Some Important Formulas

(1) Let X be a rv $\ni$ $P(X>0)=1$
Then $E(X^r) = \int_0^\infty r x^{r-1}(1-F(x))\,dx$

(2) X and Y are said to be independently distributed if $F_{X,Y}(x,y) = F_X(x)F_Y(y) \ \forall (x,y)$

(3) X and Y are said to be independently distributed iff $P_{X,Y}(x,y) = P_X(x)P_Y(y) \ \forall (x,y)$
iff $f_{X,Y}(x,y) = f_X(x)f_Y(y) \ \forall (x,y)$

(4) $V\left(\sum_1^n l_i X_i\right) = \sum_1^n l_i^2 V(X_i) + \sum_{i\neq j} l_i l_j \operatorname{cov}(X_i, X_j)$

(5) $\operatorname{cov}\left(\sum_1^m a_i X_i, \sum_1^n b_j Y_j\right) = \sum_{i=1}^m \sum_{j=1}^n a_i b_j \operatorname{cov}(X_i, Y_j)$

(6) For independent rvs X and Y
$E(XY) = E(X)E(Y)$

(7) $E(E(Y|X)) = E(Y)$
$V(Y) = E(V(Y|X)) + V(E(Y|X))$

(8) If the support of joint distn
$\neq$ (the " " X) $\times$ (the support of Y)

Then X and Y aren't independently distributed

(9) Chebyshev's inequality

$$P(|X-a| < t) \geq 1 - \frac{E(X-a)^2}{t^2}, \quad t > 0$$

$$P(|X-\mu| < t) \geq 1 - \frac{V(X)}{t^2}, \quad t > 0$$

(10) Markov's Inequality

$P(X > 0) = 1$ with $E(X) = \mu < \infty$

$$\Rightarrow P(X \geq t) \leq \frac{E(X)}{t}, \quad t > 0$$

(11) Cantelli's Inequality

X : rv, F : df, $\mu = E(X)$, $\sigma^2 = V(X) < \infty$

Then $$F(x) \leq \frac{\sigma^2}{\sigma^2 + (x-\mu)^2} \quad \forall \; x \leq \mu$$

$$F(x) \geq \frac{(x-\mu)^2}{\sigma^2 + (x-\mu)^2} \quad \forall \; x \geq \mu$$

(12) One-sided Chebyshev's Inequality

X : RV, $\mu = E(X)$, $\sigma^2 = V(X) < \infty$

$$P(X \geq \mu + t\sigma) \leq \frac{1}{1+t^2} \quad \text{[By "]}, \; t > 0$$

(13) If $P_X(t)$ be the pgf of a rv X then

$P_X(1) = 1$ (obvious)

(14) Suppose X is non-negative and integer-valued and moments of all orders exist.

Then, $$P_X(t) = \sum_{k=0}^{\infty} \left\{ E(X)_k \right\} \frac{(t-1)^k}{k!}, \quad |t| \leq 1$$

(15) For any integer-valued rv x

$$\sum_{n=0}^{\infty} z^n P(x \geq n) = (1-z)^{-1} P_x(z)$$

where $P_x(z)$ is the pgf of x.

(16) Continuity Theorem

If $\{A_n\}$ be an expanding or contracting sequence of events then

$$\lim_{n\to\infty} P(A_n) = P\left(\lim_{n\to\infty} A_n\right)$$

(17) $\cancel{x()}$ $V(x) \leq E(x^2)$

i.e. $V(x)$ exists if $\mu_2' < \infty$

(18) x be a continuous rv $\ni P(x \geq 0) = 1$ and $E(x)$ exists then (1) $\lim_{x\to\infty} x(1-F(x)) = 0$

(2) $E(x) = \int_0^{\infty}(1-F(x))dx$

(19) For non-negative integer valued rv x; $E(x)$ exists. $E(x) = \sum_{i=0}^{\infty} (1-F(i))$

(20) For a continuous rv $\ni E(x)$ exists.

$x(1-F(x)) \to 0$ as $x \to \infty$

$x F(x) \to 0$ as $x \to -\infty$.

$$E(x) = \int_0^{\infty} (1-F(x)-F(-x))\, dx$$

$$V(x) = \int_0^{\infty} 2x(1-F(x)+F(-x))dx - \mu_2^2$$

(21) For integer-valued rv $X \geq 0$, $E(X)$ exists.

$$E(X) = \sum_{j=1}^{\infty} \left(1 - F(j-0) - F(j)\right)$$

(22) X is said to be symmetrically distributed about a if

$$P(X \leq a-x) = P(X \geq a+x) \quad \forall x$$

or $P(X \geq a-x) = P(X \leq a+x) \quad \forall x$

i.e. $F(a-x) + F(a+x-0) = 1$ or $F(a-x-0) + F(a+x) = 1$

Discrete Case :- iff $P(a-x) = P(a+x) \; \forall x$

Continuous Case :- iff $f(a-x) = f(a+x) \; \forall x$

(23) X : continuous rv symmetrically distributed about 'a'.

$$Y = \begin{cases} 1 & \text{if } X > a \\ 0 & \text{o.w.} \end{cases}$$

$|X-a|$ and Y are independent of each other.

