## Chapter 10 Exercises Gallian's Book on Abstract Algebra

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## Lemma 1

Let H be a proper subgroup of G. Then for all  $g \in G - H$  and all  $h \in H$ ,  $gh \in G - H$ .

Suppose  $gh = h' \in H$ . Then  $g = h'h^{-1} \in H$ , which is a contradiction. Therefore,  $gh \in G - H$ .

## Lemma 2

Let N be a normal subgroup of a group G. Then for any  $g \in G$  and any  $n \in N$ , there exists  $n' \in N$  such that gn = n'g or such that ng = gn'.

## Exercise 39

If K is a subgroup of G and N is a normal subgroup of G, prove that  $K/(K \cap N)$  is isomorphic to KN/N.

Notice that the normality of the subgroup  $K \cap N$  in K is implied in the problem statement. (I do not believe that  $(K \cap N)$  is normal in K generally.)

It can be shown, however, that KN is a group, and if a group, clearly N is normal in KN. Let  $x \in KN$ . Then x = kn for some  $k \in K$  and  $n \in N$ . But then by Lemma 2 above,  $x = n'k \in NK$  for some  $n' \in N$ . It follows that  $KN \subseteq NK$ . Similarly, we can show that  $NK \subseteq KN$ , so NK = KN. It then follows by Exercise 6 of the supplementary exercises for chapters 5 through 8 that NK is a group.

We now let  $\phi: K/(K \cap N) \to KN/N$  be a function defined as

$$\phi(k(K \cap N)) = kN,$$

and show that it is a homomorphism. Let us first verify that this is a well defined function. Let  $a, b \in K$  such that  $a(K \cap N) = b(K \cap N)$ . Then  $ab^{-1} \in K \cap N \subseteq N$ , showing that aN = bN. Then, by the normality of N and  $N \cap K$ , we see that

$$\phi(a(K \cap N)b(K \cap N))$$

$$= \phi(ab(K \cap N))$$

$$= abN = aNbN$$

$$= \phi(a(K \cap N))(\phi(b(K \cap N)),$$

showing that  $\phi$  is operation preserving.

We now consider the kernel of  $\phi$ . Notice that

$$\ker \phi = \{k(K \cap N) \in K/(K \cap N) | k \in N\},$$
  
= \{k(K \cap N) \in K/(K \cap N) | k \in K \cap N\},  
= \{K \cap N\}.

It follows that  $\phi$  is an isomorphism by Property 9 of Theorem 10.2.