Section 2.6 Exercises
Hertein's Topics In Algebra

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Thoughts

Remembering Gallian's book, he shows that the operation Herstein introduces here is well-defined. Let N be a normal subgroup of G and define, for any $a, b \in G$,

$$(Na)(Nb) = N(ab).$$

Is this a well-defined operation? Well, $Na = Nb$ if and only if $ab^{-1} \in N$. So let $a', b' \in G$ such that $Na = Na'$ and $Nb = Nb'$ and write

$$(Na')(Nb') = N(a'b').$$

Can we show that

$$ab(b')^{-1}(a')^{-1} = ab(a'b')^{-1} \in N?$$

Well, clearly $n_b = b(b')^{-1} \in N$. Now since $n_a = a(a')^{-1} \in N$, we have

$$ab(b')^{-1}(a')^{-1} = an_b(a')^{-1} = n_a[(a')n_b(a')^{-1}] \in N,$$

since N is normal.

Problem 6

Show that every subgroup of an abelian group is normal.

Let H be a subgroup of an abelian group G. Then, for any $h \in H$ and $g \in G$, observe that

$$ghg^{-1} = gg^{-1}h = h \in H,$$

showing that H is normal in G.

Problem 7

Is the converse of Problem 6 true?

The converse would read: if every subgroup of a group is normal, then the group is abelian.

I can't find a counter-example, but I'm willing to bet the converse is false.

When $gN = Ng$, this does not require that $gn = ng$ for all $n \in N$.

Problem 9

Suppse H is the only subgroup of order $|H|$ in the finite group G. Prove that H is a normal subgroup of G.

This would follow from proving the following statement. If $\{H_i\}$ is a finite set of subgroups of G, each of order n, then for all $g \in G$, and every integer i, there exists an integer j, such that

$$gH_i = H_j g.$$

Interestingly, this presents the idea of two subgroups of G being co-normal. Neither is necessarily normal by themselves, but together, they're co-normal.

Here's an idea. Let H be a subroup of G. Then, for any $g \in G$, let K be the set given by

$$K = \{g^{-1}hg | h \in H\}.$$

It is clear that $gK = Hg$. We now show, whether or not H is a normal subgroup of G, that K is a subgroup of G having the same order as H.

Clearly $e \in K$, so K is non-empty. Closure is trivial, for

$$(g^{-1}h_1g)(g^{-1}h_2g) = g^{-1}h_1h_2g \in K$$

since $h_1h_2 \in H$. And then

$$(g^{-1}hg)^{-1} = g^{-1}h^{-1}g \in K$$

since $h^{-1} \in H$.

To show now that $|K| = |H|$, let $\phi_g(h) = g^{-1}hg$ and write

$$g^{-1}h_1g = g^1h_2g \implies h_1 = h_2,$$

showing that ϕ_g is one-to-one. Then since H is finite, ϕ_g is also onto K. It follows that $|K| = |H|$.

Returning to the original problem, we see that H must be normal in G, because H is the only subgroup of G of its order. (That is, we must have $K = H$.)

Now, if two subgroups of co-normal, can we make a group out of the set of cosets shared between the two subgroups? I don't see how. There are two identity elements.

Problem 10

If H is a subgroup of G, let $N(H) = \{g \in G | gHg^{-1} = H\}$.

Part A

Prove $N(H)$ is a subgroup of G.

So $e \in N(H)$ since $eHe^{-1} = H$. Let $x, y \in N(H)$.

$$xyH(xy)^{-1} = xyHy^{-1}x^{-1} = xHx^{-1} = H,$$

so $xy \in N(H)$. Similarly,

$$xHx^{-1} = H \implies xH = Hx \implies H = x^{-1}Hx,$$

so $x^{-1} \in N(H)$. Note that H did not need to be a normal group for this. We got the desired properties by virtue of x's membership in $N(H)$.

Part B

Prove H is normal in $N(H)$.

Clearly $H \subseteq N(H)$ since for all $h \in H$, $hHh^{-1} = H$ by the cancelation property of groups. (For if $hah^{-1} = hbh^{-1}$, then $a = b$.) So H is a subgroup of $N(H)$. Now for $x \in N(H)$ and $h \in H$, there exists $h' \in H$ such that $xhx^{-1} = h'$ since $xHx^{-1} = H$.

Part C

Prove that if H is a normal subgroup of the subgroup K in G, then $K \subseteq N(H)$ (that is, $N(H)$ is the largest subgroup of G in which H is normal.)

(Recall that normality is not a transitive proprety of subgroups. That is, A normal in B, and B normal in C does not imply that A is normal in C.)

For $x \in K$ and $h \in H$, we know that $xHx^{-1} = H$ by Lemma 2.6.1, and so this qualifies x for membership in $N(H)$.

Part D

Prove that H is normal in G if and only if $N(H) = G$.

Let $g \in G$. If H is normal in G, then $gHg^{-1} = H \implies g \in N(H) \implies G \leq N(H)$. But now clearly $N(H) \leq G$, so $G = N(G)$. Now if $N(H) = G$, then for all $g \in G$, $gHg^{-1} = H$ implies that H is normal in G by Lemma 2.6.1.

Problem 11

If N and M are normal subgroups of G, prove that NM is also a normal subgroup of G.

Cleaty $e \in NM$, so it is non-empty. Now for $n, n' \in N$ and $m, m' \in M$, we have

$$nmn'm' = nn''mm' \in NM,$$

showing closure. Here we used the normal proprety of M to say that there must exist $n'' \in N$ such that $mn' = n''m$. We then have

$$(nm)^{-1} = m^{-1}n^{-1} = n'm^{-1} \in NM.$$

By the normal property of M again, there must exist $n' \in N$ such that $m^{-1}n^{-1} = n'm^{-1}$. Lastly, we have, for any $g \in G$,

$$gnmg^{-1} = gng^{-1}gmg^{-1} = n'm' \in NM,$$

by the normality of N and M.

Problem 12

Suppose that N and M are two normal subgroups of G and that $N \cap M = \{e\}$. Show that for any $n \in N$, $m \in M$, $nm = mn$.

Consider the commutator $nmn^{-1}m^{-1}$. See that $nmn^{-1} \in M$ since M is normal; and therefore, the commutator is in M. Similarly, see that

$mn^{-1}m^{-1} \in N$ since N is normal; and therefore, the commutator is in N. It follows that $nmn^{-1}m^{-1} = e$, from which the result follows.

Problem 15

If N is normal in G and $a \in G$ is of order $o(a)$, prove that the order, m, of Na in G/N is a dividsor of $o(a)$.

Clearly $(Na)^{|Na|} = N$ by definition of $|Na|$. Now observe that $(Na)^{|a|} = Na^{|a|} = Ne = N$. We then see that $|Na|$ divides $|a|$ since

$$(Na)^{|Na|} = (Na)^{|a|} = N.$$