(24) X : symm about 'a', g : odd func^n ;

Then $E(g(X-a)) = 0$ provided expectation exists.

(25) Jensen's Inequality

g : convex func^n $\Rightarrow E(g(X)) \geq g(E(X))$

g : concave " $\Rightarrow E(g(X)) \leq g(E(X))$

(26) C-S inequality

$$E(g^2(x))\, E(h^2(x)) \geq E^2(g(x)h(x))$$

'=' holds when $g(x) = \lambda h(x)$ w.p. 1

(27) From the knowledge of marginal distributions determination of joint distribution isn't unique.

(28) Legendre's Duplication Formula

$$\Gamma(n)\,\Gamma(n+\tfrac{1}{2}) = \frac{\Gamma(2n)\cdot\sqrt{\pi}}{2^{2n-1}}$$

(29) Poincaré's Theorem

If A_i's are events in $\mathcal{A}$, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = S_1 - S_2 + S_3 - \cdots + (-1)^{n-1} S_n$$

$$S_k = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} P(A_{i_1} A_{i_2} \cdots A_{i_k})$$

(30) Probability of occurrence of exactly m out of n events $A_1, A_2, \ldots, A_n$.

$$\boxed{P_{[m]} = \sum_{r=0}^{n-m} (-1)^r \binom{m+r}{m} S_{m+r}}$$

(31) Probability of occurrence of atleast m out of n events is $P_{\langle m \rangle} = \sum_{r=0}^{n-m} (-1)^r \binom{m+r-1}{m-1} S_{m+r}$

(32) Imp inequalities

$$P\left(\bigcup_{i}^{n} A_i\right) \leq \sum_{i}^{n} P(A_i) \quad \text{[Boole's]}$$

$$P\left(\bigcap_{i}^{n} A_i\right) \geq \sum_{i}^{n} P(A_i) - (n-1) \quad \text{[Bonferroni's]}$$

Discrete Distributions

① Binomial ② Negative Binomial ③ Geometric
④ Poisson ⑤ Hypergeometric

1 Binomial Distribution

$$P(X=x) = \begin{cases} \binom{n}{x} p^x q^{n-x} ; & 0<p<1, \; q=1-p \\ & x=0(1)n \\ 0 & : \text{o.w.} \end{cases}$$

$\mu_{[r]}' = E(X)_r = (n)_r p^r$

$\mu_1' = E(X) = np$

$\mu_2 = \text{variance} = npq$

\# Skewness $= \sqrt{\beta_1} = \dfrac{\mu_3}{\mu_2^{3/2}} = \dfrac{1-2p}{\sqrt{npq}}$

+vely skewed if $p < \frac{1}{2}$
-vely " " $p > \frac{1}{2}$
symmetric " $p = \frac{1}{2}$.

\# Kurtosis $= \gamma_2 = \beta_2 - 3 = \dfrac{\mu_4}{\mu_2^2} - 3$
$= \dfrac{1-6pq}{npq}$

leptokurtic if $pq < \frac{1}{6}$
mesokurtic " $pq = \frac{1}{6}$
platykurtic " $pq > \frac{1}{6}$

mean > variance

$v(x) \leq \frac{n}{4}$ '=' holds when $p = q = \frac{1}{2}$

$\frac{x}{n} = f$ proportion of success.

$E(f) = p$

$v(f) = \frac{pq}{n}$

$v(f) \leq \frac{\frac{n}{4}}{n^2} = \frac{1}{4n}$

$Cov\left(\frac{x}{n}, \frac{n-x}{n}\right) = -v\left(\frac{x}{n}\right) = \frac{-pq}{n}$

$P(X=x) = \frac{n-x+1}{xq}\, p\, P(X=x-1)$

let $P(X=x) = p_x$

$\frac{p_1}{p_0} \geq \frac{p_2}{p_1} \geq \cdots \geq \frac{p_n}{p_{n-1}}$

i.e. $\frac{P(X=x)}{P(X=x-1)}$ is a $\downarrow$ func^n of x.

