7. Chapter: Analysis of frequency dependent structures – 2nd order filters

Time of study: 6 hours

Goals: the student should be able to

- define ideal transfer functions of 2nd order filters: Low Pass (LP); High Pass (HP); Band Pass (BP) and Band Reject (BR; band stop, notch)
- define basic properties of common approximation functions (Butterworth, Chebyshev, Bessel)
- analyze 2nd order filter properties – with an ideal amplifiers
- judge an influence of real amplifier properties on LP filters

Text

Filters are classified according to the functions they are to perform, in terms of ranges of frequencies, as ***pass bands*** and ***stop bands***.

A LOWPASS (LP) filter characteristic is one in which the pass band extends from $\omega = 0$ to $\omega = \omega_0$, where ω_0 is known as the ***cutoff frequency*** – fig. 1.

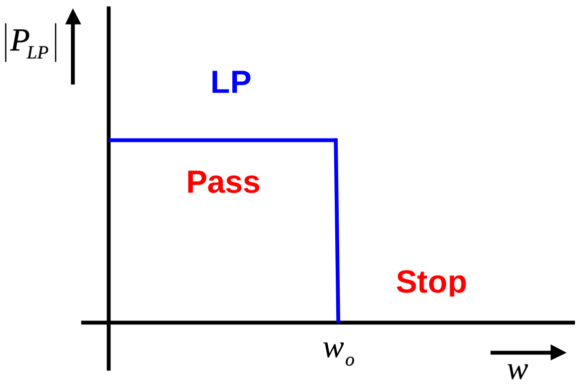


Fig. 1 Magnitude response $|P_{LP}|$ of an ideal lowpass-brick wall

A HIGHPASS (HP) filter is the complement to the lowpass filter in that the frequency range from 0 to ω_0 is a stop band, while from ω_0 to infinity is a pass band – fig. 2.

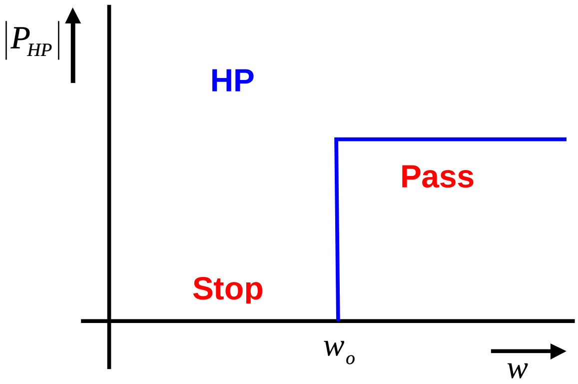


Fig. 2 Magnitude response $|P_{HP}|$ of an ideal highpass-brick wall

A BANDPASS (BP) filter is one which frequencies extending from ω_1 to ω_2 are passed, while other frequencies are stopped – fig. 3.

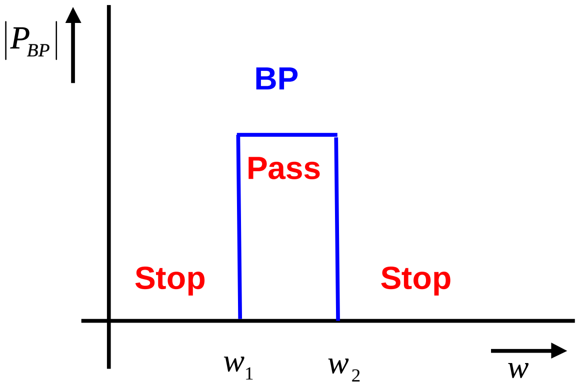


Fig. 3 Magnitude response $|P_{BP}|$ of an ideal bandpass-brick wall

A BANDSTOP (BS) filter is the complement of the bandpass filter where the frequencies from ω_1 to ω_2 are stopped and others are passed – fig. 3. These filters are sometimes known as ***notch*** filters or as ***rejec***t filters.

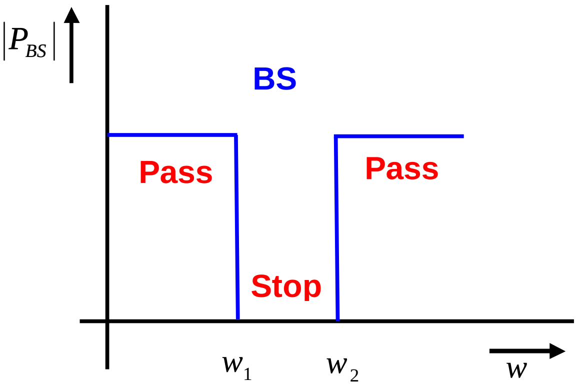


Fig. 4 Magnitude response $|P_{BS}|$ of an ideal bandstop-brick wall

It is not possible to realize the ideal characteristic above with a finite number of elements (it needs infinite number of elements). Realistic LP characteristic is shown in fig. 5. The sharpness of the transition from stop band to pass band can be controlled to some extent in the design of the filters.

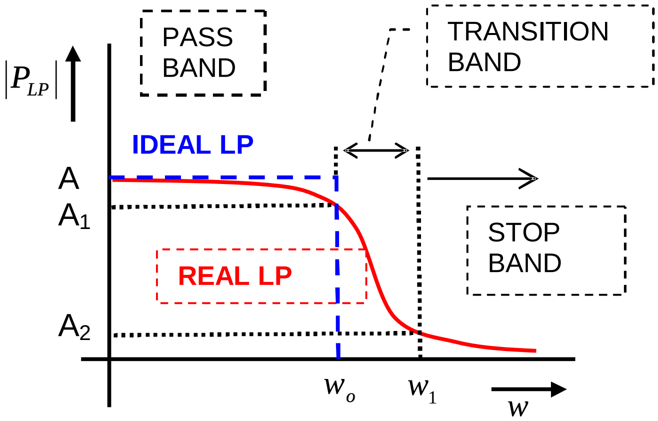


Fig. 5 Magnitude response $|P_{LP}|$ of a real lowpass

A PASS BAND – the magnitude of the transfer function $|P_{LP}|$ is always greater than a value designated A_1 [the ***attenuation*** $a_1 = 20\log(A/A_1)$ is less than a value designated as $a_{\max}$].

A STOP BAND – the magnitude of the transfer function $|P_{LP}|$ is always less than a value designated A_2 [the attenuation $a_2 = 20\log(A/A_2)$ is greater than a value designated as $a_{\min}$].

A TRANSITION BAND – a band of frequencies between the stop band and pass band.

The same definitions are analogously valid for the other above described filter types.

Approximation LP normalized functions – general – for a cascade realization:

fig. 6, tab. 1, 2, 3.

Even n (order n = 2, 4, 6,...) – cascade realization by means n/2 second order filters.

$$P_{LP}(s) = \prod_{k=1}^{n/2} A_k \frac{b_k}{s^2 + a_k s + b_k}$$

Odd n (order n = 1, 3, 5,...) – cascade realization by means one first order filter and (n-1)/2 second order filters.

$$P_{LP}(s) = \frac{A_0}{s+b_0} \times \prod_{k=1}^{(n-1)/2} A_k \frac{b_k}{s^2 + a_k s + b_k}$$

If we need denormalized LP we substitute $s = p/w_0$.

If we need denormalized HP we substitute $s = w_0/p$.

Here ω_0 is generally the "all filter frequency".

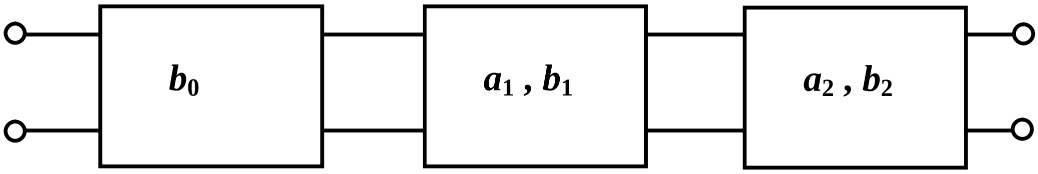


Fig. 6 An example of cascade filter realization – n = 5 (5^{th} order filter)

Three commonly used filter types (approximation) are:

- Butterworth
- Chebyshev
- Bessel

<u>Butterworth LP</u> (maximally flat magnitude); $b_k = 1$

$$a_k = 2 \cdot \sin \frac{(2k-1)p}{2n}; \quad Q_k = 1/a_k$$ ***- a quality factor***

This filter has the flattest possible pass-band magnitude response. Attenuation is – 3 dB at the design cutoff frequency, always. Attenuation above the cutoff frequency is a moderately steep 20-dB per decade per pole (per "every one order") – fig.7. The pulse response of the Butterworth filter has moderate overshoot and ringing.

$|P_{LP}| = 1/\sqrt{1+(w/w_0)^{2n}}$; n – Given number of poles (filter order)

$|P_{LP}(w_0)| = 1/\sqrt{1+(1)^{2n}} = 1/\sqrt{2} \quad \Rightarrow |P_{LP}(w_0)|_{dB} = 20 \cdot \log(2)^{-0,5} = -3\,dB$ - ***<u>always</u>***

$$|P_{LP}(w \geq w_0)| \cong 1/\sqrt{(w/w_0)^{2n}} = (w/w_0)^{-n} \Rightarrow$$

$|P_{LP}(w \geq w_0)|_{dB} = -n \cdot 20 \cdot \log(w/w_0) \quad \Rightarrow$ slope is $-n \times 20\,dB/dec$

Approximation LP function – for cascade realization

Even n (order n = 2, 4, 6,...) – cascade realization by means n/2 second order filters.

$$P_{LP}(p) = P_{LP}(s = p/w_0) = \prod_{k=1}^{n/2} \frac{A_k}{(p/w_0)^2 + a_k(p/w_0) + 1}$$

$$P_{LP}(p) = \prod_{k=1}^{n/2} \frac{A_k w_0^2}{p^2 + a_k w_0 p + w_0^2} = \prod_{k=1}^{n/2} \frac{A_k w_0^2}{p^2 + p w_0 / Q_k + w_0^2}$$

Odd n (order n = 1, 3, 5,...) – cascade realization by means one first order filter and (n-1)/2 second order filters – analogously we get

$$P_{LP}(p) = \frac{A_0 w_0}{p + w_0} \times \prod_{k=1}^{(n-1)/2} \frac{A_k w_0^2}{p^2 + a_k w_0 p + w_0^2}$$

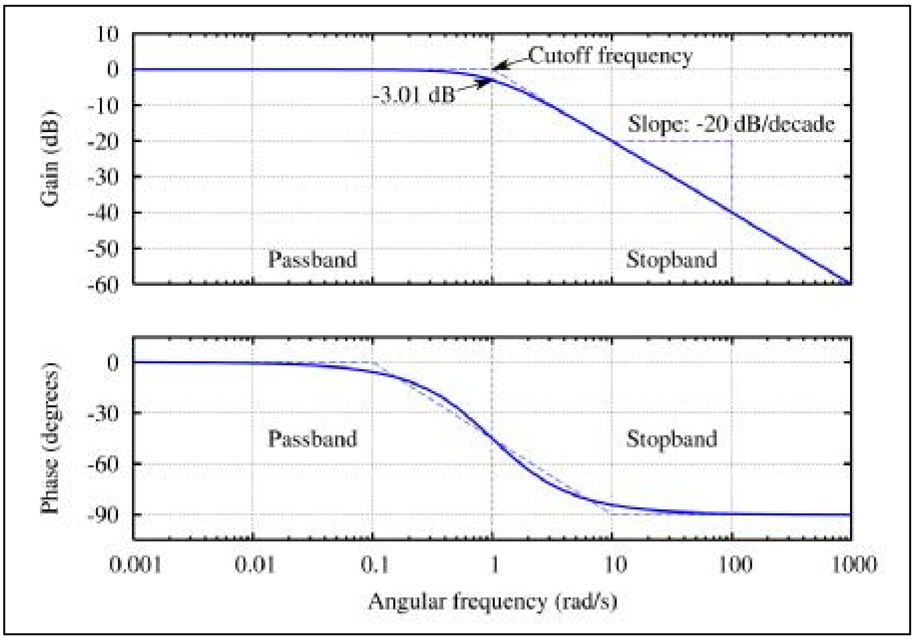


Fig. 7 The 1st order Butterworth lowpass

Example 1:

Derive a_1 and a_2 if you need Butterworth LP, the 4th order.

Solution:

It is even n = 4, thus k = 1 to n/2 = 1 to 2, thus

$k = 1$: $a_1 = 2 \cdot \sin\frac{(2k-1)p}{2n} = 2 \cdot \sin(p/8) = 0{,}7653668$

$k = 2$: $a_2 = 2 \cdot \sin\frac{(2 \cdot 2-1)p}{2n} = 2 \cdot \sin(3p/8) = 1{,}847759$

n	b_0	a_1	b_1	a_2	b_2
2	-	1,414 214	1, 000 000	-	-
3	1, 000 000	1, 000 000	1, 000 000	-	-
4	-	0,765 367	1, 000 000	1,847 759	1, 000 000
5	1, 000000	0,618 034	1, 000 000	1,618 034	1, 000 000

Tab.1: Butterworth filter

Determining needed n for the Butterworth lowpass

To determine n, we start with equation $|P_{LP}| = A/\sqrt{1+(w/w_0)^{2n}}$ - see fig. 5. We require $A(w) \le A_2$ if $w \ge w_1$. Attenuation on the frequency w_1 is $a = a_2 = 20 \cdot \log(A/A_2)$ (in dB). Further it means that $A_2 = |P_{LP}(w = w_1)| = A/\sqrt{1+(w_1/w_0)^{2n}}$. We substitute value of the A_2 and we get

$$a = 20\log(A/A_2) = 20\log\left(\frac{A}{A/\sqrt{1+(w_1/w_0)^{2n}}}\right) = 10\log\left(1+(w_1/w_0)^{2n}\right) \Rightarrow$$

$$a/10 = \log\left(1+(w_1/w_0)^{2n}\right) \Rightarrow \qquad 10^{a/10} = \left(1+(w_1/w_0)^{2n}\right) \Rightarrow$$

$$10^{a/10} - 1 = (w_1/w_0)^{2n} \Rightarrow \quad \log(10^{a/10} - 1) = 2\cdot n\cdot\log(w_1/w_0) \Rightarrow$$

$$n = \frac{\log(10^{a/10} - 1)}{2\cdot\log(w_1/w_0)}$$

This is the needed Butterworth filter order (n) if we require the attenuation value α on the frequency ω_1 – fig. 5.

Butterworth HP (maximally flat magnitude) – fig. 8

If we need Butterworth HP models we just substitute $s \to w_0/p$. We get

$|P_{HP}| = 1/\sqrt{1+(w_0/w)^{2n}}$; n – Given number of poles (filter order)

$|P_{HP}(w_0)| = 1/\sqrt{1+(1)^{2n}} = 1/\sqrt{2} \Rightarrow |P_{LP}(w_0)|_{dB} = 20\cdot\log(2)^{-0,5} = -3\,dB$ - ***always***

$|P_{HP}(w_0 \geq w)| \cong 1/\sqrt{(w_0/w)^{2n}} = (w/w_0)^n \Rightarrow$

$|P_{HP}(w_0 \geq w)|_{dB} = +n\cdot 20\cdot\log(w/w_0) \Rightarrow$ slope is $+n\times 20\,dB/dec$

$$P_{HP}(p) = P_{HP}(s = w_0/p) = \prod_{k=1}^{n/2}\frac{A_k}{(w_0/p)^2 + a_k(w_0/p) + 1} = \prod_{k=1}^{n/2}\frac{A_k\cdot p^2}{p^2 + a_k p + w_0^2}$$

and accordingly

$$P_{LP}(p) = \frac{A_0\cdot p}{p+1}\times\prod_{k=1}^{(n-1)/2}\frac{A_k\cdot p^2}{p^2 + a_k p + w_0^2}$$

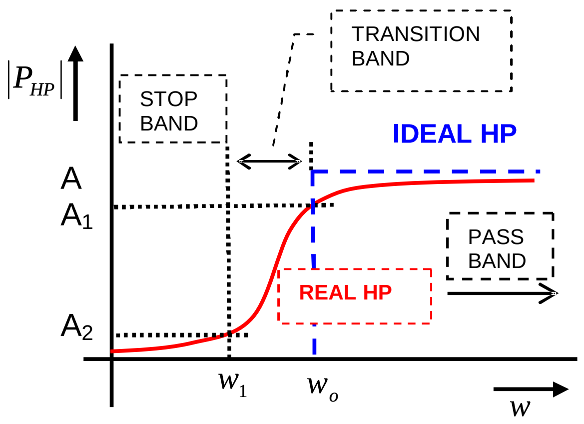


Fig. 8 Magnitude response $|P_{HP}|$ of a real highpass

Chebyshev (equal ripple magnitude), tab 2a, b, c, d

Chebyshev cutoff frequency is defined as the frequency at which the response falls below the ripple band. For a given number of poles (filter order), a steeper cutoff can be achieved by allowing more bandpass ripple – fig. 9. The Chebyshev has even more ringing in its pulse response than the Butterworth.

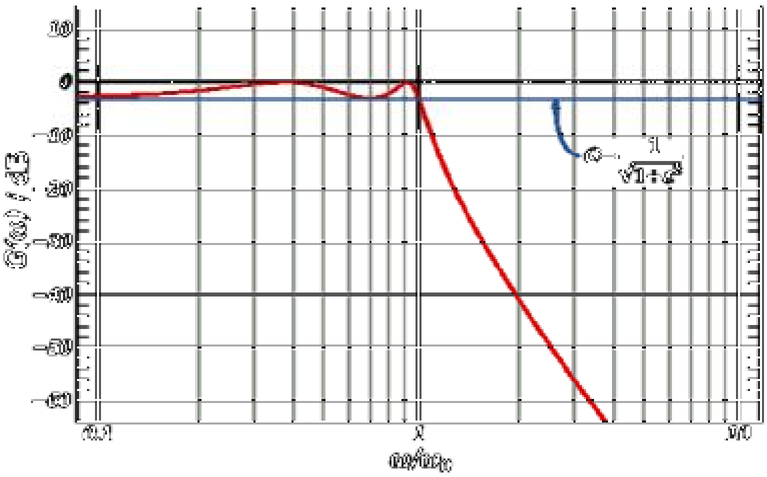


Fig. 9 An example of the Chebyshev magnitude response

n	b_0	a_1	b_1	a_2	b_2
2	-	1,425 625	1,516 203	-	-
3	0,626 456	0,626 456	1,142 448	-	-
4	-	0,350 706	1,063 519	0,846 680	0,356 412
5	0,362 320	0,223 926	1,035 784	0,586 245	0,476 676
Tab.2a: Chebyshev filter – bandpass ripple G = 0, 5 dB					

n	b_0	a_1	b_1	a_2	b_2
2	-	1,097 734	1,102 510	-	-
3	0,494 171	0,494 171	0,994 205	-	-
4	-	0,279 072	0,986 505	0,673 739	0,279 398
5	0,289 493	0,178 917	0,988 315	0,468 410	0,429 298
Tab.2b Chebyshev filter – bandpass ripple G = 1dB					

n	b_0	a_1	b_1	a_2	b_2
2	-	0,803 816	0,823 060	-	-
3	0,368 911	0,368 911	0,886 095	-	-
4	-	0,209 775	0,928 675	0,506 440	0,221 568
5	0,218 308	0,134 922	0,952 167	0,353 230	0,393 150
Tab.2c Chebyshev filter – bandpass ripple G = 2dB					

Example 2:

Derive denormalized *Chebyshev* HP transfer functions, the 3th order, bandpass ripple 2 dB
Solution:

We use Tab. 2c and substitution $s = w_0 / p$:

$$P_{HP}(p) = P_{HP}(s = w_0 / p) = \frac{A_0 b_0}{w_0 / p + b_0} \times \prod_{k=1}^{(3-1)/2} \frac{A_k b_k}{w_0^2 / p^2 + a_k w_0 / p + b_k} =$$

$$= \frac{A_0 p}{p + w_0 / b_0} \cdot \frac{A_1 b_1}{w_0^2 / p^2 + a_1 w_0 / p + b_1} = \frac{A_0 p}{p + w_0 / b_0} \cdot \frac{A_1 \cdot p^2}{p^2 + w_0 p a_1 / b_1 + w_0^2 / b_1} =$$

$$= \frac{A_0 p}{p + w_0 / 0{,}368911} \cdot \frac{A_1 \cdot p^2}{p^2 + w_0 p \cdot 0{,}368911 / 0{,}886095 + w_0^2 / 0{,}886095} =$$

$$= \frac{A_0 p}{p + w_0 \cdot 2{,}7107} \cdot \frac{A_1 \cdot p^2}{p^2 + w_0 p \cdot 0{,}4163 + w_0^2 \cdot 1{,}1285}$$

Thus we must realize the cascade connection of the 1st order HP filter with a partial characteristic frequency $w_p = w_0 / b_0 = w_0 \cdot 2{,}7107$ and of the 2nd order HP filter with a partial characteristic frequency $w_p = \sqrt{w_0^2 / b_1} = w_0 / \sqrt{b_1} = w_0 \cdot 1{,}0623$ because generally model of the second order denominator is always $p^2 + p \cdot w_p / Q_p + w_p^2$ - its quality factor we get from equation $w_0 a_1 / b_1 = w_p / Q_p$ thus

$$Q_p = w_p b_1 / (w_0 a_1) = (w_0 / \sqrt{b_1}) \cdot b_1 / (w_0 a_1) = \sqrt{b_1} / a_1 = 2{,}5516$$

It is valid that 3-dB frequency of LP Chebyshev filters is

$$\frac{w_3}{w_0} = \cosh\left(\frac{\operatorname{arg}\cosh(1/e)}{n}\right); \quad \operatorname{arg}\cosh x = \ln(x + \sqrt{x^2 - 1})$$

where we know that bandpass ripple (in dB) is $G = 20 \cdot \log \sqrt{1 + e^2} = 10 \cdot \log(1 + e^2)$.

For HP Chebyshev is valid reciprocal formula $\frac{w_0}{w_3} = \cosh\left(\frac{\operatorname{arg}\cosh(1/e)}{n}\right)$

Example 3:

Determine the 3-dB frequency of the:

a) 3rd order LP Chebyshev filter with bandpass ripple G = 1 dB.
b) 5th order LP Chebyshev filter with bandpass ripple G = 0, 5 dB.
c) 3rd order HP Chebyshev filter with bandpass ripple G = 1 dB.

Solution:

Ad a) $G = 10 \cdot \log(1 + e^2) \quad \Rightarrow \quad e = \sqrt{10^{G/10} - 1} \quad \Rightarrow \quad e = \sqrt{10^{0{,}1} - 1} = 0{,}5088$

$$\frac{w_3}{w_0} = \cosh\left(\frac{\operatorname{arg}\cosh(1/0{,}5088)}{3}\right) = \cosh\left(\frac{\operatorname{arg}\cosh 1{,}96541}{3}\right) = \cosh\left(\frac{\ln(3{,}6574)}{3}\right) = 1{,}095$$

Ad b) $G = 10 \cdot \log(1 + e^2) \quad \Rightarrow \quad e = \sqrt{10^{G/10} - 1} \quad \Rightarrow \quad e = \sqrt{10^{0,05} - 1} = 0,3493$

$$\frac{w_3}{w_0} = \cosh\left(\frac{\arg\cosh(1/0,3493)}{5}\right) = \cosh\left(\frac{\arg\cosh 2,8628}{5}\right) = \cosh\left(\frac{\ln(5,5452)}{5}\right) = 1,059$$

Ad c) We do reciprocal calculus for the HP – we use result of a) solution:

$$\frac{w_0}{w_3} = 1,095 \quad \Rightarrow \quad w_3 = w_0 / 1,095$$

See Tab. 2d, too.

