CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 00, Pages 000–000 (Xxxx XX, XXXX)
S 1088-4173(XX)0000-0

THE MOTHER MINKOWSKI ALGEBRA OF ORDER m

SPENCER T. PARKIN

ABSTRACT. It is found that all polynomials of up to degree m have an encoding as m-vectors in a geometric algebra referred to as the mother Minkowski algebra of order m. It is then shown that all conformal transformations may be applied to these m-vectors, the results of which, when converted back into polynomial form, give us the transformed surfaces in terms of the zero sets of the original and final polynomials.

1. MOTIVATION

Before presenting the Mother Minkowski algebra of order m, we lead up to it here with some background and motivation.[1] We begin by recalling that an algebraic set is any subset of an n-dimensional euclidean space $\mathbb{R}^n$ that is also the zero set of one or more polynomials, each in n independent variables. Given a geometric algebra $\mathbb{G}$, we can represents such sets using blades $B \in \mathbb{G}$ as the set of all points $x \in \mathbb{R}^n$ such that

$$p(x) \cdot B = 0,$$

where $p : \mathbb{R}^n \to \mathbb{V}$ maps points in $\mathbb{R}^n$ to a vector space $\mathbb{V}$ generating our geometric algebra $\mathbb{G}$. Though not necessary, $\mathbb{R}^n$ is often embedded in $\mathbb{V}$; but regardless of this, the function p is necessarily defined in such a way that the expression $p(x) \cdot B$ is a polynomial in the vector components of x when $B \in \mathbb{V}$.

Letting $\mathbb{B}$ denote the set of all blades found in $\mathbb{G}$, and letting $P(\mathbb{R}^n)$ denote the power set of $\mathbb{R}^n$, we will find it useful to define the mapping $\dot{g} : \mathbb{B} \to P(\mathbb{R}^n)$ as

$$\dot{g}(B) = \{x \in \mathbb{R}^n | p(x) \cdot B = 0\}. \tag{1.1}$$

To see that $\dot{g}(B)$ is an algebraic set, we first observe that when $B \in \mathbb{V}$, $\dot{g}(B)$ is the zero set of a polynomial in the vector components of x. Secondly, we observe that if $\bigwedge_{i=1}^k b_i$ is a factorization of the k-blade B, each b_i being in $\mathbb{V}$, then

$$p(x) \cdot B = -\sum_{i=1}^{k} (-1)^i (p(x) \cdot b_i) B_i, \tag{1.2}$$

where B_i is given by

$$B_i = \bigwedge_{j=1, j \neq i}^{k} b_j,$$

2010 *Mathematics Subject Classification.* Primary 14J70; Secondary 14J29.

[1] In all equations to follow, we let the outer product take precedence over the inner product, and the geometric product take precedence over the inner and outer products. The inner product used in this paper is the Hestenes inner product.

and therefore, since $\{B_i\}_{i=1}^k$ is a linearly independent set, we have

$$\dot{g}(B) = \bigcap_{i=1}^k \dot{g}(b_i).$$

This model of representing algebraic sets using blades of a geometric algebra presents some interesting properties. To begin, if $A, B \in \mathbb{B}$ are blades with $A \wedge B \neq 0$, then

$$\dot{g}(A) \cap \dot{g}(B) = \dot{g}(A \wedge B).$$

In this way, the outer product serves to take the intersection of two surfaces. But we can also look at the outer product in a different light as an operator that takes at least the union of its two given surfaces. To see this, we must consider an alternative interpretation of blades $B \in \mathbb{B}$ as algebraic sets. Defining $\hat{g} : \mathbb{B} \to P(\mathbb{R}^n)$ as

$$\hat{g}(B) = \{x \in \mathbb{R}^n | p(x) \wedge B = 0\}, \tag{1.3}$$

we see that $\hat{g}(B) = \dot{g}(BI)$, where I is the unit psuedo-scalar of $\mathbb{G}$, showing that the image of $\hat{g}$, like $\dot{g}$, consists of algebraic sets. Under this new interpretation, we find that for blades $A, B \in \mathbb{B}$, we have

$$\hat{g}(A) \cup \hat{g}(B) \subseteq \hat{g}(A \wedge B).$$

Exactly what surface we get from $A \wedge B$ in terms of $\hat{g}$ can be deduced by considering the surface $(A \wedge B)I$ in terms of $\dot{g}$. This is because the set of all possible surfaces can be generated through the use of $\dot{g}$.