PGF $= (q+zp)^n$, MGF $= (q+pe^t)^n$

① when $(n+1)p$ is an integer

then mode $= (n+1)p$ or $(n+1)p-1$

② when $(n+1)p$ is not an integer

mode $= [(n+1)p]$

If $X \sim Bin(2s, \frac{1}{2})$, $\frac{1}{2\sqrt{s}} < P(X=s) \leq \frac{1}{\sqrt{2s+1}}$

$P(X=k) < \frac{1}{k-np}$ when $k > np$

If $np = \lambda$, then $\frac{\lambda^x}{x!}\left(1-\frac{x}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^n \leq Bin(x; n, p) \leq \frac{\lambda^x}{x!}\left(1-\frac{\lambda}{n}\right)^{n-x}$

Mean Deviation about mean.

$$MD_\mu = 2(n-k)\, p\, P(X=k) \simeq \sqrt{\frac{2npq}{\pi}}$$

$$\left[k = [np] \right]$$

$F(x) = I_q(n-x, x+1)$

$= P(Z \leq q)$ $\quad Z \sim Beta(n-x, x+1)$

$$I_q(n-x, x+1) = \frac{1}{B(n-x, x+1)} \int_0^q z^{n-x-1}(1-z)^x \, dz$$

[2]

Negative Binomial Distribution

$$P(X=x) = \binom{x+r-1}{x} p^r q^x, \quad x = 0, 1, 2, \ldots$$

$$E(X)_k = (r+k-1)_k \left(\frac{q}{p}\right)^k$$

Mean $= \mu_1' = \frac{rq}{p}$

variance $= \mu_2 = \frac{rq}{p^2}$

Mean < Variance

$$\frac{P(X=x)}{P(X=x-1)} = \frac{r+x-1}{x} q \quad ; \; x = 1, 2, \ldots$$

\# When $(r-1)\frac{q}{p} = k$ is an integer then mode $= (r-1)\frac{q}{p} - 1$ and $(r-1)\frac{q}{p}$.

When $\left[(r-1)\frac{q}{p}\right] = k$ and $(r-1)\frac{q}{p}$ is n.t an integer then mode $= \left[(r-1)\frac{q}{p}\right]$

\# $MD_\mu = 2\frac{k+1}{p} P(X=k+1) \qquad k = [np].$

\# $P(X \le k) = I_p(r, k+1)$

\# $P(X=x) = \binom{-r}{x} p^r (-q)^x$

$= \binom{-r}{x} (-P)^x Q^{-(r+x)}$

where $p = \frac{1}{Q}$, $q = \frac{P}{Q}$, $Q - P = 1$

$M_X(t) = (Q - Pe^t)^{-r}$

$= \left(\frac{1}{p} - \frac{q}{p}e^t\right)^{-r}$ | $P_X(t) = \left(\frac{1}{p} - \frac{q}{p}t\right)^{-r}$

\# X : number of failures preceding r th success in a sequence of Bernoulli trials with success probability p.

$P(X \le k) = P(X+r \le k+r) = P(N \le k+r) = P(Z \ge r)$

N : number of trials required to get rth success.

Z = number of success out of $(k+r)$ trials.

$Z \sim Bin(k+r, p)$

\# $\sum_{k=0}^{b-1} \binom{a+k-1}{k} q^k = \sum_{k=0}^{b-1} \binom{a+b-1}{k} q^{b-1-k} p^k$

\# Let $X \sim Bin(n, p)$ and $Y \sim NB(r, p)$ denotes the number of trials required to get r th success in a sequence of independent Bernoulli trials then.

$F_X(r-1) = 1 - F_Y(n)$

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Geometric Distribution

$$P(X=x) = \begin{cases} pq^x, & x = 0,1,2,\ldots \quad \text{(Model I)} \\ pq^{x-1}, & x = 1,2,3,\ldots \quad \text{(Model II)} \end{cases}$$

$0<p<1, \quad q = 1-p$

Considering Model I,

$$E(X) = \frac{q}{p}$$

$$V(X) = \frac{q}{p^2}$$

$$\sum_{x=1}^{\infty} x q^{x-1} = \frac{1}{(1-q)^2}$$

$$\Rightarrow \frac{d}{dq}\left[\sum_{x=1}^{\infty} x q^{x-1}\right] = \frac{d}{dx}\frac{1}{(1-q)^2}$$

$$\Rightarrow \sum_{x=2}^{\infty} x(x-1) q^{x-2} = \frac{2}{(1-q)^3}$$

Mean & Variance

PGF, $P_X(t) = \left(\frac{1}{p} - \frac{q}{p}t\right)^{-1}$

$$P(1+t) = p\left[1 - q(1+t)\right]^{-1}$$

$$= \sum_{x=0}^{\infty} \left(\frac{qt}{p}\right)^x$$

$$\mu'_{[x]} = x!\left(\frac{q}{p}\right)^x$$

MGF. $M_x(t) = \left(\frac{1}{p} - \frac{q}{p}e^{t}\right)^{-1}$

Loss of Memory Property.