Chebyshev LP filter order n	bandpass ripple G		
	0,5 dB	1 dB	2 dB
2	1,390	1,218	1,074
3	1,168	1,095	1,033
4	1,093	1,053	1,018
5	1,059	1,034	1,012
	ω_3 / ω_0		

Tab. 2d 3-dB frequencies of some Chebyshev LP filters (HP is reciprocal)

Bessel (maximally flat time delay; also called Thomson), tab 3a, b

Due to its linear phase response, this filter has excellent pulse response (minimal overshoot and ringing) – its group delay $t = -dj\ /dw$ is constant, ideally $t = 1/w_0$, really see fig.10 - $t\,(w = w_0) = 0,9231/w_0$ for n =2; $t\,(w = w_0) = 0,9964/w_0$ for n =3; and $t\,(w = w_0) = 0,9999/w_0$ for n =4. For a given number of poles (filter order), its magnitude response is neither as flat, nor its attenuation beyond the -3 – dB cutoff frequency as steep as the Butterworth.

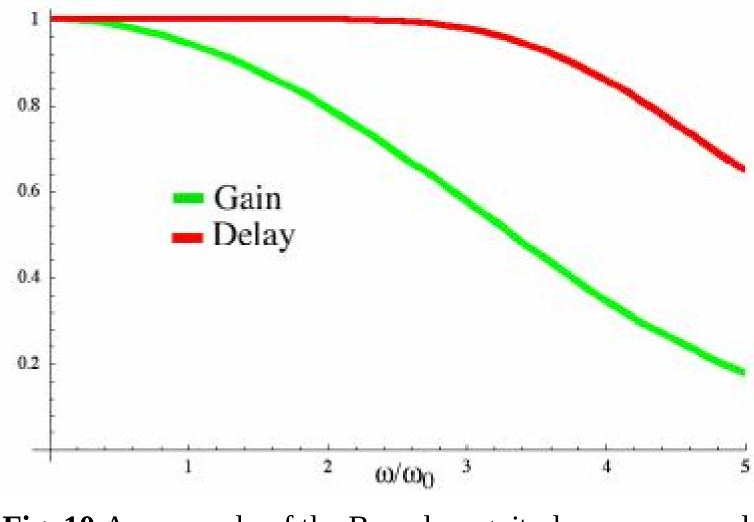


Fig. 10 An example of the Bessel magnitude response and group delay

n	b_0	a_1	b_1	a_2	b_2
2	-	3,000 000	3,000 000	-	-
3	2,322 185	3,677 815	6,459 433	-	-
4	-	5,792 421	9,140 131	4,207 579	11,487 800
5	3,646 739	6,703 913	14,272 481	4,649 349	18,156 315
Tab.3a Bessel filter					

Denormalized Bessel transfer functions we get analogously as above.

Example 4:

Derive denormalized *Bessel* LP transfer functions, the 4th order.
Solution:
We use Tab. 3 and substitution $s = p / w_0$:

$$P_{LP}(p) = P_{LP}(s = p/w_0) = \prod_{k=1}^{2} A_k \frac{b_k}{(p/w_0)^2 + a_k p/w_0 + b_k} =$$

$$= \frac{A_1 b_1 w_0^2}{p^2 + pa_1 w_0 + b_1 w_0^2} \times \frac{A_2 b_2 w_0^2}{p^2 + pa_2 w_0 + b_2 w_0^2} =$$

$$= \frac{A_1 \cdot 9{,}140131 \cdot w_0^2}{p^2 + p \cdot 5{,}792421 \cdot w_0 + 9{,}140131 \cdot w_0^2} \times \frac{A_1 \cdot 11{,}487800 \cdot w_0^2}{p^2 + p \cdot 4{,}207579 \cdot w_0 + 11{,}487800 \cdot w_0^2}$$

Thus we must realize the cascade connection of two 2nd orders LP filters. A generally model of the second order denominator is always $p^2 + p \cdot w_p / Q_p + w_p^2$ thus:

The first second order filter: $w_{p1} = w_0 \sqrt{b_1} = w_0 \cdot 3{,}0323$;

$a_1 w_0 = w_{p1} / Q_{p1} \Rightarrow Q_{p1} = \sqrt{b_1} / a_1 = 0{,}5219$

The second second order filter: $w_{p2} = w_0 \sqrt{b_2} = w_0 \cdot 3{,}3894$;

$a_2 w_0 = w_{p2} / Q_{p2} \Rightarrow Q_{p2} = \sqrt{b_2} / a_2 = 0{,}8055$

3-dB frequencies of LP Bessel filters are in Tab. 3b. For the HP Bessel filter we use reciprocal data.

filter order n	2	3	4	5
w_3 / w_0	1,36	1,75	2,11	2,42
Tab.3b 3-dB frequencies of Bessel filters				

Summary

Butterworth Response

Advantages: It provides maximally flat magnitude response in the pass-band. It has good all-around performance. Its pulse response is better than Chebyshev. Its rate of attenuation is better than that of Bessel.

Disadvantages: Some overshoot and ringing is exhibited in step response.

Chebyshev Response

Advantages: It provides better attenuation beyond the pass-band than Butterworth.

Disadvantages: Ripple in pass-band may be objectionable. There is considerable ringing in step response.

Bessel Response

Advantages: It provides best step response: very little overshoot or ringing.

Disadvantages: It exhibits slower rate of attenuation beyond the pass-band than Butterworth.

See fig. 11, too.

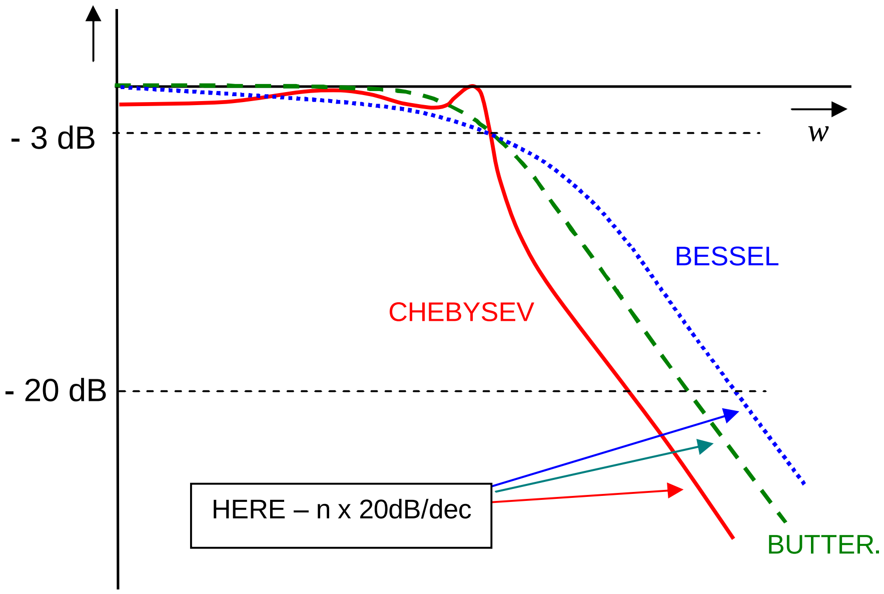


Fig. 11 Qualitative comparison of filter magnitudes for the same 3 dB frequency and the same filter order n

Basic low-pass network functions

Single pole (1st order LP) – the single-pole low-pass transfer function in the complex frequency variables is

$$\hat{P}_{LP1}(p = jw) = \frac{A_0 w_p}{p + w_p}$$

w_p ***- is just the characteristics frequency of this 1st pole LP (not of the "all filter" if we use cascade realization).***

The **<u>magnitude of the transfer</u>** function for the response to sinusoidal steady-state excitation is

$$P_{LP1}(w) = \sqrt{\frac{A_0^2 w_p^2}{w^2 + w_p^2}}$$

The **<u>phase is</u>**

$$j\,(w) = \arg \frac{A_0 w_p}{jw + w_p} = -\mathrm{arctg}(w / w_p)$$

The **<u>group delay</u>** is generally (always) $t = -\mathrm{d}j\,(w) / \mathrm{d}w$

<u>Complex Conjugate Pole Pair (2^nd order LP)</u> – the complex-conjugate-pole-pair LP transfer function and the sinusoidal steady-state magnitude function are

$$\hat{P}_{LP2}(p = jw) = A_0 \cdot \frac{w_p^2}{p^2 + aw_p p + w_p^2} = A_0 \cdot \frac{w_p^2}{p^2 + pw_p / Q + w_p^2}$$

$A_0;\ a = 1/Q;\ w_p$ ***- just the properties of this 2^nd pole LP (not of the "all filter" if we use cascade realization).***

$$P_{LP2}(w) = \sqrt{\frac{A_0^2 w_p^4}{(w_p^2 - w^2)^2 + (aw_p w)^2}} = \left| x = \frac{w}{w_p} \right| = \frac{A_0}{\sqrt{(1 - x^2)^2 + (ax)^2}}$$

Differentiation with respect to x gives (it is enough to use denominator)

$$\frac{d}{dx}\left[(1 - x^2)^2 + (ax)^2\right] = 2 \cdot (1 - x^2)(-2x) + 2a^2 x = 2x \cdot (2x^2 - 2 + a^2)$$

Now we can determine that magnitude maximum is if (function extreme)

$$(2x_m^2 - 2 + a^2) = 0$$

So we get

$$x_m^2 = 1 - a^2/2 \quad \Rightarrow \quad \left(\frac{w_m}{w_p}\right)^2 = 1 - a^2/2 \quad \Rightarrow \quad w_m = w_p \cdot \sqrt{1 - a^2/2} = w_p \cdot \sqrt{1 - 1/(2Q^2)}$$

The ***magnitude has a peak*** at $w_m = w_p \cdot \sqrt{1 - a^2/2} = w_p \cdot \sqrt{1 - 1/(2Q^2)}$ for $a \le \sqrt{2}$, ***value of this peak*** is (fig.12)

$$P_{LP2}(w_m) = P_{LP2MAX}\left|x = x_m\right| = \frac{A_0}{\sqrt{(1-x_m^2)^2 + (ax_m)^2}} = \frac{A_0}{\sqrt{(1-(1-a^2/2))^2 + a^2(1-a^2/2)}} =$$

$$= \frac{A_0}{a\cdot\sqrt{1-a^2/4}} = \frac{A_0\cdot Q}{\sqrt{1-1/(4Q^2)}}$$

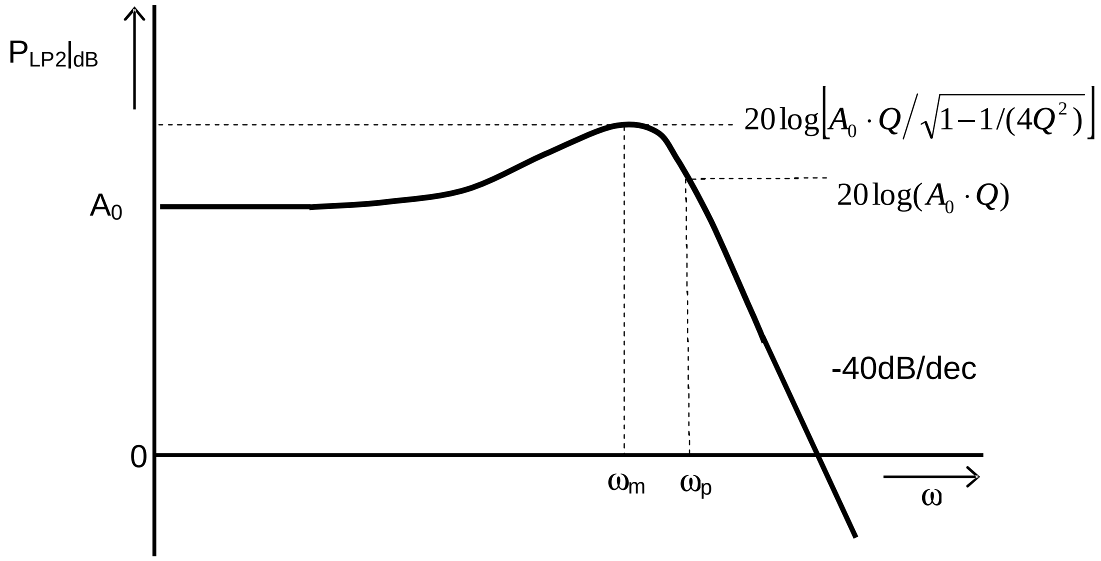


Fig. 12 Magnitude of the LP2 in dB: $P_{LP2}\big|_{dB} = 20\log P_{LP2}$

Basic high-pass network functions

Single pole (1st order HP) – the single-pole high-pass transfer function in the complex frequency variables is

$$\hat{P}_{HP1}(p = jw) = \frac{A_0 p}{p + w_p}$$

w_p ***- is just the characteristics frequency of this 1st pole HP (not of the "all filter" if we use cascade realization).***

The **magnitude of the transfer** function for the response to sinusoidal steady-state excitation is

$$P_{HP1}(w) = \sqrt{\frac{A_0^2 w^2}{w^2 + w_p^2}}$$

The **phase is**

$$j(w) = \arg\frac{A_0 jw}{jw + w_p} = p/2 - \mathrm{arctg}(w/w_p)$$

The **group delay** is generally (always) $t = -\mathrm{d}j(w)/\mathrm{d}w$

Complex Conjugate Pole Pair (2nd order HP) – the complex-conjugate-pole-pair HP transfer function and the sinusoidal steady-state magnitude are

$$\hat{P}_{HP2}(p = jw) = A_0 \cdot \frac{p^2}{p^2 + aw_p p + w_p^2} = A_0 \cdot \frac{p^2}{p^2 + pw_p / Q + w_p^2}$$

$A_0;\ a = 1/Q;\ w_p$ - ***just the properties of this 2nd pole HP (not of the "all filter" if we use cascade realization).***

$$P_{HP2}(w) = \sqrt{\frac{A_0^2 w^4}{(w_p^2 - w^2)^2 + (aw_p w)^2}} = \left| x = \frac{w_p}{w} \right| = \frac{A_0}{\sqrt{(x^2-1)^2 + (ax)^2}} = \frac{A_0}{\sqrt{(1-x^2)^2 + (ax)^2}}$$

Now we easy get again that

$$x_m^2 = 1 - a^2/2 \quad \Rightarrow \quad \left(\frac{w_p}{w_m}\right)^2 = 1 - a^2/2 \quad \Rightarrow \quad w_m = w_p / \sqrt{1 - a^2/2} = w_p / \sqrt{1 - 1/(2Q^2)}$$

The ***magnitude has a peak*** at $w_m = w_p / \sqrt{1 - a^2/2} = w_p / \sqrt{1 - 1/(2Q^2)}$ for $a \le \sqrt{2}$, ***value of this peak*** is (fig.13)

$$P_{HP2}(w_m) = P_{HP2MAX} \left| x = x_m \right| = \frac{A_0}{\sqrt{(1-x_m^2)^2 + (ax_m)^2}} = \frac{A_0}{\sqrt{(1-(1-a^2/2))^2 + a^2(1-a^2/2)}} =$$

$$= \frac{A_0}{a \cdot \sqrt{1 - a^2/4}} = \frac{A_0 \cdot Q}{\sqrt{1 - 1/(4Q^2)}}$$

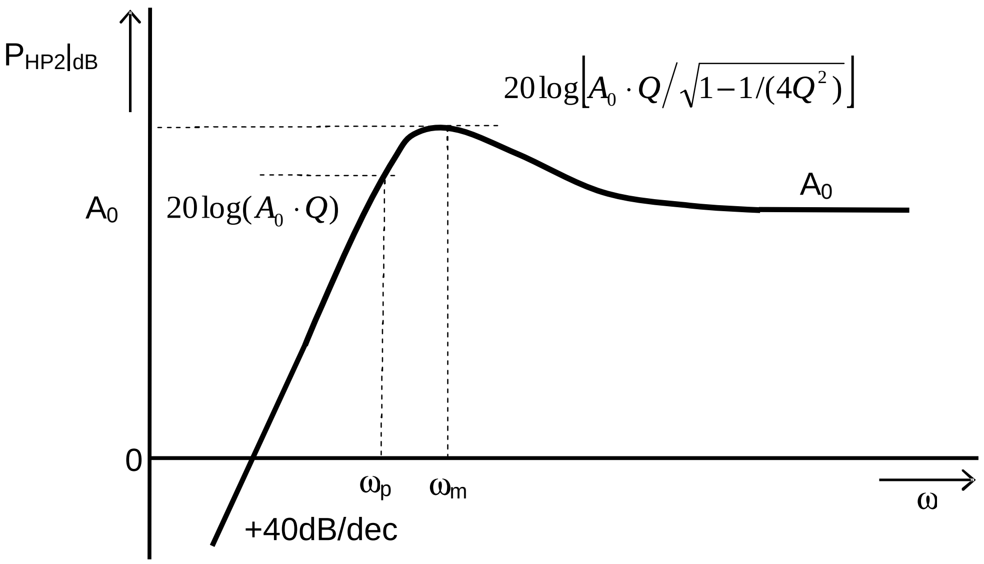


Fig. 13 Magnitude of the HP2 in dB: $P_{HP2}\big|_{dB} = 20\log P_{HP2}$

Properties described above we use for tuning partial active filter stages to get needed properties of the "all cascade filter connection".

Band-pass network function (2nd order), fig. 14

The complex-conjugate-pole-pair BP transfer function and the sinusoidal steady-state magnitude are

$$P_{BP} = \frac{A_0 p w_p / Q}{p^2 + p w_p / Q + w_p^2}$$

$$|P_{BP}| = \frac{A_0 w w_p / Q}{\sqrt{(w_p^2 - w^2)^2 + (w w_p / Q)^2}} = \frac{A_0}{\sqrt{\dfrac{(w_p^2 - w^2)^2 + (w w_p / Q)^2}{(w w_p / Q)^2}}} = \frac{A_0}{\sqrt{\left(\dfrac{w_p^2 - w^2}{w w_p}\right)^2 Q^2 + 1}} =$$

$$= \left| x = \frac{w}{w_p} \right| = \frac{A_0}{\sqrt{(1/x - x)^2 Q^2 + 1}}$$

$$|P_{BP}(x=1)| = \frac{A_0}{\sqrt{(1/1 - 1)^2 Q^2 + 1}} = A_0$$

$$|P_{BP}(x \le 1)| \cong \frac{A_0}{\sqrt{(1/x)^2 Q^2}} = A_0 \cdot x / Q \quad \Rightarrow$$ the first asymptote is + 20 dB/dec

$$|P_{BP}(x \ge 1)| \cong \frac{A_0}{\sqrt{(-x^2)^2 Q^2}} = A_0 / (x \cdot Q) \quad \Rightarrow$$ the second asymptote is - 20 dB/dec

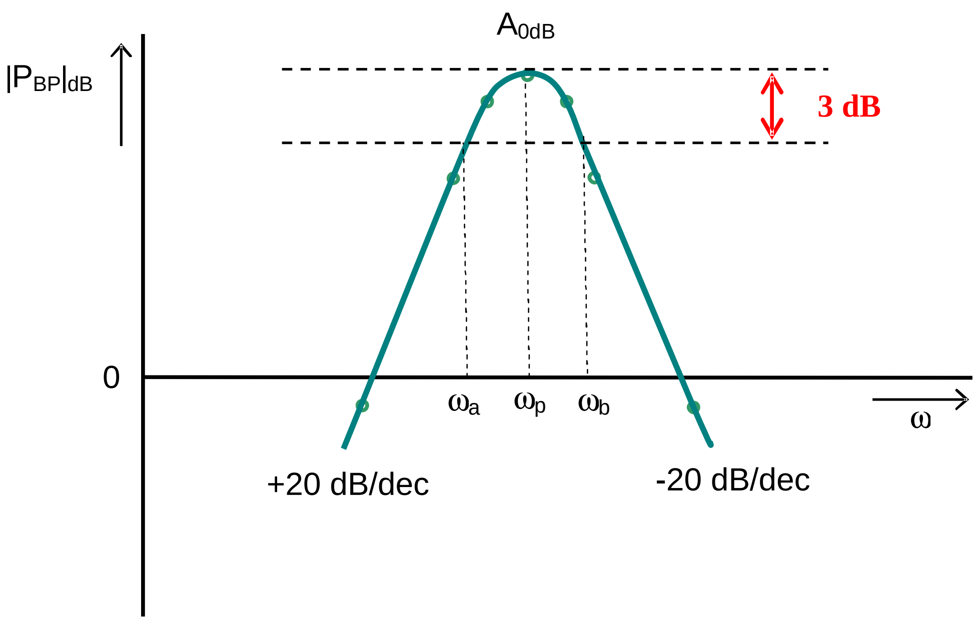


Fig. 14 Qualitative depiction of the BP magnitude

We next compute the 3-dB relative frequencies x_3 for the bandpass response. It means that it must be

$$\left|P_{BP}\right|_{3dB} = \frac{A_0}{\sqrt{(1/x_3 - x_3)^2 Q^2 + 1}} = \frac{A_0}{\sqrt{2}} \quad \Rightarrow \quad (1/x_3 - x_3)^2 Q^2 + 1 = 2 \quad \Rightarrow$$

$$(1/x_3 - x_3)^2 = 1/Q^2 \quad \Rightarrow \quad (1/x_3 - x_3) = \pm 1/Q \quad \Rightarrow \quad x_3^2 \pm x_3/Q - 1 = 0 \quad \Rightarrow$$

We get only two physically (positive) right roots (two negative roots we neglect):

$$x_{3a} = \sqrt{1 + \left(\frac{1}{2Q}\right)^2} - \frac{1}{2Q} \qquad \text{and} \qquad x_{3b} = \sqrt{1 + \left(\frac{1}{2Q}\right)^2} + \frac{1}{2Q}$$

Thus is valid:

$$\frac{w_a}{w_p} = \sqrt{1 + \left(\frac{1}{2Q}\right)^2} - \frac{1}{2Q} \quad \Rightarrow \quad w_a = w_p \cdot \left(\sqrt{1 + \left(\frac{1}{2Q}\right)^2} - \frac{1}{2Q}\right)$$

$$\frac{w_b}{w_p} = \sqrt{1 + \left(\frac{1}{2Q}\right)^2} + \frac{1}{2Q} \quad \Rightarrow \quad w_b = w_p \cdot \left(\sqrt{1 + \left(\frac{1}{2Q}\right)^2} + \frac{1}{2Q}\right)$$

These frequencies are identified in fig. 14. We easy determine that

$$w_a \cdot w_b = w_p^2$$

and frequency difference of these frequencies defines the bandwidth (BW)

$$BW = w_b - w_a = \frac{w_p}{Q} \quad \Rightarrow \quad Q = \frac{w_p}{BW} = \frac{w_p}{w_b - w_a}$$

Band-stop network function (2nd order), fig.15

The complex-conjugate-pole-pair BS transfer function and the sinusoidal steady-state magnitude are

$$P_{BS} = A_0 \frac{p^2 + w_o^2}{p^2 + p w_o / Q + w_o^2}$$

$$\left|P_{BS}\right| = A_0 \cdot \frac{\left|-w^2 + w_p^2\right|}{\sqrt{(w_p^2 - w^2)^2 + (w w_p / Q)^2}}$$

$$\left|P_{BS}(w \le w_p)\right| = A_0 \cdot \frac{-w^2 + w_p^2}{\sqrt{(w_p^2 - w^2)^2 + (w w_p / Q)^2}} = \left| x = \frac{w}{w_p} \right| = A_0 \cdot \frac{1 - x^2}{\sqrt{(1 - x^2)^2 + (x/Q)^2}}$$

$$\left|P_{BS}(x=1)\right| = A_0 \frac{1-1^2}{\sqrt{(1-1^2)^2+(1/Q)^2}} = 0$$

$$\left|P_{BS}(x=0)\right| = A_0 \frac{1-0^2}{\sqrt{(1-0^2)^2+(0/Q)^2}} = A_0$$

$$\left|P_{BS}(w \ge w_p)\right| = A_0 \cdot \frac{+w^2-w_p^2}{\sqrt{(w_p^2-w^2)^2+(ww_p/Q)^2}} = \left| x = \frac{w}{w_p} \right| = A_0 \cdot \frac{x^2-1}{\sqrt{(1-x^2)^2+(x/Q)^2}}$$

$$\left|P_{BS}(x \to \infty)\right| = A_0 \cdot \lim_{x\to\infty} \frac{x^2-1}{\sqrt{(1-x^2)^2+(x/Q)^2}} = A_0$$

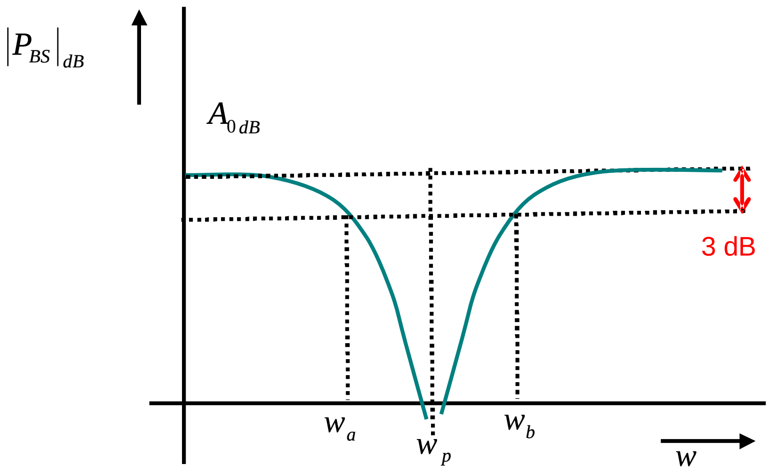


Fig. 15 Qualitative depiction of the BS magnitude

We next compute the 3-dB relative frequencies x_3 for the bandstop response. It means that it must be

$$\left|P_{BS}\right|_{3dB} = A_0 \cdot \frac{1-x_3^2}{\sqrt{(1-x_3^2)^2+(x_3/Q)^2}} = \frac{A_0}{\sqrt{2}} \quad \Rightarrow \quad \frac{1-x_3^2}{\sqrt{(1-x_3^2)^2+(x_3/Q)^2}} = \frac{1}{\sqrt{2}} \quad \Rightarrow$$

$$\frac{1}{\sqrt{\frac{(1-x_3^2)^2}{(1-x_3^2)^2}+\frac{(x_3/Q)^2}{(1-x_3^2)^2}}} = \frac{1}{\sqrt{2}} \quad \Rightarrow \qquad \frac{(x_3/Q)^2}{(1-x_3^2)^2} = 1 \quad \Rightarrow$$

$$\frac{(x_3/Q)}{(1-x_3^2)} = \pm 1 \quad \Rightarrow \qquad \pm x_3^2 + x_3/Q \pm 1 = 0$$

This equation gives the same physically right roots as it was for BP, so

$$w_a = w_p \cdot \left(\sqrt{1 + \left(\frac{1}{2Q} \right)^2} - \frac{1}{2Q} \right)$$

$$w_b = w_p \cdot \left(\sqrt{1 + \left(\frac{1}{2Q} \right)^2} + \frac{1}{2Q} \right)$$

These frequencies are identified in fig. 15. We easy determine that

$$w_a \cdot w_b = w_p^2$$

and frequency difference of these frequencies defines the bandwidth (BW) of the bandstop filter now:

$$BW = w_b - w_a = \frac{w_p}{Q} \quad \Rightarrow \quad Q = \frac{w_p}{BW} = \frac{w_p}{w_b - w_a}$$

Examples of second order filter realization

A useful structure you can see in fig. 16.

