A Variation Of The Quadric Model Of Geometric Algebra

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Abstract. A variation of the quadric model set forth in [6] is found in which the rigid body motions are represented by versors applicable to any quadric surface. Extending this variation of the original model to include a specific form of quartic surface, we find that such surfaces are closed under the application of all conformal transformations. Results of a computer program implementing this new model are presented.

1. Introduction

In the original paper [6], a model for quadric surfaces was presented based upon the ideas of projective geometry. What was unfortunate about this model, however, was its lack of support for the rigid body transformations. It was predicted in the conclusion of that paper that a better model for quadric surfaces may exist that is more like the conformal model of geometric algebra, here-after abbreviated as CGA. The present paper details what may be such a model. We'll find that the rigid body transformations can be incorporated into the model by using an alternative method of encoding the quadric form. An extension to this quadric form will then allow us to support the conformal transformations at the expense of expanding our model to necessarily include a specific form of quartic surfaces. The new model and its extension will both use the same geometric algebra to be given as follows.

2. The Geometric Algebra

We begin here with a description of the structure of the geometric algebra upon which our model will be imposed. This geometric algebra will contain

the following vector spaces.

Notation	Basis	
$\mathbb{V}^e$	$\{e_i\}_{i=1}^n$	
$\mathbb{V}^o$	$\{o\} \cup \{e_i\}_{i=1}^n$	(2.1)
$\mathbb{V}^\infty$	$\{e_i\}_{i=1}^n \cup \{\infty\}$	
$\mathbb{V}$	$\{o\} \cup \{e_i\}_{i=1}^n \cup \{\infty\}$	

The set of vectors $\{e_i\}_{i=1}^n$ forms an orthonormal set of basis vectors for the n-dimensional Euclidean vector space $\mathbb{V}^e$, which we'll use to represent n-dimensional Euclidean space. The vectors o and ∞ are the familiar null-vectors representing the points at origin and infinity, respectively, taken from CGA. An inner-product table for these basis vectors is given as follows, where $1 \le i < j \le n$.

$\cdot$	o	e_i	e_j	∞	
o	0	0	0	-1	
e_i	0	1	0	0	(2.2)
e_j	0	0	1	0	
∞	-1	0	0	0	

We will now let $\mathbb{G}(\mathbb{V})$ denote the Minkowski geometric algebra generated by $\mathbb{V}$. For each vector space in table (2.1), we will let an over-bar above this vector space denote an identical copy of that vector space. The vector space $\mathbb{W}$ will denote the smallest vector space containing each of $\mathbb{V}$ and $\overline{\mathbb{V}}$ as vector subspaces. In symbols, one may write

$$\mathbb{G}(\mathbb{W}) = \mathbb{G}(\mathbb{V} \oplus \overline{\mathbb{V}}). \tag{2.3}$$

We will use over-bar notation to distinguish between vectors taken from $\mathbb{V}$ with vectors taken from $\overline{\mathbb{V}}$. Then, so that we may use the over-bar notation in a well defined manner to distinguish between elements of $\mathbb{G}(\mathbb{V})$ and $\mathbb{G}(\overline{\mathbb{V}})$, we will define it as an outermorphic function that is also an isomorphism between $\mathbb{G}(\mathbb{V})$ and $\mathbb{G}(\overline{\mathbb{V}})$. Doing so, we see that for any element $E \in \mathbb{G}(\mathbb{V})$, we may define $\overline{E} \in \mathbb{G}(\overline{\mathbb{V}})$ as

$$\overline{E} = SES^{-1}, \tag{2.4}$$

where S is the versor given by

$$S = (1 + e_-\overline{e}_-)(1 - e_+\overline{e}_+) \prod_{i=1}^n (1 - e_i\overline{e}_i). \tag{2.5}$$

This definition is non-circular if we let the over-bars in equation (2.5) be purely notation. The vectors e_- and e_+, taken from [3], are defined as

$$e_- = \frac{1}{2}\infty + o, \tag{2.6}$$

$$e_+ = \frac{1}{2}\infty - o. \tag{2.7}$$

The vectors $\overline{e}_-$ and $\overline{e}_+$ are defined similarly in terms of $\overline{o}$ and $\overline{\infty}$. Defined this way, realize that, like the over-bar function defined in [6], here we have the property that for any vector $v \in \mathbb{V}$, we have $\overline{\overline{v}} = -v$.

