Miscellaneous

- If $X \sim \gamma(\alpha)$, $Y \sim \gamma(\beta)$ indep,

  Then $\frac{X}{X+Y} \sim \beta_1(\alpha, \beta)$ & $\frac{X}{Y} \sim \beta_2(\alpha, \beta)$.

- If $U_j \sim$ iid $R(0,1)$, $\rightarrow -2\ln U \sim \chi^2_2 \; [\overset{\exp(2)}{\equiv} \gamma(1,2)]$

  Then $-\beta \sum_{j=1}^{\alpha} \ln U_j \sim \text{Gamma}(\alpha, \beta)$

- $f(x) = |x|^p$ is (1) diff'ble everywhere if $p > 1$

  (2) cont, but not diff'ble at $x=0$ if $0 < p \leq 1$

- $f(x) = \begin{cases} x^k \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is $\rightarrow$

- $E[\max_i X_i] = \sum E[X_i] - \sum_{i<j} E[\min(X_i, X_j)] + \cdots$

  $\cdots + (-1)^{n+1} E[\min(X_1, \ldots, X_n)]$

- for large p,

  $\sqrt{2\chi^2_p} - \sqrt{2p-1} \sim N(0,1)$

- Legendre Duplication formula,

$$\Gamma(n)\,\Gamma\!\left(n+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}}.$$

- $ax \equiv b \pmod{n}$ is solvable, only when $\gcd(a,n) \mid b$.

As $n \to \infty$,

- Stirling:- $\underline{\lvert n} \simeq \sqrt{2\pi}\, e^{-n} (n)^{n+\frac{1}{2}}$.

- For r.s from $N(0,1)$, $V(S^2) = \dfrac{2\sigma^4}{n-1}$.

- For polar transforma^n $x = r\cos\theta$; $y = r\sin\theta$

  $r = \sqrt{x^2+y^2}$; $\theta = \tan^{-1}\dfrac{y}{x}$; $\boxed{dx\,dy = r\,dr\,d\theta}$

- Diagonalisa^n:-

A be a real symmetric matrix.

Then $\exists$ orthogonal P st $\underline{D = P^{-1}AP}$ is diagonal

•) For large sample, [GK, P! 17.51]

$$Z = \frac{\frac{\partial}{\partial\theta}\ln L}{\sqrt{Var\left(\frac{\partial}{\partial\theta}\ln L\right)}} \sim N(0,1)$$

C.I. : $P(|Z| \leq \tau_\alpha) = 1-\alpha$. [use suffi. statistic]

•) $\lim_{n\to\infty} \frac{\Gamma(n+k)}{\Gamma n} = n^k$.

•) $(X,Y) \sim BN(0,0,1,1,\rho)$

$\Rightarrow M_{X,Y}(t_1,t_2) = e^{\frac{1}{2}(t_1^2+t_2^2+2\rho t_1 t_2)}$.

•) Rearrangement Inequality: $a_1 \leq a_2 \leq \cdots \leq a_n$, $b_1 \leq b_2 \leq \cdots \leq b_n$

Then, $\max_{i,j} \sum a_i b_j = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$

& $\min \sum a_i b_j = a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1$

•) A.M. $\leq$ RMS $\left[= \sqrt{(x_1^2+\cdots+x_n^2)/n}\right]$

↳ (follows from Tchebycheff's, putting $a_i = b_i$)

•) Tchebycheff's Ineql :-

If $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$ and $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n$,

then $\frac{a_1 + \cdots + a_n}{n} \times \frac{b_1 + \cdots + b_n}{n} \leq \frac{a_1 b_1 + \cdots + a_n b_n}{n}$

equality :- all a_i's are equal / all b_i's are equal

(*) In two vars, $f_x = f_y = 0 \Rightarrow$ Stationary Pt ($\to$ critical pt ?)

If $D = f_{xx} f_{yy} - f_{xy}^2 < 0 \to$ saddle pt.

- Pt of Inflection: point where concavity changes. If f'' is cont, then a necessary (not suff) condition is $f''(a) = 0$, $f'''(a) \neq 0$ (but f'' may not exist: $y = x^{1/3}$ )

No. Theory [Euler $\Rightarrow$ Fermat (if n is prime)]

- Euler :- $a^{\phi(n)} \equiv 1 \pmod n$, if $\gcd(a,n) = 1$. So, $7^{16} \equiv 1 \pmod{40}$.