What's further a benefit of using blades to represent surfaces is that of the many transformations performable on such geometries through the use of outermorphisms; in particular, outermorphisms $f : \mathbb{B} \to \mathbb{B}$ of the form

$$f(B) = VBV^{-1},$$

where V is a versor of $\mathbb{G}$. Given such a function, we wish to compare $\dot{g}(B)$ with $\dot{g}(f(B))$. Interestingly, to understand the latter in terms of the former, we need only understand the mapping from $\mathbb{R}^n \to \mathbb{R}^n$, if any, induced by V through p as being each point $x \in \mathbb{R}^n$ mapped to a point $y \in \mathbb{R}^n$ satisfying the condition

$$V^{-1}p(x)V = \lambda p(y), \tag{1.4}$$

λ being some non-zero scalar in $\mathbb{R}$. This is, of course, only a well defined mapping, provided that for every point $x \in \mathbb{R}^n$, there exists such a point $y \in \mathbb{R}^n$, and that it is unique. Assuming that V and p collectively meet these requirements, and so do indeed induce such a mapping $h : \mathbb{R}^n \to \mathbb{R}^n$, we can show that

$$\dot{g}(f(B)) = h^{-1}(\dot{g}(B)).$$

Notice that by the symmetry of equation (1.4), any argument that can be used to show that h is a well defined mapping can also be used to show that h^{-1} exists. We now need only show that

$$\dot{g}(VBV^{-1}) = \{x \in \mathbb{R}^n | V^{-1}p(x)V \cdot B = 0\}.$$

To this end, we begin by factoring the k-blade B as

$$B = \bigwedge_{i=1}^k b_i.$$

Then, by substituting $V^{-1}p(x)V$ for $p(x)$ in equation (1.2), we see that

$$V^{-1}p(x)V \cdot B = 0$$

if and only if for all integers $i \in [1, k]$, we have

$$0 = V^{-1}p(x)V \cdot b_i = p(x) \cdot Vb_iV^{-1},$$

since the set of $(k-1)$-blades $\{B_i\}_{i=1}^k$ is a linearly independent set. Then, by applying equation (1.2) again to obtain

$$p(x) \cdot VBV^{-1} = -\sum_{i=1}^{k}(-1)^i(p(x) \cdot Vb_iV^{-1})VB_iV^{-1},$$

we see that for all integers $i \in [1, k]$, we have $p(x) \cdot Vb_iV^{-1} = 0$ if and only if $p(x) \cdot VBV^{-1} = 0$, because the set $\{VB_iV^{-1}\}_{i=1}^k$ is also linearly independent, which linear independence follows from that of the set $\{B_i\}_{i=1}^k$. It follows that $V^{-1}p(x)V \cdot B = 0$ if and only if $p(x) \cdot VBV^{-1} = 0$, which is what we wanted to show.

2. The Mother Minkowski Algebra of Order m

Up to this point, we have kept the definition of the function p open to interpretation, because the set of all possibilities for p, in terms of the types of geometry we can consequently do, remains an open question. What might be the most interesting and significant definition of p thus far proposed is found in [3] and given by

$$p(x) = o + x + \frac{1}{2}x^2\infty.$$

Here, the vector space $\mathbb{V}$ is generated by the set of basis vectors $\{o, \infty\} \cup \{e_i\}_{i=1}^n$, where the set of n euclidean vectors $\{e_i\}_{i=1}^n$ span $\mathbb{R}^n$ as an orthonormal basis for that space, and the vectors o and ∞ are the null vectors representing the points at origin and infinity, respectively. The geometric algebra generated by $\mathbb{V}$ is called a Minkowski algebra, and the resulting model of geometry imposed upon this algebra by p using functions (1.1) and (1.3) is known as the conformal model of geometric algebra. It has been shown in [3, 4, 2] that the versors of $\mathbb{G}$ generated by $\mathbb{V}$ induce the set of all conformal transformations through p. The induced mappings are well defined and invertible.