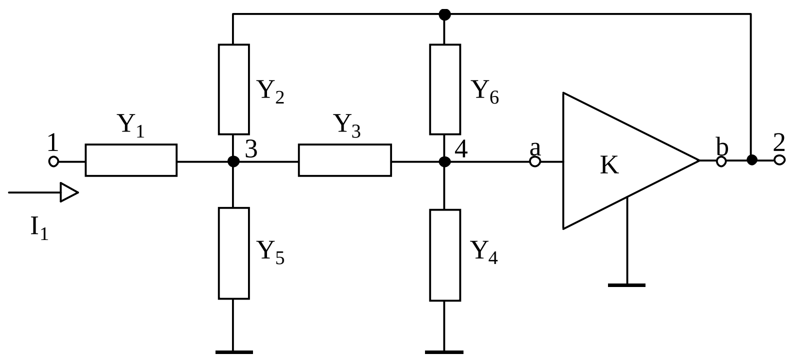


Fig. 16 The Bridgman-Brennar multiple-loop feedback biquad

We know the admittance model of the amplifier *K* – see Chapter 3:

$$\begin{array}{c|cc|} & \mathbf{a} & \mathbf{b} \\ \hline \mathbf{a} & 0 & 0 \\ \mathbf{b} & -KG_o & G_o \\ \hline \end{array} \begin{bmatrix} U_a \\ U_b \end{bmatrix} = \begin{bmatrix} I_a \\ I_b \end{bmatrix}$$

So we can easy determine an admittance model of the Bridgman-Brennar multiple-loop feedback biquad:

$$\begin{array}{c|cccc|}
 & 1 & 2(b) & 3 & 4(a) \\
\hline
1 & Y_1 & 0 & -Y_1 & 0 \\
2(b) & 0 & Y_2+Y_6+(G_o) & -Y_2 & -Y_6+(-KG_o) \\
3 & -Y_1 & -Y_2 & Y_1+Y_2+Y_3+Y_5 & -Y_3 \\
4(a) & 0 & -Y_6 & -Y_3 & Y_3+Y_4+Y_6
\end{array} \; x \; \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} I_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From this system of equations we derive, after some algebraic simplification, the ***basic transfer function*** (we suppose zero output resistance, thus $G_0 \to \infty$):

$$\left.\frac{U_2}{U_1}\right|_{G_o\to\infty} = \frac{KY_1Y_3}{Y_4(Y_1+Y_2+Y_3+Y_5)+Y_3(Y_1+Y_5)+(1-K)[Y_2Y_3+Y_6(Y_1+Y_2+Y_3+Y_5)]}$$

If we **choose** $K \to -\infty$ (an ideal inverting operational amplifier) we get

$$\left.\frac{U_2}{U_1}\right|_{\substack{G_o\to\infty \\ K\to-\infty}} = \frac{-Y_1Y_3}{[Y_2Y_3+Y_6(Y_1+Y_2+Y_3+Y_5)]}$$

In this case the admittance Y_4 has no effect on the transfer function (zero voltage across it).

If we **choose $K > 0$** we get filters with finite gain – ***Sallen and Key circuit***. If $Y_6 = Y_5 = 0$ we get

$$\left.\frac{U_2}{U_1}\right|_{G_o\to\infty} = \frac{KY_1Y_3}{Y_1Y_3+Y_4(Y_1+Y_2+Y_3)+Y_2Y_3(1-K)}$$

If $Y_6 = 0$ only we get

$$\left.\frac{U_2}{U_1}\right|_{G_o\to\infty} = \frac{KY_1Y_3}{Y_4(Y_1+Y_2+Y_3+Y_5)+Y_3(Y_1+Y_5)+Y_2Y_3(1-K)}$$

An appropriate choice of admittances gives us desired transfer functions.

Sallen-Key LP (2nd order)

The basic circuit you can see in fig. 17. It is evident that $Y_6 = Y_5 = 0$ and

$Y_1 = G_1$, $Y_2 = p\, C_A$, $Y_3 = G_3$, $Y_4 = p\, C_B$

An operational amplifier acts as noninverting amplifier with a finite gain K.

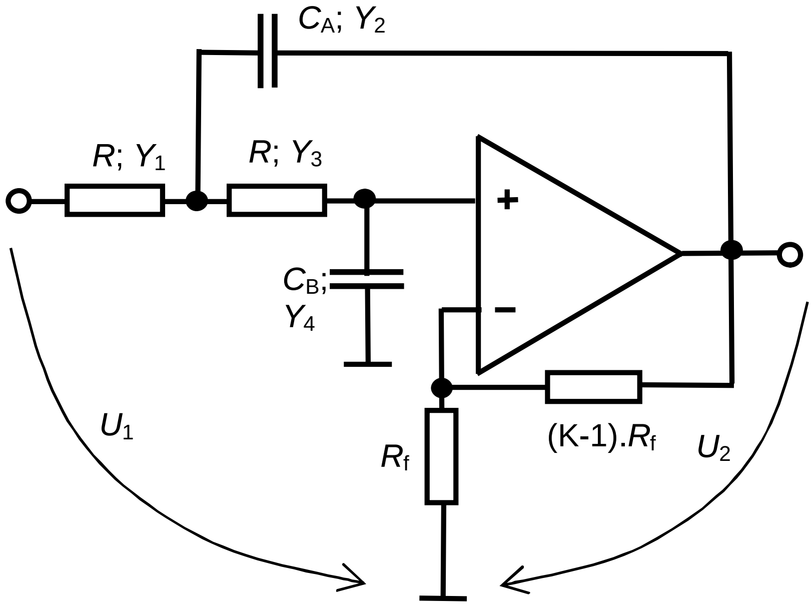


Fig. 17 One possible Sallen- Key LP configuration

Thus we can get (after some algebraic simplification)

$$\hat{P}_{LP2}(p = jw) = A_0 \cdot \frac{w_p^2}{p^2 + pw_p/Q + w_p^2}$$

where

$$w_p^2 = \frac{1}{R_1 R_3 C_A C_B}$$

$$2x = 1/Q = \sqrt{\frac{R_3 C_B}{R_1 C_A}} + \sqrt{\frac{R_1 C_B}{R_3 C_A}} + \sqrt{\frac{R_1 C_A}{R_3 C_B}} \cdot (1 - K)$$

$$A_o \equiv K$$

The first usual choice is: ***$R_1 = R_3 = R$ and $C_B = C_A = C$***. In this case we get

$$w_p^2 = \frac{1}{R^2 C^2}$$

$$2x = 1/Q = 3 - K$$

This filter is stable only if $K < 3$.

We can not define beforehand the filter gain, now. From these simple equations we can easy determine formulas suitable for this type ***filter design*** (we want ω_p and Q):

I. We chose value of the C.
II. We determine $R = 1/(\omega_p C)$
III. We determine needed $K = 3 - 1/Q = 3 - 2\xi$. We chose suitable value of the R_f and determine corresponding value (K-1).R_f

The second usual choice is: **$R_1 = R_3 = R$ and $K = 1$**; (K-1).R_f – short circuit; R_f – open.

In this case we get

$$w_p^2 = \frac{1}{R^2 C_A C_B}$$
$$2x = 1/Q = 2 \cdot \sqrt{C_B / C_A}$$

If we chose R, we can determine that

$$C_A = \frac{2Q}{w_p R}; \qquad C_B = \frac{1}{2Q w_p R}$$

This filter is stable always – theoretically.

BOX

$$w_p^2 = \frac{1}{R^2 C_A C_B} \Rightarrow C_A = \frac{1}{R^2 w_p^2 C_B}; \quad 1/Q = 2 \cdot \sqrt{C_B / C_A} = 2 \cdot \sqrt{\frac{C_B}{1/(R^2 w_p^2 C_B)}} =$$
$$= 2 \cdot \sqrt{C_B^2 R^2 w_p^2} \Rightarrow C_B = \frac{1}{2QRw_p}$$
$$C_A = \frac{1}{R^2 w_p^2 C_B} = \frac{1}{R^2 w_p^2 /(2QRw_p)} = \frac{2Q}{Rw_p}$$

The ***design procedure*** is very simple

I. We chose value of the R.

II. We determine $C_A = \dfrac{2Q}{Rw_p}$ *and* $C_B = \dfrac{1}{2QRw_p}$

Inverting LP (2nd order)

If we choose $K \to -\infty$ and $Y_1 = G_1$, $Y_2 = G_2$, $Y_3 = G_3$, $Y_5 = pC_5$ a $Y_6 = pC_6$ (fig. 18) we easy get

$$P_{LP2} = U_2 / U_1 = -\frac{1/(R_1 R_3 C_5 C_6)}{p^2 + p\dfrac{1}{C_5}\left(\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}\right) + \dfrac{1}{R_2 R_3 C_5 C_6}} = |rearranging| =$$

$$= -\frac{R_2}{R_1} \cdot \frac{1/(R_2 R_3 C_5 C_6)}{p^2 + p\dfrac{1}{C_5}\left(\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}\right) + \dfrac{1}{R_2 R_3 C_5 C_6}}$$

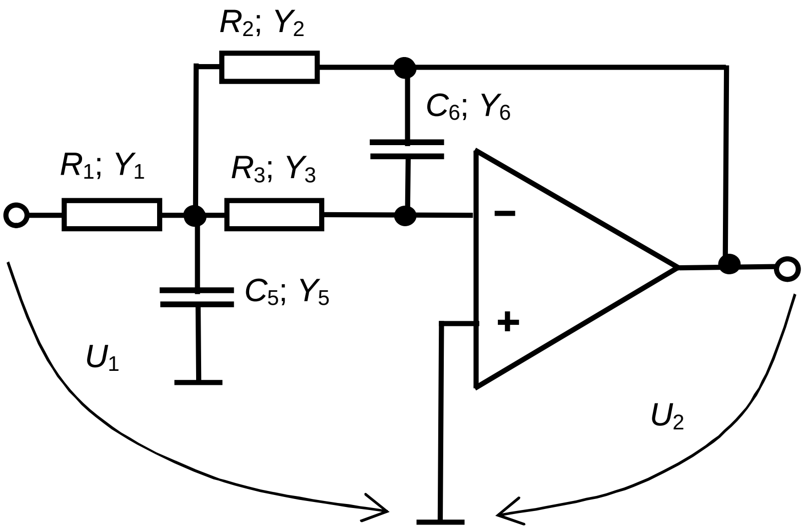


Fig. 18 One possible inverting LP configuration

It is evident now that – compare relation $\hat{P}_{LP2}(p=jw) = A_0 \cdot \dfrac{w_p^2}{p^2 + pw_p/Q + w_p^2}$ -

$$w_p^2 = \frac{1}{R_2 R_3 C_5 C_6}$$

$$A_o = -\frac{R_2}{R_1}$$

$$2xw_p = w_p/Q = \frac{1}{C_5}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

After rearranging we get

$$1/Q = 2x = \sqrt{\frac{C_6}{C_5}} \cdot \left(\sqrt{\frac{R_2 R_3}{R_1^2}} + \sqrt{\frac{R_3}{R_2}} + \sqrt{\frac{R_2}{R_3}}\right)$$

From these equations we can determine formulas suitable for this type ***filter design*** (we want A_o, ω_p and Q). We have just three equations. But we must determine five circuit elements. Thus we must choose two elements. Usually we choose C_5 = C and C_6 = m.C. Now we can determine that

$$w_p^2 = \frac{1}{R_2 R_3 m C^2}; \qquad A_o = -\frac{R_2}{R_1}; \qquad 1/Q = \sqrt{m} \cdot \left(\sqrt{\frac{R_2 R_3}{R_1^2}} + \sqrt{\frac{R_3}{R_2}} + \sqrt{\frac{R_2}{R_3}}\right)$$

and from these equations we derive

$$R_1 = \frac{1}{2\cdot|A_o|\cdot Q\cdot m\cdot w_p\cdot C}\left[1\pm\sqrt{1-4m(|A_o|+1)Q^2}\right]$$
$$R_2 = |A_o|\cdot R_1$$
$$R_3 = \frac{1}{w_p^2 R_2 m C^2}$$

BOX

$$A_o = -\frac{R_2}{R_1} \quad => \quad R_2 = |A_o|\cdot R_1; \quad w_p^2 = \frac{1}{R_2 R_3 m C^2} \quad => \quad R_3 = \frac{1}{R_2 w_p^2 m C^2};$$

$$1/Q = \sqrt{m}\cdot\left(\sqrt{\frac{R_2R_3}{R_1^2}}+\sqrt{\frac{R_3}{R_2}}+\sqrt{\frac{R_2}{R_3}}\right) = \sqrt{m}\cdot\left(\sqrt{\frac{\frac{1}{w_p^2mC^2}}{R_1^2}}+\sqrt{\frac{1}{R_2^2w_p^2mC^2}}+\sqrt{R_2^2w_p^2mC^2}\right)$$

$$1/Q = \sqrt{m}\cdot\left(\sqrt{\frac{1}{w_p^2mC^2R_1^2}}+\sqrt{\frac{1}{|A_o|^2R_1^2w_p^2mC^2}}+\sqrt{|A_o|^2R_1^2w_p^2mC^2}\right) \quad |rearranging\ gives|$$

$R_1^2 - \frac{R_1}{|A_o|\cdot m\cdot w_p\cdot C\cdot Q}+\frac{|A_o|+1}{|A_o|^2\cdot m\cdot w_p\cdot C^2}=0$. Now we easy determine that

$$R_{1a,b} = \frac{\frac{1}{|A_o|\cdot m\cdot w_p\cdot C\cdot Q}\pm\sqrt{\left(\frac{1}{|A_o|\cdot m\cdot w_p\cdot C\cdot Q}\right)^2-\frac{4\cdot(|A_o|+1)}{|A_o|^2\cdot m\cdot w_p\cdot C^2}}}{2}$$

$$R_{1a,b} = \frac{1}{2\cdot|A_o|\cdot Q\cdot m\cdot w_p\cdot C}\left[1\pm\sqrt{1-4m(|A_o|+1)Q^2}\right]$$

Physically right solution is $R_1 > 0$ (positive value of the R_1) – so it must be

$$1-4m(|A_o|+1)Q^2 > 0$$

Often we choose just extreme case $1-4m(|A_o|+1)Q^2 = 0$, thus

$$m = \frac{1}{4\cdot(|A_o|+1)\cdot Q^2}$$

The ***design procedure*** is now:

I. We chose appropriate value of the $C_6 = m\,C = m\,C_5$

II. We determine $C = C_5 = C_6/\,m = C_6 \cdot \left[4 \cdot \left(|A_o| + 1\right) \cdot Q^2\right]$

III. We determine

$$R_1 = \frac{1}{2 \cdot |A_o| \cdot Q \cdot m \cdot w_p \cdot C} = \frac{1}{2 \cdot |A_o| \cdot Q \cdot w_p \cdot C_6}$$

$$R_2 = |A_o| \cdot R_1 = \frac{1}{2 \cdot Q \cdot w_p \cdot C_6}$$

$$R_3 = \frac{1}{w_p^2 R_2 m C^2} = \frac{1}{w_p^2 \cdot (R_2 m C) \cdot C} = \left| R_2 m C = R_2 C_6 = \frac{1}{2 \cdot Q \cdot w_p} \right| = \frac{1}{2 \cdot Q \cdot (|A_o| + 1) \cdot w_p \cdot C_6}$$

Inverting BP (2nd order)

If we choose $K \to -\infty$ and $Y_1 = G_1$, $Y_2 = Y_3 = pC$, $Y_5 = 0$ a $Y_6 = G_6$ (fig. 19) we get

$$\hat{P}_{BP2}(p = jw) = A_0 \cdot \frac{p w_p / Q}{p^2 + p w_p / Q + w_p^2}$$

where

$$w_p^2 = \frac{1}{R_1 R_6 C^2}; \quad A_o = -R_6 /(2R_1); \quad 2x = 1/Q = 2 \cdot \sqrt{\frac{R_1}{R_6}}$$

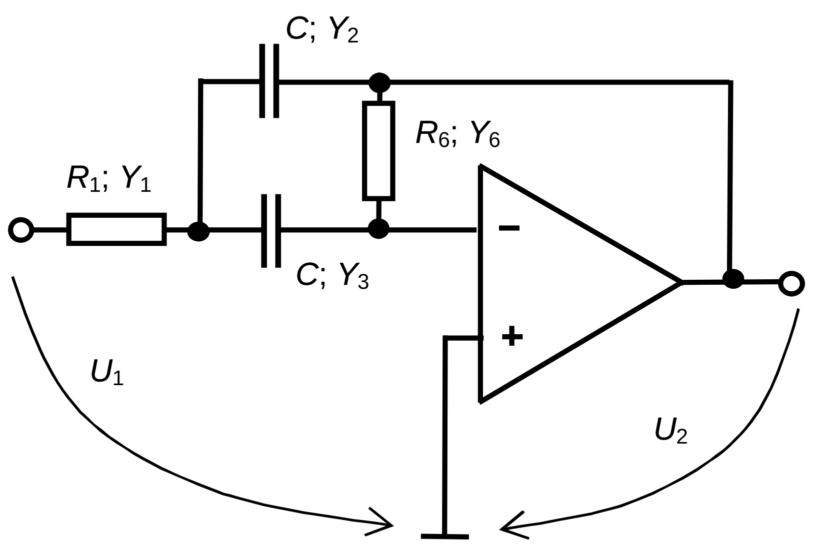


Fig. 19 One possible inverting BP configuration

BOX

$$\left.\frac{U_2}{U_1}\right|_{\substack{Y5=0\\K\to-\infty}} = \frac{-Y_1Y_3}{Y_2Y_3+Y_6(Y_1+Y_2+Y_3)} = \frac{-G_1pC}{p^2C^2+G_6(G_1+pC+pC)} =$$

$$= \frac{-G_1pC}{p^2C^2+2pCG_6+G_1G_6} = -\frac{G_1C}{C^2}\cdot\frac{p}{p^2+p\dfrac{2G_6}{C}+\dfrac{G_1G_6}{C^2}} =$$

$$= -\frac{G_1C}{C^2}\cdot\frac{p\cdot\dfrac{2G_6}{C}\cdot\dfrac{C}{2G_6}}{p^2+p\cdot\dfrac{2G_6}{C}+\dfrac{G_1G_6}{C^2}} = -\frac{G_1C}{C^2}\cdot\frac{C}{2G_6}\cdot\frac{p\cdot\dfrac{2G_6}{C}}{p^2+p\cdot\dfrac{2G_6}{C}+\dfrac{G_1G_6}{C^2}} =$$

$$= -\frac{R_6}{2R_1}\cdot\frac{p\cdot\dfrac{2}{CR_6}}{p^2+p\cdot\dfrac{2}{CR_6}+\dfrac{1}{R_1R_6C^2}} \Rightarrow w_p^2 = \frac{1}{R_1R_6C^2};\quad A_0 = -\frac{R_6}{2R_1}$$

$$\frac{w_p}{Q} = \frac{2}{CR_6} \Rightarrow Q = \frac{w_pCR_6}{2} = \frac{1}{2}\cdot\sqrt{\frac{R_6}{R_1}} \Rightarrow R_6 = 4Q^2R_1 \Rightarrow$$

$A_0 = -2Q^2 \Rightarrow$ we can not choose this parameter in this simple circuit

$$w_p^2 = \frac{1}{R_1R_6C^2} \Rightarrow C^2 = \frac{1}{R_1R_6w_p^2} \Rightarrow C = \frac{1}{2QR_1w_p}$$

The ***design procedure*** is now:

I. We chose appropriate value of the R_1
II. We determine $R_6 = 4Q^2R_1$
III. We determine $C = \dfrac{1}{2QR_1w_p}$; the filter $A_0 = -2Q^2$ we can not change in this simple circuit (fig. 19)

If we need define the A_0, too, we must choose more complex circuit in fig. 20. The other one resistor R_5 allows us to do it.

