Chapter 7 Homework

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Problem 1

Let $H = \{(1), (12)(34), (13)(24), (14)(23)\}$. Find the left cosets of H in A_4.

By Theorem 3.3, H is a subgroup of A_4. Then by Corollary 1 of Theorem 7.1, there are $|A_4 : H| = |A_4|/|H| = 12/4 = 3$ cosets for us to find. H is one of them. The other two are given by...

$$(123)H = \{(123), (134), (243), (142)\}$$
$$(132)H = \{(132), (234), (124), (143)\}$$

Problem 2

Let H be as in Problem 1. How many left cosets of H in S_4 are there?

Since H is a subgroup of S_4, the answer is $|S_4 : H| = |S_4|/|H| = 4!/4 = 3! = 6$.

Problem 3

Let $H = \{0, \pm 3, \pm 6, \pm 9, \dots\}$. Find al the left cosets of H in Z.

There are 3 of them: $0 + H$, $1 + H$, and $2 + H$.

Problem 4

Rewrite the condition $a^{-1}b \in H$ given in property 4 of the lemma on page 138 in additive notation. Assume that the group is Abelian.

$aH = bH$ if and only if $b - a \in H$.

Problem 5

Let H be as in Problem 3. Use Problem 4 to decide whether or not the some cosets of H are the same.

Notice that H is Abelian. Is it true that $11 + H = 17 + H$? Well, $17 - 11 = 6 \in H$, so yes. Is it true that $-1 + H = 6 + H$? Well, $6 - (-1) = 7 \notin H$, so no. Is it true that $7 + H = 23 + H$? Well, $23 - 7 = 16 \notin H$, so no.

Problem 6

Let n be a positive integer. Let $H = \{0, \pm n, \pm 2n, \pm 3n, \dots\}$. Find all left cosets of H in $\mathbb{Z}$. How many are there?

There are n cosets: $\{k + H\}_{k=0}^{n-1}$.

Problem 7

Find all of the left cosets of $\{1, 11\}$ in $U(30)$.

They are $\{1, 11\}$, $\{7, 17\}$, $\{13, 23\}$, and $\{19, 29\}$.

Problem 8

Suppose that a has order 15. Find all of the left cosets of $\langle a^5 \rangle$ in $\langle a \rangle$.

Let $G_k = \{a^k, a^{5+k}, a^{10+k}\}$. Then the set of cosets is $\{G_k\}_{k=0}^4$.

Problem 9

Let $|a| = 30$. How many left cosets of $\langle a^4 \rangle$ in $\langle a \rangle$ are there? List them.

There are two of them.

$$\langle a^4 \rangle = \{e, a^4, a^8, a^{12}, a^{16}, a^{20}, a^{24}, a^{28}, a^2, a^6, a^{10}, a^{14}, a^{18}, a^{22}, a^{26}\}$$
$$a\langle a^4 \rangle = \{a, a^5, a^9, a^{13}, a^{17}, a^{21}, a^{25}, a^{29}, a^3, a^7, a^{11}, a^{15}, a^{19}, a^{23}, a^{27}\}$$

Problem 12

Let $\mathbb{C}^*$ be the group of non-zero complex numbers under multiplication and let $H = \{a + bi \in \mathbb{C}^* : a^2 + b^2 = 1\}$. Give a geometric description of the cosets of H.

The geometric interpretation of complex multiplication describes the result as a diolation and rotation. H is all the points on the unit circle. Any coset of H in G would then be a circle with a different radius. Diolating a circle changes its radius, while rotating it has no affect.

Problem 14

Suppose that K is a proper subgroup of H and H is a proper subgroup of G. If $|K| = 42$ and $|G| = 420$, what are the possible orders of H?

By Lagrange's Theorem we must have $2 \cdot 3 \cdot 7$ divide $|H|$ and $|H|$ divide $2^2 \cdot 3 \cdot 5 \cdot 7$. So I believe the choices are...

$$|H| = 2 \cdot 3 \cdot 5 \cdot 7 = 210$$
$$|H| = 2^2 \cdot 3 \cdot 7 = 84$$

Problem 15

Suppose that $|G| = pq$, where p and q are prime. Prove that every proper subgroup of G is cyclic.

By Lagrange's Theorem, the possible orders of proper subgroups of G are 1, p, and q. In the first case, $G = \langle e \rangle$. In the other cases G is cyclic by Corollary 3 of Theorem 7.1. Also, every non-identity element of a group of prime order is a generator of that group.

Problem 17

Compute $5^{15} \mod 7$ and $7^{13} \mod 11$.

Using Fermat's Little Theorem...

$$5^{15} = 5(5^7)^2 \equiv 5(5^2) \equiv 5 \cdot 4 \equiv 6 \pmod 7$$
$$7^{13} = 7^{11}7^2 \equiv 7 \cdot 7^2 \equiv 7 \cdot 5 \equiv 2 \pmod{11}$$

Problem 18

Use Corollary 2 of Lagrange's Theorem (Theorem 7.1) to prove that the order of $U(n)$ is even when $n > 2$.

