The Intersection Of Rays And Algebraic Surfaces

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Abstract. Although a well known result in the traditional language of multi-variable calculus, in this paper it is shown, using the language of geometric algebra, that for any real-valued, multi-variable polynomial defined over the real numbers that the image of any line through the domain of such a function is determined entirely by all orders of the directional derivatives of this function at any one point along the line and in the direction of the line. This result has an application in the problem of casting rays through algebraic surfaces as it shows that such a problem, in all cases, reduces to the problem of finding the roots of a single-variable polynomial having an explicit formulation in terms of the multi-variable polynomial and ray in question.

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1. Introduction

This paper uses the framework laid out in [3] for the representation of algebraic surfaces, and so assumes sufficient knowledge of that work. What may be worth revisiting here at the start, however, is an additional conception of the notation used in that paper; specifically to do with the heavy use of subscripts. This paper is no exception to such use of subscripts, and so the reader must be up to speed on the notation if there is to be any hope of success in communicating the ideas to follow.

Unless otherwise specified, a subscript denotes membership in a specific sub-algebra. These sub-algebras of our "mother" algebra $\mathbb{G}$ are enumerated by the integers in $[1, m]$, and denoted by $\mathbb{G}_i$. The absence of a subscript can denote membership in an algebra or space outside of $\mathbb{G}$, usually $\mathbb{R}^n$, which is thought of as an n-dimensional euclidean vector space.

This, however, does not have to be the case, and we can work exclusively in $\mathbb{R}^n_1$, a sub-space of the vector space $\mathbb{V}_1$ generating $\mathbb{G}_1$. The omission of a subscript, (or, as usual, the presence of the subscript 1), can denote membership in $\mathbb{G}_1$, while the presence of a subscript can be thought of as the application of an outermorphism[1] that takes us from $\mathbb{G}_1$ to the corresponding element in $\mathbb{G}_i$, i being the subscript in question. This being the conception of our use of subscripts, they may begin to play a role in our performance of certain algebraic manipulations, such as

$$\begin{aligned}(a+b)_i &= a_i + b_i,\\ (a\wedge b)_i &= a_i \wedge b_i,\\ a\cdot b &= a_i\cdot b_i,\\ a_i\cdot a_j &= 0,\\ a_i\wedge a_j &\neq 0,\\ (a_i)_i &= -a,\end{aligned}$$

with $a, b \in \mathbb{V}_1$ and $i \neq j$. Except for the last, these are all important properties of the algebra that make the representation scheme work. In the remainder of this paper, you may think of $\mathbb{R}^n$ as $\mathbb{R}^n_1$ if you would like. In any case, the reader must understand the application of subscripts to otherwise unsubscripted variables and expressions. This is covered in detail in [3].

2. An Approach To The Problem

Letting $f : \mathbb{R}^n \to \mathbb{R}$ be any polynomial equation in n variables and up to degree m, it was shown in [3] that the function $\bigwedge_{i=1}^m p_i(x)$ may be factored out of this polynomial in terms of the inner product as

$$f(x) = \bigwedge_{i=1}^{m} p_i(x)\cdot B, \tag{2.1}$$

where B is an m-vector of our geometric algebra with $\infty_i \cdot B = 0$ for all integers $i \in [1, m]$, and where the function $p_i : \mathbb{R}^n \to \mathbb{V}_i$ is given by

$$p_i(x) = o_i + x_i + \frac{1}{2}x_i^2\infty_i, \tag{2.2}$$

having its origins in the paper [2]. Given a point $x \in \mathbb{R}^n$ and a direction vector $v \in \mathbb{R}^n$, we wish to find the set of all scalars $\lambda \in \mathbb{R}$ such that $f(x + \lambda v) = 0$. Utilizing equation (2.1) for this purpose, we easily find that

$$f(x+\lambda v) = \bigwedge_{i=1}^{m}(p_i(x) + \lambda v_i)\cdot B, \tag{2.3}$$

because we can ignore the $\frac{1}{2}x_i^2\infty_i$ term in equation (2.2). Looking at equation (2.3), it is immediately clear that its expansion is that of a polynomial in λ of up to degree m. What we're going to show in this paper is that an explicit formula for this polynomial can be found in terms of all orders of directional derivatives of f at x and in the direction of v.