3. The Form Of Quadric Surfaces In $\mathbb{G}(\mathbb{W})$

We now give a formal definition under which elements $E \in \mathbb{G}(\mathbb{W})$ are representative of n-dimensional quadric surfaces in our present variation of the original model.

Definition 3.1. Referring to an element $E \in \mathbb{G}(\mathbb{W})$ as a quadric surface, it is representative of such an n-dimensional surface as the set of all points $p \in \mathbb{V}^o$ such that

$$0 = p \wedge \bar{p} \cdot E. \tag{3.1}$$

From this definition it can be seen that the general form of a quadric $E \in \mathbb{G}(\mathbb{W})$ is given by

$$E = \sum_{i=1}^{k} a_i \bar{b}_i, \tag{3.2}$$

where each of $\{a_i\}_{i=1}^{k}$ and $\{b_i\}_{i=1}^{k}$ is a sequence of k vectors taken from $\mathbb{V}^{\infty}$. To see why, realize that the form (3.2) can always be reduced to the form

$$E \equiv \sum_{i=1}^{n} \sum_{j=i}^{n} \lambda_{ij} e_i \bar{e}_j + \sum_{i=1}^{n} \lambda_i e_i \overline{\infty} + \lambda \infty \overline{\infty}, \tag{3.3}$$

where each of λ_{ij}, λ_i, and λ are scalars, in the sense that this reduced form represents the same surface as that in equation (3.2) under Definition 3.1. We then see that this form (3.3), when it is substituted into equation (3.1), produces to a polynomial equation of degree 2 in the vector components of $p + (p \cdot \infty)o$. Doing so with $p = o + x$, where $x \in \mathbb{V}^e$, we get the equation

$$0 = -\sum_{i=1}^{n} \sum_{j=i}^{n} \lambda_{ij} (x \cdot e_i)(x \cdot e_j) + \sum_{i=1}^{n} \lambda_i (x \cdot e_i) - \lambda, \tag{3.4}$$

which we may recognize as the general equation of an n-dimensional quadric surface.

In practice, a computer program might take such a bivector of the form (3.2) and extract from it the coefficients of the quadric polynomial (3.4) it represents. It could then render the surface using traditional methods, such as those used to render the traced surfaces in Figure 1 far below, or the meshed surfaces in Figure 2 yet further below.

Of course, using geometric algebra on paper, it might be undesirable and unnecessary to think of quadrics in terms of polynomial equations. A, perhaps, better way to think of quadrics is in terms of an element of a geometric algebra whose decomposition produces the parameters characterizing the quadric surface. For example, many common quadrics are the solution set in $\mathbb{V}^e$ of the equation

$$0 = -r^2 + (x - c)^2 + \lambda((x - c) \cdot v)^2, \tag{3.5}$$

in the variable x. (An explanation of the parameters r, c, v and λ was given in [6].) Then, factoring out $-p\bar{p}$, we see that the element $E \in \mathbb{G}(\mathbb{W})$, given

by

$$\Omega + \lambda v \overline{v} + 2(c + \lambda(c \cdot v)v)\overline{\infty} + (c^2 + \lambda(c \cdot v)^2 - r^2)\infty\overline{\infty} \qquad (3.6)$$

is representative of this very same quadric by Definition 3.1, where Ω is defined as

$$\Omega = \sum_{i=1}^{n} e_i \overline{e}_i. \qquad (3.7)$$

Canonical forms similar to (3.6) can be found for specific geometries, such as planes, spheres, plane-pairs, circular cylinders, circular conical surfaces, and so on.

4. Transformations Supported By The Model

The main result of this section will depend upon the following lemma.