Notice that for all $n > 2$ we have $\gcd(n-1, n) = 1$ so that $n-1$ is in $U(n)$. Then $(n-1)^2 = n^2 - 2n + 1 \equiv 1 \pmod{n}$, showing that $|(n-1)| = 2$, since $n > 2$. So by Corollary 2 of Lagrange's Theorem, 2 divides $|U(n)|$.

Problem 20

Suppose H and K are subgroups of a group G. If $|H| = 12$ and $|K| = 35$, find $|H \cap K|$.

Notice that $e \in H \cap K$ so that it's non-empty. Now let $a, b \in H \cap K$. Then $ab \in H$ and $ab \in K$ implies that $ab \in H \cap K$ so that by Theorem 3.3, $H \cap K$ is a group. Further more, since $H \cap K \subseteq H$ and $H \cap K \subset K$, we see that $H \cap K$ is a subgroup of each of these. So by Lagrange's Theorem, $|H \cap K|$ must divide $|H|$ and $|K|$. But $\gcd(|H|, |K|) = 1$. Therefore, $|H \cap K| = 1$ and we see that $H \cap K = \{e\}$.

Problem 24

Let G be a group of order 25. Prove that G is cyclic or $g^5 = e$ for all $g \in G$.

If G is cyclic, then we're done. Suppose G is not cyclic. By Lagrange's Theorem, $|g|$ divides $|G| = 5^2$ for all $g \in G$. We also know that $5^0 < |g| < 5^2$ for all non-identity elements of G, since G does not have a generator. It follows that all non-identity elements of G have order 5. Then since $e^5 = e$, we have $g^5 = e$ for all $g \in G$.

Problem 27

Let H and K be subgroups of a finite group G with $H \subseteq K \subseteq G$. Prove that $|G : H| = |G : K||K : H|$.

This follows directly from Corollary 1 of Lagrange's Theorem.

$$|G : H| = \frac{|G|}{|H|} = \frac{|G|}{|K|} \cdot \frac{|K|}{|H|} = |G : K||K : H|$$

Problem 31

Let $G = \{(1), (12)(34), (1234)(56), (13)(24), (1432)(56), (56)(13), (14)(23), (24)(56)\}$. Find stabalizers and oribts.

$$\begin{aligned}
\text{stab}_G(1) &= \{(1), (24)(56)\} \\
\text{orb}_G(1) &= \{1, 2, 3, 4\} \\
\text{stab}_G(3) &= \{(1), (24)(56)\} \\
\text{orb}_G(3) &= \{3, 4, 2, 1\} \\
\text{stab}_G(5) &= \{(1), (12)(34), (13)(24), (14)(23)\} \\
\text{orb}_G(5) &= \{5, 6\}
\end{aligned}$$

Problem 32

Prove that 3, 5, and 7 are the only three consecutive odd integers that are prime.

After declaring 3 as prime, every third number is divisible by 3. Enumerating the odd integers starting with 3, we have $3 + 2k$ for $k = 0, 1, 2, \ldots$. Since k is divisible by 3 every third integer, $3|(3 + 2k)$ every third odd integer for $k = 0, 1, 2, \ldots$. This leaves only the possibility of two consecutive odd integers being prime.

Problem 36

Let G be a finite Abelian group and let n be a positive integer that is relatively prime to $|G|$. Show that the mapping $a \to a^n$ is an automorphism of G.

Let $a, b \in G$. The mapping is operation preserving by the Abelian property of G since $(ab)^n = a^n b^n$. Now let $a^n = b^n$. This implies that $a^n(b^{-1})^n = (ab^{-1})^n = e$ by the Abelian property of G. We now know that $n = k|ab^{-1}|$ for some integer k. But by Lagrange's Theorem, $|ab^{-1}|$ divides $|G|$ while by the problem statement, we must have $\gcd(k|ab^{-1}|, |G|) = 1$. Suppose $ab^{-1} \neq e$. Then $|ab^{-1}| > 1$ and therefore $\gcd(k|ab^{-1}|, |G|) > 1$. So by contradiction $ab^{-1} = e$. This then implies that $a = b$, and we see that the mapping is one-to-one. Since the mapping is one-to-one from a finite set to itself, it is also onto. We've now shown enough to say that the mapping is an automorphism of G.

Problem 40

Let G be the group of rotations of a plane about a point P in the plane. Thinking of G as a group of permutations of the plane, describe the orbit of a point Q in the plane.

If $Q \neq P$, then would the orbit of Q be the circle in the plane determined by P and Q where P is the center and Q is on the perimeter. If $Q = P$, then the orbit of Q would just be P.