[1] An explicit formula for this outermorphism was given in the appendix of [3].

3. The Result

We begin by rewriting equation (2.3) as

$$f(x+\lambda v) = \sum_{i=0}^{m} T_i(x), \tag{3.1}$$

where $T_i(x)$ will denote the i^{th} term collecting all multiples of λ^i in the series expansion of (2.3). Carefully formulating this term, we get

$$T_i(x) = \lambda^i \sum_{j=1}^{\binom{m}{i}} W_{j,i}(x) \cdot B,$$

where $W_{j,i}$ is the j^{th} way to write an outer product involving i vectors taken from $\{v_k\}_{k=1}^m$ and $m-i$ vectors taken from $\{p_k(x)\}_{k=1}^m$ in an order having ascending sub-scripts. The following example helps clarify this in the case $m=3$.

$$\begin{aligned}
W_{1,0} &= p_1 \wedge p_2 \wedge p_3 \\
W_{1,1} &= p_1 \wedge p_2 \wedge v_3 \\
W_{2,1} &= p_1 \wedge v_2 \wedge p_3 \\
W_{3,1} &= v_1 \wedge p_2 \wedge p_3 \\
W_{1,2} &= p_1 \wedge v_2 \wedge v_3 \\
W_{2,2} &= v_1 \wedge p_2 \wedge v_3 \\
W_{3,2} &= v_1 \wedge v_2 \wedge p_3 \\
W_{1,3} &= v_1 \wedge v_2 \wedge v_3
\end{aligned}$$

Having now come to terms, (no pun intended), with the general expansion of equation (2.3), we proceed now to fearlessly take the directional derivative of T_i at x and in the direction of v. Doing so, we get

$$\nabla_v T_i(x) = \lambda^i \sum_{j=1}^{\binom{m}{i}} \lim_{\delta\to 0} \frac{W_{j,i}(x+\delta v) - W_{j,i}(x)}{\delta} \cdot B,$$

knowing that each individual limit will exist. What we must realize now is that the term $W_{j,i}(x)$ will get canceled in the expansion of $W_{j,i}(x+\delta v)$, leaving only terms that are multiples of positive powers of δ. Furthermore, it is only those remaining terms that are multiples of δ itself that will survive the limit process. We are therefore left to deduce these terms in an evaluation of the limit. What we find is that all such terms are of the form $\delta W_{j,i+1}(x)$, but we need to determine just how many of them there are. Realizing that $\binom{m}{i}$ old terms will each contribute $m-i$ new terms of this form, of which there should be $\binom{m}{i+1}$, but that no type of term will be produced any more or less than any other, we see that

$$\frac{(m-i)\binom{m}{i}}{\binom{m}{i+1}} = i+1$$

is the number of such terms of the form $\delta W_{j,i+1}(x)$, and we may write

$$\begin{aligned}\nabla_v T_i(x) &= \lambda^i \sum_{j=1}^{\binom{m}{i+1}} \lim_{\delta\to 0} \frac{(i+1)\delta W_{j,i+1}(x)}{\delta} \cdot B \\ &= \lambda^i (i+1) \sum_{j=1}^{\binom{m}{i+1}} W_{j,i+1}(x) \cdot B \\ &= \frac{i+1}{\lambda} T_{i+1}(x). \end{aligned} \tag{3.2}$$

Returning to equation (3.1), and realizing that $T_0(x) = f(x)$, we can now finally deduce the expansion of (2.3) using the recurrence relation of equation (3.2) as

$$f(x+\lambda v) = \sum_{i=0}^{m} \frac{\lambda^i}{i!} \nabla_v^i f(x), \tag{3.3}$$

where $\nabla_v^i f(x)$ is the i^{th} order directional derivative of f at x in the direction of v with $\nabla_v^0 f(x) = f(x)$.