Lemma 4.1. *For any versor $V \in \mathbb{G}(\mathbb{W})$, and any four vectors $a, b, c, d \in \mathbb{V}$, we have*

$$V^{-1}aV \wedge \overline{V^{-1}bV} \cdot c \wedge \overline{d} = a \wedge \overline{b} \cdot V\overline{V}(c \wedge \overline{d})(V\overline{V})^{-1}. \qquad (4.1)$$

Proof. We begin by first establishing that

$$V^{-1}aV \wedge \overline{V^{-1}bV} \cdot c \wedge \overline{d} \qquad (4.2)$$

$$= -(V^{-1}aV \cdot c)(V^{-1}bV \cdot d) \qquad (4.3)$$

$$= -(a \cdot VcV^{-1})(b \cdot VdV^{-1}) \qquad (4.4)$$

$$= a \wedge \overline{b} \cdot VcV^{-1} \wedge \overline{VdV^{-1}}. \qquad (4.5)$$

We now notice that

$$VcV^{-1} \qquad (4.6)$$

$$= V\overline{VV^{-1}}cV^{-1} \qquad (4.7)$$

$$= (-1)^m V\overline{V}c\overline{V^{-1}}V^{-1} \qquad (4.8)$$

$$= (-1)^m V\overline{V}c(V\overline{V})^{-1}, \qquad (4.9)$$

where m is the number of vectors taken together in a geometric product to form V. We then notice that

$$\overline{VdV^{-1}} \qquad (4.10)$$

$$= VV^{-1}\overline{VdV^{-1}} \qquad (4.11)$$

$$= (-1)^{m^2} V\overline{V}V^{-1}\overline{dV^{-1}} \qquad (4.12)$$

$$= (-1)^{m^2+m} V\overline{V}d\overline{V^{-1}}\,\overline{V^{-1}} \qquad (4.13)$$

$$= (-1)^{2m^2+m} V\overline{V}d\overline{V^{-1}}V^{-1} \qquad (4.14)$$

$$= (-1)^m V\overline{V}d(V\overline{V})^{-1}. \qquad (4.15)$$

It now follows that

$$a \wedge \overline{b} \cdot VcV^{-1} \wedge \overline{VdV^{-1}} = a \wedge \overline{b} \cdot V\overline{V}(c \wedge \overline{d})(V\overline{V})^{-1}. \qquad (4.16)$$

$\square$

We're now ready to prove the main result as follows.

Theorem 4.2. *Letting $E \in \mathbb{G}(\mathbb{W})$ be a bivector of the form* (3.2), $p, p' \in \mathbb{V}^o$ *be a pair of points related by a versor $V \in \mathbb{G}(\mathbb{V})$ by the equation*

$$p' = o \cdot V^{-1} p V \wedge \infty, \tag{4.17}$$

and $E' \in \mathbb{G}(\mathbb{W})$ a bivector given by

$$E' = V\overline{V}E(V\overline{V})^{-1}, \tag{4.18}$$

the set of all points $p \in \mathbb{V}^o$ such that

$$0 = p' \wedge \overline{p}' \cdot E \tag{4.19}$$

is exactly the set of all points $p \in \mathbb{V}^o$ such that

$$0 = p \wedge \overline{p} \cdot E'. \tag{4.20}$$

Proof. The theorem goes through by the following chain of equalities.

$$(o \cdot V^{-1} p V \wedge \infty) \wedge \overline{(o \cdot V^{-1} p V \wedge \infty)} \cdot E \tag{4.21}$$

$$= V^{-1} p V \wedge \overline{V^{-1} p V} \cdot E \tag{4.22}$$

$$= p \wedge \overline{p} \cdot (V\overline{V})E(V\overline{V})^{-1}. \tag{4.23}$$

The first equality holds by the fact that E is of the form (3.2), while the second equality holds by Lemma 4.1. $\square$

A corrollary to Theorem 4.2 immediately follows.

Corollary 4.3. *If $V \in \mathbb{G}(\mathbb{V})$ is a versor such that E' in equation* (4.18) *is of the form* (3.2), *then the versor $V\overline{V}$ represents a transformation closed in the set of all quadric surfaces.*

The key motivation behind Theorem 4.2 is the observation that the desired transformation of E by V is given by the algebraic set of equation (4.19), because an understanding of how V^{-1} transforms p gives us an understanding of what type of geometry we get from equation (4.19) in terms of E and V. The theorem then shows that this is also the algebraic set of equation (4.20), thereby giving us a means of performing desired transformations on elements representative of quadric surfaces in $\mathbb{G}(\mathbb{W})$. By Corrollary 4.3, what we get from such a transformation is also a quadric surface, provided that V is a versor such that E' in (4.18), like E, is also a bivector of the form (3.2).