$x \sim Geo(p)$

Then $P(x \leq t+k \mid x \geq t) = P(x \leq k)$

This property characterizes ~~Geo~~ Geometric Distn

$P(X > i+j) = P(X > i)\, P(X \geq j)$

If x_1, x_2 are iid following $G(p)$. Then the conditional distribution $x_1 \mid x_1 + x_2$ is uniform.

$P(x_1 = x_1 \mid x_1 + x_2 = x) = \frac{1}{x+1}$, $x_1 = 0, 1, 2, \dots, x$
[For Model I]

For Model II, $P(x_1 = x_1 \mid x_1 + x_2 = x) = \frac{1}{x-1}$ $I_{x_1 = 1, 2, \dots, x-1}$

1

Poisson Distribution

$$P(X=x) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad x=0,1,2,\ldots, \quad \lambda>0$$

$$E(X)_r = \lambda^r$$

$$E(X) = V(X) = \lambda.$$

Skewness $= \gamma_1 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\lambda}{\lambda^{3/2}} > 0$

$\Rightarrow$ Poisson Distn is +vely skewed.

Kurtosis $= \gamma_2 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{3\lambda^2+\lambda}{\lambda^2} - 3 = \frac{1}{\lambda} > 0$

$\Rightarrow$ Poisson Distn is leptokurtic.

MGF, $M_X(t) = e^{\lambda(e^t-1)}$

PGF, $P_X(t) = e^{\lambda(t-1)}$

$P(X=x) = \frac{\lambda}{x}P(X=x-1), \quad x=1,2,3,\ldots$

$MD_\mu(x) = E|X-\mu|$

$$= 2\lambda\left(\frac{e^{-\lambda}\lambda^{x_0}}{x_0!}\right), \quad x_0 = [\lambda]$$

$$\rightarrow \sqrt{\frac{2\lambda}{\pi}}$$

$X \sim P(\lambda = 1) \Rightarrow MD_{\mu}(x) = \frac{2}{e} \times s.d.$

$E\left(\frac{1}{1+X}\right) = \frac{1}{\lambda}\left(1 - e^{-\lambda}\right) \quad ; \quad X \sim P(\lambda)$

$E(X \cdot g(X)) = \lambda E[g(X+1)]$ if $X \sim P(\lambda)$

If $X \sim P(\lambda)$, then $E(X^{r}) = \lambda E(X+1)^{r-1}$

$\mu_{r+1} = \lambda \sum_{i=1}^{r} \binom{r}{i} \mu_{r-i}$

$P(X \leq k) = 1 - \frac{\int_0^{\lambda} e^{-z} z^{k} \, dz}{\Gamma k+1}$

If $X \sim P(\lambda)$, then $P(X \geq n) < \frac{\lambda^{n}}{n!}$

If $X_1, X_2, \ldots X_n$ are independently distributed poisson variates and $X_i \sim P(\lambda_i)$ $\forall$ $i = 1(1)n$, then $S_n = \sum_{1}^{n} X_i \sim P\left(\sum_{i} \lambda_i\right)$

Suppose X_1 and X_2 are independently distributed poisson variates $\ni$ $X_1 \sim P(\lambda_1)$, $X_2 \sim P(\lambda_2)$, then

$$X_1 \mid X_1 + X_2 = k \sim Bin\left(k, \frac{\lambda_1}{\lambda_1 + \lambda_2}\right)$$

If $X \sim P(\lambda)$ and $Y \mid X = x \sim Bin(x, p)$, then $Y \sim P(\lambda p)$

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Hypergeometric Distribution

$$P(X=x) = \frac{\binom{Np}{x}\binom{Nq}{n-x}}{\binom{N}{n}} = \binom{n}{x}\frac{(Np)_x\,(Nq)_{n-x}}{(N)_n}$$

$x = 0, 1, 2, \ldots, n$

$0<p<1$

\# $Hyp(N; n, p) \to Bin(n, p)$ when $N \to \infty$.

\# $$E(X)_r = \frac{(n)_r\,(Np)_r}{(N)_r}$$

\# Mean : $\mu_1' = np$

\# variance $= npq\,\frac{N-n}{N-1} \to npq$ as $N \to \infty$

\# If $\frac{(n+1)(Np+1)}{N+2} = k$ is an integer then there are two modes at $k-1$ and at k.

If k isn't an integer then the mode is unique at $[k]$.

\# $MD_\mu = 2(n-k)\left(p-\frac{k}{N}\right)P(X=k)$ Here $k=[np]$

\# $V(X) \le \frac{n}{4}$.