Building upon the ideas presented in [1], we will now consider a new model of geometry based upon a geometric algebra $\mathbb{G}$ generated by a vector space $\mathbb{V}$ described in set builder notation as

$$\mathbb{V} = \left\{ \sum_{i=1}^{m} v_i \,\middle|\, v_i \in \mathbb{V}_i \right\},$$

where for each $\mathbb{V}_i$, the geometric algebra generated by $\mathbb{V}_i$ is a Minkowski algebra. For all $i \neq j$, we have $\mathbb{V}_i \cap \mathbb{V}_j = \{\vec{0}\}$, the singleton set containing the zero vector. Moreover, for all $i \neq j$, if $a \in \mathbb{V}_i$ and $b \in \mathbb{V}_j$, we have $a \cdot b = 0$. We will refer to $\mathbb{G}$ as the mother Minkowski algebra of order m.

Letting $\mathbb{B}$ denote the set of all blades taken from $\mathbb{G}$, we now define the function $\dot{G} : \mathbb{B} \to P(\mathbb{R}^n)$ as

$$\dot{G}(B) = \left\{ x \in \mathbb{R}^n \,\middle|\, \bigwedge_{i=1}^{m} p_i(x) \cdot B = 0 \right\}, \tag{2.1}$$

where we define $p_i : \mathbb{R}^n \to \mathbb{V}_i$ as

$$p_i(x) = o_i + x_i + \frac{1}{2}x_i^2\infty_i, \tag{2.2}$$

where $o_i, \infty_i \in \mathbb{V}_i$ are null vectors, and x_i denotes the embedding of x in the n-dimensional euclidean sub-space of $\mathbb{V}_i$. If more precision is needed here, we can let $\mathbb{B}_i$ denote the set of all blades generated by $\mathbb{V}_i$, let $\mathbb{R}^n_i$ denote the n-dimensional euclidean sub-space of $\mathbb{V}_i$, and then work exclusively in $\mathbb{R}^n_1$ by defining an outermorphism that takes any blade in $\mathbb{B}_1$ to its corresponding blade in $\mathbb{B}_i$. The function p_i can then be defined in terms of this outermorphism. Interestingly, an explicit formula for this outermorphism can be found and carried through all of the equations we'll present in the remainder of this paper, but there is no need to formally introduce it, because the equations still go through in its absense.

What is immediately clear from the definition of $\dot{G}$ in equation (2.1) is that unless for all integers $i \in [1, m]$, a vector $v \in \mathbb{V}_i$ exists such that $v \wedge B = 0$, we must have $\dot{G}(B) = \mathbb{R}^n$. We will therefore limit our attemtion to those blades $B \in \mathbb{B}$ having factorizations involving a representative from each $\mathbb{B}_i$. Doing so, we write B as

$$B = \bigwedge_{i=1}^m B_i, \tag{2.3}$$

where each B_i is in $\mathbb{B}_i$, and then see that

$$\bigwedge_{i=1}^m p_i(x) \cdot B = (-1)^k \bigwedge_{i=1}^m p_i(x) \cdot B_i, \tag{2.4}$$

where the integer k is given by

$$k = \sum_{i=1}^m \text{grade}(B_i) \left(\sum_{j=1, j\neq i}^m \text{grade}(B_j) - m + i \right). \tag{2.5}$$

Subscripting equation (1.1) as

$$\dot{g}_i(B_i) = \{x \in \mathbb{R}^n | p_i(x) \cdot B_i = 0\},$$

what we now find is that

$$\dot{G}(B) = \bigcup_{i=1}^m \dot{g}_i(B_i).$$

This shows that we can represent any union of up to m surfaces taken from the conformal model, (let $B_i = \infty_i$ to fill any remaining and unused blade factors), but if we extend our function $\dot{G}$ to the set of all m-vectors, we can do even better. To see why, we need only show that any monomial in up to n variables and at most degree m can be represented by the expression on the right-hand side of equation (2.4). The n variables are taken from the components of the point $x \in \mathbb{R}^n$. Let each B_i be a vector in $\mathbb{V}_i$ with $B_i \cdot \infty_i = 0$. The expression then becomes

$$(-1)^k \prod_{i=1}^m p_i(x) \cdot B_i.$$

For an appropriate choice of each vector B_i, we can formulate any monomial in the components of x. Letting B be a general m-vector, (which is not necessarily an m-blade), we see now that the expression that is the left-hand side of equation

(2.4) represents any polynomial of at most degree m in the vector components of x. Of course, if $B_i \cdot \infty_i = 0$ for not all vector factors of the blades in B, what we get is a polynomial of at most degree $2m$ by the squaring that occures in equation (2.2), but we cannot represent all polynomials of up to this degree. If polynomials of a higher degree are needed, simply go to a mother Minkowski algebra of higher order.

