dx\,dy$ iii) $\iint_D xy\,dx\,dy$

where D is the region bounded by $(ax^2+2hxy+by^2)\le R^2$

where $a>0$, $b>0$, $h^2-ab<0$.

Hint. $ax^2+2hxy+by^2 = a\left(x+\frac{h}{a}y\right)^2+\left(b-\frac{h^2}{a}\right)y^2$

$= a\left(x+\frac{h}{a}y\right)^2+\frac{ab-h^2}{a}y^2 = u^2+v^2$

where $u = \sqrt{a}\left(x+\frac{h}{a}y\right)$ & $v = \frac{\sqrt{ab-h^2}}{\sqrt{a}}y$.

Then $ax^2+2hxy+by^2\le R^2$

$\Rightarrow u^2+v^2\le R^2$

Let $u = r\cos\theta$, $v = r\sin\theta$. where $0<r<R$, $0<\theta<2\pi$.

$$(x,y) \longrightarrow (u,v) \longrightarrow (r,\theta)$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(r,\theta)} \quad (\text{why?})$$

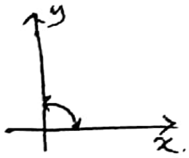
Again, $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix} \sqrt{a} & h/\sqrt{a} \\ 0 & \dfrac{\sqrt{ab-h^2}}{\sqrt{a}} \end{vmatrix}} = \dfrac{1}{\sqrt{ab-h^2}}$

and $\dfrac{\partial(u,v)}{\partial(r,\theta)} = r$

$$J = \frac{r}{\sqrt{ab-h^2}}$$

i) $\displaystyle\iint_D dx\,dy = \int_0^{2\pi}\int_0^{R} \frac{r}{\sqrt{ab-h^2}}\,dr\,d\theta = \frac{R^2}{2\sqrt{ab-h^2}}\cdot 2\pi = \frac{\pi R^2}{\sqrt{ab-b^2}}$

Ex. Evaluate $\displaystyle\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ using polar transformation. hence evaluate $\displaystyle\int_0^\infty e^{-x^2}\,dx$.




Soln: Here $0 < x, y < \infty$.

let $x = r\cos\theta$, $y = r\sin\theta$ where $0<r<\infty$, $0<\theta<\pi/2$.

Here $|J| = r$.

Hence,
$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\cdot r\,dr\,d\theta$$

$$= \left(\int_0^\infty e^{-r^2}\cdot r\,dr\right)\left(\int_0^{\pi/2} d\theta\right) = \left(\int_0^\infty e^{-u}\frac{du}{2}\right)\pi/2 = \pi/4.$$

Now,
$$\frac{\pi}{4} = \left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-y^2}dy\right) = \left(\int_0^\infty e^{-x^2}dx\right)^2$$

$$\therefore \int_0^\infty e^{-x^2}dx = \cancel{\sqrt{\pi}/\sqrt{2}}\ \frac{\sqrt{\pi}}{2}$$

i) $\pi/3$ ii) $\pi/\sqrt{3}$ iii) $\frac{\sqrt{2}\pi}{3}$ iv) $\frac{2\pi}{\sqrt{3}}$

Soln. $I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left\{(x+y/2)^2 + \frac{3y^2}{4}\right\}} dx\, dy$

$= \int_{-\infty}^{\infty} e^{-3/4 y^2}\left\{\int_{-\infty}^{\infty} e^{-(x+y/2)^2} dx\right\} dy$

$= \sqrt{\pi}\frac{2}{\sqrt{3}}\int_{-\infty}^{\infty} e^{-\left(\frac{\sqrt{3}}{2}y\right)^2} d\left(\frac{\sqrt{3}}{2}y\right) = \frac{2\pi}{\sqrt{3}}$

1) Evaluate $\int_0^\infty\int_0^\infty e^{-(ax^2+2hxy+by^2)} dx\, dy$ where $a>0$, $b>0$, $ab-h^2>0$

2) Evaluate $\iint_D \frac{xy}{x^2+y^2} dx\, dy$, $D = \{(x,y): x^2+y^2 \le 4, x\ge 0, y\ge 0\}$

3) ~~Evaluate~~ Show that $\iint_E \frac{dx\, dy}{xy} = \log\frac{a'}{a} + \log\frac{b'}{b}$ where E is the region bounded by the circles $x^2+y^2 = ax$, $a'x$, $by$, $b'y$

4) Evaluate $\int_1^\infty\int_0^{y-1} e^{-\frac{y}{x+1}-x} dx\, dy$.

5) Evaluate $\iint_D e^{\frac{x^2+y^2}{2}} dx\, dy$ where D is the region bounded by $y=0$, $y=x$ & $x^2+y^2=1$, $x^2+y^2=2$.