Continuous Distributions

① Uniform Distn ② Exponential Distn
③ Gamma " ④ Beta "
⑤ Normal "

① Uniform Distn

r.v. $X \sim U(a, b)$ or $R(a, b)$

$$f_X(x) = \frac{1}{b-a} I_{a<x<b}$$

$$F_X(x) = \begin{cases} 0 & , x \le a \\ \frac{x-a}{b-a} & , a < x < b \\ 1 & , x \ge b \end{cases}$$

$\mu_1' = E(X) = \frac{a+b}{2}$.

$var(X) = E(X^2) - E^2(X) = \frac{b^3 - a^3}{3(b-a)} - \left(\frac{b+a}{2}\right)^2$

$$= \frac{(b-a)^2}{12}$$

$E(X^r) = \frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}$

Let X be a continuous r.v., with PDF $f(x)$, then $Y = F(X) = \int_{-\infty}^{x} f(t)\,dt$ has uniform distn over $[0, 1]$

$MGF = M_X(t) = \begin{cases} \frac{e^{tb} - e^{ta}}{t(b-a)} & , t \ne 0 \\ 1 & , t = 0 \end{cases}$

$X \sim U(0,n)$, $n \in \mathbb{N}$. Let $Y = X - [X]$. Then $Y \sim U(0,1)$

(2) Exponential Distⁿ

For $X \sim$ Exp with mean θ,

$$f_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} I_{0<x<\infty}$$

$E(X) = \theta$.

$$E(X^r) = \int_0^\infty r x^{r-1} (1 - F(x))\,dx = \lfloor r \, \theta^r$$

$var(X) = \theta^2$.

Lack of Memory Property :-

$X \sim$ Exp then $P[X > a+b \mid X > a] = P[X > b]$

for $a > 0$, $b > 0$.

Only Exponential Distⁿ possesses Lack of Memory Property.

If X is a rv $\ni$ $P[X \geq 0] = 1$ and X is absolutely continuous with df $F(x) \ni$

$$\frac{F'(x)}{1 - F(x)} = \text{a constant}, \quad \forall x \geq 0$$

then X must follow a exp distⁿ.

[Iff Condition]

Let $X_i \sim$ Exponential Distⁿ with mean θ, $i = 1(1)n$ independently. Then

$$Y = \min_{i=1(1)n} \{X_i\} \sim \text{Exp with mean } \frac{\theta}{n}.$$

SOURCE: GITHUB.COM/SOURAVSTAT

If $X \sim \text{Exp}$ with mean $\frac{1}{\theta}$, then median$(X) = \frac{\ln 2}{\theta}$.

and $E[|X-a|] \geq \frac{\ln 2}{\theta}$ [$\because MD_{median} \leq MD_{\bar{x}}$]

Shifted Exponential Distn

$$f_X(x) = \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}} I_{x>\mu}$$

$F_X(x) = \begin{cases} 0 & \text{if } x \leq \mu \\ 1 - e^{-\frac{x-\mu}{\sigma}} & \text{if } x > \mu \end{cases}$

The pth quantile ξ_p is,
$\xi_p = \mu - \sigma \ln(1-p)$

$E(X-\mu)^r = \sigma^r \underline{|r} = \sigma^r \Gamma(r+1)$

$E(X) = \mu + \sigma$
$V(X) = \sigma^2$

Doesn't possess lack of memory property

Double Exponential Distn / Laplace Distn

$X \sim D.E.(\mu, \sigma)$

$$\Rightarrow f_X(x) = \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}} I_{x \in \mathbb{R}}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$

$E(x) = \mu$ and $\mu_{2r-1} = 0$.

$$\mu_{2r} = \sigma^{2r} \lfloor 2r$$

$F_x(x) = \begin{cases} \frac{1}{2} e^{+\frac{x-\mu}{\sigma}} & \text{if } x \leq \mu \\ 1 - \frac{1}{2} e^{-\frac{x-\mu}{\sigma}} & \text{if } x > \mu \end{cases}$

For $z \sim DE(0,1)$, $f_z(z) = \frac{1}{2} e^{-|z|}$, $z \in \mathbb{R}$

$M_z(t) = \frac{1}{1-t^2}$, $|t| < 1$

For $x \sim DE(\mu, \sigma)$

$M_x(t) = e^{t\mu} \frac{1}{1-t^2\sigma^2}$, $|t| < \frac{1}{\sigma}$

③ Beta Distribution

$$f_X(x) = \frac{x^{m-1}(1-x)^{n-1}}{B(m,n)} \; I_{0<x<1} \quad , \quad \begin{matrix} m>0 \\ n>0 \end{matrix}$$

Let G be the Geometric Mean of $B_1(m,n)$, then,

$$\ln_e G = \frac{\partial}{\partial m} \ln \frac{\Gamma(m)}{\Gamma(m+n)}$$

$E(X) = \dfrac{a}{a+b}$ $\qquad X \sim B_1(a, b)$

$$V(X) = \frac{ab}{(a+b)^2(a+b+1)}$$

$$V(X) \leq \frac{1}{4}$$

$$\text{mode} = \frac{a-1}{a+b-2}$$

MGF of ~~$B_1(a,b)$~~ $B_1(a,b)$ exist.