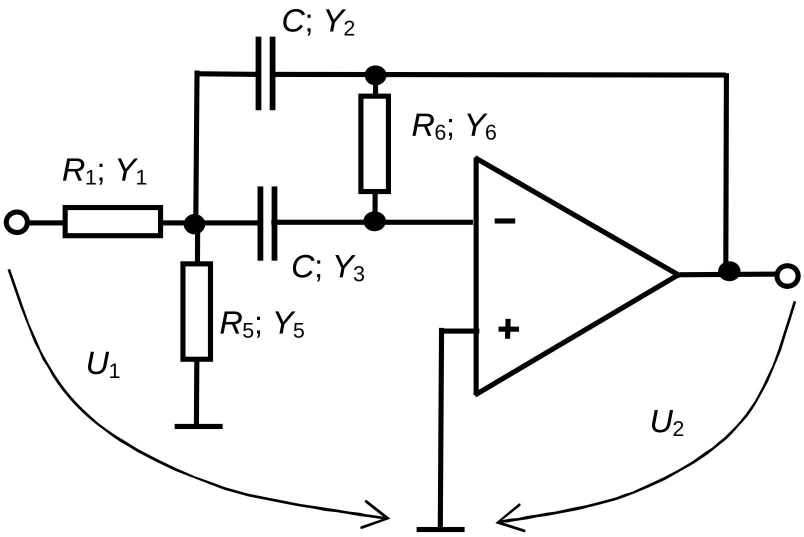


Fig. 20 More complex inverting BP configuration

In this case we get ($Y_5 = G_5$):

$$w_p^2 = \frac{1}{(R_1 // R_5) R_6 C^2}$$

$$A_o = -R_6 / (2R_1)$$

$$2x = 1/Q = 2 \cdot \sqrt{\frac{R_1 // R_5}{R_6}}$$

This set of equations allows us determine formulas needed to ***design procedure***, if we choose appropriate value of *C*:

BOX

$$1/Q = 2 \cdot \sqrt{\frac{R_1 // R_5}{R_6}} \quad \Rightarrow \quad R_1 // R_5 = \frac{R_1 R_5}{R_1 + R_5} = \frac{R_6}{4Q^2} \quad \Rightarrow$$

$$w_p^2 = \frac{1}{(R_1 // R_5) R_6 C^2} = \frac{1}{\left(\frac{R_6}{4Q^2}\right) R_6 C^2} \quad \Rightarrow \quad R_6 = \frac{2Q}{w_p C}$$

$$|A_o| = R_6 / (2R_1) \quad \Rightarrow \quad R_1 = \frac{R_6}{2 \cdot |A_o|} \quad \Rightarrow \quad R_1 = \frac{Q}{w_p C \cdot |A_o|}$$

$$R_1 // R_5 = \frac{R_1 R_5}{R_1 + R_5} = \frac{R_6}{4Q^2} \quad \Rightarrow \quad 4Q^2 R_1 R_5 = R_6 \cdot (R_1 + R_5) \quad \Rightarrow$$

$$4Q^2 R_1 R_5 - R_6 R_5 = R_6 R_1 \quad \Rightarrow \quad R_5 = \frac{R_6 R_1}{4Q^2 R_1 - R_6} = \frac{R_6}{4Q^2 - R_6 / R_1} = \frac{\frac{2Q}{w_p C}}{4Q^2 - 2 \cdot |A_0|} \quad \Rightarrow$$

$$R_5 = \frac{Q}{(2 \cdot Q^2 - |A_0|) \cdot w_p C}$$

The ***design procedure*** is now:

I. We chose appropriate value of the C (fig. 20)
II. We determine

$$R_1 = \frac{Q}{w_p C \cdot |A_o|}; \qquad R_5 = \frac{Q}{(2 \cdot Q^2 - |A_0|) \cdot w_p C}; \qquad R_6 = \frac{2Q}{w_p C}$$

It is evident that physically right solution is $R_5 > 0$ so that we may choose $2Q^2 > |A_o|$ only if we have $C_3 = C_2 = C$.

Inverting HP (2nd order)

If we choose $K \to -\infty$ and $Y_1 = pC_1$, $Y_2 = pC_2$, $Y_3 = pC_3$, $Y_5 = G_5$, $Y_6 = G_6$ (fig. 21) we get

$$\hat{P}_{HP2}(p = jw) = A_0 \cdot \frac{p^2}{p^2 + pw_p/Q + w_p^2}$$

where $w_p^2 = \dfrac{1}{R_5 R_6 C_2 C_3}$

$$A_o = -C_1/C_2$$

$$w_p/Q = \left(\frac{C_1}{C_2 C_3} + \frac{1}{C_2} + \frac{1}{C_3}\right) \Big/ R_6$$

Usually we choose $C_1 = C_3 = C$ and we can get from above equations next useful "design formulas":

$$C_2 = C/|A_o|$$

$$R_5 = |A_o| / [(2 \cdot |K_o| + 1) Q w_p C]$$

$$R_6 = Q(2 \cdot |A_o| + 1) / (w_p C)$$

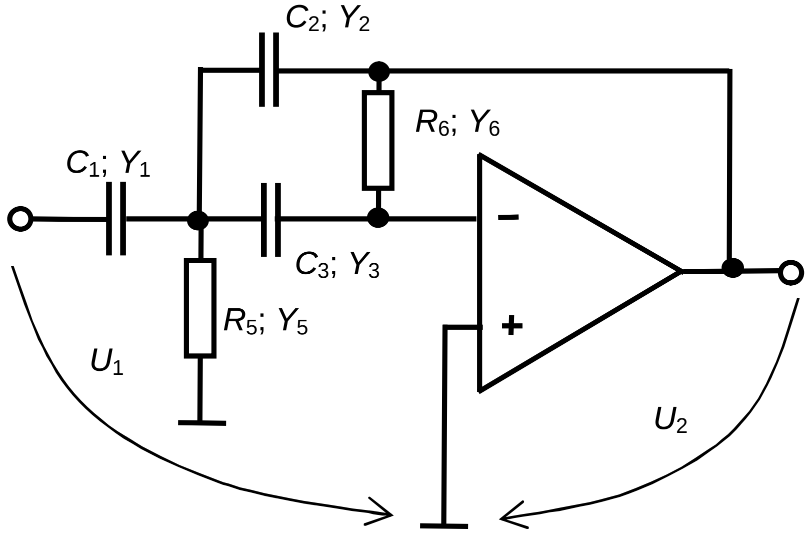


Fig. 21 Inverting HP circuit

Sallen-Key HP (2nd order)

The basic circuit you can see in fig. 22. It is evident that $Y_6 = Y_5 = 0$ and

$$Y_1 = \mathrm{p}\, C_1,\ Y_2 = G_2,\ Y_3 = \mathrm{p}\, C_3\ ,\ Y_4 = G_4$$

An operational amplifier acts as noninverting amplifier with a finite gain *K*.

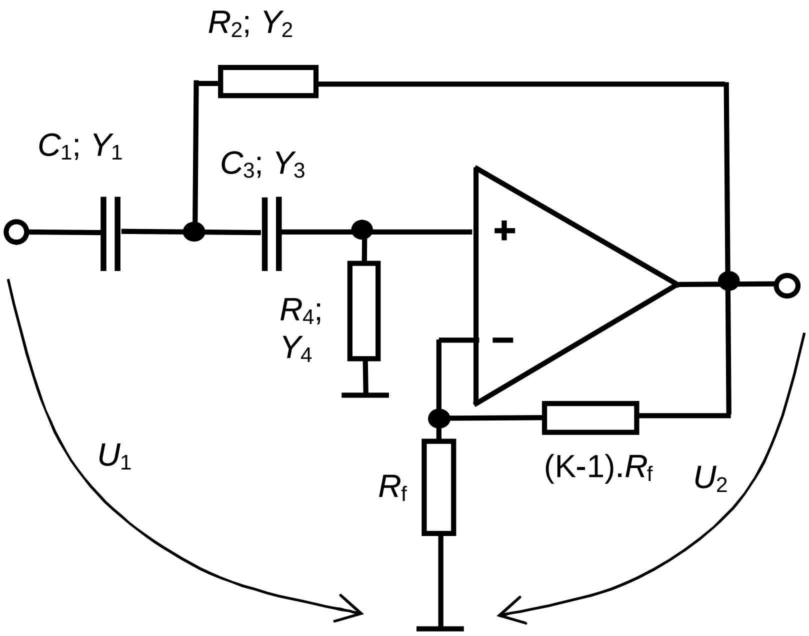


Fig. 22 One possible Sallen- Key HP configuration

If **$C_1 = C_3 = C,\ R_2 = R_4 = R$** than

$$\hat{P}_{HP2}(p = jw) = A_0 \cdot \frac{p^2}{p^2 + pw_p / Q + w_p^2}$$

where

$$w_p^2 = \frac{1}{(RC)^2}; \qquad 2x = 1/Q = 3 - K; \qquad A_o = K$$

We can not define beforehand the filter gain, now. From these simple equations we can easy determine formulas suitable for this type ***filter design*** (we want ω_p and Q):

I. We chose value of the C.
II. We determine $R = 1/(\omega_p C)$
III. We determine needed $K = 3 - 1/Q = 3 - 2\xi$. We chose suitable value of the R_f and determine corresponding value (K-1).R_f

This filter is stable only if $K < 3$.

The second usual choice is: **$C_1 = C_3 = C$ and $K = 1$;** (K-1).R_f – short circuit; R_f – open. Than is valid

$$w_p^2 = \frac{1}{C^2 R_2 R_4}; \qquad 2x = 1/Q = 2 \cdot \sqrt{R_2 / R_4}; \qquad A_o = 1$$

From these simple equations we can easy determine formulas suitable for this type ***filter design*** (we want ω_p and Q):

I. We chose value of the C.

II. We determine $R_2 = \dfrac{1}{2Qw_p C}$; $\qquad R_4 = \dfrac{2Q}{w_p C}$

Notch filter (2nd order; bandstop)

The very simple realization of the notch filter is in fig. 23. The upper operational amplifier creates LP (2nd order) filter. The bottom operational creates HP (2nd order) filter. These filters must have the same properties – it means ω_p, Q and K (thus R_d). The third operational amplifier creates the inverting adder amplifier.

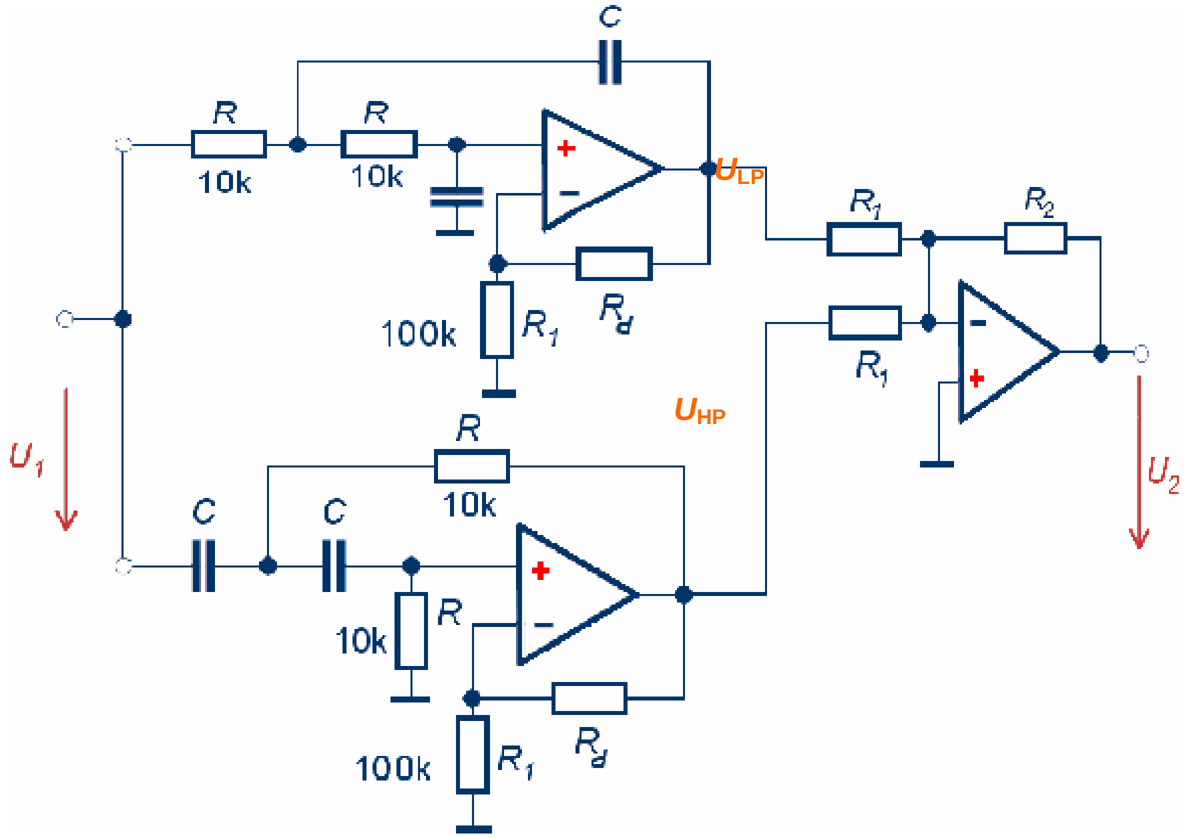


Fig. 23 One possible notch filter configuration – by means of Sallen-Key LP and HP filters – no cascade connection

It is evident that

$$U_{LP} = U_1 \cdot P_{LP2} = U_1 \cdot A_0 \cdot \frac{w_p^2}{p^2 + pw_p/Q + w_p^2}$$

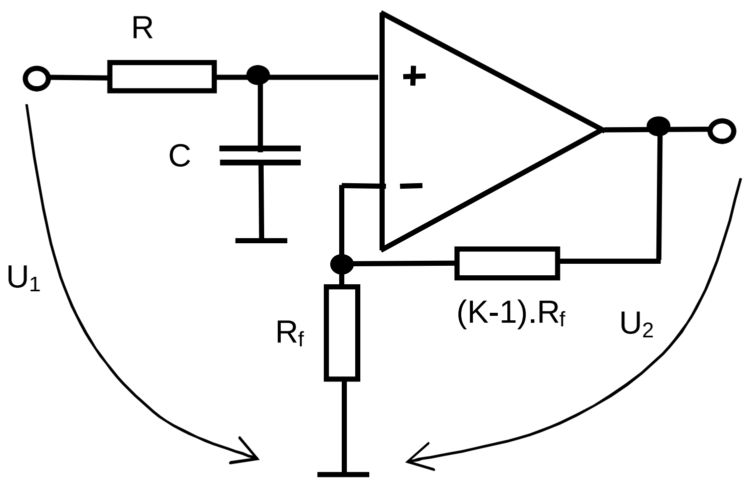
$$U_{HP} = U_1 \cdot P_{HP2} = U_1 \cdot A_0 \cdot \frac{p^2}{p^2 + pw_p/Q + w_p^2}$$

$$w_p^2 = \frac{1}{(RC)^2}; \qquad 2x = 1/Q = 3 - K; \qquad A_o = K = 1 + R_d/R_1$$

$$U_2 = -\frac{R_2}{R_1} \cdot (U_{LP} + U_{HP}) = -\frac{R_2}{R_1} \cdot K \cdot U_1 \cdot \left(\frac{w_p^2}{p^2 + pw_p/Q + w_p^2} + \frac{p^2}{p^2 + pw_p/Q + w_p^2} \right) \Rightarrow$$

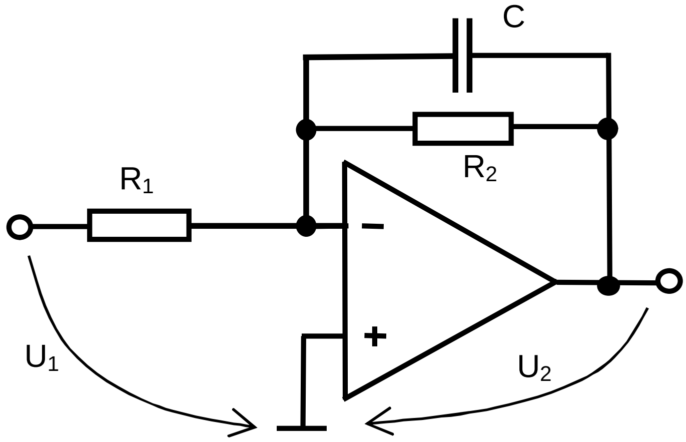
$$U_2/U_1 = -\frac{R_2}{R_1} \cdot K \cdot \frac{p^2 + w_p^2}{p^2 + pw_p/Q + w_p^2}$$ - This is the notch filter – really.

Examples of first order filter realization



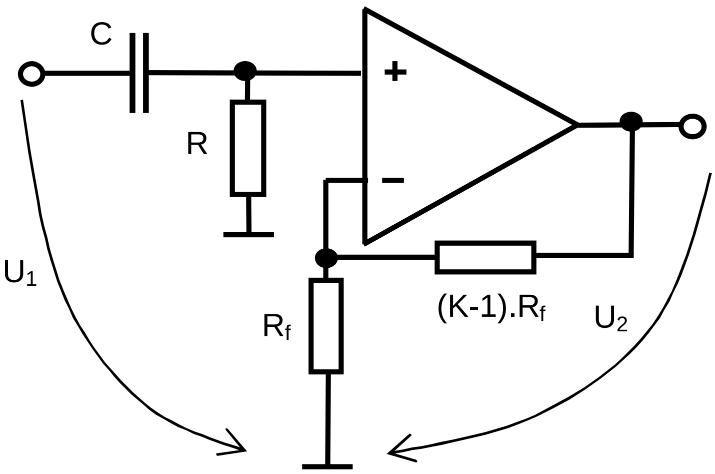
$$\frac{U_2}{U_1} = \frac{\frac{1}{pC}}{R + \frac{1}{pC}} \cdot K = \frac{1/(RC)}{p + 1/(RC)} \cdot K = \frac{K \cdot w_p}{p + w_p}; \qquad w_p = 1/(RC)$$

Fig. 24 Noninverting 1st order LP; an operational amplifier creates noninverting structure with gain K; a characteristic frequency is $w_p = 1/(RC)$; A_0 = K



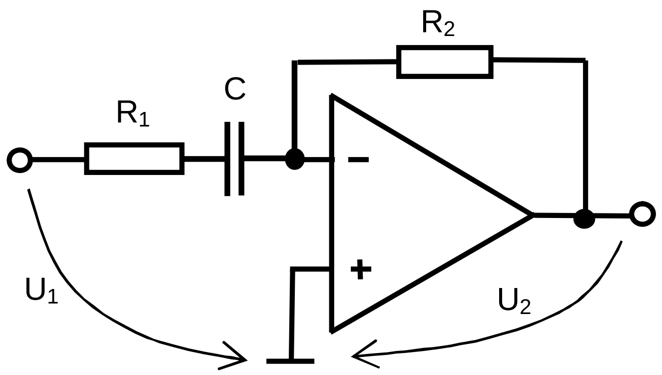
$$\frac{U_2}{U_1} = -\frac{\dfrac{R_2 \cdot \dfrac{1}{pC}}{R_2 + \dfrac{1}{pC}}}{R_1} =$$
$$= -\frac{R_2}{R_1} \cdot \frac{1/(R_2 C)}{p + 1/(R_2 C)} =$$
$$= \frac{K \cdot w_p}{p + w_p}; w_p = 1/(R_2 C); K = -\frac{R_2}{R_1}$$

Fig. 25 Inverting 1st order LP; an operational amplifier creates inverting structure with DC gain $K = - R_2/R_1$; a characteristic frequency is $w_p = 1/(RC)$; $A_0 = K$



$$\frac{U_2}{U_1} = \frac{R}{R + \dfrac{1}{pC}} \cdot K =$$
$$= \frac{p}{p + 1/(RC)} \cdot K =$$
$$= \frac{K \cdot p}{p + w_p}; \qquad w_p = 1/(RC)$$

Fig. 26 Noninverting 1st order HP; an operational amplifier creates noninverting structure with gain K; a characteristic frequency is $w_p = 1/(RC)$; $A_0 = K$



$$\frac{U_2}{U_1} = -\frac{R_2}{R_1 + \frac{1}{pC}} =$$
$$= -\frac{R_2}{R_1} \cdot \frac{1/(R_1 C)}{p + 1/(R_1 C)} =$$
$$= \frac{K \cdot w_p}{p + w_p}; w_p = 1/(R_1 C); K = -\frac{R_2}{R_1}$$

Fig. 27 Inverting 1st order HP; an operational amplifier creates inverting structure with AC gain $K = -R_2/R_1$; a characteristic frequency is $w_p = 1/(RC)$; $A_0 = K$

Basic cascade - connections

Cascade connections give us opportunity to create others transfer functions.

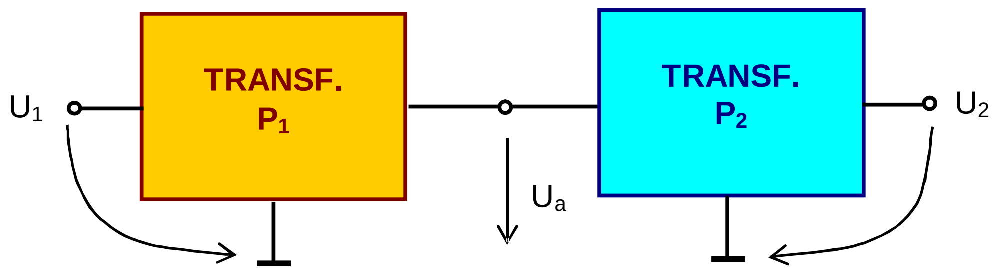


Fig. 28 Cascade connection of two filters with transfer functions $P_1 = U_a/U_1$ a $P_2 = U_2/U_a$

It is evident that

$$P = U_2/U_1 = (U_a/U_1)\cdot(U_2/U_a) = P_1 \cdot P_2$$

Further

$$|P| = |P_1 \cdot P_2| = |P_1| \cdot |P_2|$$

Thus in dB is valid

$$|P|_{dB} = 20\log(|P_1|\cdot|P_2|) = 20\log|P_1| + 20\log|P_2| = |P_1|_{dB} + |P_2|_{dB}$$

The last formula

$|P|_{dB} = |P_1|_{dB} + |P_2|_{dB}$

is very important. From this formula we can deduce that:

***If we know the "decibel description" of the transfer function modulus P_1 (dB) and P_2 (dB), the resulting modulus of cascade connection we get as a sum** $|P_1|_{dB} + |P_2|_{dB}$.*

Two LP2s cascade – fig. 29

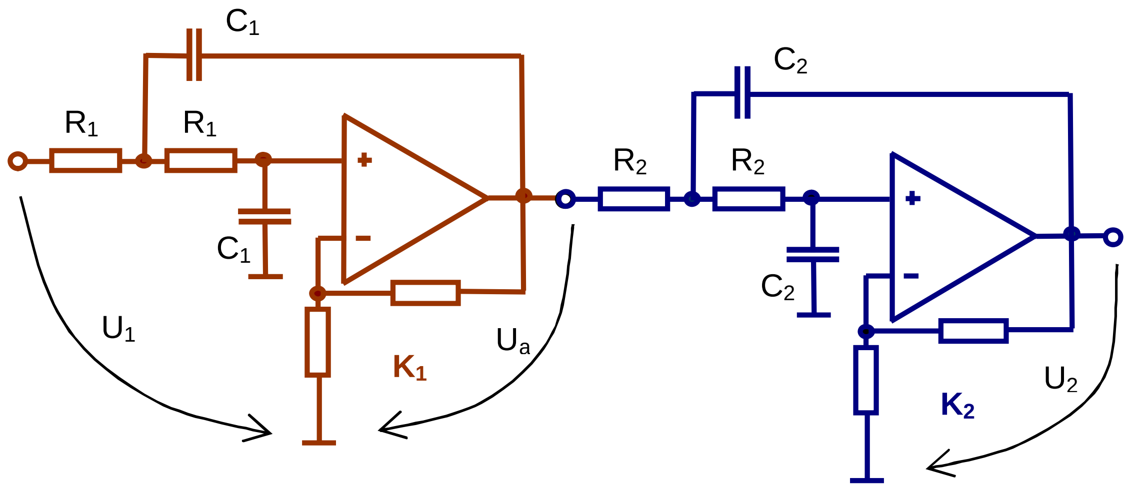


Fig. 29 Cascade connection of two LP2; LP2₁ ($w_{p1} = 1/(R_1C_1)$; $Q_1 = 1/(3-K_1)$) and LP2₂ ($w_{p2} = 1/(R_2C_2)$; $Q_2 = 1/(3-K_2)$)

If we choose suitable $w_{p1} = 1/(R_1C_1)$, $Q_1 = 1/(3-K_1)$, $w_{p2} = 1/(R_2C_2)$ and $Q_2 = 1/(3-K_2)$ we can get different frequency responses. A qualitative description you can see in fig. 30; for w_{p2} > w_{p1} and Q_2 > Q_1 .

