

# Chapter 10 Exercises

## Gallian's Book on Abstract Algebra

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### Lemma 1

Let  $H$  be a proper subgroup of  $G$ . Then for all  $g \in G - H$  and all  $h \in H$ ,  $gh \in G - H$ .

Suppose  $gh = h' \in H$ . Then  $g = h'h^{-1} \in H$ , which is a contradiction. Therefore,  $gh \in G - H$ .

### Lemma 2

Let  $N$  be a normal subgroup of a group  $G$ . Then for any  $g \in G$  and any  $n \in N$ , there exists  $n' \in N$  such that  $gn = n'g$  or such that  $ng = gn'$ .

### Lemma 3

Let  $G$  be a group and let  $n$  be a positive integer. Then the number of elements in  $G$  of order  $n$ , if any, is divisible by  $\phi(n)$ , the totient of  $n$ .

Suppose  $G$  has one or more elements of order  $n$ . Let  $N$  be the set  $\{x \in G \mid |x| = n\}$ . Then, for any pair of elements  $a, b \in N$ , let  $a \sim b$  if and only if  $a \in \langle b \rangle$ . This defines an equivalence relation on  $N$ , since  $a \in \langle a \rangle$  gives us the reflexive property, since  $a \in \langle b \rangle \implies b \in \langle a \rangle$  gives us the symmetric property, and since  $a \in \langle b \rangle$  and, for  $c \in N$ ,  $b \in \langle c \rangle$  implies that  $a \in \langle c \rangle$ , giving us the transitive property. We now note that by Theorem 4.4, the size of each equivalence class is  $\phi(n)$ . It follows that the number of elements of order  $n$  in  $G$  is  $s\phi(n)$ , where  $s$  is the number of equivalence classes.

Oh, I had already read this in the book as the Corollary to Theorem 4.4.

## Lemma 4

If  $N$  is a normal subgroup of a group  $G$  and  $gN$  for some  $g \in G$  is a coset in  $G/N$ , then for any  $g' \in gN$ , we have  $gN = g'N$ .

If  $g' \in gN$ , then there exists  $n \in N$  such that  $g' = gn$ . Then  $g'g^{-1} = gng^{-1} \in N$  by the normality of  $N$  in  $G$ , and it follows that  $gN = g'N$  by Property 4 of the Lemma on cosets in Chapter 7. Thus any member of a coset can act as a representative of the coset.

## Exercise 38

For each pair of positive integers  $m$  and  $n$ , we can define a homomorphism from  $Z$  to  $Z_m \oplus Z_n$  by  $x \rightarrow (x \bmod m, x \bmod n)$ . What is the kernel when  $(m, n) = (3, 4)$ ? What is the kernel when  $(m, n) = (6, 4)$ ? Generalize.

Let  $\phi : Z \rightarrow Z_m \oplus Z_n$  be the homomorphism. Seeing that

$$\begin{aligned}\ker \phi &= \{x \in Z \mid x \equiv 0 \pmod{m} \text{ and } x \equiv 0 \pmod{n}\} \\ &= \{zm \mid z \in Z\} \cap \{zn \mid z \in Z\},\end{aligned}$$

it follows that

$$\ker \phi = \{z \operatorname{lcm}(m, n) \mid z \in Z\}.$$

## Exercise 39

If  $K$  is a subgroup of  $G$  and  $N$  is a normal subgroup of  $G$ , prove that  $K/(K \cap N)$  is isomorphic to  $KN/N$ .

Notice that the normality of the subgroup  $K \cap N$  in  $K$  is proven by the problem similar to Exercise 50 in Chapter 9.

We now show that  $KN$  is a group. Let  $x \in KN$ . Then  $x = kn$  for some  $k \in K$  and  $n \in N$ . But then by Lemma 2 above,  $x = n'k \in NK$  for some  $n' \in N$ . It follows that  $KN \subseteq NK$ . Similarly, we can show that  $NK \subseteq KN$ , so  $NK = KN$ . It then follows by Exercise 6 of the supplementary exercises for chapters 5 through 8 that  $NK$  is a group.

Is  $N$  normal in  $KN$ ?

We now let  $\phi : K/(K \cap N) \rightarrow KN/N$  be a function defined as

$$\phi(k(K \cap N)) = kN,$$

and show that it is a homomorphism. Let us first verify that this is a well defined function. Let  $a, b \in K$  such that  $a(K \cap N) = b(K \cap N)$ . Then  $ab^{-1} \in K \cap N \subseteq N$ , showing that  $aN = bN$ .

We now show that  $\phi$  is operation preserving. By the normality of  $N$  and  $N \cap K$ , we see that

$$\begin{aligned} & \phi(a(K \cap N)b(K \cap N)) \\ &= \phi(ab(K \cap N)) \\ &= abN = aNbN \\ &= \phi(a(K \cap N))(\phi(b(K \cap N))), \end{aligned}$$

showing that  $\phi$  is operation preserving.