If $X \sim B_1(a, b)$ then

$$E\left[\left\{b - \frac{(a-1)(1-X)}{X}\right\} g(X)\right] = E\left[(1-X)g'(X)\right]$$

Suppose $X \sim B_2(m, n)$

$$f_x(x) = \frac{x^{m-1}}{(1+x)^{m+n}} \frac{1}{B(m,n)} \quad I_{x>0}$$

Then $\frac{X}{1+X} \sim B_1(m, n)$

$$E(X) = \frac{m}{n-1}$$

$$V(X) = \frac{m(n+m-1)}{(n-1)^2(n-2)}$$

MGF of X doesn't exist.

① Gamma Distribution

$$f_X(x) = \frac{\theta^n}{\Gamma n} e^{-\theta x} x^{n-1} \quad I_{x>0}$$

$\theta > 0$
$n > 0$

Suppose $X \sim \text{Gamma}(n, \theta)$

$$E(X) = \frac{n}{\theta}$$

$$V(X) = \frac{n}{\theta^2}$$

$$M_X(t) = \left(\frac{\theta}{\theta - t}\right)^n.$$

$$HM = \frac{n-1}{\theta} \quad , \text{if } n > 1$$

$$\text{mode}(X) = \frac{n-1}{\theta}$$

\# If $X \sim P(\lambda)$ then

$$P[X \geq k+1] = \Gamma_\lambda (k+1)$$

where $$\Gamma_\lambda (k+1) = \frac{\int_0^\lambda e^{-t} t^k \, dt}{\Gamma(k+1)}$$

\# If $X_1 \sim \text{Gamma}(n_1, \theta)$, $X_2 \sim \text{Gamma}(n_2, \theta)$

then $X_1 + X_2 \sim \text{Gamma}(n_1 + n_2, \theta)$

⑤ Normal Distribution

$X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} I_{x \in \mathbb{R}}$$

mean = median = mode = μ

$\mu_{2r-1} = 0$

$\mu_{2r} = \sigma^{2r}(2r-1)(2r-3)\cdots 5.3.1$

$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$E|X-\mu| = \sqrt{\frac{2}{\pi}}\,\sigma$

$$\frac{MD_X(\mu)}{SD_X} = \sqrt{\frac{2}{\pi}} \quad \text{(Geary's Ratio)}$$

\# For $x > 0$

$$\left(\frac{1}{x} - \frac{1}{x^3}\right)\phi(x) \le 1 - \Phi(x) \le \frac{\phi(x)}{x}$$

\# $$\lim_{x \to \infty} \frac{x\{1 - \Phi(x)\}}{\phi(x)} = 1$$

\# If $X \sim N(\mu, 1)$ then,

$$E\left\{\frac{1 - \Phi(x)}{\phi(x)}\right\} = \frac{1}{\mu}.$$

$X \sim N(0,1) \Rightarrow E[X] = -\frac{1}{2}$

If $X \sim (\mu, \sigma^2)$, $\mu \neq 0$ then $E\,\Phi(X) = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)$

$\cancel{\int_{-\infty}^{\infty}} \int x\,\phi(x)\,dx = -\phi(x)$

Chi-Square

$X \sim \chi^2_n \iff X \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$

$$f_X(x) = \frac{e^{-x/2} \, x^{n/2 - 1}}{\Gamma\frac{n}{2} \, 2^{n/2}} \, I_{x>0}, \quad n>0$$

$E(\chi^2_n) = n$

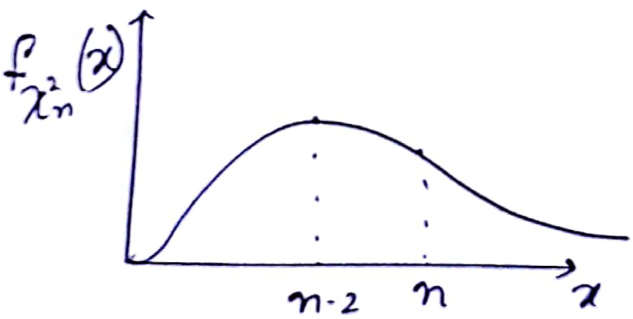
$V(\chi^2_n) = 2n$

$$M_{\chi^2_n}(t) = \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{\frac{n}{2}} = \left(\frac{1}{1-2t}\right)^{n/2}, \; t < \frac{1}{2}$$

