Chapter 10 Exercises
Gallian's Book on Abstract Algebra

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Lemma 1

Let H be a proper subgroup of G. Then for all $g \in G - H$ and all $h \in H$, $gh \in G - H$.

Suppose $gh = h' \in H$. Then $g = h'h^{-1} \in H$, which is a contradiction. Therefore, $gh \in G - H$.

Lemma 2

Let N be a normal subgroup of a group G. Then for any $g \in G$ and any $n \in N$, there exists $n' \in N$ such that $gn = n'g$ or such that $ng = gn'$.

Lemma 3

Let G be a group and let n be a positive integer. Then the number of elements in G of order n, if any, is divisible by $\phi(n)$, the totient of n.

Suppose G has one or more elements of order n. Let N be the set $\{x \in G||x| = n\}$. Then, for any pair of elements $a, b \in N$, let $a \sim b$ if and only if $a \in \langle b \rangle$. This defines an equivilance relation on N, since $a \in \langle a \rangle$ gives us the reflexive property, since $a \in \langle b \rangle \implies b \in \langle a \rangle$ gives us the symmetric property, and since $a \in \langle b \rangle$ and, for $c \in N$, $b \in \langle c \rangle$ implies that $a \in \langle c \rangle$, giving us the transitive property. We now note that by Theorem 4.4, the size of each equivilance class is $\phi(n)$. It follows that the number of elements of order n is G is $s\phi(n)$, where s is the number of equivilance classes.

Exercise 39

If K is a subgroup of G and N is a normal subgroup of G, prove that $K/(K \cap N)$ is isomorphic to KN/N.

Notice that the normality of the subgroup $K \cap N$ in K is proven by the problem similar to Exercise 50 in Chapter 9.

We now show that KN is a group. Let $x \in KN$. Then $x = kn$ for some $k \in K$ and $n \in N$. But then by Lemma 2 above, $x = n'k \in NK$ for some $n' \in N$. It follows that $KN \subseteq NK$. Similarly, we can show that $NK \subseteq KN$, so $NK = KN$. It then follows by Exercise 6 of the supplementary exercises for chapters 5 through 8 that NK is a group.

Is N normal in KN?

We now let $\phi : K/(K \cap N) \to KN/N$ be a function defined as

$$\phi(k(K \cap N)) = kN,$$

and show that it is a homomorphism. Let us first verify that this is a well defined function. Let $a, b \in K$ such that $a(K \cap N) = b(K \cap N)$. Then $ab^{-1} \in K \cap N \subseteq N$, showing that $aN = bN$.

We now show that ϕ is operation preserving. By the normality of N and $N \cap K$, we see that

$$\begin{aligned}
&\phi(a(K \cap N)b(K \cap N)) \\
=&\phi(ab(K \cap N)) \\
=&abN = aNbN \\
=&\phi(a(K \cap N))(\phi(b(K \cap N)),
\end{aligned}$$

showing that ϕ is operation preserving.

We now consider the kernel of ϕ. Notice that

$$\begin{aligned}
\ker \phi &= \{k(K \cap N) \in K/(K \cap N) | k \in N\}, \\
&= \{k(K \cap N) \in K/(K \cap N) | k \in K \cap N\}, \\
&= \{K \cap N\}.
\end{aligned}$$

It follows that ϕ is an isomorphism by Property 9 of Theorem 10.2.

Exercise 40

If M and N are normal subgroups of G and $N \leq M$, prove that $(G/N)/(M/N) \approx G/M$.