Clearly this is a Taylor series, and so a much simpler derivation could have probably been found, perhaps using the concept of integration along the ray. In any case, we have been able to show the promised result.

4. Making Use Of The Result

In its present form, equation (3.3) lacks ease of use, because knowledge of the vector v is needed to calculate all orders of the directional derivative. To solve this problem, we need to decouple the knowledge of this vector from the limit processes by generalizing the idea of the gradient to higher orders. This has already been done in [] with the use of k-forms. The manifestation of such things in the algebra $\mathbb{G}$ that we are using may be as follows.

We begin with the gradient of f, usually written ∇f. But this is ambiguous in our algebra, because we need to specify the sub-algebra over which ∇f will be taken. We will do this with an integral subscript as $\nabla_i f$, not to be confused with the directional derivative, which uses a vector subscript. The gradient can now be defined similarly to it's definition in [] as

$$\nabla_i = \sum_{j=1}^{n} e_{i,j} \nabla_{e_j},$$

where $\{e_{i,j}\}_{j=1}^n$ is an orthonormal basis for $\mathbb{R}_i^n$. Having done this, we have, for any integer $i \in [1, m]$,

$$\nabla_v = v_i \cdot \nabla_i,$$

which is a well-known result. Generalizing this, we find that

$$\nabla_v^i = -(-1)^i \bigwedge_{j=1}^{i} v_j \cdot \bigwedge_{j=1}^{i} \nabla_j,$$

where the function operator $\bigwedge_{j=1}^{i} \nabla_j$ is defined as

$$\bigwedge_{j=1}^{i} \nabla_j = \sum_{j_1=1}^{n} \cdots \sum_{j_i=1}^{n} \bigwedge_{k=1}^{i} e_{k,j_k} \prod_{k=0}^{i-1} \nabla_{e_{j_{i-k}}},$$

realizing that, as was assumed throughout [3], the outer and inner products bind tighter than the geometric product. Here, the function operator being applied to f is an i^{th} order partial derivative.

The decoupling of equation (3.3) can now be written as

$$f(x+\lambda v) = -\sum_{i=0}^{m} \frac{(-\lambda)^i}{i!} \bigwedge_{j=1}^{i} v_j \cdot \left(\bigwedge_{j=1}^{i} \nabla_j \right) f(x), \tag{4.1}$$

where the outer product $\bigwedge_{j=1}^{i} v_j$ is one in the case $i = 0$, and where the function operator $\bigwedge_{j=1}^{i} \nabla j$ is the identity operator in the same case.

If all orders of the gradient of f are available to us, equation (4.1) becomes a convenient way to calculate the coeficients of the polynomial whose zeros give us the parameters of the intersection points of our ray with the a given algebraic surface.

Looking back again at equation (2.3), this is certainly a usable form if a symbolic calculator is available. Use of symbolic calculation was made in [1]. If no such thing is available, then (4.1) allows us to come up with the polynomial through literal evaluation; provided, again, that we have all gradients available to us. For algebraic surfaces of degrees greater than or equal to five[2], we may lose the need to come up with the polynomial expansion of $f(x + \lambda v)$ altogether in favor of root-finding methods that need know nothing more about $f(x + \lambda v)$ other than that it is continuous.[3]

In any case, (4.1) appears to be, if nothing more, an interesting result about algebraic surfaces that may find applications in other areas.

References

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[2]There is no closed form solution to a general polynomial of degree greater than or equal to five in terms of elementary functions. See[].
[3]In such a case, the intermediate value theorem applies. Recall that the composition of two continuous functions is continuous.

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