$\phi(40) = 40\left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right) = 16$, $2^3 \times 5$

- Fermat :- $a^{p-1} \equiv 1 \pmod p$ [$p \nmid a$]

$p \to$ prime. So, $2^6 \equiv 1 \pmod 7$

- Wilson :-

for prime p, $(p-1)! \equiv -1 \pmod p$

e.g., $12! + 1$ is divisible by 13

Math Results

- If $f(tx, ty) = t^n f(x, y)\ \forall t > 0$,

then $f(x, y)$ is called Homogeneous f^n of deg n.

Euler's Thm :- for homogeneous $f(x, y, z)$ of deg n with cont 1st order partial derivatives, $x f_x + y f_y + z f_z = n f$.

3D

- **Max-Min** :- $f_x = 0$, $f_y = 0$ → Solve for (a, b).

$D = f_{xx} f_{yy} - (f_{xy})^2$. See if $D(a, b) > 0$ [Max/Min exists]

Then if $f_{xx} > 0$ → min

< 0 → max.

2D:

- **Double pts** :- (α, β) double pt of $f(x, y) = 0$

if → (i) $f(\alpha, \beta) = 0$, $f_x = f_y = 0$ at (α, β)

Identification :-

At $(\alpha, \beta) \longrightarrow D$ (See above)

$$D \begin{cases} > 0 \rightarrow \text{isolated pt} \\ = 0 \rightarrow \text{cusp} \\ < 0 \rightarrow \text{node} \end{cases}$$

•) <u>Chain Rule</u> :- $z = f(x, y)$ be a diff'ble f^n of x & y, and $x = x(t)$, $y = y(t)$ be diff'ble f^n of t,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \times \frac{dx}{dt} + \frac{\partial z}{\partial y} \times \frac{dy}{dt} .$$

<u>Differential Eq^n</u>

① <u>Exact eq^n</u> :- $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ in $M dx + N dy = 0$ → (i)

Sol^n is, $\int \underset{(y \text{ Const})}{M \, dx} + \int \begin{pmatrix} \text{terms of } N \text{ not} \\ \text{having } x \end{pmatrix} dy = C$

•) If (i) is homogeneous, [terms are of same deg, e.g. $x^2y + x^3 + y^3$] and $Mx + Ny \neq 0$, then IF is $\frac{1}{Mx + Ny}$.

•) If (i) is of form $f_1(xy) \cdot y \, dx + f_2(xy) \, x \, dy = 0$, then $IF = \frac{1}{Mx - Ny}$, provided $D^n \neq 0$.

•) If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is f^n of x alone, $f(x)$, then $IF = e^{\int f(x) dx}$.

•) If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \to f(y)$, $IF = e^{\int f(y) dy}$.

↳ <u>turning into exact</u>

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(2) Linear eqn :-

$$\frac{dy}{dx} + Py = Q \quad [P, Q \to f(x)/\text{const}]$$

Soln is, $y e^{\alpha} = \int Q e^{\alpha} dx + c \quad [\alpha = \int P dx$, $IF = e^{\alpha}]$.

Bernoulli :- $\frac{dy}{dx} + Py = Q y^n \Rightarrow$ Put $y^{1-n} = v$.

(3) Clairaut's eqn :-

$y = px + f(p)$, $p = \frac{dy}{dx} \Rightarrow$ G.S is, $y = cx + f(c)$.

(*) G.S arises from factor having $\frac{dp}{dx}$ on soln.

Other factor gives S.S [by elimination of p with original eqn]

(4) Homogeneous Linear eqn, [Euler-Cauchy]

$$x^n y^{(n)} + P_1 x^{n-1} y^{(n-1)} + \dots + P_{n-1} x y^{(1)} + P_n y = X$$

$y^{(n)} \to \frac{d^n y}{dx^n}$; $X \to f(x)$.

$\underline{x = e^z}$ gives, $x\frac{dy}{dx} = Dy \quad \left[D \to \frac{d}{dz}, D^2 \to \frac{d^2}{dz^2}\right]$

$$x^n y^{(n)} = D(D-1) \dots (D-n+1) y.$$

•) If $(a+bx)^n y^{(n)} + \dots$, Put $a + bx = u$.

Then $u = e^z$.