We now consider the kernel of  $\phi$ . Notice that

$$\begin{aligned} \ker \phi &= \{k(K \cap N) \in K/(K \cap N) \mid k \in N\}, \\ &= \{k(K \cap N) \in K/(K \cap N) \mid k \in K \cap N\}, \\ &= \{K \cap N\}. \end{aligned}$$

It follows that  $\phi$  is an isomorphism by Property 9 of Theorem 10.2.

## Exercise 40

If  $M$  and  $N$  are normal subgroups of  $G$  and  $N \leq M$ , prove that  $(G/N)/(M/N) \approx G/M$ .

Notice that  $M/N$  is a subgroup of  $G/N$ . To see that  $M/N$  is normal in  $G/N$ , let  $g \in G$  and let  $m \in M$ , and see that

$$gNmN(gN)^{-1} = gmNg^{-1}N = gmg^{-1}N \in M/N,$$

since  $gmg^{-1} \in M$  by the normality of  $M$  in  $G$ .

Now consider the mapping  $\phi : (G/N)/(M/N) \rightarrow G/M$ , defined as

$$\phi(xN(M/N)) = yM,$$

where  $y$  is any element in the coset  $xN$ . Let us now show that this is a well defined mapping. Let  $a, b \in G$  such that  $aN(M/N) = bN(M/N)$ . It follows that  $aN(bN)^{-1} = ab^{-1}N \in M/N \implies ab^{-1} \in M$ . Now let  $aN(M/N)$  map to  $a'M$  and  $bN(M/N)$  map to  $b'M$ . Now if  $a' \in aN \subseteq aM$ , then  $a'M = aM$ . Similarly, if  $b' \in bN \subseteq bM$ , then  $b'M = bM$ . But now since  $ab^{-1} \in M$ , we see that  $aM = bM$ , so  $a'M = b'M$ .

Notice that the proof that  $\phi$  is well defined also lets us simplify its usage. That is, for any  $x \in G$ , we can let  $xN(M/N)$  map to  $xM$ . This will greatly ease the remainder of our proof.

We now show that  $\phi$  is operation preserving. Letting  $a, b \in G$ , we have

$$\begin{aligned} & \phi(aN(M/N)bN(M/N)) \\ &= \phi(aNbN(M/N)) \\ &= \phi(abN(M/N)) \\ &= abM = aMbM \\ &= \phi(aN(M/N))\phi(bN(M/N)). \end{aligned}$$

We now consider the kernel of  $\phi$ . We have

$$\begin{aligned} \ker \phi &= \{gN(M/N) | g \in G \text{ and } \phi(gN) = M\} \\ &= \{gN(M/N) | g \in M\}. \end{aligned}$$

Now let  $a, b \in M$  and consider  $aN(M/N)$  and  $bN(M/N)$ . Since  $a, b \in M$ , we have  $ab^{-1}N \in M/N$ , which, in turn, implies that  $aN(bN)^{-1} \in M/N \implies aN(M/N) = bN(M/N)$ . It follows that  $|\ker \phi| = 1$ , and therefore,  $\phi$  is an isomorphism.

## Exercise 47

Suppose that for each prime  $p$ ,  $Z_p$  is the homomorphic image of a group  $G$ . What can we say about  $|G|$ ? Give an example of such a group.

By Property 6 of Theorem 10.2, we see that  $|\phi(G)|$  divides the order of  $|G|$ . So, since  $\phi(G) = Z_p$ , we see that  $p$  divides  $|G|$ .

An automorphism of  $Z_p$  may be a trivial example.

After reading the answer in the back of the book, I'm wrong, because I did not understand the problem statement. For *every* prime  $p$ ,  $Z_p$  is a homomorphic image of *the* group  $G$ . So by Property 6 of Theorem 10.2, every prime  $p$  divides  $|G|$ ; and since there are infinitely many primes,  $|G| = \infty$ .

## Exercise 49

Let  $N$  be a normal subgroup of a group  $G$ . Use property 7 of Theorem 10.2 to prove that every subgroup of  $G/N$  has the form  $H/N$ , where  $H$  is a subgroup of  $G$ .

For every subgroup  $H$  of  $G$  with  $N \leq H$ , it is clear that  $N$  is normal in  $H$  and that  $H/N \leq G/N$ . Now let's consider what is somewhat the converse of this. For every subgroup  $K$  of  $G/N$ , does there exist a subgroup  $H$  of  $G$  such that  $K = H/N$ ?

Let  $\phi : G \rightarrow G/N$  be defined as  $\phi(g) = gN$ . This is well defined and operation preserving, so it is a homomorphism from  $G$  to  $G/N$ . Then, by property 7 of Theorem 10.2, we see that  $\phi^{-1}(K)$  is a subgroup of  $G$ . Now notice that if  $n \in N$ , then  $\phi(n) = nN = N \in K$ , showing that  $N \leq \phi^{-1}(K)$ . It follows that  $N$  is normal in  $\phi^{-1}(K)$ . Letting  $H = \phi^{-1}(K)$ , what remains to be shown now is that  $H/N = K$ . Letting  $g \in G$ , we have

$$gN \in \phi^{-1}(K)/N \iff g \in \phi^{-1}(K) \iff \phi(g) \in K \iff gN \in K.$$

It follows that  $H/N = K$ .