$\text{mode}(\chi^2_n) = n-2 \quad \text{if } n > 2$

\# Skewness

$$Sk = \frac{\text{mean} - \text{mode}}{\text{s.d.}} = \sqrt{\frac{2}{n}} > 0$$



\# If $X \sim \chi^2_2$ and $p = P[X > x_0]$ then,

$x_0 = -2 \ln p$

\# $P[\chi^2_n > n] < \frac{1}{2}$ $\quad$ [$\because$ mean > median]

$x_i \overset{iid}{\sim} N(0,1) \Rightarrow \sum_{1}^{n} x_i^2 \sim \chi_n^2$.

$X \sim \chi^2_{n_1}$ and $Y \sim \chi^2_{n_2}$ independently

$\Rightarrow X+Y \sim \chi^2_{n_1+n_2}$

$\Rightarrow \dfrac{X}{X+Y} \sim B_1\left(\dfrac{n_1}{2}, \dfrac{n_2}{2}\right)$

$\Rightarrow \dfrac{X}{Y} \sim B_2\left(\dfrac{n_1}{2}, \dfrac{n_2}{2}\right)$

T Distribution

$T \sim t_n \Rightarrow f_T(t) = \frac{1}{\sqrt{n}\, B\left(\frac{1}{2}, \frac{n}{2}\right)} \cdot \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$ $\quad t \in \mathbb{R}$

$t_1 \equiv C(0,1)$

\# $T \sim t_n \Rightarrow \mu_r'$, $r \geq n$ doesn't exist.

\# T is a symmetric distn.

$\mu_{2r-1} = 0$

\# $E(T) = 0$, $V(T) = \frac{n}{n-2}$ if $n > 2$

\# $X, Y \overset{iid}{\sim} N(0,1) \Rightarrow \frac{X}{|Y|} \sim C(0,1)$.

o Let X and Y be independently distributed rvs where $X \sim N(0,1)$ and $Y \sim \chi^2_n$

Define $t = \frac{X}{\sqrt{\frac{Y}{n}}}$

Then $t \sim t_n$

o Suppose $\left.\begin{array}{l} X \sim \chi^2_m \\ Y \sim \chi^2_n \end{array}\right\}$ independently

Then $F = \frac{X/m}{Y/n} \sim F_{m,n}$

F - distribution

$F \sim F_{n_1, n_2}$

$$\Rightarrow f_F(f) = \frac{\left(\frac{n_1}{n_2} f\right)^{\frac{n_1}{2}-1} \left(\frac{n_1}{n_2}\right)}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2} f\right)^{\frac{n_1+n_2}{2}}} I_{f>0}$$

\# $\frac{n_1}{n_2} F = \frac{\chi^2_{n_1}}{\chi^2_{n_2}} \sim B_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$

\# $E(F) = \frac{n_2}{n_2-2}$ if $n_2 > 2$

$V(F) = \frac{n_2^2(2n_1+2n_2-4)}{n_1(n_2-2)^2(n_2-4)}$ if $n_2 > 4$.

$\text{mod}(F) = \frac{n_2(n_1-2)}{n_1(n_2+2)}$ if $n_1 > 2$

$t_n^2 \stackrel{D}{=} F_{1,n}$

\# $F \sim F_{n_1, n_2} \Rightarrow \frac{1}{F} \sim F_{n_2, n_1}$

\# If q_p is the pth quantile of F_{n_1,n_2}

and q_p' " " p th " " F_{n_2, n_1}

then $q_p' = \frac{1}{q_{1-p}}$

Bivariate Normal Distribution

$(X, Y) \sim BN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right\}}$$

$I(x,y) \in \mathbb{R}^2$

$|\rho| < 1$

$\Rightarrow X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$

$$Y \mid X = x \sim N\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1),\ \sigma_2^2(1-\rho^2)\right)$$

\# $(X, Y) \sim BN$ then X and Y are independently distributed iff $\rho(X,Y) = 0$.

