Section 3.7 Exercises
Herstein's Topics In Algebra

Spencer T. Parkin

April 14, 2016

Thoughts

The terms *unit* and *unit element* are, in my opinion, confusing and unfortunate choices. If I understand correctly, a unit element x of a ring R is such that for all $r \in R$, $rx = xr = r$. A unit, on the other hand, is an element (but not to be confused with a *unit element*) that has a multiplicative inverse. I think that Gallian uses the term *unity* instead of *unit element*.

In an integral domain, we can show that if there is a unit element, then it is unique. Let $x_1, x_2 \in D$ be such elements. Then for any $0 \neq r \in D$, we have $rx_1 = rx_2 \implies r(x_1 - x_2) = 0 \implies x_1 - x_2 = 0 \implies x_1 = x_2$. Something similar may be said about units. If $x \in D$ is a unit, suppose $y, z \in D$ such that $xy = 1$ and $xz = 1$, yet $y \neq z$. But then $xy = xz \implies x(y - z) = 0 \implies y - z = 0 \implies y = z$.

It's now worth mentioning that if a ring is said to contain a unit, then this must necessarily imply the existence of a unit element. The converse, however, does not necessarily hold, as far as I know.

Problem 1

In a commutative ring with unit element prove that the relation a is an associate of b is an equivalence relation.

We say $a \sim b$ if and only if there exists a unit $u \in R$ such that $a = ub$. Clearly any unit element is a unit, so we can easily conclude that $a \sim a$ since $a = (1)a$. If $a \sim b$, then $a = ub \implies b = u^{-1}a \implies b \sim a$. If $a \sim b$

and $b \sim a$, then $a = ub$ and $b = vc$, so $a = uvc \implies a \sim c$, since clearly $(uv)^{-1} = v^{-1}u^{-1}$ is a unit of R.

Problem 2

In a Euclidean ring prove that any two greatest common divsors of a and b are associates.

Let A be the ideal of our ring R that is generated by a and b as $\{ra + sb | r, s \in R\}$. Then, as shown in the proof of Lemma 3.7.1, $A = \langle d \rangle$, where d is any greatest common divisor of a and b. So, letting $d_1, d_2 \in A$ be any two such elements, we must have $d_1 = xd_2$ and $d_2 = yd_1$ for some pair of elements $x, y \in R$. But then $d_1 = xyd_1 \implies 1 = xy$, since we're in an integral domain. So x is a unit and therefore, $d_1 \sim d_2$.

Problem 3

Prove that a necessary and sufficient condition that the element a in the Euclidean ring be a unit is that $d(a) = d(1)$.

Let $d(a) = d(1)$, and suppose a is not a unit. Then by Lemma 3.7.3, $d(1) < d((1)a)$, which is clearly a contradiction. So a is a unit. Now if a is a unit, then $d(a) \leq d(aa^{-1}) = d(1)$. But if $R = \langle 1 \rangle$, then there does not exist $a \in R$ such that $d(a) < d(1)$. So $d(a) = d(1)$. (See the proof of Lemma 3.7.3 to get this last part.)

Problem 5

Prove that if an ideal U of a ring R contains a unit of R, then $U = R$.

Let $u \in U$ be a unit. Now let $r \in R$ such that $r = u^{-1}$. Then $x = uu^{-1} = ur \in U$ and x is a unit element of R. But now $r = xr \in U$ for all $r \in R$. So $U = R$.