We can now apply Theorem 4.2 to show that the rigid body transformations are supported in our new variation of the original model. Letting $\pi \in \mathbb{V}$ be a dual plane of CGA, given by

$$\pi = v + (c \cdot v)\infty, \tag{4.24}$$

where $v \in \mathbb{V}^e$ is a unit-length vector indicating the norm of the plane, and where $c \in \mathbb{V}^e$ is a vector representing a point on the plane, we see that for any homogenized point $p \in \mathbb{V}^o$, we have

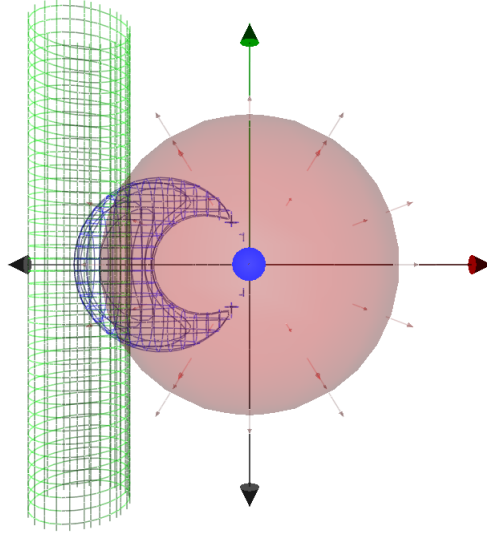$$-\pi p \pi^{-1} = o + x - 2((x - c) \cdot v)v + \lambda\infty, \tag{4.25}$$

FIGURE 1. The inversion of a cylinder in a sphere. Traces in various planes were used to render the cylinder and its inversion.

where $p = o + x$ with $x \in \mathbb{V}^e$, and where the scalar $\lambda \in \mathbb{R}$ is of no consequence. Letting $V = \pi$, the point $p' \in \mathbb{V}^o$ of consequence here is given by equation (4.17), from which we can recognize an orthogonal reflection about the plane π. It now follows by Theorem 4.2 that $\pi\overline{\pi}$ is a versor capable of reflecting any quadric surface about the plane π. Being able to perform planar reflections of any quadric in any plane, it now follows that we can always find a versor $V \in \mathbb{G}(\mathbb{W})$ capable of performing any rigid body motion applicable to any quadric surface. The development of the rigid body motions, (combinations of translations and rotations), by planar reflections, is well known, and can be found in section 2.7 of [3].

In retrospect, what we have done to find the rigid body motions of quadric surfaces is similar to what was done in [4]; and according to [7], we can state more generally that what we have done is at least similar to finding an isomorphism between quadratic spaces.

5. Extending The New Model

Interestingly, if we were not content with the rigid body motions of quadrics, then we really could find what is, for example, the spherical inversion of, say, an infinitely long cylinder in a sphere. To do this, we change Definition 3.1 into the following definition.
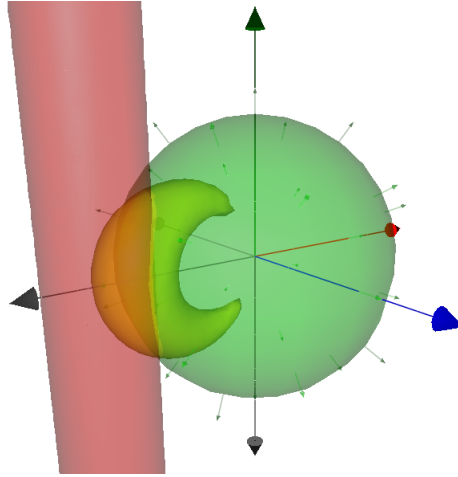
FIGURE 2. The inversion of a cylinder in a sphere. A surface mesh generation algorithm was used to skin the cylindrical and inverted surfaces. The inverted mesh suffers where the curvature becomes extreme.

