Section 2.7 Exercises
Herstein's Topics In Algebra

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Problem 3

Let G be a finite abelian group of order $|G|$ and suppose the integer n is relatively prime to $|G|$. Prove that every $g \in G$ can be written as $g = x^n$ with $x \in G$.

I'm doubtful I could have done this problem without the hint Herstein provides.

Let $\phi : G \to G$ be defined as $\phi(y) = y^n$. We then easily see that

$$\phi(xy) = (xy)^n = x^n y^n = \phi(x)\phi(y),$$

since G is abelian, and so ϕ is a homomorphism. Consider the kernel of ϕ. Note that

$$y^n = e \implies y = e,$$

since the order of $y \neq e$ cannot divide a number n that is coprime with $|G|$, by Lagrange's theorem. It follows that ϕ is an isomorphism. Now since G is finite, we can easily claim that every element has the form x^n for some $x \in G$. (To find such an x for a given g, just let $x = \phi^{-1}(g)$.)

Problem 4

Part A

Given any group G and a subset U, let $\hat{U}$ be the smallest subgroup of G which contains U. Prove there is such a subgroup $\hat{U}$ in G.

I would write

$$\hat{U} = \{g \in G | g \in \prod_{w \in W} w, W \subseteq V\},$$

where the set V is given by

$$V = \{u^z | u \in U, z \in \mathbb{Z}\}.$$

I believe this is the smallest subgroup of G containing U, because no element is added unnecessarily.

One drawback of this formulation, however, is that it makes it difficult to write the form of a general element of the group. If we restrict ourselves to finite or even countably infinite subset U of G, then we can write a general element $u \in \hat{U}$ as

$$u = \prod_i u_i^{z_i},$$

where $\{u_i\} \subseteq U$ and $\{z_i\} \subseteq \mathbb{Z}$ are each finite or countably infinite sequences.

Part B

If $gug^{-1} \in U$ for all $g \in G$, $u \in U$, prove that $\hat{U}$ is a normal subgroup of G.

Let $u \in \hat{U}$. We then see that

$$gug^{-1} = \prod_i gu_i^{z_i}g^{-1} = \prod_i (gu_ig^{-1})^{z_i} \in \hat{U}.$$

1 Problem 5

Let $U = \{xyx^{-1}y^{-1} | x, y \in G\}$. In this case $\hat{U}$ is usually written G' and is called the *commutator subroup of* G.

Part A

Prove that G' is normal in G.

By part B of problem 4, we need only show that for any commuator $u \in U$, and any $g \in G$, we have $gug^{-1} \in U$. Let $u = xyx^{-1}y^{-1}$, and see that

$$gug^{-1} = gxyx^{-1}y^{-1}g^{-1} = (gxg^{-1})(gyg^{-1})(gxg^{-1})^{-1}(gyg^{-1})^{-1} \in U.$$

Part B

Prove that G/G' is abelian.

For $a, b \in G$, we have

$$G'aG'b(G'a)^{-1}(G'b)^{-1} = Gaba^{-1}b^{-1} = G \implies G'aG'b = G'bG'a.$$

Part C

If G/N is abelian, prove that $N \supseteq G'$.

For all $a, b \in G$,

$$Naba^{-1}b^{-1} = N \implies aba^{-1}b^{-1} \in N \implies G' \subseteq N.$$

I have to try to say something here concerning the significance of commutator groups. For a non-abelian group, this shows that the largest abelian factor group we can find is found by mod-ing out by G'. And I remember reading somewhere that the order of this factor group in comparison to the order of G is somehow a measure of "how abelian" the group G is.

Part D

Prove that if H is a subgroup of G and $H \supseteq G'$, then H is normal in G.

If $H = G'$, we're done. So let $H \supset G'$. Now if $h \in G'$ and $g \in G$, clearly $ghg^{-1} \in G' \subset H$ by the normality of G', so let $h \in H - G'$. Now since $c = ghg^{-1}h^{-1} \in G'$, we have $ghg^{-1} = ch \in H$ by closure in H.

Problem 17

Let G be the group of real numbers under addition and let N be the subgroup of G consisting of all the integers. Prove that G/N is isomorphic to the group of all complex numbers of absolute vlaue 1 under multiplication.

Let $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be defined by

$$\phi(\mathbb{Z} + r) = \exp(2\pi ri).$$

Now for any $a, b \in \mathbb{R}$, we have

$$\mathbb{Z} + a = \mathbb{Z} + b \iff a = b + z,$$

for some integer $z \in \mathbb{Z}$. We then have

$$\exp(2\pi ai) = \exp(2\pi(b+z)i) = \exp(2\pi bi)\exp(2\pi zi) = \exp(2\pi bi).$$

Thus far we have shown that ϕ is well-defined and onto-to-one. Clearly ϕ is onto $\mathbb{C}$. Is ϕ operation preserving?

$$\begin{aligned}\phi(\mathbb{Z}+a+\mathbb{Z}+b) &= \phi(\mathbb{Z}+a+b) = \exp(2\pi(a+b)i)\\ &= \exp(2\pi ai)\exp(2\pi bi) = \phi(\mathbb{Z}+a)\phi(\mathbb{Z}+b)\end{aligned}$$