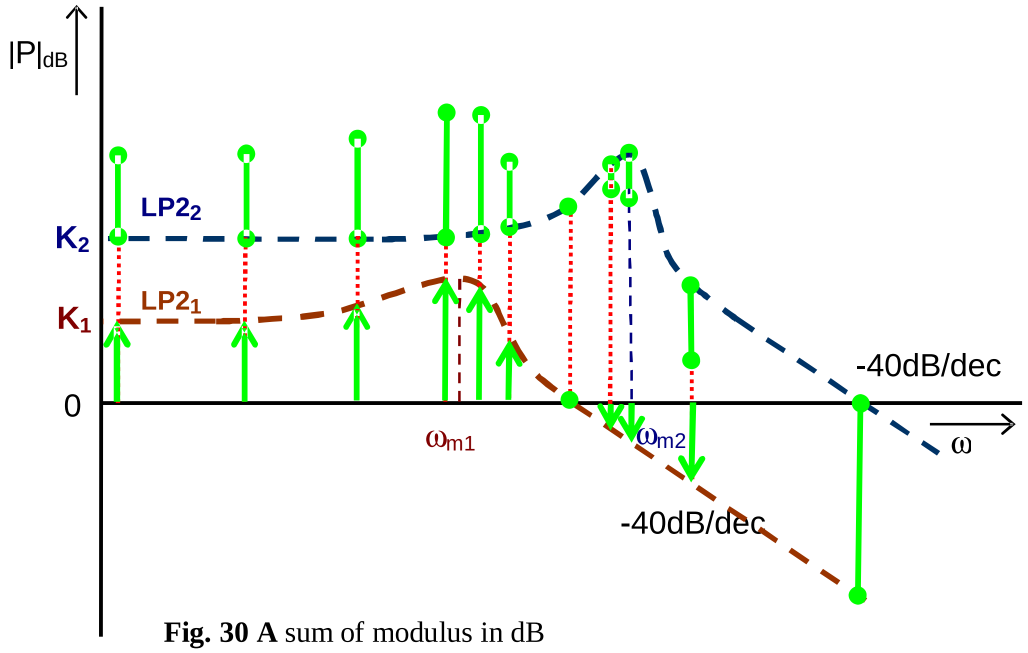


Fig. 30 A sum of modulus in dB

The final modulus we can see in fig. 31.

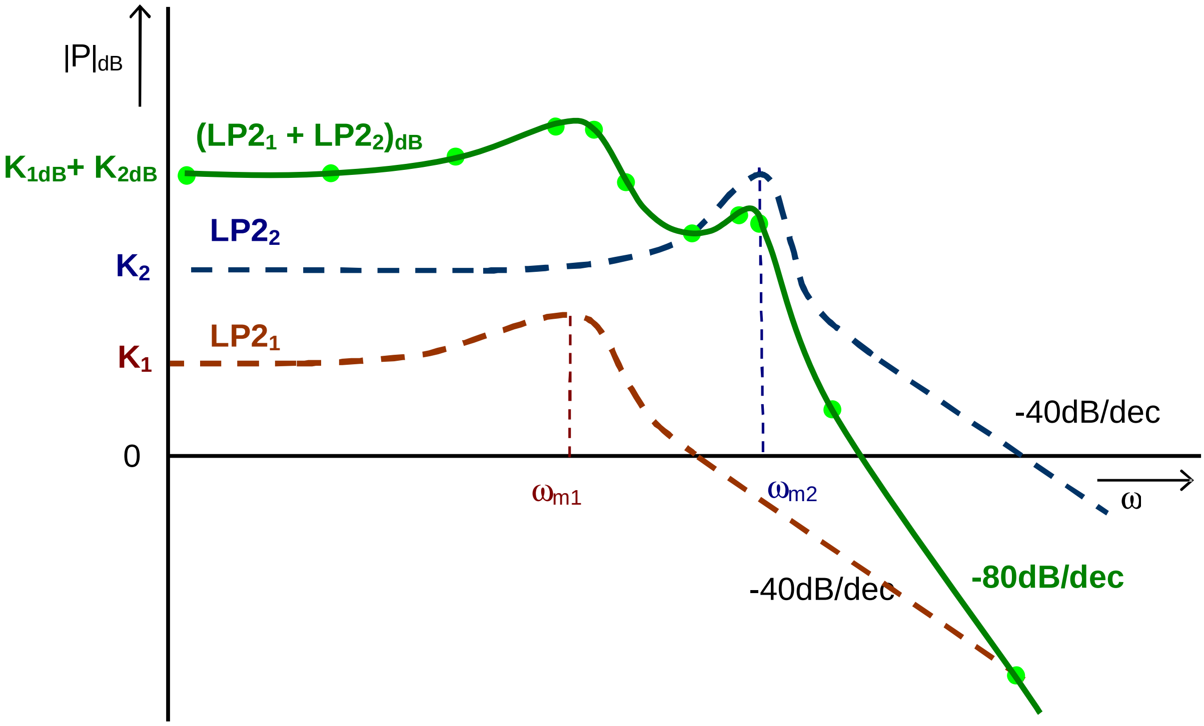


Fig. 31 A final sum of modulus in dB $\left|P\right|_{dB} = \left|P_{DP21}\right|_{dB} + \left|P_{DP22}\right|_{dB}$

If we increase ω_{o1} and decrease quality factor Q_1 (K_1), we can get more friendly response – gray line depicts previous situation – fig. 32.

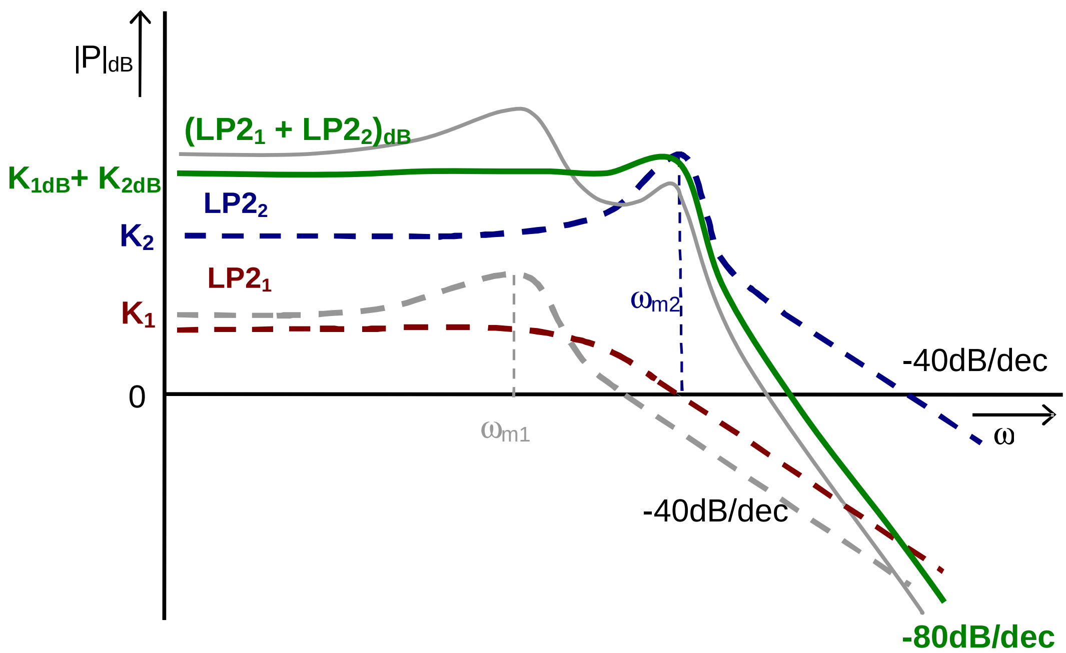


Fig. 32 A sum of modulus after increasing ω_{o1} and decreasing Q_1 (K_1)

If we decrease quality factor Q_2 (K_2) now, we can another improvement of the modulus – gray line depicts previous situation – fig. 33.

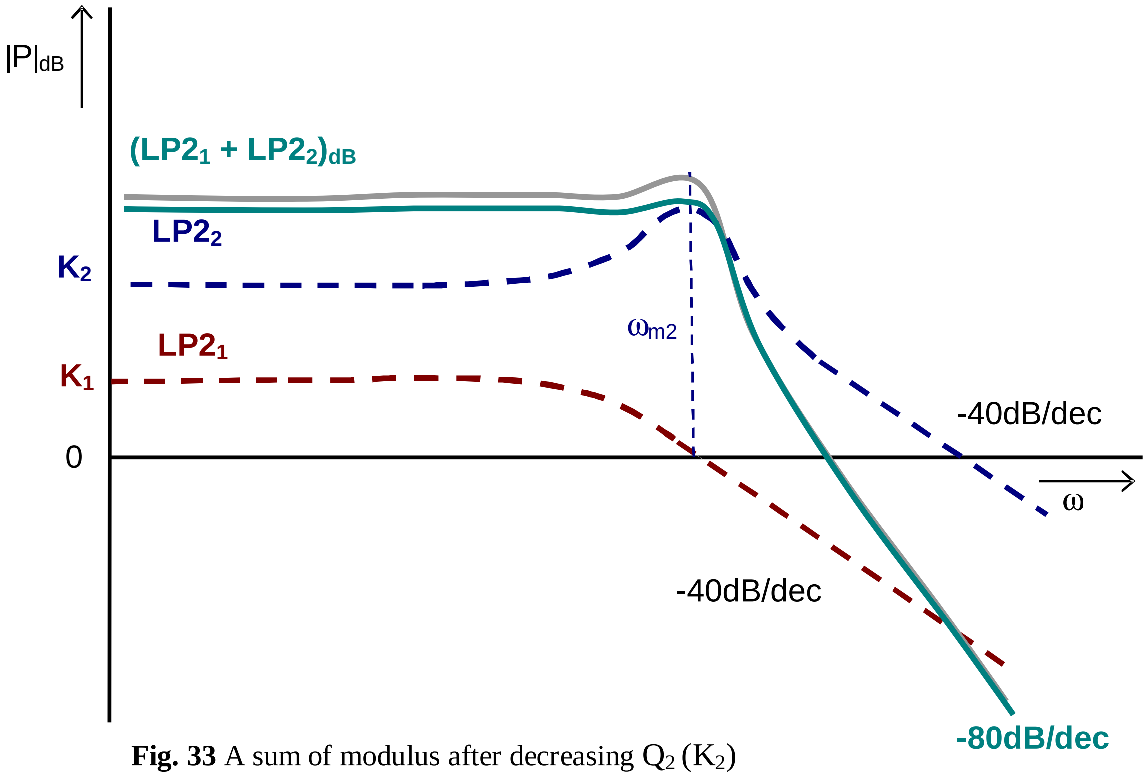


Fig. 33 A sum of modulus after decreasing Q_2 (K_2)

The all history of the filter adjustment you can see in fig. 34.

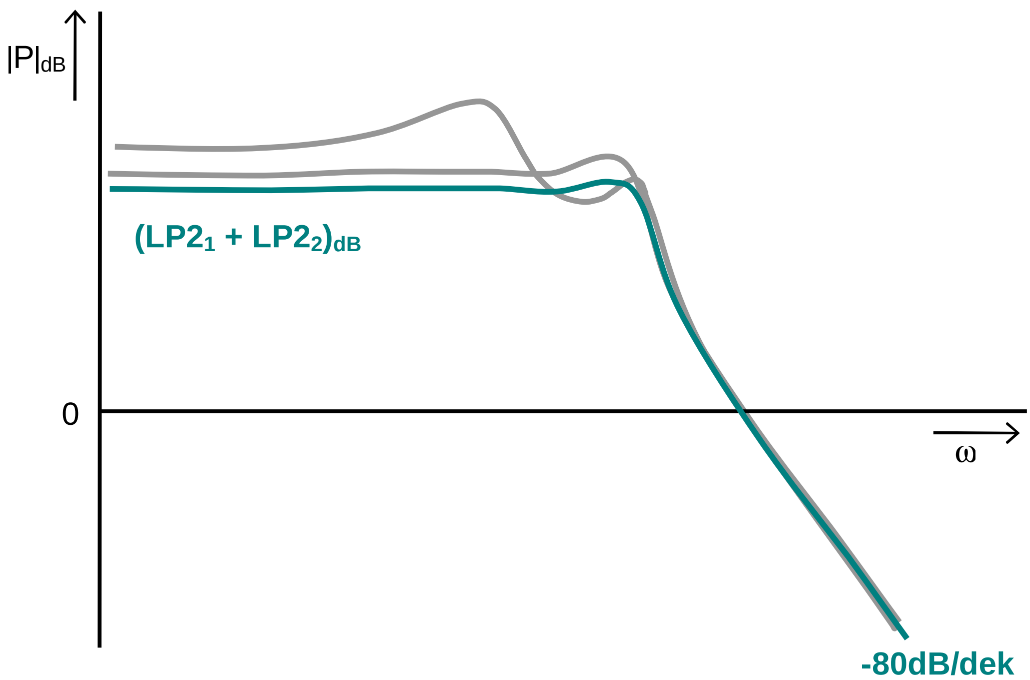


Fig. 33 A qualitative depiction of the filter adjustment history:

$$|P|_{dB} = |P_{LP21}|_{dB} + |P_{LP22}|_{dB}$$

We can see that a final filter slope is – 80 dB/dec – this means we have LP filter of 4th order. It was created intuitively. Needed ω_{p1}, ω_{p2} and Q_1, Q_2, we can get from graphs in figures.

If we take these parameters from known approximation functions, we get filter properties in accordance with used approximation.

Two HPs cascade – fig. 34

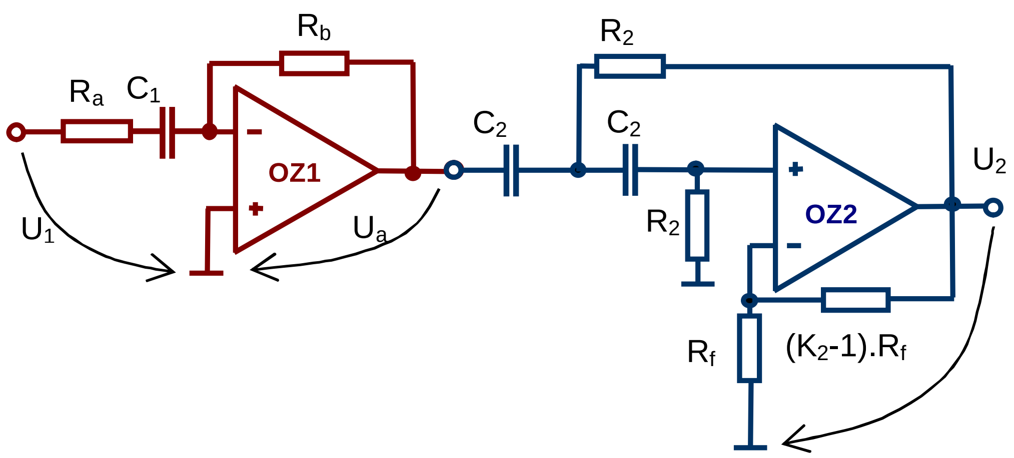


Fig. 34 Cascade connection of two HPs; HP1 (OZ1) – 1st order inverting high pass; HP2 (OZ2) – 2nd order noninverting highpass

It is evident that

$$P_1 = P_{HP1} = \frac{U_a}{U_1} = -\frac{R_b}{R_a + 1/(pC_1)} = -\frac{R_b}{R_a} \cdot \frac{p}{p + 1/(R_a C_1)} \Rightarrow$$

$$K_1 = -\frac{R_b}{R_a}; \qquad w_{p1} = 1/(R_a C_1)\,.$$

$$P_2 = P_{HP2}; K_2; w_{p2} = 1/(R_2 C_2)$$

A qualitative depiction of modulus summing is in fig. 35. If we decrease quality factor Q_2 (it means K_2) we improve the final frequency response – fig. 36. For low frequencies are filter slope +60 dB/des – it is 3rd order HP.

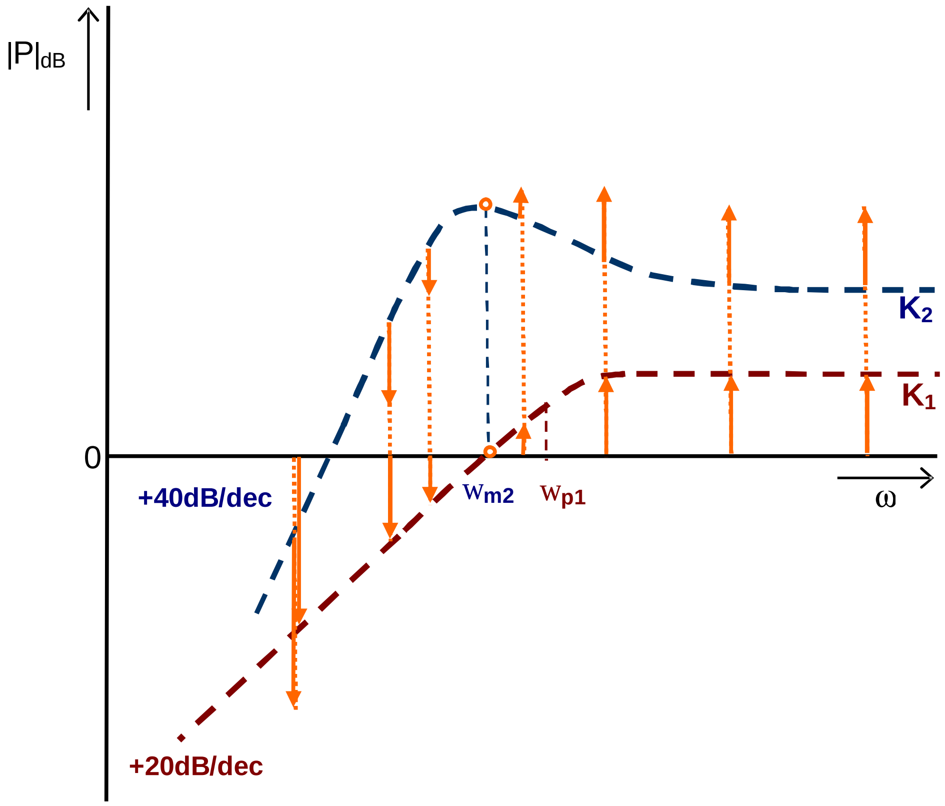


Fig. 35 A qualitative depiction of HP modulus summing

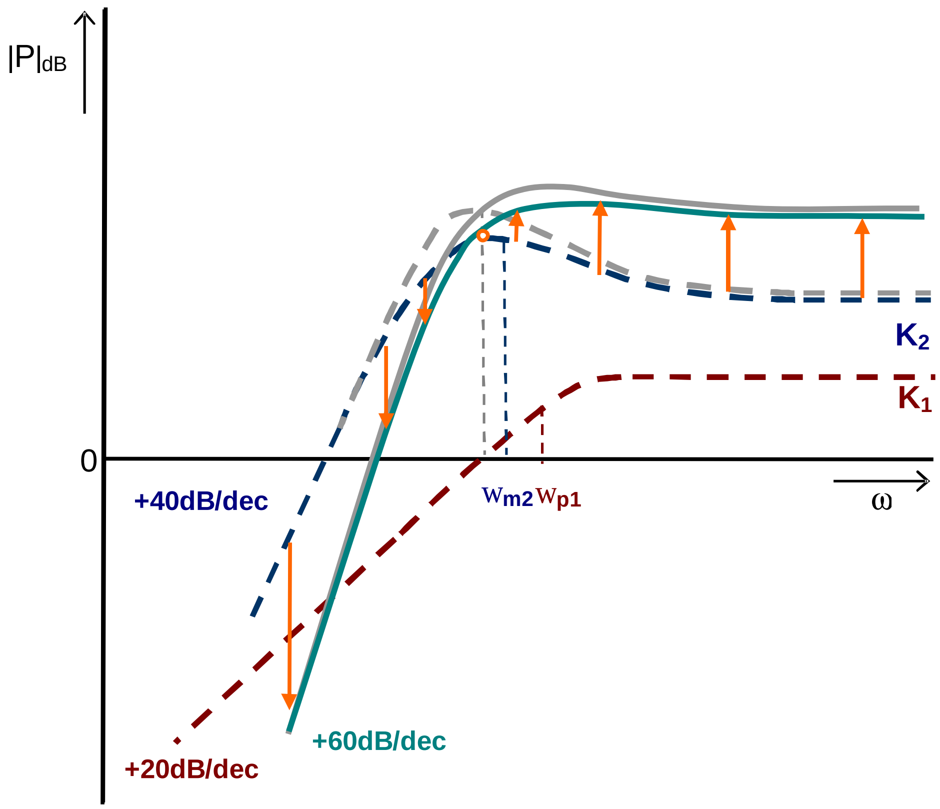


Fig. 36 A qualitative depiction of HP modulus summing

$|P|_{dB} = |P_{HP1}|_{dB} + |P_{HP2}|_{dB}$ - after adjustment of HP2

Cascade connection of LP2 and HP2 – fig. 37

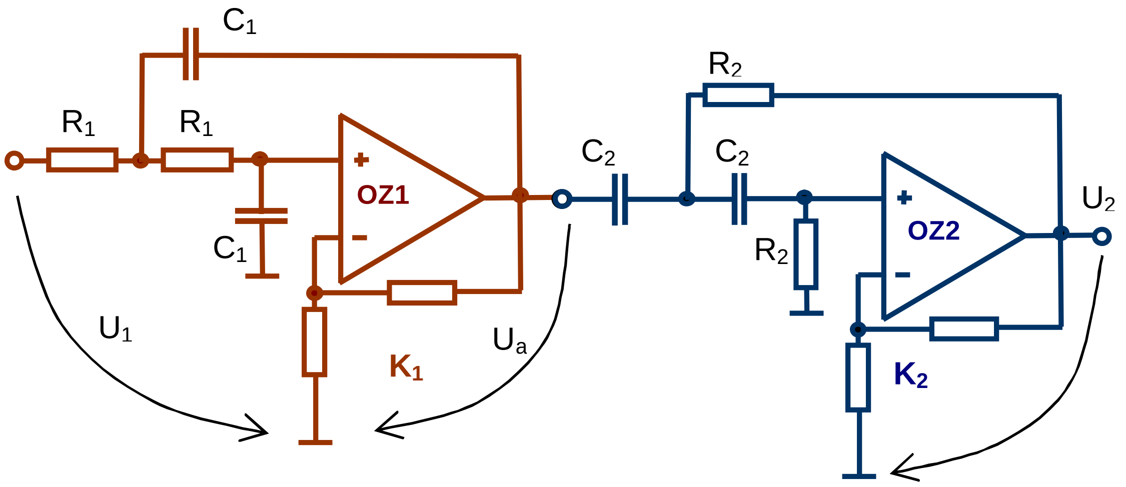


Fig. 37 Cascade connection of LP2 (($w_{p1} = 1/(R_1C_1)$; $Q_1 = 1/(3-K_1)$) and HP2 (($w_{p2} = 1/(R_2C_2)$; $Q_2 = 1/(3-K_2)$)

Let us choose $K_1 = K_2 = K$ (thus $Q_1 = Q_2 = Q$). A situation for $w_{p1} = w_{p2} = w_p$ is depicted in fig. 38.

