Nailing Down The Directed Integral

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1 Motivation

2 Defining The Directed Integral

We shall let $\mathbb{R}^n$ denote n-dimensional euclidean space and let this space be represented by a vector space that is denoted by the same symbol $\mathbb{R}^n$. We will let $\mathbb{G}$ denote the geometric algebra that is generated by $\mathbb{R}^n$. The euclidean metric shall be assumed on $\mathbb{R}^n$, which, for any pair of vectors (points) $a, b \in \mathbb{R}^n$, may be taken as $|a - b|$. The whole of $\mathbb{R}^n$ then becomes a metric space under this measure of distance between points. As important as it is that $\mathbb{R}^n$ be a metric space, we assume a metric on the whole of $\mathbb{G}$ that turns it into a metric space. For any two multivectors $A, B \in \mathbb{G}$, the norm $|A - B|$ is taken as a measure of the distance between A and B.

We shall assume the usual topology on $\mathbb{R}^n$ for open sets.

Definition 2.1 (Tangent Vector). *Given any subset S of $\mathbb{R}^n$ and a point $x \in S$, we call a vector $t \in \mathbb{R}^n$ a tangent vector of S at x if there exists a sequence of points $\{x_i\}_{i=1}^{\infty} \subseteq S$ such that for any real number $\epsilon > 0$, there exists an integer $j > 0$ such that for all $i \geq j$, we have $|x_i - x| < \epsilon$ and*

$$\left| \frac{t}{|t|} - \frac{x_i - x}{|x_i - x|} \right| < \epsilon.$$

In light of Definition 2.1, we shall let $T(x)$ denote the set of all tangent vectors of S at the point x.

Definition 2.2 (Surface). *A subset S of $\mathbb{R}^n$ is a k-dimensional surface if for all points $x \in S$, the set $T(x) \cup \{0\}$ is a vector space of dimension k.*

With Definition 2.2 in place, it is easy to imagine examples of surfaces in $\mathbb{R}^n$, such as a hollow sphere or plane, although the typical surface may not really be anything like what we would or could imagine.

Given a surface $S \subseteq \mathbb{R}^n$, we will, for any point $x \in S$, let $G(x)$ denote the geometric algebra generated by the tangent space $T(x)$ at x.

If S is an orientable surface, then there exists a function $v : S \to \mathbb{G}$ giving, for each point $x \in S$, a consistent unit psuedo-scalar for the tangent algebra $G(x)$. The unit psuedo-scalar $v(x)$ is referred to as the tangent of S at x, while its principle dual, the normal of S at x.

Definition 2.3 (Surface Covering)**.** *Given a surface S, a surface covering of S of radius r is a set C of least possible cardinality of open balls centered on points of S, each of radius r, with the property that for any point $x \in S$, there exists an open ball $b \in C$ such that $x \in b$.*

Letting $\text{ball}(x, r)$ denote an open ball of radius r centered at a point x, notice that if a surface S is compact, then, by the Heine-Borel property, (see []), we can always take the covering $\{\text{ball}(x, r) | x \in S\}$ and reduce it to a finite sub-cover. That is, find a finite subset of this cover that is also a cover of S. A surface cover of S is then a cover of this form of smallest possible cardinality.

If C is a surface covering of S, then we are going to let C' denote the set of open ball centers of all open balls in C.

Definition 2.4 (Directed Integral)**.** *Let S be a compact surface upon which is defined a multivector field f. Then the directed integral of f over S, if it exists, is a multivector $L \in \mathbb{G}$, and we write*

$$L = \int_S dv f(x),$$

if for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that if C is a surface covering of S of radius $r < \delta$, then

$$\left| L - \sum_{x \in C'} r v(x) f(x) \right| < \epsilon.$$

We will characterize the set of all integrable functions on a compact surface S as those defined on such a surface, and for which the integral of Definition 2.4 exists over that surface.

Lemma 2.1. *The directed integral of Definition 2.4, as a function, is well defined.*

Proof. Letting f be an integrable function on S, we must show here that there are no two multivectors $L_0 \neq L_1$ of $\mathbb{G}$ that are both integrals of f over S. To that end, we begin by letting $D = |L_0 - L_1|$, choose $\epsilon = \frac{D}{2}$, and define the function

$$F(C) = \sum_{x \in C'} rv(x)f(x).$$

Since L_0 is an integral of f over S, there exists $\delta_0 > 0$ such that if C is a surface covering of S of radius $r < \delta_0$, we have $|L_0 - F(C)| < \epsilon$. Similarly, since L_1 is an integral of f over S, there exists $\delta_1 > 0$ such that if C is a surface covering of S of radius $r < \delta_1$, we have $|L_1 - F(C)| < \epsilon$. Now letting $\delta = \min\{\delta_0, \delta_1\}$, we see that if C is a surface covering of S of radius $r < \delta$, we have $|L_0 - F(C)| < \epsilon$ and $|L_1 - F(C)| < \epsilon$. We then see that

$$D = |L_0 - L_1| \leq |L_0 - F(C)| + |F(C) - L_1| < 2\epsilon = D,$$

which is an impossibility. Having reached this contradiction, we can conclude that there does not exist a pair of multivectors $L_0 \neq L_1$ that are both integrals of f over S. $\square$

Having defined our integral only over compact surfaces, notice that we can integrate over hollow spheres, but not planes. Also note that not all closed and bounded subsets of $\mathbb{R}^n$ are compact.

3 Using The Directed Integral

The directed integral becomes useful to us when we can find a relationship between it and an anti-derivative of the function it integrates. Without this, there is no clear way to evaluate the integral for a given integrable function. The goal of this section, therefore, is to find such a relationship.