Chapter 16 Exercises
Gallian's Book on Abstract Algebra

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Exercise 5

Prove Corollary 1 of Theorem 16.2.

Let F be a field, $a \in F$, and $f(x) = F[x]$. Then $f(a)$ is the remainder in the division of $f(x)$ by $x - a$.

Proof: The case when $\deg f = 0$ is easily verified. Now let $\deg f > 0$ and suppose the statement is true for all polynomials of degrees $\deg f - 1$. Let $f(x) = a_n x^n + \cdots + a_0$. As we begin to divide $x - a$ into f using the defintion algorithm, we get $f(x) = (x - a)a_n x^{n-1} + g(x)$, where $g(x) = f(x) - (x - a)a_n x^{n-1}$. Now since $\deg g = \deg f - 1$, we see, by our inductive hypothesis, that for a quotient $q(x)$, we have $g(x) = (x - a)q(x) + g(a)$. It now follows that

$$\begin{aligned} f(x) &= (x-a)a_n x^{n-1} + g(x) \\ &= (x-a)a_n x^{n-1} + (x-a)q(x) + g(a) \\ &= (x-a)(a_n x^{n-1} + q(x)) + f(a). \end{aligned}$$

Here, the quotient upon dividing $f(x)$ by $x - a$ is $a_n x^{n-1} + q(x)$ and the remainder is $f(a)$, as claimed in the statement of the theorem.

Exercise 7

Prove Corollary 2 of Theorem 16.2.

Let F be a field, $a \in F$, and $f(x) \in F[x]$. Then a is a zero of $f(x)$ if and only if $x - a$ is a factor of $f(x)$.

Proof: by Corollary 1 of Theorem 16.2, there exists a quotient $q(x)$ such that

$$f(x) = (x - a)q(x) + f(a).$$

Using this equation, it is clear that if a is a zero of f, then $x - a$ is a factor of f. Conversely, if $x - a$ is a factor of f, then a is a zero of f.

Exercise 10

If the rings R and S are isomorphic, show that $R[x]$ and $S[x]$ are isomorphic.

Let ϕ be an isomorphism between the rings R and S. Now define the function $\Psi : R[x] \to S[x]$ as

$$\Psi(f) = \phi(a_n)x^n + \phi(a_{n-1})x^{n-1} + \cdots + \phi(a_0),$$

where $f \in R[x]$ is the polynomial given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0.$$

Clearly Ψ is onto $S[x]$ since ϕ is onto S, and Ψ is one-to-one since ϕ is one-to-one. Lastly, Ψ preserves the addition and multiplication, because ϕ does. It can be shown, but I'm going to be lazy.

Exercise 16

Show that Corollary 3 of Theorem 16.2 is false for any commutative ring that has a zero divisor.

Let R be such a ring, and consider $R[x]$. Let $a \in R$ be a zero-divisor. Then there exists a non-zero element $b \in R$ such that $ab = 0$. Now consider the polynomial $f(x) = ax$ in $R[x]$, and realize that $\deg f = 1$, yet 0 and b are distinct zeros of f.

Exercise 18

Prove that the ideal $\langle x \rangle$ in $Q[x]$ is maximal.

Let $\phi : Q[x] \to Q$ be defined as $\phi(f) = f(0)$. This is a homomorphism from $Q[x]$ to Q and $\ker \phi = \langle x \rangle$. It then follows by Theorem 15.3, that

$$Q[x]/\langle x \rangle = Q[x]/\ker \phi \approx \phi(Q[x]) = Q.$$

Now since Q is a field, we know that $Q[x]/\langle x \rangle$ is a field. We can now claim that $\langle x \rangle$ is a maximal ideal of $Q[x]$ by Theorem 14.4.

Exercise 24

Let $f(x) \in R[x]$. Suppose that $f(a) = 0$ but $f'(a) \neq 0$, where $f'(x)$ is the derivative of $f(x)$. Show that a is a zero of $f(x)$ of multiplicity 1.

It follows immediately that

$$f(x) = (x-a)^k q(x),$$

where k is the multiplicity of a as a zero of f. (The integer k here in this equation is as large as it can be so that there exists such a quotient $q(x) \in R[x]$ with no remainder.) The derivative of f is then given by

$$f'(x) = k(x-a)^{k-1}q(x) + (x-a)^k q'(x) = (x-a)^{k-1}(kq(x) + (x-a)q'(x)).$$

But a is not a zero f', so we must have $k = 1$.

Exercise 26

Show that Corollary 3 of Theorem 16.2 is true for polynomials over integral domains.

Revisiting the proof of this theorem in the text, it only used the fact that a field is an integral domain. (No where did the proof depend upon properties of a field that set it apart from an integral domain.) The theorem therefore holds for integral domains as well.

Exercise 30

Find infinitely many polynomials $f(x)$ in $Z_3[x]$ such that $f(a) = 0$ for all a in Z_3.

Consider the set of polynomials $\{x^k(x-1)(x-2)\}_{k=1}^{\infty}$.

Exercise 36

If I is an ideal of a ring R, prove that $I[x]$ is an ideal of $R[x]$.

Let $f \in R[x]$ and $g \in I[x]$. Then $fg \in I[x]$ and $gf \in I[x]$, since all coefficients of fg and gf are in I, since I is an ideal of R.

Exercise 44

For any field F, recall that $F(x)$ denotes the field of quotients of the ring $F[x]$. Prove that there is no element in $F(x)$ whose square is x.

Elements of $F(x)$ are of the form $f(x)/g(x)$ where $f(x), g(x) \in F[x]$ with $g(x) \neq 0$. Now see that $(f(x)/g(x))^2 = x$ if and only if $f(x)^2 = xg(x)^2$. But $\deg f^2$ is even while $\deg xg^2$ is odd, so this can't happen.