\# $$M_{X,Y}(t_1, t_2) = E\left(e^{t_1 X + t_2 Y}\right)$$
$$= e^{t_1\mu_1 + t_2\mu_2 + \frac{1}{2}\left\{t_1^2\sigma_1^2 + t_2^2\sigma_2^2 + 2\rho\sigma_1\sigma_2 t_1 t_2\right\}}$$

Some important transformations

- $X_1, X_2 \overset{iid}{\sim} C(0,1) \Rightarrow X_1 + X_2 \sim C(0,2)$
- $x_1, x_2, \cdots, x_n$: r.s. from $C(\mu, 1)$

  Then $\Sigma x_i \sim C(n\mu, n)$ and $\bar{x} \sim C(\mu, 1)$
- AM $(A) = \bar{x}$ and HM $(H) = \dfrac{n}{\sum^{n} \frac{1}{x_i}}$ has the same distn if $x_1, x_2, \cdots, x_n \overset{iid}{\sim} C(0,1)$
- $\left.\begin{array}{l} X \sim \gamma(m) \\ Y \sim \gamma(n) \\ Z \sim \gamma(p) \end{array}\right\}$ independently

  Then $X+Y+Z \sim \gamma(m, n, p)$

  $\dfrac{Y}{Y+Z} \sim \beta_1(n, p)$

  $\dfrac{X}{X+Y+Z} \sim \beta_1(m, n+p)$

  $\dfrac{X+Y}{X+Y+Z} \sim \beta_1(m+n, p)$
- $\left.\begin{array}{l} X \sim \gamma(n) \\ Y \sim \gamma(n+\frac{1}{2}) \end{array}\right\}$ independent

  Then $2\sqrt{XY} \sim \gamma(2n)$

○ $X_1 \sim \beta_1(n_1, n_2)$
$X_2 \sim \beta_1(n_1+\frac{1}{2}, n_2)$ > independent $\Rightarrow \sqrt{X_1 X_2} \sim \beta_1(2n_1, 2n_2)$

○ $X \sim \beta_1(a, b)$
$Y \sim \beta_1(c, d)$ > independent
and $a = c+d$ then $XY \sim \beta_1(c, b+d)$

○ $X \sim R(0,1)$
$Y \sim R(0,1)$ > independent
$U = \sqrt{-2\ln X}\cos 2\pi Y$, $V = \sqrt{-2\ln X}\sin 2\pi Y$
Then $U, V \overset{iid}{\sim} N(0,1)$

○ $X, Y \overset{iid.}{\sim} N(0,1)$
Then $\frac{XY}{\sqrt{X^2+Y^2}}, \frac{X^2-Y^2}{2\sqrt{X^2+Y^2}} \overset{iid.}{\sim} N(0, \frac{1}{4})$

○ Suppose $X_i \overset{iid}{\sim} N(0,1)$, $i = 1(1)4$.
Then $M_{X_1 X_2}(t) = \frac{1}{\sqrt{1-t^2}}$, $|t|<1$
and $M_{X_1X_2 - X_3X_4}(t) = \frac{1}{1-t^2}$, $|t|<1$

i.e. $X_1X_2 - X_3X_4 \sim$ Standard Laplace Distn

Order Statistics

\# The df of the rth order statistic $X_{(r)}$ is,

$$F_{X_{(r)}}(x) = \sum_{j=r}^{n} \binom{n}{j} (F(x))^j (1-F(x))^{n-j}$$

$$\left(\begin{aligned} \because F_r(x) &= P(\text{atleast } r \text{ of the } X_i\text{'s are} \le x) \\ &= P(Y \ge r) = \sum_{j=r}^{n} P(Y=j) \\ &= \sum_{j=r}^{n} \binom{n}{j} \{F(x)\}^j \{1-F(x)\}^{n-j} \end{aligned} \right)$$

So, $F_r(x) = I_{F(x)}(r, n-r+1)$

<u>Discrete Case</u>

$$P_r(x) = I_{F(x)}(r, n-r+1) - I_{F(x-0)}(r, n-r+1)$$

<u>Continuous Case</u>

$$f_r(x) = \frac{n!}{(r-1)!\,(n-r)!} (F(x))^{r-1} (1-F(x))^{n-r} f(x)$$

\# $X_1, X_2, \ldots, X_n$: R.S. from $R(0,1)$

$X_{(r)} \sim \beta_1(r, n-r+1)$

$$E(X_{(r)}) = \frac{r}{n+1}$$

\# $F(X_1), F(X_2), \ldots, F(X_n)$: iid $R(0,1)$

$$\Rightarrow E(F(X_{(r)})) = \frac{r}{n+1}$$

Joint distn of $X_{(r)}, X_{(s)}$

Continuous Case

$$f_{r,s}(x,y) = \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}\,(F(x))^{r-1}\,(F(y)-F(x))^{s-r-1}\,(1-F(y))^{n-s}\,f(x)\,f(y)$$

Discrete Case

$$p_{r,s}(x,y) = F_{r,s}(x,y) - F_{r,s}(x-0,y) - F_{r,s}(x,y-0) + F_{r,s}(x-0,y-0)$$

$R(0,1) \rightarrow R.S.: x_1, x_2, \ldots, x_n$

$E(R) = \frac{n-1}{n+1}$ where $R = X_{(n)} - X_{(1)} \sim \beta_1(n-1, 2)$