3. Conformal Transformations

While this new model certainly expands upon the set of all possible surfaces that may be represented by the conformal model, not all of the nice properties discussed in the motivating section carry over very easily, if at all. What we will show in this paper, however, is that all of the conformal transformations are available in the new model. It may be worth comparing this method of applying such transformations to surfaces not native to the confromal model to those found in [6, 5].

Letting $\{V_i\}_{i=1}^m$ be a set of m versors, each V_i taken from the geometric algebra generated by $\mathbb{V}_i$, and each representing the same conformal transformation, what we simply need to show is that

$$\dot{G}(VBV^{-1}) = \left\{ x \in \mathbb{R}^n \,\middle|\, \bigwedge_{i=1}^m V_i^{-1} p_i(x) V_i \cdot B = 0 \right\}, \tag{3.1}$$

where the versor V is given by

$$V = \prod_{i=1}^m V_i.$$

It is clear that the right-hand side of equation (3.1) is the surface $\dot{G}(B)$ having undergone the transformation represented by each V_i. Our result will show that this is also the surface $\dot{G}(VBV^{-1})$.

To begin, we notice that by the linearity of the outer product, there is no loss in generality in letting B be an m-blade instead of a general m-vector. We will therefore precede by factoring B as we have in equation (2.3) with each $B_i \in \mathbb{V}_i$. We then have

$$\begin{aligned}
\bigwedge_{i=1}^m V_i^{-1} p_i(x) V_i \cdot B &= (-1)^k \prod_{i=1}^m V_i^{-1} p_i(x) V_i \cdot B_i \\
&= (-1)^k \prod_{i=1}^m p_i(x) \cdot V_i B_i V_i^{-1} \\
&= \bigwedge_{i=1}^m p_i(x) \cdot \bigwedge_{i=1}^m V_i B_i V_i^{-1} && (3.2) \\
&= \bigwedge_{i=1}^m p_i(x) \cdot \bigwedge_{i=1}^m V B_i V^{-1} && (3.3) \\
&= \bigwedge_{i=1}^m p_i(x) \cdot VBV^{-1},
\end{aligned}$$

where here, the integer k is given by equation (2.5). The step taking us from (3.2) to (3.3) deserves some explanation. Removing the subscript i from V_i is done by inserting each of the factors $V_j V_j^{-1} = 1$ with $j \neq i$ into the appropriate position, and then commuting the V_j^{-1} to the other side of B_i into its appropriate position. Finding the net change in sign is an exercise in combinatorics, and ends up being

$$(-1)^{m(m-1)} = 1.$$

This result suggests the following process of transforming a polynomial f into another polynomial f' by a conformal transformation represented by a versor V.

$$\begin{tikzcd}
B \arrow[r] & VBV^{-1} \arrow[d] \\
f \arrow[u] \arrow[r, dashed] & f'
\end{tikzcd}$$

Here, the polynomial f is converted into an m-vector B, the versor V is applied to this m-vector as the m-vector VBV^{-1}, which in turn is converted back into the polynomial f'. The conversion process is straightforward and can be easily handled by a computer algebra system.

References

1. C. Doran, D. Hestenes, F. Sommen, and N. Van Acker, *Lie groups as spin groups*, J. Math. Phys. **34** (1993), 8.
2. L. Dorst, D. Fontijne, and S. Mann, *Geometric algebra for computer science*, Morgan Kaufmann, 2007.
3. D. Hestenes, *Old wine in new bottles: A new algebraic framework for computational geometry*, Advances in Geometric Algebra with Applications in Science and Engineering (2001), 1–14.
4. L. Hongbo, D. Hestenes, and A. Rockwood, *Generalized homogeneous coordinates for computational geometry*, Geometric Computing with Clifford Algebra **Volume 24., Berlin Heidelberg, Springer-Verlag** (2001), 27–60.
5. A. Lasenby, *Recent applications of conformal geometric algebra*, IWMM 2004, LNCS 3519 Springer-Verlag (2005).
6. G. Sobczyk, *Conformal mappings in geometric algebra*, AMS Notices **Volume 59** (2012), 264–5.

102 W. 500 S., Salt Lake City, UT 84101
E-mail address: `spencerparkin@outlook.com`