## Exercise 50

Show that a homomorphism defined on a cyclic group is completely determined by its action on a generator of the group.

Let  $\phi : G \rightarrow \overline{G}$  be a homomorphism of the cyclic group  $G = \langle g \rangle$ . Then  $\phi(g^k) = \phi(g)^k$ .

## Exercise 51

Use the First Isomorphism Theorem to prove Theorem 9.4.

We want to show that for any group  $G$ ,  $G/Z(G) \approx \text{Inn}(G)$ . To that end, let  $\phi_g(x) = gxg^{-1} \in \text{Inn}(G)$ , and define  $\psi(g) = \phi_g$ . It is clear that  $\psi$  is a homomorphism from  $G$  onto  $\text{Inn}(G)$ . Now realize that  $\psi(g) = \phi_e$  if and only if  $gxg^{-1} = e$ , showing that  $\ker \psi = Z(G)$ . It now follows from Theorem 9.4 that  $G/G(Z) = G/\ker \psi \approx \psi(G) = \text{Inn}(G)$ .

## Exercise 52

Let  $\alpha$  and  $\beta$  be group homomorphisms from  $G$  to  $\overline{G}$  and let  $H = \{g \in G \mid \alpha(g) = \beta(g)\}$ . Prove or disprove that  $H$  is a subgroup of  $G$ .

Clearly  $e \in H$  by Property 1 of Theorem 10.1. Now let  $a, b \in H$ . We then have

$$\alpha(ab^{-1}) = \alpha(a)\alpha(b)^{-1} = \beta(a)\beta(b)^{-1} = \beta(ab^{-1}),$$

showing that  $ab^{-1} \in H$ . So I think it's a subgroup of  $G$ .

## Exercise 54

If  $H$  and  $K$  are normal subgroups of  $G$  and  $H \cap K = \{e\}$ , prove that  $G$  is isomorphic to a subgroup of  $G/H \oplus G/K$ .

Let  $\phi(g) = (gH, gK)$ . It is clear that  $\phi$  is a homomorphism from  $G$  to  $G/H \oplus G/K$ . Then, by property 1 of Theorem 10.2, we see that  $\phi(G)$  is a subgroup of  $G/H \oplus G/K$ . We will now show that  $G \approx \phi(G)$ , and do so by showing that  $\phi$  is an isomorphism between  $G$  and  $\phi(G)$ . Notice that  $\phi$  is clearly onto  $\phi(G)$ , of course. We already know it is operation preserving. All that remains to be shown then is that  $\phi$  is one-to-one. So, let  $a, b \in G$  such that  $(aH, aK) = (bH, bK)$ . Then  $aH = bH$  and  $aK = bK$ , so  $ab^{-1} \in H$  and  $ab^{-1} \in K$ . It follows that  $ab^{-1} \in H \cap K \implies ab^{-1} = e \implies a = b$ , showing that  $\phi$  is one-to-one.

## Exercise 55

Suppose that  $H$  and  $K$  are distinct subgroups of  $G$  of index 2. Prove that  $H \cap K$  is a normal subgroup of  $G$  of index 4 and that  $G/(H \cap K)$  is not cyclic.

It is easy to show that both  $H$  and  $K$  are normal in  $G$ . By symmetry of the problem, we need only show that  $H$  is normal in  $G$ . (The proof for  $K$  is similar.) Let  $g \in G - H$ . Then  $gH \neq H$  and  $Hg \neq H$ . But  $gH$  is the only remaining of the 2 cosets of  $H$  in  $G$ , so  $gH = Hg$ , and therefore  $H$  is normal in  $G$ .

We then see that  $H \cap K$  is normal in  $G$  by Exercise 50 of chapter 9.

At this point I had to peak at the back of the book to get some hints. By Exercise 39 in this chapter, we see that  $K/(K \cap H) \approx KH/H$ , so  $|K|/|K \cap$

$|H| = |KH|/|H|$ . But  $|H| = |K| = |G|/2$ , so we see that  $|G|^2/4 = |KH||K \cap H|$ . Now realize that  $|KH| = |G|$ , because  $K \neq H \implies |HK| > |G|/2$ , and  $KH = HK$  implies that  $HK$  is a subgroup of  $G$ , and so  $|HK|$  divides  $|G|$ .

Lastly, the back of the book says that  $G/(H \cap K)$  has two subgroups of order 2, which clearly means that it can't be cyclic by the Fundamental Theorem of Cyclic Groups. To which two subgroups of  $G/(H \cap K)$  is Gallian referring? I think it's  $\{(H \cap K), h(H \cap K)\}$  where  $h \in H$ , but  $h \notin K$ , and then something similar for the other group. I need to think about it more.