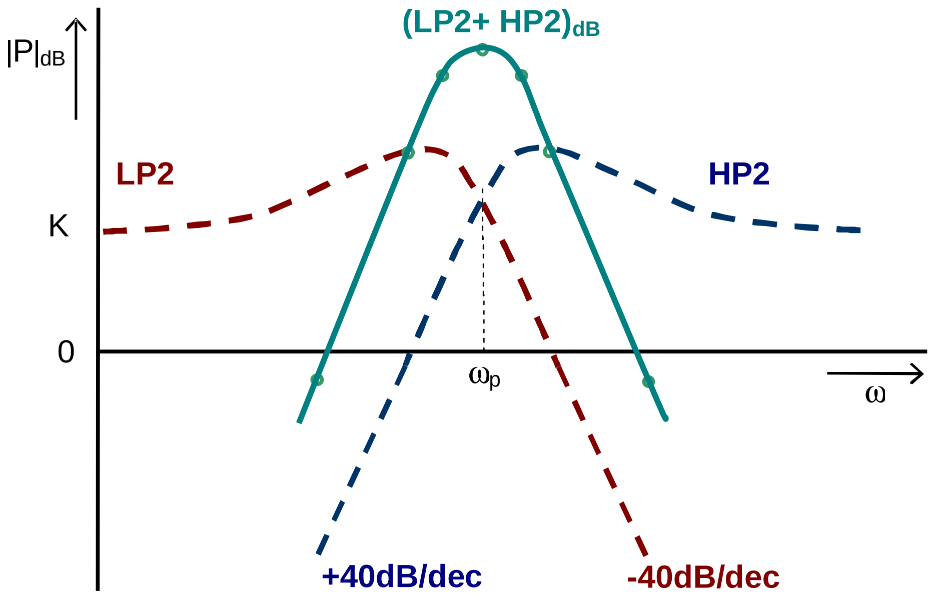


Fig. 38 A qualitative depiction of modulus summing: $|P|_{dB} = |P_{LP2}|_{dB} + |P_{HP2}|_{dB}$ if $w_{p1} = w_{p2} = w_p$ and $K_1 = K_2 = K$ ($Q_1 = Q_2 = Q$)

This frequency response describes 4th order bandpass:

$$P = P_1 \cdot P_2 = P_{LP2} \cdot P_{HP2} = \frac{K w_p^2}{p^2 + p w_p / Q + w_p^2} \cdot \frac{K p^2}{p^2 + p w_p / Q + w_p^2} \Rightarrow$$

$$P = K^2 Q^2 \cdot \left(\frac{p w_p / Q}{p^2 + p w_p / Q + w_p^2} \right)^2$$

$$P(w = w_p) = K^2 Q^2 \cdot \left(\frac{j w_p \cdot w_p / Q}{-w_p^2 + j w_p \cdot w_p / Q + w_p^2} \right)^2 = K^2 Q^2$$

$$P(w \langle\langle w_p) \to K^2 Q^2 \cdot \left(\frac{j w \cdot w_p / Q}{w_p^2} \right)^2 = -K^2 (w / w_p)^2$$ - ***asymptote with slope***

+40 dB/dec

$$P(w \rangle\rangle w_p) \to K^2 Q^2 \cdot \left(\frac{j w \cdot w_p / Q}{(jw)^2} \right)^2 = -K^2 (w_p / w)^2$$ - ***asymptote with slope***

-40 dB/dec

Let us choose $K_1 = K_2 = K$ (thus $Q_1 = Q_2 = Q$) and $w_{p1} \rangle w_{p2}$ - we get wideband band-pass – fig. 39.

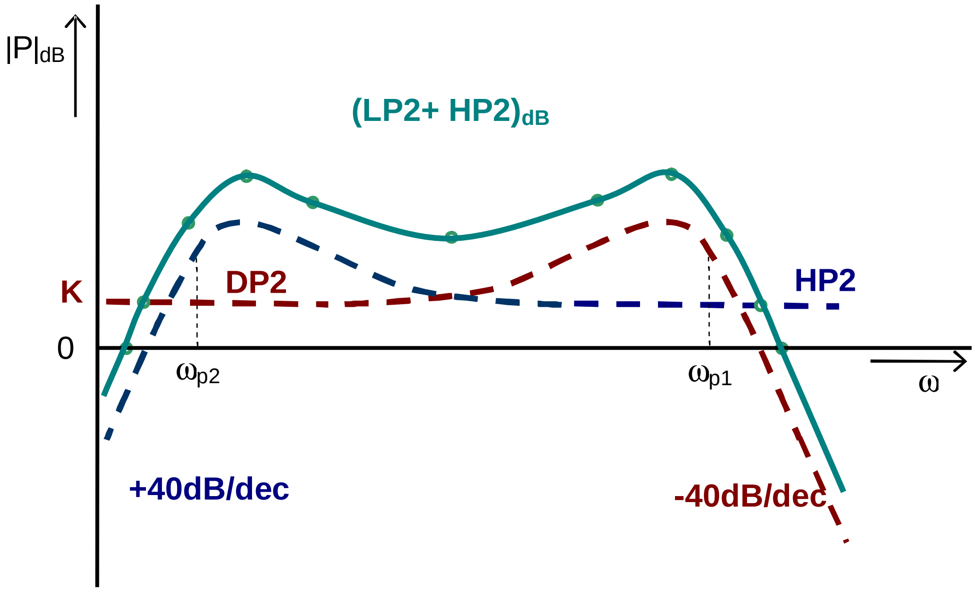


Fig. 39 A qualitative depiction of modulus summing: $|P|_{dB} = |P_{LP2}|_{dB} + |P_{HP2}|_{dB}$ if $w_{p1} \rangle w_{po2}$ and $K_1 = K_2 = K$ ($Q_1 = Q_2 = Q$)

If we adjust quality factors we change frequency response, again.

Cascade connection of LP2 and BS (notch) – fig. 40

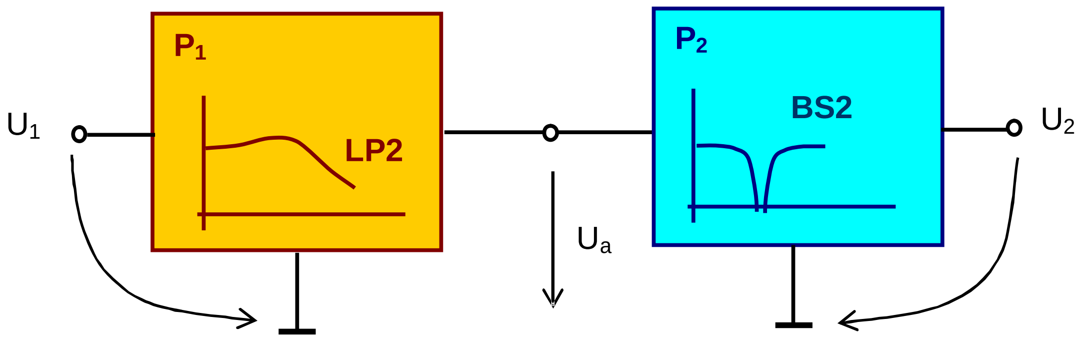


Fig. 40 Cascade connection of LP2 and bandstop filter

That way we get a **low-pass filter with „zero frequency"** - fig. 41.

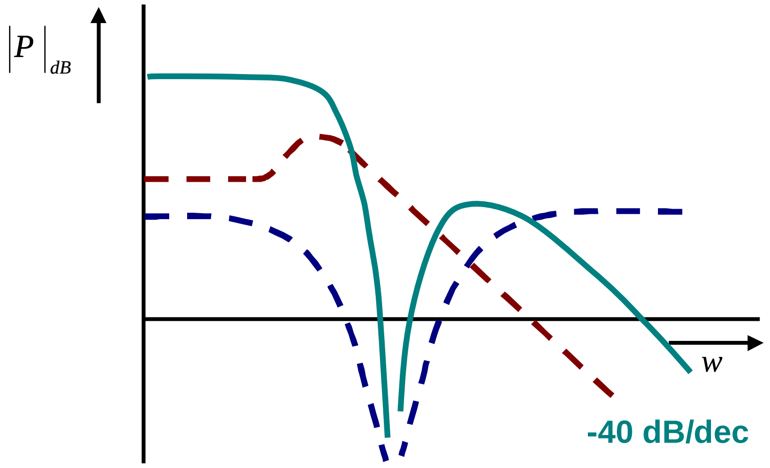


Fig. 41 A qualitative depiction of modulus of cascade connection LP2 and BS

Cascade connection of HP2 and BS (notch) – fig. 42

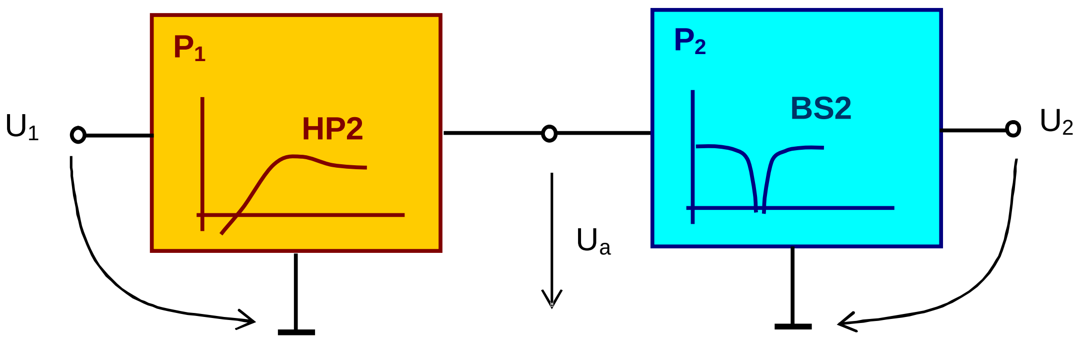


Fig. 42 Cascade connection of HP2 and bandstop filter

That way we get a **high-pass filter with „zero frequency”** - fig. 43.

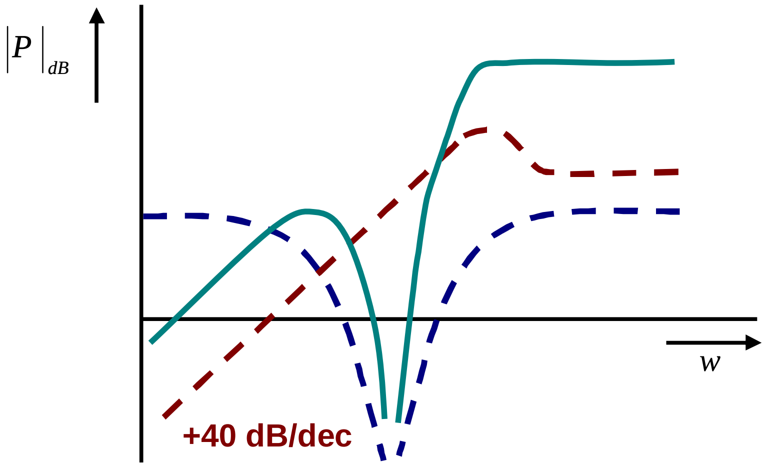


Fig. 41 A qualitative depiction of modulus of cascade connection HP2 and BS

No ideal operational amplifiers

An operational amplifier is not ideal circuit component. The most degradation is a change of its gain as frequency increases – fig. 42; a real operational amplifier modulus of gain A.

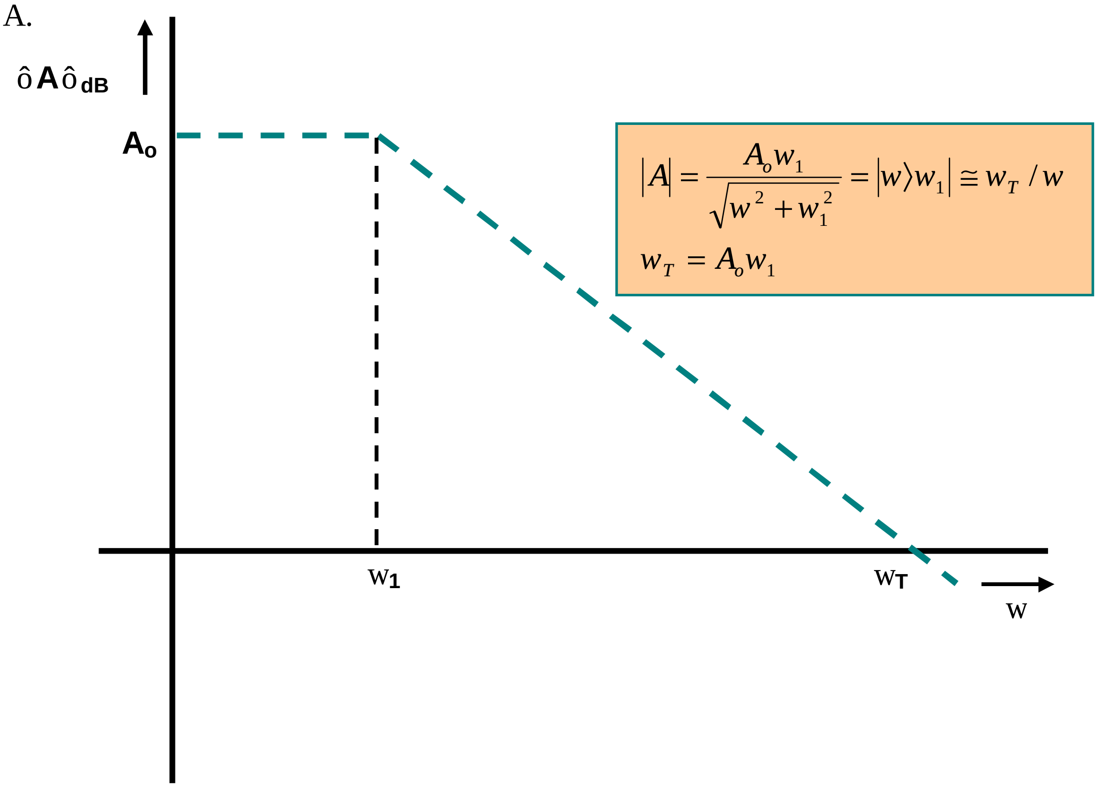


Fig. 42 A real operational amplifier modulus of gain A.

All above done outcomes are valid if needed modulus of transfer function is much less as the operational amplifier modulus |A| - preferably for 40 dB (and more).

In fig.43 we can see **blue lines** – these mete the above demands. **Red lines** define transfers which no mete the above demands **("graphical criterion").** It is evident that in real circuitry we are not able to construct an ideal HP or BS structures – high frequencies are problematic in any case (with any real operational amplifier).

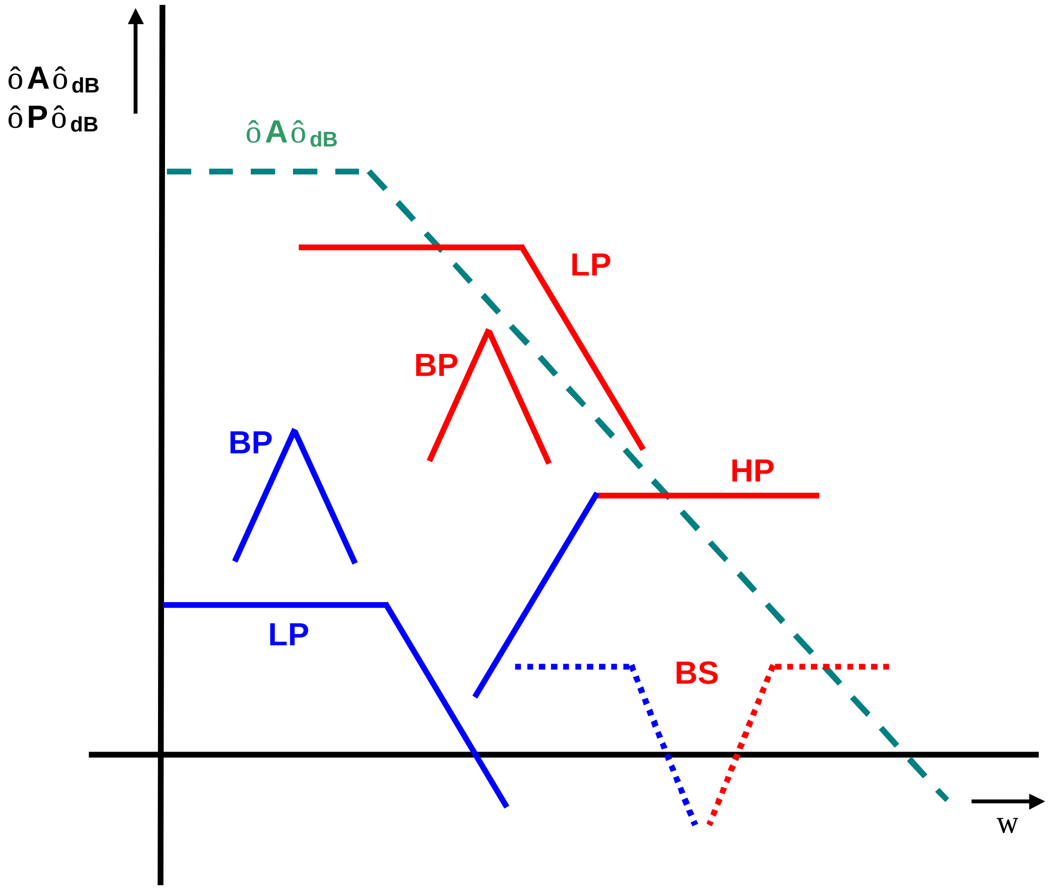


Fig. 43 A **„Graphical criterion of operational amplifier usability": modulus of needed transfer function is much less than modulus of OPA gain – blue lines – satisfactory functions; red lines - not satisfactory functions**

Basic texts

Other text

Questions

Answers you find in this text

1. Define mathematical models of 2nd order filters (LP, HP, BP, BS – notch).
2. How many elements needs an ideal LP filter (brick-wall frequency response)?
3. What approximation has maximally flat magnitude?
4. How we can gat a BS (notch) filter from LP and HP filters (the same characteristic frequency and Q)?
5. What is it a “peak” of magnitude (LP2, HP2) - how is it connected with quality factor Q?

Problems

1. It is required that we design the second order low pass filter. However, only one op – amp is available and the stockroom has only 0,1 μF capacitors. It does have a good stock of resistors. You are to design a filter to meet the following specifications:
(a) The response is to be Butterworth.
(b) The characteristic frequency f_0 is 159,15 Hz.
(c) You can choose any low frequency gain.

2. It is required that we design the third order low pass filter. You are to design a filter to meet the following specifications:
(a) The response is to be Butterworth.
(b) The characteristic frequency f_0 is 159,15 Hz.
(c) We need low frequency gain just 1.

3. It is required that we design the third order low pass filter. You are to design a filter to meet the following specifications:
(a) The response is to be Butterworth.
(b) The characteristic frequency f_0 is 1591,5 Hz.
(c) We need low frequency gain just 1.

4. It is required that we design the third order low pass filter. You are to design a filter to meet the following specifications:
(a) The response is to be Butterworth.
(b) The characteristic frequency f_0 is 15,915 Hz.
(c) We need low frequency gain just 1.

Problems key

Ad 1) See Tab.1: Butterworth filter; a_1 =1,414 214 thus Q = 1/ a_1 = 0,7071; b_1 = 1 so $\omega_p = b_1\omega_0 = 2\pi f_0 = 1000$ rad.s^{-1}. We can choose the circuit in fig.17 for example – fig.44.

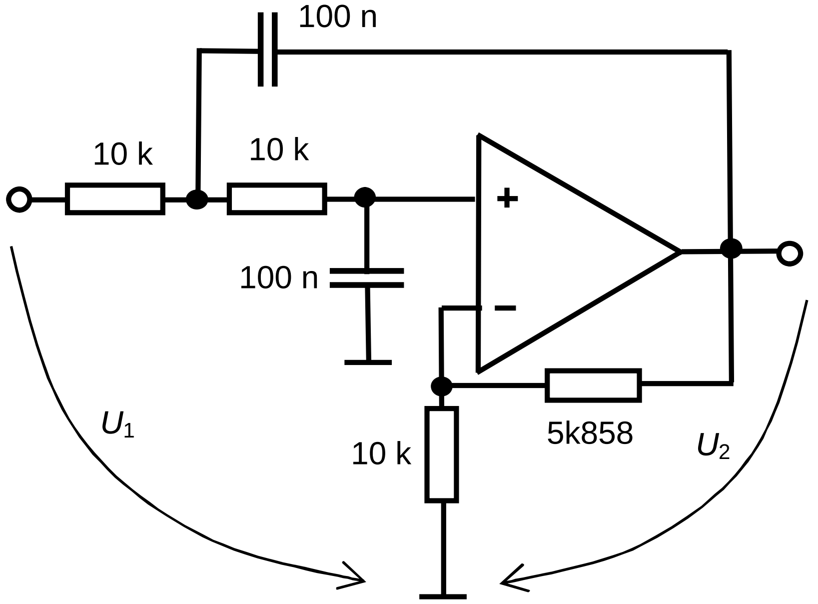


Fig. 44 One possible solution of the problem 1

Then

I. We chose value of the C = 100 nF.
II. We determine $R = 1/(\omega_p C) = 1/(10^3.10^{-7}) = 10$ kΩ.
III. We determine needed $K = 3 - 1/Q = 3 - 1/0{,}7071 = 1{,}5858$. We chose suitable value of the R_f = 10 kΩ and determine corresponding value (K-1).R_f = 0,5858.R_f = 5,858 kΩ.

Ad 2) See Tab.1: Butterworth filter;

n	b_0	a_1	b_1
3	1, 000000	1, 000000	1, 000000

Thus we must use a cascade connection of one the second order LP filter: a_1 =1,000000; b_1 =1,000000 → Q_1 = 1/ a_1 = 1 and $\omega_p = b_1\omega_0 = 2\pi f_0 = 1000$ rad.s^{-1} and Q = 0,7071; and one the first order LP with $\omega_p = b_1\omega_0 = 2\pi f_0 = 1000$ rad.s^{-1}.
For the second order filter part we can choose the circuit in fig.17 for example – with K = 1 – see fig. 45 – and a usual choice ***$R_1 = R_3 = R$.***

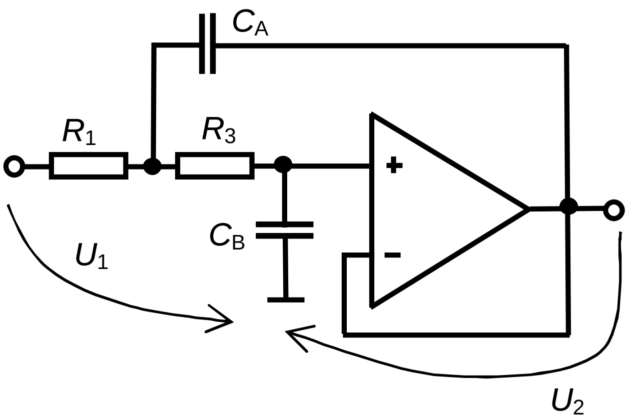


Fig. 45 The second order part of the all filter with low pass gain 1

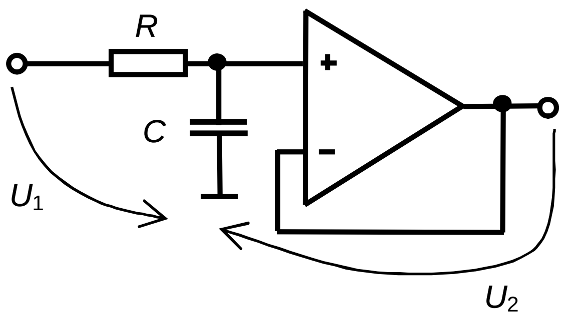
Then

I. We chose value of the R = 10 kΩ.

II. We determine $$C_A = \frac{2Q_1}{Rw_p} = \frac{2}{Rw_p} = \frac{2}{10^4 \cdot 10^3} = 2 \cdot 10^{-7} F \quad \text{(200 nF)}$$

$$C_B = \frac{1}{2Q_1 Rw_p} = \frac{1}{2 \cdot Rw_p} = \frac{C_A}{4} = 0{,}5 \cdot 10^{-7} F \quad \text{(50 nF)}$$

For the first order filter part we can choose the circuit in fig.24 for example – with K = 1 – see fig. 46.



$$\frac{U_2}{U_1} = \frac{\frac{1}{pC}}{R + \frac{1}{pC}} = \frac{1/(RC)}{p + 1/(RC)} =$$

$$= \frac{w_p}{p + w_p}; \qquad w_p = 1/(RC)$$

Fig. 46 The first order part of the all filter with low pass gain 1

I. We chose value of the R = 10 kΩ.

II. We determine $C = \frac{1}{Rw_p} = 10^{-7} F$ (100 nF)

Then the “whole” 3rd order Butterworth LP we can see in fig. 47.