⑤ 2nd Order (Special form) → i) $\frac{d^2y}{dx^2} = f(y)$ → multiply both sides by $2\frac{dy}{dx}$.

⑥ $f(y^{(n)}, y^{(n-1)}, \dots, y^{(1)}, x/y) = 0$

↳ Put p = lowest order derivative present.

⑦ 2nd Order Linear: $(D^2 + aD + b)y = 0$.

Roots of auxiliary eqn	G.S.
(i) m_1, m_2 →	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.
(ii) α, α →	$y = e^{\alpha x}(c_1 x + c_2)$
(iii) $\alpha \pm i\beta$ →	$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$.
	or, $y = Re^{\alpha x}\cos(\beta x + \phi)$

$\circ$) Repeated Roots [of $(D^n + a_1 D^{n-1} + \dots + a_n)y = X$]

(i) $m_1, m_2, \dots, m_n$ be roots of auxil. eqn,

G.S → $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$

(ii) m_1 repeated k times, & $m_{k+1}, \dots, m_n$;

G.S. → $y = e^{m_1 x}(c_1 + c_2 x + \dots + c_k x^{k-1}) + c_{k+1} e^{m_{k+1} x} + \dots + c_n e^{m_n x}$

(iii) $(\alpha \pm i\beta)$, repeated r times,

G.S → $y = e^{\alpha x}\{(A_1 + A_2 x + \dots + A_r x^{r-1})\cos\beta x + (B_1 + \dots + B_r x^{r-1})\sin\beta x\}$

$A_1, \dots, A_r, B_1, \dots, B_r$ → indep arbitrary consts.

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$\cdot$) Inverse Operator: $\frac{1}{f(D)}X$ stands for the soln (y) of $f(D)y = X$.

eg, $\frac{1}{D}x^4 = \frac{x^5}{5}$; $\frac{1}{D-\alpha}X = e^{\alpha x}\int X e^{-\alpha x}dx$

Result:- (i) $f(D)e^{ax} = f(a)e^{ax}$

(ii) $f(D)\, e^{ax} V = e^{ax} f(D+a) V$ (V: fn of x)

(iii) $f(D^2) \begin{matrix}\sin \\ \cos\end{matrix}(ax+b) = f(-a^2) \begin{matrix}\sin \\ \cos\end{matrix}(ax+b)$

(*) Particular Integral ⇒

~~(i) $\frac{1}{f(D)}\{e^{ax}V\} = e^{ax}\frac{1}{f(D+a)}V$ ... fn of x ... en lecture~~

(ii) $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$, provided $f(a) \neq 0$.

(iii) $\frac{1}{(D-a)^n\,\psi(D)}e^{ax} = \frac{e^{ax}}{\psi(a)} \cdot \frac{x^n}{\lfloor n}$ $(\psi(a) \neq 0)$

(iv) $\frac{1}{f(D)}xV = \left\{x - \frac{1}{f(D)}f'(D)\right\}\frac{1}{f(D)}V$.

(*) for $f(D)y = V$, GS = CF + PI,

where $PI = \frac{1}{f(D)}V$; CF is soln of $f(D)y = 0$.

Rectification (Arc-length)

i) Carlesian :- Length of $y=f(x)$ from $A(a,f(a))$ to $B(b,f(b)) = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

provided $f'(x)$ is cont. on $[a,b]$.

ii) Parametric :- If smooth curve is represented as $x=f(t)$, $y=\phi(t)$, then

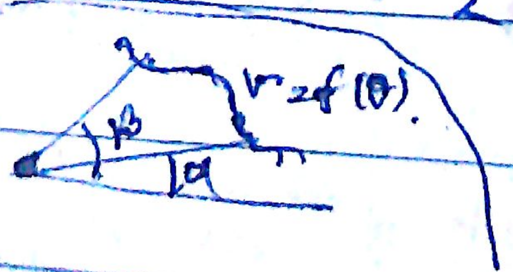
length $= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ $\left[\begin{array}{l} t=t_1 \text{ at } A \\ \quad = t_2 \text{ at } B \end{array}\right]$

iii) Polar :- eq^n be $r=f(\theta)$, $\left[\begin{array}{l} A\to\theta_1 \\ B\to\theta_2 \end{array}\right]$