Soln. $ax^2+2hxy+by^2 = a\left(x+\frac{h}{a}y\right)^2 + \frac{ab-h^2}{a}y^2 = u^2+v^2$

where $u = \sqrt{a}\left(x+\frac{h}{a}y\right)$ $v = \sqrt{\frac{ab-h^2}{a}}\cdot y$

let $u = r\cos\theta$, $v = r\sin\theta$ where $0<r<\infty$, $0<\theta<\pi/2$

$(x,y)\to(u,v)\to(r,\theta)$

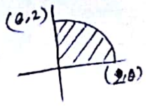
$\therefore J = \frac{r}{\sqrt{ab-h^2}}$

$\therefore I = \int_0^\infty\int_0^\infty e^{-(ax^2+2hxy+by^2)} dx\, dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\frac{r\, dr\, d\theta}{\sqrt{ab-h^2}}$

$= \frac{1}{\sqrt{ab-h^2}}\pi/2\int_0^\infty e^{-r^2} r\, dr = \frac{\pi}{4\sqrt{ab-h^2}}$

Soln. 2 Let $x = r\cos\theta$ $y = r\sin\theta$ where $0 < r \le 2$ & $0 < \theta < \pi/2$

$|J| = r$



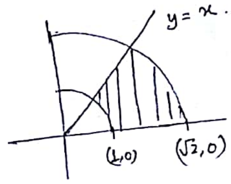

$$I = \iint_D \frac{xy}{x^2+y^2}\, dx\, dy = \int_0^{\pi/2}\int_0^{2} \frac{r^2\cos\theta\cdot\sin\theta\cdot r\cdot dr\, d\theta}{r^2}$$

$$= \left(\int_0^2 r\, dr\right)\left(\int_0^{\pi/2} \cos\theta\cdot\sin\theta\cdot d\theta\right) = 2\cdot\frac{1}{2}\cdot\int_0^{\pi/2}\sin 2\theta\cdot d\theta = 2.$$

Soln. 5/ Let $x = r\cos\theta$ $y = r\sin\theta$.

$|J| = r$.

where $1 < r < \sqrt{2}$, $0 < \theta < \pi/4$.




$$\therefore I = \iint_D e^{-\frac{x^2+y^2}{2}}\, dx\, dy.$$

$$= \int_0^{\pi/4}\int_1^{\sqrt{2}} e^{-r^2/2}\, r\, dr\cdot d\theta. = \frac{\pi}{4}\left\{-\left[e^{-r^2/2}\right]_1^{\sqrt{2}}\right\}$$

$$= \frac{\pi}{4}\left(\frac{1}{\sqrt{e}} - \frac{1}{e}\right) = \frac{\pi}{4}\left(\frac{\sqrt{e}-1}{e}\right).$$

$J = \frac{1}{4}\int_5^7$

Ex-8

Ex. 6.

SOURCE: GITHUB.COM/SOURAVSTAT

Let $u = y - x$, $v = 3y + x$ then the region becomes

$$D = \{(u,v) : -3 < u < 1,\ 5 < v < 7\}$$

$\therefore$ $y = \dfrac{u+v}{4}$ & $x = \dfrac{v-3u}{4}$

$$\therefore J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -3/4 & 1/4 \\ 1/4 & 1/4 \end{vmatrix} = -\frac{4}{16} = -1/4$$

$$I = \frac{1}{4}\int_5^7\int_{-3}^1 u\,du\cdot dv = \frac{1}{4}\int_5^7 \left[\frac{u^2}{2}\right]_{-3}^1 dv = \frac{1}{4}\left(\frac{1}{2} - \frac{9}{2}\right)\times 2 = -2. \text{ (Ans)}$$

Ex. 5 Using Dirichlet integral

a) $l = 3/2$ $m = 3/2$ $p = 1$

$$\therefore \iint_E \sqrt{xy}\,dx\,dy = \frac{\Gamma(3/2)\Gamma(3/2)\Gamma(1)}{\Gamma(3/2+3/2+1)} = \frac{1/2\cdot 1/2\left(\Gamma(1/2)\right)^2}{\Gamma 4} = \frac{1}{4}\cdot\frac{\pi}{6} = \frac{\pi}{24}.$$

b) $l = 3/2$ $m = 3/2$ $p = 3/2$

$$\therefore \iint_E \sqrt{xy(1-x-y)}\,dx\,dy = \frac{\Gamma(3/2)\Gamma(3/2)\Gamma(3/2)}{\Gamma(3/2+3/2+3/2)} = \frac{(1/2)^3\cdot\pi\sqrt{\pi}}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{2\pi}{105}.$$

Ex. 6. Let $u = x - y$ $v = x + y$. Then $x = \dfrac{u+v}{2}$ $y = \dfrac{v-u}{2}$

$\therefore -v < u < v$ & $0 < v < 1$.

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = 1/4 + 1/4 = 1/2.$$

$$\therefore I = \frac{1}{2}\int_0^1\int_{-v}^{v} \sin\left(\frac{u}{v}\right) du\,dv = 0 \quad \text{[sin being an odd func.]}$$