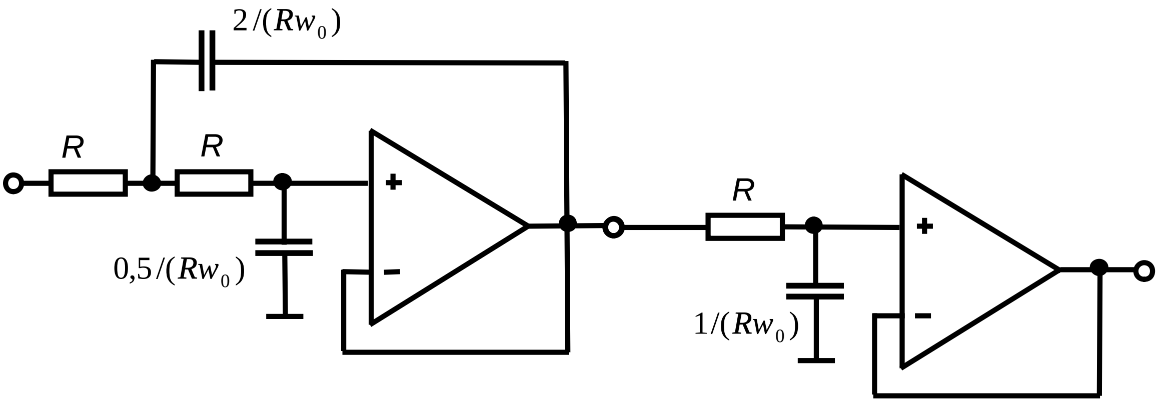


Fig. 47 The whole third order Butterworth LP with low pass gain 1; if $\omega_0 = 10^3$ and $R = 10^4$ then $1/(R\,\omega_0) = 10^{-7}$

Ad 3) We can use the solution of the problem 2. It is evident that needed new characteristic frequency is ten times larger than 159, 15 Hz. Thus it is enough to divide all capacitances by 10 (or it is enough to divide all resistances by 10) to get needed characteristic frequency. Now is valid $1/(R\,\omega_0) = 10^{-8}$. Or we can to divide all capacitances by 2 and all resistances by 5, for example.

Ad 4) We can use the solution of the problem 2. It is evident that needed new characteristic frequency is ten times smaller than 159, 15 Hz. Thus it is enough to multiple all capacitances by 10 (or it is enough to multiple all resistances by 10) to get needed characteristic frequency. Now is valid $1/(R\,\omega_0) = 10^{-6}$. Or we can to multiple all capacitances by 2 and all resistances by 5, for example.

Recommendation

If you can solve and answer more than circa 60 % of the problems and questions, you may continue your study.

1. 7. 2009

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Josef PUNČOCHÁŘ

ANALYSIS OF SALLEN AND KEY LOW-PASS FILTERS WITH REAL OPERATIONAL AMPLIFIERS

Abstract

Generalized nodal voltage analysis is very simple and useful in the analysis of linear active networks with active n-terminals, thus it is useful for the analysis of Sallen and Key filters, too. In the first order analysis of filters it is most convenient to assume the op amp to be ideal. In exacting applications, the design must be based on a more accurate op amp model. It has been found that the op amp´s bandwidth f_T and output resistance R_o are the most limiting factors for Sallen and Key second-order low-pass filters.

1. INTRODUCTION

Generalized nodal voltage analysis is very simple and useful in the analysis of linear active networks with active n-terminals, thus, it is useful for the analysis of Sallen and Key filters, too.

Sallen and Key second-order filters (Fig.1) are very important because higher-order filters can be designed using them. In the first order analysis of filters it is most convenient to assume the op amp to be ideal. If the op amp is not ideal, it´s parameters (A_0, ω_1, R_o) can have a drastic effect on filter response.

2. GENERALIZED NODAL VOLTAGE ANALYSIS

Let us revise basic rules of the generalized nodal voltage analysis [10, 11]:

1) First we determine the admittance matrix of the circuit „without“ active n-terminal („passive part“ of the circuit); I_i are exciting currents, U_i are nodal voltages, Y_{11}, ..., Y_{mm} are elements of the admittance matrix; Y_{kk} are sums of admittances of elements connected to the k-th node - they are always positive; Y_{rs} are sums of admittances of elements connected between the r-th and s-th nodes - all these elements are negative.

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2) Now we rewrite matrices of the active n-terminals in the same system of nodes (in the admittance matrix „without“ active n-terminals).

3) In „places of coincidences“ we add the respective matrix elements from the matrices of active n-terminals.

4) The resultant admittance matrix, thus, the circuit equations, describe the linear active network and we can solve the problem - the analysis of circuit with active n-terminals by means of Cramer’s rule.

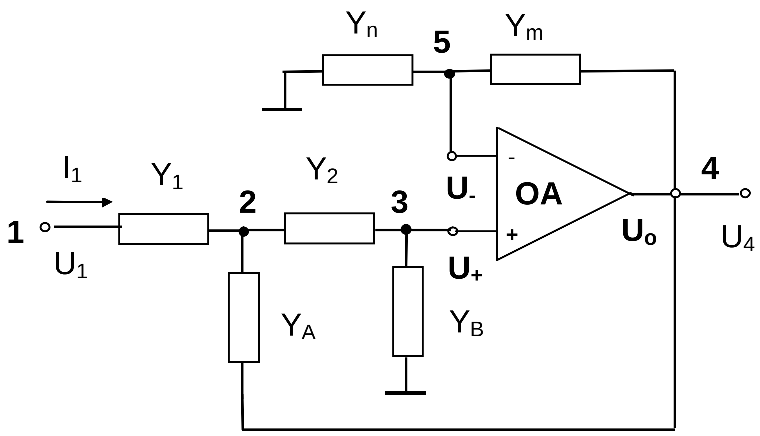

Fig.1. Sallen and Key filters

It is evident that for the generalized voltage analysis of Sallen and Key second-order low-pass filters (Fig.1) it would be known an **admittance matrix** **of real operational amplifier**.

2.1 A matrix description of op amps

The assignment and sign convention for input and output voltages and currents of operational amplifiers is shown in Fig.2.

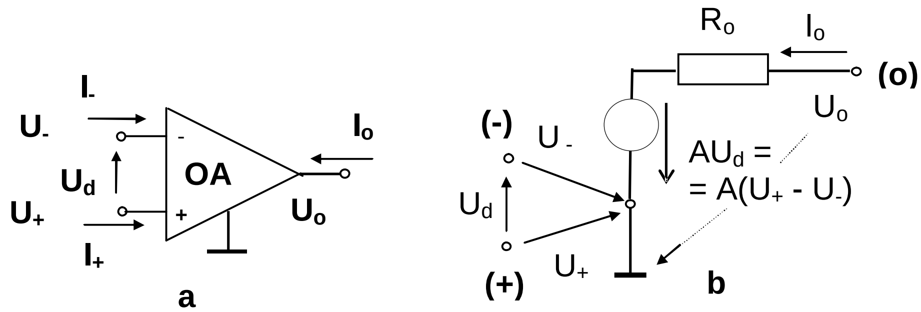

Fig.2. a) Symbol for an operational amplifier ; b) signal model of OA

The ground (reference) terminal provides a reference point for the three others (Fig. 2b):

- noninverting input (+)
- inverting input (-)
- output (o)

The simplified signal model (for $I_+ = I_- = 0$) is shown in Fig.2b. We can easy determine equations

$I_+ = 0.U_+ + 0.U_- + 0.U_o$
$I_- = 0.U_+ + 0.U_- + 0.U_o$
$I_o = [U_o - A(U_+ - U_-)]/R_o = -AG_oU_+ + AG_oU_- + G_oU_o$

A matrix form (model) of the equations is

$$\begin{array}{c|ccc|} & (+) & (-) & (o) \\ (+) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ (-) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ (o) & \mathbf{-AG_o} & \mathbf{AG_o} & \mathbf{G_o} \end{array} \begin{bmatrix} \mathbf{U_+} \\ \mathbf{U_-} \\ \mathbf{U_o} \end{bmatrix} = \begin{bmatrix} \mathbf{I_+} \\ \mathbf{I_-} \\ \mathbf{I_o} \end{bmatrix} \qquad (1)$$

The eq.(1) describes the matrix admittance model of the operational amplifier with the output resistance R_o (= $1/G_o$) and voltage gain A.

2.2 Results for real and ideal op amps

Now we can to analyze the linear electronic circuit in Fig.1. First we number nodes. There is only one signal current source I_1 in the circuit. We use the admittance matrix from the eq. (1) and basic rules of generalized nodal voltage analysis - we get:

$$\begin{array}{c|ccccc|} & 1 & 2 & 3\,(+) & 4\,(o) & 5\,(-) \\ 1 & Y_1 & -Y_1 & 0 & 0 & 0 \\ 2 & -Y_1 & Y_1+Y_A+Y_2 & -Y_2 & -Y_A & 0 \\ 3\,(+) & 0 & -Y_2 & Y_2+Y_B+(0) & 0+(0) & 0+(0) \\ 4\,(o) & 0 & -Y_A & 0+(-AG_o) & Y_A+Y_m+(G_o) & -Y_m+(AG_o) \\ 5\,(-) & 0 & 0 & 0+(0) & -Y_m+(0) & Y_m+Y_n+(0) \end{array} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = \begin{bmatrix} I_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

ROW COL.

( ) ↔ ( ) - „coincidences“; we add respective matrix elements from the eq.(1);
e.g.: (o) ↔ (-) give us ... +(AG_o), (-) ↔ (o) give us ... +(0)

This set of equations defines the electronic circuit in the Fig.1 with the real operational amplifier (we suppose zero input currents only).

In general, the frequency response of the op amp is determined by many poles and zeros; however, in order to assure stability in closed-loop feedback configurations, most modern op amps are designed to have a dominant real pole at $p = -\omega_1$, so the suitable model of an op amp voltage gain [1, 2, 8] is ($p = j\omega$ for steady state solution)

$$A(p) = \omega_T/(p + \omega_1) \cong \omega_T/p \qquad (2)$$

consequently

$$A(s) = (\omega_T/\omega_0)/(p/\omega_0) = \gamma/s \qquad (3)$$

where ω_T is gain-bandwidth product defined as $\omega_T = A_0\omega_1$; A_0 is the op amp´s dc gain; ω_1 is the 3-dB frequency; $\gamma = \omega_T/\omega_0$ is the normalized gain-bandwidth product; $s = p/\omega_0$ is the normalized frequency ($j\omega/\omega_0$ for steady state solution).

In reality output impedance has an effect on filters response, too. The op amp open - circuit impedance is considered to be ohmic, $R_0 = 1/G_o$ [1].

Solving matrix set of equations [3] for

$Y_1 = G_1 = 1/R_1$; $Y_2 = G_2 = 1/R_2$; $Y_A = pC_A$; $Y_B = pC_B$;
$Y_m = G_m = 1/R_m$; $Y_n = G_n = 1/R_n$

give us voltage transfer function $P(s) = U_4/U_1$ of the analyzed circuit (A - see eq.(3)):

$$P(s) = \frac{R_o/R_1}{D + R_o/R_{12}} \cdot \frac{s^3 + s^2 w_0/w_A + g\ R_1/R_o}{s^3 + s^2 \dfrac{g/H + D/Q + Hw_0/w'_A}{D + R_o/R_{12}} + s\dfrac{g/(HQ) + D}{D + R_o/R_{12}} + \dfrac{g/H}{D + R_o/R_{12}}} \qquad (4)$$

where

$\omega_o = 1/\sqrt{R_1 R_2 C_A C_B}$ is the ideal characteristic frequency
$H = 1 + R_m/R_n$ is the ideal „dc gain“

$$1/Q = \sqrt{\frac{R_1 C_B}{R_2 C_A}} + \sqrt{\frac{R_2 C_B}{R_1 C_A}} + \sqrt{\frac{R_1 C_A}{R_2 C_B}}(1 - H)$$ defines the „ideal Q“

$$D = (R_m + R_n + R_o)/(R_m + R_n) = |R_m + R_n \rangle\rangle R_o| \approx 1$$

$$w_A = 1/(R_1 C_A); \qquad R_{12} = R_1 R_2/(R_1 + R_2); \qquad 1/w'_A = (R_1 + R_o/(DH))C_A$$

For an ideal op amp ($\gamma \to \infty$) we can derive the known „ideal normalized low-pass“ transfer function

$$P_{id} = U_o/U_i = H/(s^2 + s/Q + 1) \qquad (4a)$$

3. DISCUSSION OF RESULTS

The loci of **zeros** (**poles)** are obtained by factoring the numerator (denominator) of eq.(4) for different values of γ, R_o/R_1 and Q.

3.1 Zeros - feedforward transmission [3, 4, 5]

The feedforward transmission through the Sallen and Key feedback impedances („into R_o“) takes place at high frequencies as a result of the decrease in the open-loop gain A(s). The transfer function P(s) has a cubic equation in the numerator. It can be determined that for $\omega R_2 C_B >> 1$ we can write

$$s^3 + s^2 w_0 / w_A + g\ R_1 / R_o \cong s^3 + g\ R_1 / R_o \qquad (5)$$

This has three solutions for *s* which make the numerator equal to zero:

$$s_{Z1} = -\sqrt[3]{gR_1 / R_o} \quad ; \ s_{Z2,3} = |s_{Z1}| \cdot \exp(\pm jp/3)$$

It means that

$$|s_{Z1}| = |s_{Z2}| = |s_{Z3}| = \sqrt[3]{gR_1 / R_o}$$

consequently the zero frequency is ($|s_Z| = w_Z / w_0$)

$$w_Z = w_{Z1} = w_{Z2} = w_{Z3} = w_0 \cdot \sqrt[3]{gR_1 / R_o} = \sqrt[3]{w_0^2 w_T R_1 / R_o} \qquad (6)$$

In the frequency domain (Fig.3), those three zeros will manifest themselves by stopping the decrease in the closed-loop gain. If $w_Z >> w_0$ we can derive

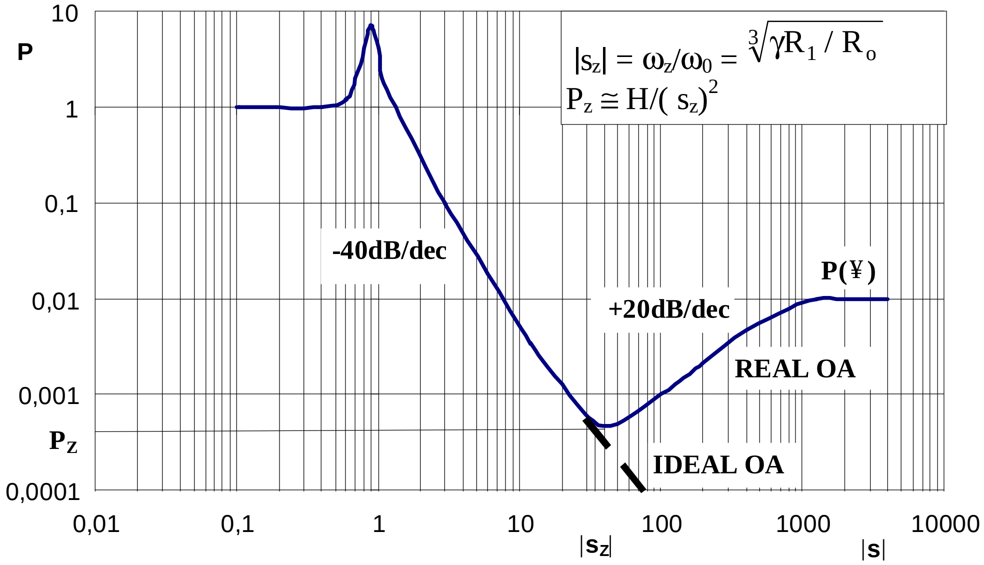

Fig.3. Frequency response of an ideal and real operational amplifier

$$P_Z \cong P_{id}(s = s_Z) = \frac{H}{s_Z^2 + s_Z / Q + 1} \cong \frac{H}{s_Z^2} = \frac{H w_0^2}{w_Z^2} \qquad (7)$$

and (refer to Fig.3)

$$P_{ZdB} = 20 \log P_Z = 20 \log H - 20 \log (w_Z / w_0)^2 = 20 \log H - \frac{40}{3} \log \left(\frac{w_T R_1}{w_0 R_o} \right) \qquad (8)$$

As the value of *s* becomes larger, the transfer function increases and approaches value ($R_1 >> R_o$)

$$P(\infty) \cong 20 \log (R_o / R_1) \qquad (9)$$

where P(∞) is the **feedforward transmission for** $|s| > w_T/\omega_0$.

The effect of op amp parameters on the over-all filter performance has been simulated by a computer program (MCII) and also was tested in the laboratory. Results show that the filter response is in accordance with previous equations - Fig.5.

3.2 Poles - real characteristic frequency [3, 12]

The loci of **poles** are obtained by factoring the denominator of eq.(4) for different values of γ, R_o/R_1 and Q. The equation

$$s^3 + bs^2 + cs + d = 0 \qquad (10)$$

has one real root [b, c, d - see denominator of eq.(4)], and if this is found, say $s = s_1$ (**high frequency pole**), the equation can be factorized into the suitable form

$$(s - s_1) \cdot \left(s^2 + s k_r / Q_r + k_r^2 \right) = 0 \qquad (11)$$

or

$$(s - s_1) \cdot (s - s_2) \cdot (s - s_3) = 0 \qquad (12)$$

where $s_{2,3} = -\alpha \pm j\beta$ are two complex-conjugate roots - **dominant poles**.

Comparing eq.(11) and eq.(12), k_r, Q_r, α and β must satisfy:

$$k_r / Q_r = 2a \quad \text{or} \quad a = k_r / (2 Q_r) \qquad (13)$$

$$k_r^2 = a^2 + b^2 \quad \text{or} \quad b^2 = k_r^2 [1 - 1/(4 Q_r^2)] \qquad (14)$$

$$Q_r = \sqrt{a^2 + b^2} / (2a) \qquad (15)$$

where $k_r = \omega_r/\omega_o$ is the normalized characteristic frequency „with real operational amplifier“ Q_r is the „real Q“.

Comparing eq.(10) and eq.(11) we get (exact equations)

$$s_1 = -b + k_r / Q_r \tag{16}$$
$$k_r^2 = d/(-s_1) \tag{17}$$
$$Q_r = \sqrt{d(-s_1)} \Big/ \left(c - k_r^2\right) \tag{18}$$

We are able to solve eq.(10) completely if we know the real root s_1.

It is evident that for the op amp with $\gamma >> 1$ is $b >> 1$, $k_r \to 1$, $Q_r \to Q$, consequently $\underline{s_1 \approx -b}$. Combining this and eqs.(17), (18) and again (16) gives

$$s_1 \approx -b + c/b - d/b^2 \tag{19}$$

Repeated combining eq.(19) and eqs.(17), (18) and (16) gives (for $b^2 >> c$)

$$s_1 \approx -b + bc/(b^2 - c) - d/b^2 \tag{20}$$

For example, we assume $H = 1$ ($R_m = 0$, $R_n \to \infty$), $R_1 = R_2 = R$. The expressions for ω_o^2, $1/Q$, R_{12}, $1/\omega'_A$ simplify to: $w_o^2 = 1/(R^2 C_A C_B)$; $1/Q = 2 \cdot \sqrt{C_B / C_A}$; $R_{12} = R/2$; $1/w'_A = (R + R_o)C_A$; consequently:

$$w_o / w'_A = 2Q(1 + R_o / R);$$

$$b = \frac{g + 1/Q + 2Q(1 + R_o / R)}{1 + 2R_o / R}; \qquad c = \frac{g/Q + 1}{1 + 2R_o / R}; \qquad d = \frac{g}{1 + 2R_o / R}$$

The results are summarized in Table 1.

4. EXPERIMENTS

To overcome (partly) the effect of the op amp output resistance we can modify the RC network. What is required is a circuit having a falling response above frekvency $f_z = \omega_z/(2\pi)$. This can be done by falling R_1 into two parts and adding a compensating capacitor C_k. In the frequency domain, this new pole manifests itself by stopping the increase in the closed-loop gain (above f_z) and by decrease in the closed-loop gain for frequencies above f_T/H.

Table 1. Some results of analysis - poles
Notes: ECS - Exact Cardan’s Solution; HNE - Has Not Effect; NE - No Exist;
IOA - Ideal Op Amp; ROA - Real Op Amp

EQUATIONS→			(20)	(17)	(18)	ECS	(17)	(18)	NO-TES
Q	γ	R_o/R	$-s_1$	k_r	Q_r	$-s_1$	k_r	Q_r	
	∞	HNE	NE	1	0,5	NE	1	0,5	IOA
		0	101,2	→	→	101,2	0,9949	0,5025	
	100	0,1	83,933	→	→	83,933	0,9964	0,5023	ROA
0,5		0,5	49,751	→	→	49,750	1,0025	0,5013	
		0	36,059	→	→	36,058	0,9852	0,5073	
C_A/C_B	35	0,1	29,799	0,9893	0,5067	29,797	0,9894	0,5066	ROA
= 1		0,5	17,258	1,0070	0,5039	17,251	1,0072	0,5038	
		0	16,142	0,9640	0,5175	16,136	0,9642	0,5174	
	15	0,1	13,206	0,9729	0,5163	13,198	0,9732	0,5161	ROA
		0,5	7,2913	1,0141	0,5110	7,2564	1,0166	0,5099	
0,6			30,042	→	→	30,042	0,9853	0,6107	
1	35	0,1	30,891	→	→	30,893	0,9717	1,0331	ROA
3			34,664	→	→	34,664	0,9173	3,2732	
10			47,517	→	→	47,517	0,7835	11,871	
3	200		172,12	0,9840	3,0561	←	←	←	
	100	0,1	88,799	→	→	88,800	0,9687	3,1086	ROA
C_A/C_B	15		18,038	0,8325	3,4864	18,039	0,8324	3,4864	
=36	8		12,237	0,7381	3,5975	12,240	0,7380	3,5977	
10	200		184,99	→	→	184,99	0,9492	10,578	
	100	0,1	101,67	→	→	101,67	0,9053	11,027	ROA
	15		30,921	0,6358	11,709	30,862	0,6364	11,703	
C_A/C_B	200	0	220,1	→	→	220,1	0,9532	10,443	ROA
=400	200	0,5	114,97	→	→	114,97	0,9326	11,134	

The practical configuration of the Sallen and Key low-pass filtr is in Fig.4. Table 2 and Fig.5 summarize properties of this filtr.

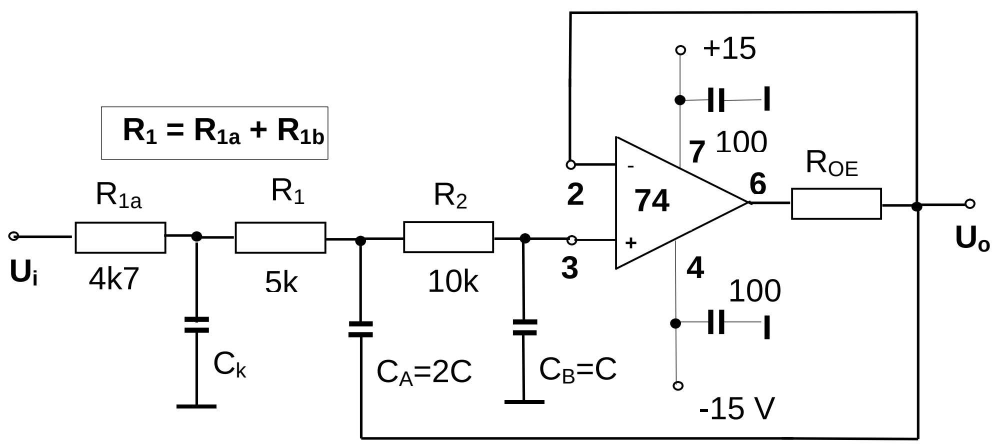

Fig.4. Practical realization of the Sallen and Key low-pass filter with the „741“ [$R_o \cong 70\ \Omega$; $f_T \cong 0{,}6$ MHz - f_T measured]

It is evident that

$H = 1 + 0/\infty = 1$

$R_1 = R_2 = R = 10\ k\Omega$

$C_A = 2C_B = 2C$

consequently

$f_0 = 1/(2\pi\sqrt{2}\ RC)$

$Q = 0{,}5\ (C_A/C_B)^{1/2} = 0{,}707$ ....Butterworth approximation.