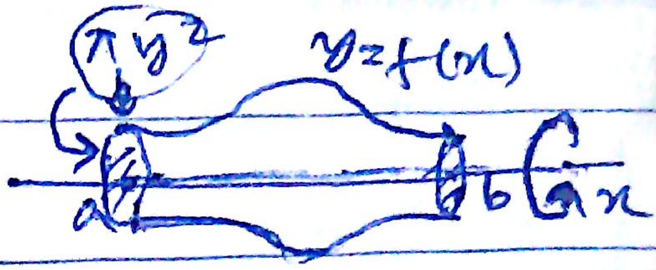
length $= \int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

Quadrature (Area)

Area bounded by $r=f(\theta)$ betw^n $\theta=\alpha$ and $\theta=\beta$ is $\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$ (f cont & finite in (α,β))



•) Vol^m of Revolution



(a) $= \pi \int_a^b \{f(x)\}^2 \, dx$

(b) Surface of Revolution

(b) $= 2\pi \int_a^b f(x) \sqrt{1+\{f'(x)\}^2} \, dx$

•) For polar coordinates, [Ex. Deg Math P: 4-159]

(a) $= \pi \int_{\theta_1}^{\theta_2} r^2 \sin^2\theta \, d(r\cos\theta)$

(b) $= 2\pi \int_{\theta_1}^{\theta_2} r\sin\theta \sqrt{(dr)^2 + r^2(d\theta)^2}$

Leibnitz Rule (nth order differentiation)

f & g be diff'ble n times at a. Then,

$$(fg)^n(a) = \sum_{r=0}^{n} \binom{n}{r} D^{n-r} f(a) \, D^r g(a)$$

i.e, $(fg)^n \equiv f^n g + \binom{n}{1} f^{n-1} g^1 + \binom{n}{2} f^{n-2} g^2 \cdots + f g^n$

Taylor's theorem :-

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k), \quad 0<\theta<1$$

Confidence Interval $\rightarrow$ eg, $P[\chi^2 > \chi^2_\alpha] = \alpha$; $P[|t| > t_{\frac{\alpha}{2}}] = \alpha$

$P(Z > Z_\alpha) = \alpha$; [Z → standardised]

$$P\left(-Z_{\alpha/2} < \frac{\bar{x} - \mu_x}{\sigma_x/\sqrt{n}} < Z_{\alpha/2}\right) = 1 - \alpha$$

∘) So, $100(1-\alpha)\%$ confidence interval for μ_x is: $\left(\bar{x} - Z_{\alpha/2}\frac{\sigma_x}{\sqrt{n}}, \bar{x} + Z_{\alpha/2}\frac{\sigma_x}{\sqrt{n}}\right)$.

i.e., → $\bar{x} \pm Z_{\alpha/2}\, \sigma_{\bar{x}}$.

[SCHAUM] (NVI)

∘) Confidence limits for population mean are, $\bar{x} \pm Z_c \frac{\sigma_{\bar{x}}}{\cancel{\sqrt{n}}}$ (eg, 95% confidence limits: $\bar{x} \pm 1.96\, \sigma_{\bar{x}}$)

Z_c	Confidence Level $= 2\Phi(Z_c) - 1$
1.00	68.27%
1.645 →	90%
✓ 1.96	95%
2.33 →	98%
✓ 2.58	99%

Normal Pop^n

⇒ 95% confidence interval for variance:

$$\chi^2_{0.025} \le \frac{(n-1)\hat{S}^2}{\sigma^2} \le \chi^2_{.975}$$

$$\Leftrightarrow \frac{\hat{S}\sqrt{n-1}}{\chi_{.975}} \le \sigma \le \frac{\hat{S}\sqrt{n-1}}{\chi_{0.025}}$$

• 98% C.I for $\frac{\sigma_1^2}{\sigma_2^2}$ [samples of size m & n from two normally distd popⁿs with var σ_1^2, σ_2^2]

$$F_{0.01} \le \frac{\hat{S}_1^2/\sigma_1^2}{\hat{S}_2^2/\sigma_2^2} \le F_{0.99}$$

$$\Leftrightarrow \frac{1}{F_{.99}}\frac{\hat{S}_1^2}{\hat{S}_2^2} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{1}{F_{.01}}\frac{\hat{S}_1^2}{\hat{S}_2^2}.$$

Gupta - Kapoor

• Having obtained value of statistic t from sample, we want to guess parameter θ of popⁿ.
Choose c_1, c_2 st $P[c_1 < \theta < c_2 \mid t] = 1-\alpha$
Here $c_1, c_2 \rightarrow$ confidence limits / fiducial limits
$[c_1, c_2] \rightarrow$ confidence interval; $1-\alpha \Rightarrow$ confidence coefficient
↳ $100(1-\alpha)\%$

eg, $Z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

So, $\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$ are 95% confidence limits for μ.
$= 100(1-0.05)\%$.

