Chapter 9 Exercises
Gallian's Book on Abstract Algebra

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Exercise 1

Let $H = \{(1), (12)\}$. Is H normal in S_3.

No, $(123)H \neq H(123)$, because $(123)(12) = (13) \neq (23) = (12)(123)$.

Exercise 2

Prove that A_n is normal in S_n.

Let $\alpha \in A_n$ and let $\beta \in S_n$. Now notice that $\beta\alpha\beta^{-1} \in A_n$, in the case that β is an even permutation, or an odd permuation. It then follows by Theorem 9.1 that A_n is a normal subgroup of S_n.

Exercise 3

Show that if G is the internal direct product of $H_1, H_2, \dots, H_n$ and $i \neq j$ with $1 \leq i \leq n$, $1 \leq j \leq n$, then $H_i \cap H_j = \{e\}$.

Without loss of generality, let $i < j$. Now notice that

$$H_i \subseteq H_1 H_2 \dots H_i \dots H_{j-2} H_{j-1}$$

and that $H_1 H_2 \dots H_i \dots H_{j-2} H_{j-1} \cap H_j = \{e\}$. It follows that $H_i \cap H_j = \{e\}$.

Finishing Theorem 9.6

We are given $\phi(h_1 h_2 \dots h_n) = (h_1, h_2, \dots, h_n)$. It is immediately clear that ϕ is onto $H_1 \oplus H_2 \oplus \cdots \oplus H_n$. By the uniqueness of representation of elements in $H_1 H_2 \dots H_n$ already proven, it follows that ϕ is one-to-one. That ϕ is operation preserving follows from the commutativity among disjoint subgroups. For all integers $i \in [1, n]$, for all $a_i, b_i \in H_i$, we have

$$\begin{aligned}
&\phi(a_1 a_2 \dots a_n b_1 b_2 \dots b_n)\\
=&\phi(a_1 b_1 a_2 b_2 \dots a_n b_n)\\
=&(a_1 b_1, a_2 b_2, \dots, a_n b_n)\\
=&(a_1, a_2, \dots, a_n)(b_1, b_2, \dots, b_n)\\
=&\phi(a_1 a_2 \dots a_n)\phi(b_1 b_2 \dots b_n).
\end{aligned}$$

Exercise 7

Prove that if H has index 2 in G, then H is normal in G.

The cosets of H in G are H and xH for any $x \in G - H$. Now consider Hx. This is H or xH. But it can't be H, because $x \notin H$. It must, therefore, be xH.

Exercise 10

Prove that a factor group of a cyclic group is cyclic.

Let H be a normal subgroup of a cyclic group $G = \langle g \rangle$. (All cyclic groups are Abelian and therefore, all subgroups of a cyclic group are normal.) We then see that

$$G/H = \{aH | a \in G\} = \{g^k H | k \in \mathbb{Z}\} = \langle gH \rangle.$$

Exercise 11

Let H be a normal subgroup of G. If H and G/H are Abelian, must G be Abelian?

No. Consider $G = D_4$ and H as the subgroup of rotations in D_4.

Exercise 12

Prove that a factor group of a cyclic group is cyclic.

Let $a, b \in G/H$ with $a = xH$ and $b = yH$ for $x, y \in G$. We then have

$$ab = xHyH = xyH = yxH = yHxH = ba.$$

Exercise 44

If $|G| = pq$, where p and q are primes that are not necessarily distinct, prove that $|Z(G)| = 1$ or pq.

If $Z(G) = \{e\}$, we're done. So suppose $Z(G) \neq \{e\}$. Notice that G must have an element of order q, since if $a \in G$ and $a \neq e$, we have $(a^p)^q = e \implies |a^p| = q$ by Legrange's Theorem and by Corollary 2 of Theorem 4.1. Similarly, G must have an element of order p.

Suppose $p \neq q$. Let $a, b \in G$ be elements of orders p and q, respectively. Then $\langle a \rangle$ is a normal subgroup in G, since for all $g \in G - \langle a \rangle$, and any integer k, the element ga^kg^{-1} has the same order as a^k, so we have $g\langle a \rangle g^{-1} \subseteq \langle a \rangle$. Similarly, $\langle b \rangle$ is normal in G. Furthermore, it is clear that $\langle a \rangle \cap \langle b \rangle = \{e\}$. Now if $x, x' \in \langle a \rangle$ and $y, y' \in \langle b \rangle$, and if $xy = x'y'$, then $(x')^{-1}x = y'y^{-1} \in \langle a \rangle \cap \langle b \rangle$, so $x = x'$ and $y = y'$, and we see that $\langle a \rangle \langle b \rangle = G$. We can now invoke Theorem 9.6, and say that

$$G \approx \langle a \rangle \oplus \langle b \rangle \approx Z_p \oplus Z_q,$$

and it follows that $Z(G) = G$ and therefore $|Z(G)| = pq$.

Now suppose $p = q$. If there exists an element $g \in G$ with order pq, we're done, since $G \approx Z_{p^2} \implies Z(G) = G$. So suppose that G has no such element. Go on...

Exercise 46

Let G be an Abelian group and let H be the subgroup consisting of all elements of G that have finite order. Prove that every nonidentity element in G/H has infinite order.

Let $a \in G/H$ be a non-identity element. Then there exists $g \in G$ such that $a = gH$. Clearly $g \notin H$ by Property 2 of the Lemma for Theorem 7.1. It follows that $|g| = \infty$.

Exercise 61

Suppose that H is a normal subgroup of a finite group G. If G/H has an element of order n, show that H has an element of order n. Show, by example, that the assumption that G is finite is necessary.

The case $n = 1$ is trivial, so let $n > 1$. Let $a \in G$ such that $|aH| = n$. Clearly $a \neq e$. It follows that the mapping $\phi : H \to H$, given by $\phi(h) = a^n h$ is a non-trivial permutation of the elements of H and so ϕ is a member of the group of permutations of H. We then see that $a^{|\phi|n} = e$. But it is easy to see that for all integers $i \in [1, |\phi|n - 1]$, we have $a^i \neq e$. So $|a^{|\phi|}| = n$.

Exercise 62

Do it...

Exercise 65

If $|G| = 30$ and $|Z(G)| = 5$, what is the structure of $G/Z(G)$?.

Note that $|G/Z(G)| = 30/5 = 6 = 2 \cdot 3$. It follows from Theorem 7.2 that $G/Z(G)$ is isomorphic to Z_6 or D_3. Erf...which one? Think about it.