# Chapters 9-11 Supplementary Exercises
# Gallian's Book on Abstract Algebra

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## Exercise 8

Let $k$ be a divisor of $n$. The factor gruop $(Z/\langle n\rangle)/(\langle k\rangle/\langle n\rangle)$ is isomorphic to some very familiar group. What is the group?

By Exercise 40 of Chapter 10 (The Third Isomorphism Theorem), we see that $(Z/\langle n\rangle)/(\langle k\rangle/\langle n\rangle) \approx Z/\langle k\rangle$. What more is there to say?

## Exercise 30

Let $G$ be a group and let $\phi : G \to G$ be a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1g_2g_3 = e = h_1h_2h_3$. Prove that there exists an element of $a$ in $G$ such that $\Psi(x) = a\phi(x)$ is a homomorphism.

If $a = e \in G$ and $u, v \in G$, then

$$\Psi((uv)^{-1})\Psi(u)\Psi(v) = e \implies \Psi(v^{-1}u^{-1}) = \Psi(v)^{-1}\Psi(v)^{-1}.$$

Hmmm... Can we somehow show that $\Psi$ is a homomorphism here? We cannot use homomorphic properties of $\Psi$ before we know that it's a homomorphism.

# Exercise 36

A proper subgroup $H$ of a group $G$ is called *maximal* if there is no subgroup $K$ such that $H \subset K \subset G$. Prove that $Q$ under addition has no maximal subgroups.

This is a very difficult problem, and after giving it a great deal of thought, the best I can come up with so far is the following hypothesis.

Let $H$ be any subgroup of $Q$ and let $B$ be a subset of $H$ of smallest possible cardinality such that

$$H = \{z_1 b_1 + \cdots + z_k b_k | z_i \in Z, b_i \in B, k \in Z^+\}.$$

Call such a subset $B$ of $H$ a basis for $H$. If $B$ is of finite cardinality $k$, then we may write

$$H = \langle b_1, \ldots, b_k \rangle.$$

It is easy to show that if $B$ is of finite cardinality, then $H$ is a proper subgroup of $Q$. Also, if $B$ is a basis for $Q$, then $B$ is infinite. The converse of either of these statements, however, is not obvious. Let's suppose for the moment, however, that $H$ is a proper subgroup of $Q$ if and only if $B$ is of finite cardinality. If then $H$ is a proper subgroup of $Q$ and we let $q \in Q - H$, it is clear that

$$\langle b_1, \ldots, b_k \rangle + \langle q \rangle = \langle b_1, \ldots, b_k, q \rangle$$

is a subgroup of $Q$ that properly contains $H$. That it is a proper subgroup of $Q$ follows from our assumption above and a realization that a basis for $H + \langle q \rangle$ is $B \cup \{q\}$, which is clearly finite.

Notice that all subgroups of $Q$ of the form $\langle q \rangle$ for some non-zero $q \in Q$ are minimal subgroups of $Q$ isomoprhic to $Z$.

Can an example be found that disproves the assumption? Can the assumption be proved? One approach to proving it is to take a subgroup $H$ of $Q$ having an infinite basis $B$ and showing that for any $q \in Q$, we have $q \in H$. This would show that $H = Q$. It may be easy to show that $q$ is always a limit point of $H$, but this does not imply membership in $H$.