Rank

- $R(AA') = R(A'A) = R(A) = R(A')$
- $R(AB) \leq \min\{R(A), R(B)\}$.

$R(A) = R(PAQ)$, P, Q are non singular

- $\dim[\mathcal{N}(A)] = n - R(A)$.
- $R(A+B) \leq R(A) + R(B)$
- Sylvester :- $R(AB) \geq R(A) + R(B) - n$
- Frobenius :- $R(ABC) \geq R(AB) + R(BC) - R(B)$
- $|A^T A| \geq 0$, for any A. $(A^2 = A)$

⊛ For idempotent A, $R(A) = Tr(A)$

- $Tr(AB) = Tr(BA)$ [Trace].

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P K Guin Highlights

Det ⇒ $\cdot\ a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13} = \Delta$,

$\cdot$ (A) but $a_{12} A_{11} + a_{22} A_{21} + a_{32} A_{31} = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\downarrow$ Cofactor

- $\text{Adj}\,\Delta = \Delta^{n-1}$, So $\Delta^{-1} = \frac{\text{Adj}\,\Delta}{\Delta^n}$

Mat ⇒ $\cdot$ idempotent $(A^2 = A)$ $\cdot$ nilpotent $(A^m = 0)$

- Involutory $(A^2 = I)$
- $A \cdot \text{Adj}\,A = \text{Adj}\,A \cdot A = |A| I_n$
- $\text{Adj}(A^T) = (\text{Adj}\,A)^T$; $\text{Adj}(kA) = k^{n-1}\,\text{Adj}\,A$
- $|\text{Adj}\,A| = |A|^{n-1}$; $|\text{adj}(\text{adj}\,A)| = |A|^{(n-1)^2}$

$A^{-1} = \frac{\text{Adj}\,A}{|A|}$ → Transpose of the cofactor mat.

- $(\text{Adj}\,A)^{-1} = \frac{A}{|A|} = \text{Adj}(A^{-1})$, if $|A| \neq 0$
- $\text{Adj}(\text{Adj}\,A) = |A|^{n-2} A$.
- Orthogonal → $AA^T = I \Leftrightarrow |A| = \pm 1$

$\qquad\qquad\qquad\qquad\qquad\Leftrightarrow A^{-1} = A^T$.

- Consistency : (System of eqn) $r(A) = r(A|b) \begin{cases} = n \to \text{unique} \\ < n \to \infty \end{cases}$.

$A^{m\times n} \underset{\sim}{x}^{n\times 1} = \underset{\sim}{b}^{m\times 1}$; $[m \geq n]$

- Consistent, iff $R(A) = R(A|b) = r$

(A) $r = n \leq m \rightarrow$ unique solⁿ.

(B) $r < \min(m, n) \rightarrow \infty$ solⁿs.

Pol $\Rightarrow$ • If $f(x)$ is divided by $x-h$, remainder will be $f(h)$

- Polⁿ $[f(x)]$ of deg n vanishes for $> n$ values of $x \Rightarrow$ all coefficients are zero.

Interpolation: Given (x_i, y_i), $i = 0(1)n \Rightarrow y = f(x)$

- Newton's forward formula $\longrightarrow f(x) \simeq \sum_{r=0}^{n} \binom{u}{r} \Delta^r y_0$

- " backward " $\longleftarrow f(x) \simeq \sum_{r=0}^{n} \binom{u+r-1}{r} \nabla^r y_{n-r}$

Where $u = \frac{x - x_0 / x_n}{h}$ $[x_r = x_0 + rh]$; equal spacing

- Lagrange's formula, $f(x) \simeq \phi(x) \sum \frac{y_i}{D_i}$

where $\phi(x) = (x-x_0)(x-x_1) \dots (x-x_n)$

$D_i = \cancel{(x-x_i)\,\phi'(x_i)} = (x_i - x_0) \dots (x_i - x_{i-1}) \cancel{(x-x_i)}$
$(x_i - x_{i+1}) \dots (x_i - x_n)$

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