Table 2. Some results of the circuit in Fig.4 analysis and measurement

f	$P(f) = U_o/U_i$ (U_i = 1 V)				
Hz	[- /dB]				
40	0,948/ -0,46	0,946/ -0,48	0,947/ -0,47	-	-
60	0,800/ -1,94	0,798/ -1,96	0,800/ -1,94	-	-
80	0,603/ -4,40	0,599/ -4,45	0,600/ -4,43	-	-
100	0,432/ -7,30	0,431/ -7,32	0,433/ -7,28	-	-
200	0,119/ -18,5	0,119/ -18,5	0,117/ -18,6	-	-
500	0,0193/ -	0,0191/ -34,4	0,0195/-34,2	-	-
1 k	$4{,}62.10^{-3}$/ -	$4{,}52.10^{-3}$/-	$4{,}57.10^{-3}$/ -	0,990/ -0.09	0,987/ -0,11
2 k	$1{,}15.10^{-3}$/ -	$1{,}00.10^{-3}$/ -	$9{,}77.10^{-4}$/ -	0,979/ -0,18	0,971/ -0,26
3 k	$5{,}62.10^{-4}$/ -	$5{,}69.10^{-4}$/ -	**$6{,}26.10^{-4}$/ -**	0,882/ -1,09	0,869/ -1,21
4 k	$3{,}80.10^{-4}$/ -	**$3{,}55.10^{-4}$/ -**	**$4{,}90.10^{-4}$/ -**	0,750/ -2,50	0,748/ -2,52
5 k	$2{,}51.10^{-4}$/ -	**$2{,}60.10^{-4}$/-**	**$5{,}31.10^{-4}$/ -**	0,541/ -5,34	0,530/ -5,51
7 k	**$1{,}41.10^{-4}$/-**	**$2{,}99.10^{-4}$/ -**	$7{,}08.10^{-4}$/ -	0,299/ -10,5	0,295/ -10,6
8 k	**$1{,}26.10^{-4}$/ -**	$3{,}35.10^{-4}$/ -	$7{,}94.10^{-4}$/ -	0,211/ -13,5	0,209/ -13,6
10 k	**$1{,}58.10^{-4}$/ -**	$4{,}12.10^{-4}$/ -	$1{,}05.10^{-3}$/ -	0,155/ -16,2	0,150/ -16,5
20k	$2{,}24.10^{-4}$/ -	$7{,}50.10^{-4}$/ -	$2{,}11.10^{-3}$/ -	0,0376/ -	0,0372/ -
50k	$4{,}22.10^{-4}$/ -	$1{,}76.10^{-3}$/ -	$5{,}01.10^{-3}$/ -	$5{,}96.10^{-3}$/-	$5{,}89.10^{-3}$/ -
100 k	$7{,}76.10^{-4}$/ -	$3{,}80.10^{-3}$/ -	$1{,}00.10^{-2}$/ -	**$1{,}64.10^{-3}$/-**	$1{,}60.10^{-3}$/ -
200 k	$1{,}51.10^{-3}$/ -	$7{,}08.10^{-3}$/ -	0,0197/ -34,1	**$1{,}30.10^{-3}$/ -**	$1{,}26.10^{-3}$/ -
300 k	$2{,}07.10^{-3}$/ -	$1{,}00.10^{-2}$/ -	0,0316/ -30,0	$1{,}80.10^{-3}$/ -	$1{,}26.10^{-3}$/ -
500 k	$3{,}80.10^{-3}$/ -	0,0162/ -35,8	0,0468/ -26,6	$3{,}09.10^{-3}$/ -	$1{,}17.10^{-3}$/ -
700 k	$4{,}90.10^{-3}$/ -	0,0211/ -33,5	0,0442/ -27,1	$4{,}42.10^{-3}$/ -	$7{,}24.10^{-4}$/ -
1 M	$6{,}61.10^{-3}$/ -	0,0248/ -32,1	0,0432/ -27,3	$5{,}96.10^{-3}$/ -	$5{,}01.10^{-4}$/ -
1,5 M	$8{,}91.10^{-3}$/ -	0,0254/ -31,9	0,0422/ -27,5	$7{,}85.10^{-3}$/ -	$3{,}98.10^{-4}$/ -
2 M	$9{,}66.10^{-3}$/ -	0,0260/ -31,7	0,0412/ -27,7	$9{,}23.10^{-3}$/-	$3{,}16.10^{-4}$/ -
C [nF]	163			2,8	
R_o[Ω]	70	220	540	70	
C_K[pF]	no			no	440
f_0[Hz]	69			4019	
$\gamma = f_T/f_0$	8696			149,3	
$\gamma R_1/R_o$	$1{,}24.10^{6}$	$3{,}95.10^{5}$	$1{,}61.10^{5}$	$2{,}13.10^{4}$	
f_Z [kHz]-	7,412	5,063	3,754	111,407	
$P(\infty)$[dB]-	-43,1	-33,2	-25,4	-43,1	
P_Z [dB] -	-81	-74	-69	-57,7	

The output resistance of operational amplifier is changed by means of R_{OE} (see Fig.4), thus

$$R_o \cong 70\ \Omega + R_{OE}$$

where 70 Ω is „the output quality“ of 741.

From the eq.(6) we can get „zero frequency“ f_z:

$$f_Z \cong f_0 \cdot \sqrt[3]{\gamma R_1 / R_o} = \sqrt[3]{f_0^2 f_T R_1 / R_o}$$

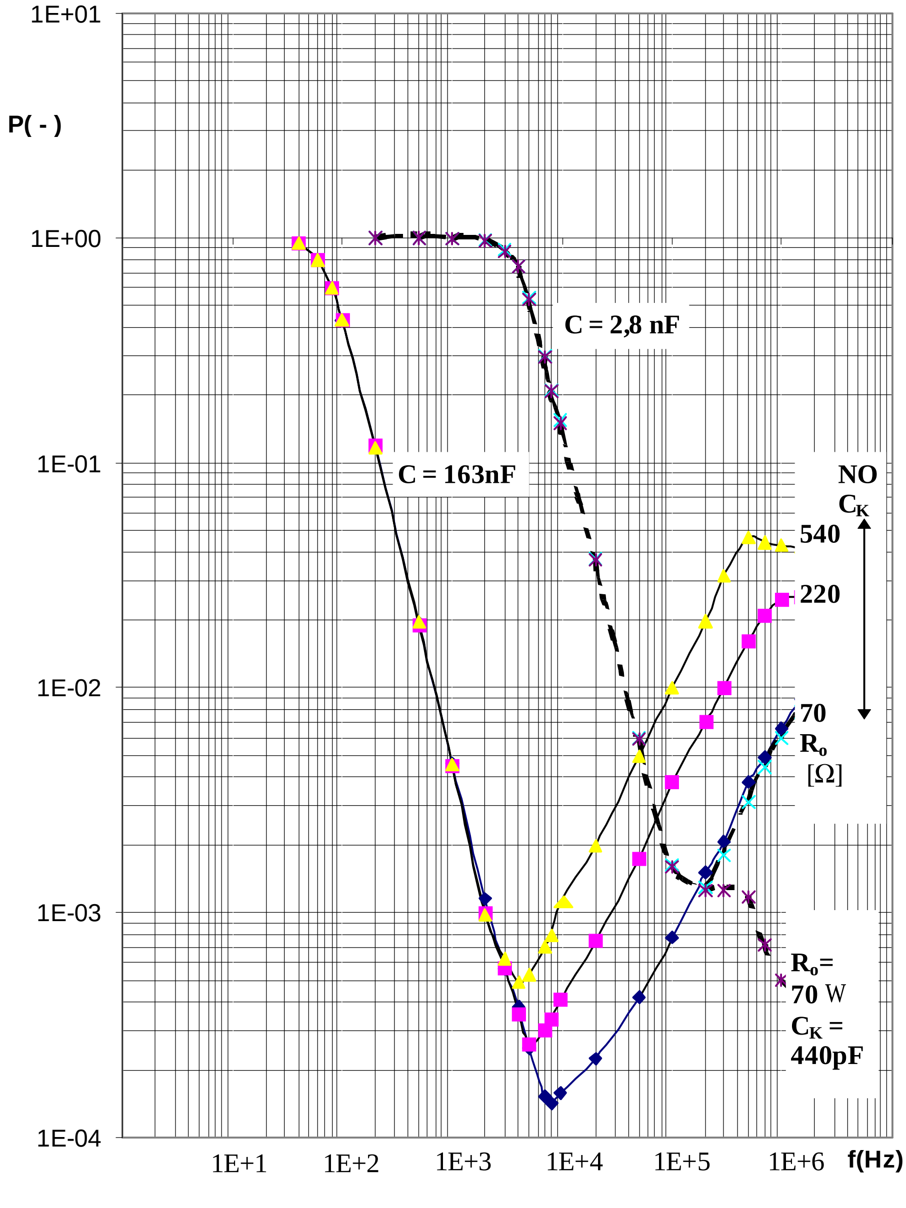

Fig.5. Measured frequency response of the circuit in Fig.4

CONCLUSION

The dominant poles are the pair of complex poles that correspond to the ideal poles but are shifted because of finite gain-bandwidth product and nonzero output resistance R_o. To see the effect graphically, we can use eqs.(13) and (14) and draw loci of the dominant poles. The results show that for $Q < 1$ the output resistance R_o compensates the negative influence of the finiteγ. For Q´s > 1 this is not valid; for increasing R_o the normalized characteristic frequency k_r invariably decreases from the ideal value 1.

Results in the Table 1 show that for determining s_1 (to our purpose) we can use eq.(20), we need not use the exact Cardan´s solution.

But the feedforward transmission (due to R_o and finite γ) is unpleasant problem, at all events. To overcome (partly) this effect, we can modify the RC network. We fall R_1 into two parts and add compensating capacitor C_k, see Fig.4. It is a good solution for the low Q´s second order Sallen &Key low-pass filters.

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APPENDIX B Band stop filters with real operational amplifiers

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Abstract: This paper deals with influence of first pole of op amp upon properties of two band stop filters. There are new equations for real characteristic frequency ω_{0r} (f_{0r}). Generalized nodal voltage analysis is used. Both finite input resistance and nonzero output resistance does not have so important influence and are omitted.

Key words: Filter, operational amplifier, real properties, admittance matrix

Generalized nodal voltage analysis

Circuits are analyzed by the means of generalized nodal voltage analysis. There are several basic rules of analysis.

1) First we determine admittance matrix of the circuit without active elements (only passive part). $Y_{11} \dots Y_{nn}$ are elements of matrix. Y_{kk} are sums of admittance of elements connected to the k-th node and they are always positive. Y_{rs} are sums of admittances of elements connected between the r-th and s-th nodes and all these elements are negative. I_i is vector of exciting currents and U_i is vector of nodal voltages.
2) Now we rewrite matrices of the active elements in the same system of nodes (in the passive matrix)
3) In the places of coincidences we add the respective matrix elements from the active matrix
4) The resultant matrix contains describes of active elements. For solving we can use Cramer's rules.

It is necessary to know admittance matrix of active elements if we can use this method. The shape of admittance matrix of op amp is [6, 7]:

$$\begin{array}{c|c|c|c|} & (+) & (-) & (o) \\ \hline (+) & 0 & 0 & 0 \\ \hline (-) & 0 & 0 & 0 \\ \hline (o) & -AG_O & AG_O & G_O \\ \hline \end{array} * \begin{array}{|c|} \hline U_+ \\ \hline U- \\ \hline U_O \\ \hline \end{array} = \begin{array}{|c|} \hline I_+ \\ \hline I\ - \\ \hline I_O \\ \hline \end{array} \qquad \textbf{(1)}$$

We have to take into account nonzero output resistance R_0, otherwise it is not possible to determine the matrix (1). In the resultant equation we can equate R_0 to zero and simplify the equation. An A is an op amp voltage gain. The gain A is not constant and real, but is frequency dependent. Generally the op amp has three parts and its amplifier characteristic has three poles, but if second and third poles are in place where the gain is lower then 1 (0 dB), they can be omitted. In this case the gain can be described by next equations:

$$\bar{A} = A_0 / (1 + j\omega / \omega_1) = |\omega \rangle\rangle \omega_1| = A_0 \omega_1 / j\omega = \omega_T / p \qquad (2)$$

where

A_0 is dc gain
p is $j\omega$ (for steady state harmonic solution)

Analysis of band stop filters

Now we can try to analyze some selected circuits. Selected circuits are a Wien bridge, and a T-bridge. The Wien bridge and T-bridge are different circuits, but its transfer function is the same, more see in [3, 4].

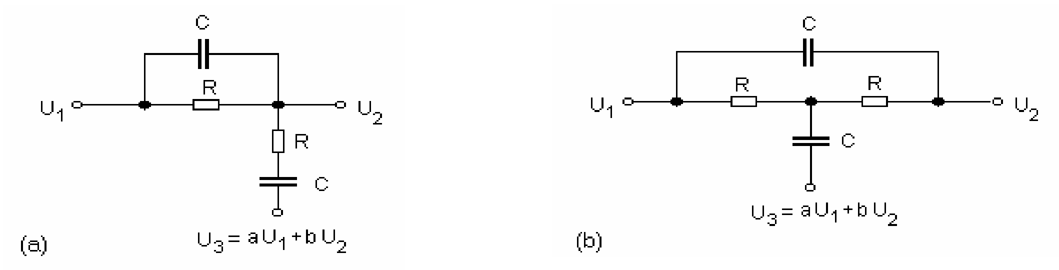


Fig.1 a) Wien bridge as band stop filter
b) T-bridge as band stop filter
(*a, b* – external transfers of circuit)

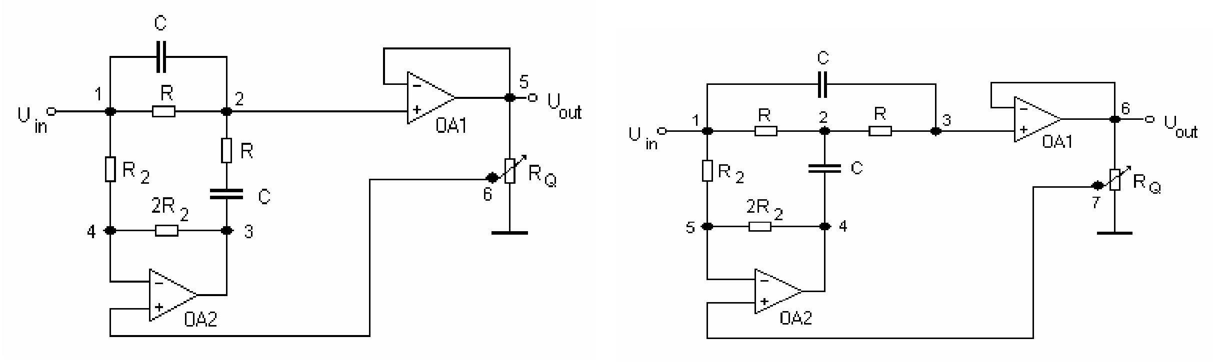
Voltage U_3 is created in external adding and amplifying circuits. These circuits can contain op amps. Theoretical equations for circuits in Fig. 1 are:

$$H(\mathrm{p}) = \frac{p^2 + p(2+a)w_0 + w_0^2}{p^2 + p(3-b)w_0 + w_0^2} \qquad (3)$$

$$\omega_0 = 1/(RC) \qquad (4)$$

$$Q = \frac{w_0}{w_2 - w_1} = \frac{1}{3-b} \qquad (5)$$

A voltage transfer on ω_0 (f_0) is zero if a = -2. Q and ω_0 (f_0) can be changed independently. Ones of the possible real configurations with op amps are in Fig. 2.

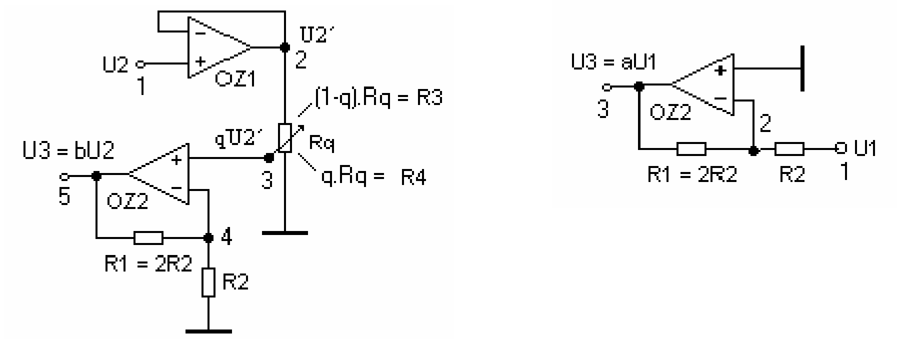


(a) (b)

Fig.2 a) Real configuration of band stop filter with Wien bridge and two OA
b) Real configuration of band stop filter with T-bridge and two OA

OA1 creates voltage follower and OA2 creates adding and amplifying circuit. Because real properties of op amps influence external transfers *a, b* only, we can calculate elements *a, b* and substitute them to the theoretical eq.(3).

In the first we must determine models for transfers *a, b*. For this we use generally known superposition theorem, see Fig.3.



(a) *(b)*

Fig.3 Circuits for determining of a) external transfer *b* (U1 = 0 - Fig.2 a,b - node 1);
b) external transfer ***a*** (U+ of OA2 is zero); OA ≡ OZ

Then we write passive matrix of circuits and consequently we add matrix description of op amps. This matrix model we will solve by means of Cramer's rules. If the properties of op amps are the same, the matrix of Fig.3a has next shape. We suppose only one exciting current I incoming to node 1.

I		0	0	0	0	0		U_2	
0		$-AG_0$	$G_3+G_0(1+A)$	$-G_3$	0	0		U_2'	
0	=	0	$-G_3$	G_3+G_4	0	0	*	$q\,U_2'$	(**6**)
0		0	0	0	G_1+G_2	$-G_2$		U_4	
0		0	0	$-AG_0$	$-G_2+AG_0$	G_2+G_O		U_3	

The result of solving is external transfer *b*. Output resistance R_0 in resultant equations is zero. After arrange-ment we get:

$$b = \frac{U_3}{U_2} = \frac{\Delta_5}{\Delta_1} = \frac{3qA^2}{(1+A)(3+A)} \qquad (7)$$

where

$$q = \frac{R_4}{R_3 + R_4}$$

is dividing rate of Rq

For $A \to \infty$ is $b = 3q$.

A solving for external transfer ***a*** is made by the same way. If we respect real properties of op amps we have next matrix model:

I		G_1	$-G_1$	0		U_1	
0	=	$-G_1$	G_1+G_2	$-G_2$	*	U_x	
0		0	$-G_2+AG_0$	G_0+G_2		U_3	(**8**)

The external transfer ***a*** is again calculated by means of Cramer's rules and in resultant equations is output resistance $R_0 = 0$.

$$a = \frac{U_3}{U_1} = \frac{\Delta 3}{\Delta 1} = -\frac{2A}{3+A} \qquad (9)$$

It is evident that for A→∞ is *a* constant and equals –2. This is requirement for zero transfer on ω_0 (f_0), how was said before. Now we substitute (2) to (7) and (9). Then we substitute (7) and (9) to theoretical eq.(3). Because output signal is taken from output of AO1 we must come into question the transfer function of OA1.

$$H(p)_{OA1}=\frac{A}{(1+A)}=\frac{w_T}{w_T+p}$$

This is transfer function of voltage follower OA1 itself. The transfer function (3) is now.

$$H(p)=\frac{p^2+2pw_0\left(1-\frac{A}{3+A}\right)+w_0^2}{p^2+3pw_0\left(1-\frac{qA^2}{(1+A)(3+A)}\right)+w_0^2}\cdot\frac{A}{1+A}=$$

$$\frac{p^2+\frac{6p^2w_0}{p+w_T}+w_0^2}{p^2+3pw_0\left(\frac{(p+w_T)(3p+w_T)-qw_T^2}{(p+w_T)(3p+w_T)}\right)+w_0^2}\cdot\frac{w_T}{p+w_T} \qquad (10)$$

The transfer function is fourth order now. The solving of this function is rather difficult, but it is possible to derive, by means of Matlab or Derive, from this function that real characteristic frequency ω_{0r} (f_{0r}) is:

$$w_{0r}=\frac{\sqrt{3}}{9}\sqrt{\sqrt{81w_0^4+756w_0^3w_T+234w_0^2w_T^2+24w_0w_T^3+w_T^4}-18w_0^2-12w_0w_T-w_T^2} \qquad (11)$$

This function good corresponds to both PC simulation and measurement with real devices. Real frequency ω_{0r} (f_{0r}) is always lower then ideal ω_0 (f_0).

A derivation of expression for real quality factor Q_r was not done, because fourth order of transfer function. Only graph obtains from numerical solution is at hand. Behavior of Qr is not so explicit. For Q = 0,66 is Q_r higher then Q. Situation is in Fig.5. Theoretical frequency f_0 is 98,9 kHz. (R = 10 kΩ, R2 = 22 kΩ, C = 160 pF). We have done numerical solution of eq.(10) in Matlab and eq.(11), too. We have simulated both circuits by means of PC simulation program MicroCap 7. Real measurement was done for the circuit Fig.2b with several types of op amps. All results are summarized in tab. 1.

Tab.1 A comparison of different solution of influence f_T upon f_0 for different types of op amps (f_T is catalog value)

	f_T = 1MHz (UA741)	f_T = 4MHz (TL072)	f_T = 15MHz (LM318)
Numerical solution of eq.(10)	78,8 kHz	92,3 kHz	97,1 kHz
Result of eq.(11)	78,0 kHz	92,0 kHz	96,9 kHz
MicroCap VII	78,3 kHz	92,5 kHz	96,8 kHz
Measurement	76,0 kHz	86,0 kHz	93,0 kHz

Values obtained from eq.(10), eq.(11) and from simulation program MicroCap 7 are very close and results of real measurement are enough close, too. Catalog value f_T need not be absolute precision.

There are courses of Q_r/Q versus f_T/f_0 in Fig.4. Value of Q_r may be higher then Q in particular district, for Q = 0,66. For Q = 3,33 (q = 0,9) is Q_r appreciably lower then Q even for f_T/f_0 = 200! Because eq.(10) is fourth order, explicit function for real quality factor Q_r was not found.

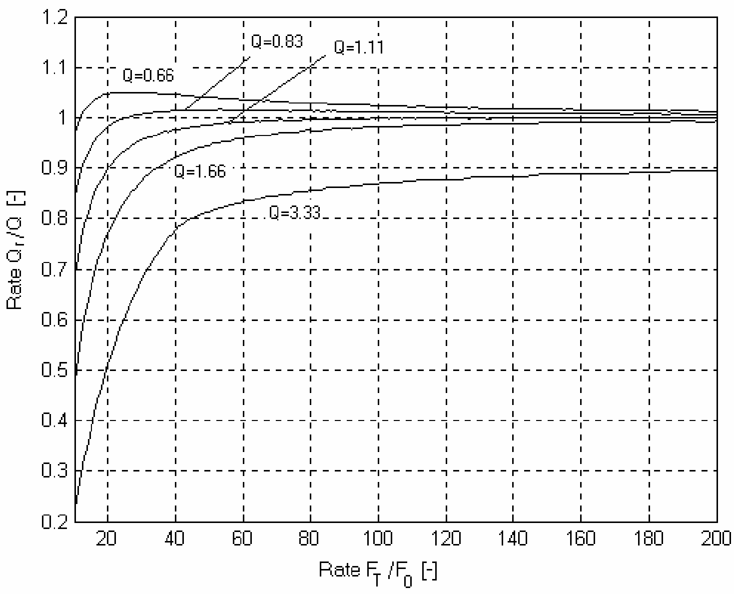


Fig.4 Rate Qr/Q versus f_T/f_0 for several values Q

Conclusion

It is evident, that for use of ideal eqs.(3-5) the rate f_T/f_0 have to be 100/1 or higher, otherwise the deviation is enough great. Moreover the attenuation on ω_0 (f_0) is not infinite and decreases with decrease of rate f_T/f_0.

One important fact was not emphasized. Transfer function over ω_T (f_T) decreases about 20 dB/dek. In this area band stop filter works as low pass filter. This is influence of output voltage follower OA1.

The results obtained by means of algebraic or numeric solution were verified in simulation program MicroCap 7. This program is very good useful for quick verification and simulations. All results of real measurement, PC simulations and mathematic calculations are in tab.1.

If we wont to have frequency ω_0 (f_0) still the same, without dependency on ω_T (f_T), we can use another type of circuit. One suitable is bridge double T. The circuit is a little more complicated, but frequency ω_0 (f_0) depends on ω_T (f_T) very few. More information is in [2, 4], for example.

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