Section 3.2 Exercises
Herstein's Topics In Algebra

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Problem 3

find the form of the binomial theorem in a general ring; in other words, find an expression for $(a+b)^n$, where n is a positive integer.

This becomes more complicated than the usual binomial theorem because we can't take commutativity for granted.

How about

$$(a+b)^n = \sum_{i=0}^{2^n-1} \prod_{j=0}^{i-1} f(\lfloor i2^{-j} \rfloor \mod 2),$$

where $x(0) = a$ and $x(1) = b$.

Problem 4

If every $x \in R$ satisfies $x^2 = x$, prove that R must be commutative.

We have

$$a + b = (a+b)^2 = a^2 + ab + ba + b^2 = a + ab + ba + b \implies 0 = ab + ba.$$

We then see that

$$ab = -ba = (-ba)^2 = (ba)^2 = ba.$$

Problem 6

If D is an integral domain and D is of finite characteristic, prove that the characteristic of D is a prime number.

Suppose the characteristic of D is composite. Then it may be written as mn where m and n are integers, each greater than one. Now since mn is the characteristic of D, there must exist at least one $a \in D$ such that $ma \neq 0$. Then for all $b \in D$, we have

$$0 = mnab = (ma)(nb) \implies nb = 0,$$

since we're workign in an integral domain. But now we've reached a contradiction since $n < mn$. It follows that the chracteristic of D is not composite, and therefore prime.

Problem 8

If D is an integral domain and if $na = 0$, for some $a \neq 0$ in D and some integer $n \neq 0$, prove that D is of finite characteristic.

Notice that for any $d \in D$, we have

$$0 = d(na) = (nd)a \implies nd = 0,$$

since $a \neq 0$. It follows that n is an upper-bound on the characteristic of D.

Problem 9

If R is a system satisfying all the conditions for a ring with unit element with the possible exception of $a + b = b + a$, prove that the axiom $a + b = b + a$ must hold in R and that R is thus a ring.

Given Herstein's hint, this problem isn't hard. We're showing that when the ring has a multiplicative identity, the additive commutativity axiom is superfluous. Indeed, we see that

$$a + a + b + b = (a + b)(1 + 1) = a + b + a + b \implies a + b = b + a.$$

Problem 10

Show that the commutative ring D is an integral domain if and only if for $a, b, c \in D$ with $a \neq 0$ the relation $ab = ac$ implies that $b = c$.

If D is an integral domain, then

$$ab = ac \implies a(b-c) = 0 \implies b - c = 0 \implies b = c,$$

since $a \neq 0$. On the other hand, let $x, y \in D$ such that $xy = 0$. If $x \neq 0$, then

$$xy = x0 \implies y = 0.$$

Similarly, we can show that if $y \neq 0$, then $x = 0$.

Problem 11

Prove that Lemma 3.2.2 is false if we drop the assumption that the integral domain is finite.

I think we can just look at the integers $\mathbb{Z}$. They're clearly an integral domain, and perhaps even the motivation behind the general idea of an integral, yet they certainly don't form a commutative division ring.

Problem 12

Prove that any field is an integral domain.

Let $a, b \in F$ such that $0 = ab$. If $a \neq 0$, then $a^{-1} \in F$ and

$$0 = a^{-1}0 = a^{-1}ab = b.$$

Similarly, if $b \neq 0$, we can show that $a = 0$.