Definition 5.1. For any element $E \in \mathbb{G}(\mathbb{W})$, we may refer to it as an n-dimensional quartic surface as the set of all points $p \in \mathbb{V}^e$ such that

$$0 = P(p) \wedge \overline{P}(p) \cdot E, \tag{5.1}$$

where $P : \mathbb{V}^e \to \mathbb{V}$ is the conformal mapping of CGA, defined in [] as

$$P(p) = o + p + \frac{1}{2}p^2 \infty. \tag{5.2}$$

A version of Theorem 4.2 is then easily found such that if $V \in \mathbb{G}(\mathbb{W})$ is any versor of CGA, and if E is a surface under Definition 5.1, then the element $E' \in \mathbb{G}(\mathbb{W})$, given by equation (4.18), must, by Definition 5.1, be representative of the desired transformation of E by the versor $V\overline{V}$. The general polynomial equation arising from the form of such elements E in Defintion 5.1 is much more involved than what we have in equation (3.4). Nevertheless, it is possible to extract a specific form of a quartic polynomial equation in the vector components of p from equation (5.1). The result being unsightly, it will not be presented here. Suffice it to say, a computerized algebra system was used to find the polynomial form. In any case, it is easy to see from equation (5.1) that the degree of the resulting polynomial will be four.

Now notice that under Definition 5.1, canonical forms such as (3.6) are still valid. This is because

$$P(p) \wedge \overline{P}(p) \cdot E = (o + p) \wedge \overline{(o + p)} \cdot E \tag{5.3}$$

in the case that E is of the form (3.2). This allows us to use what we already know about quadrics in the new model with its extension.

Putting theory into practice, the author wrote a piece of computer software that implements this CGA-like model for the special class of quartic surfaces of equation (5.1). Giving the program the following script as input, the output of the program is given in Figure 1 and rendered another way in Figure 2. The script is easy for anyone to read, even if they are not familiar with its language. It is given here to illustrate how one might use the model with the aide a computer system.

```
/*
 * Calculate the surface that is the
 * inversion of a cylinder in a sphere.
 */
do
(
    /* Make the cylinder. */
    v = e2, c = -7*e1, r = 2,
    cylinder = Omega - v^bar(v) + 2*c*nib + (c.c - r*r)*ni^nib,
    bind_quadric(cylinder),
    geo_color(cylinder,0,1,0),

    /* Make the sphere. */
    c = 0, r = 6,
    sphere = no + c + 0.5*(c.c - r*r)*ni,
    bind_dual_sphere(sphere),
    geo_color(sphere,1,0,0,0.2),

    /* Make the inversion of the cylinder in the sphere. */
    V = sphere*bar(sphere),
    inversion = V*cylinder*V~,
    bind_conformal_quartic(inversion),
    geo_color(inversion,0,0,1),
)
```

The functions beginning with the word "bind" create and bind an entity to the given element of the geometric algebra that is responsible for interpreting that element as a surface under Definition 5.1 or as a surface under the definition given by CGA. The computer program can then use traditional methods to render the surface from the extracted polynomial equation. For example, the polynomial equation in x, y and z for the inverted surface presented in Figure 1 is given by

$$
\begin{aligned}
0 = {} & 28.8x^2 + 11.2x^3 + x^4 + 11.2xy^2 + 2x^2y^2 + \\
& 11.2xz^2 + 2x^2z^2 + y^4 + 2y^2z^2 + 28.8z^2 + z^4.
\end{aligned}
\tag{5.4}
$$

It is interesting how a bit of reasoning in geometric algebra has given us such a simple means to obtaining this polynomial equation. Of course, while such equations lend themselves to computer algorithms, they are not practical on paper. This is where the canonical forms of elements might become

useful; although, admittedly, even these forms have proven to be unwieldy and impractical for the author, unlike their CGA counterparts.

6. Dual And Direct Surfaces

The goal from the beginning has been to find a model, similar to CGA, for the general set of surfaces up to degree 2, not just the specific class of surfaces, up to degree 2, that are just the spheres and planes of CGA. While this has been accomplished to some extent, one of the greatest deficiencies remaining appears to be the inability for the model to represent surfaces of up to the desired degree for all dimensions from zero to n in the same manner that this is done in CGA.[1] One possible solution to this is that of utilizing the geometric algebra that is generated by the linear space of bivectors in $\mathbb{G}(\mathbb{W})$. We could define a linear function on this space that maps it to a vector space $\mathbb{B}$. If $[\,]$ was such a function, then for any pair of 2-blades $A, B \in \mathbb{G}(\mathbb{W})$, we could define

$$[A] \cdot [B] = A \cdot B. \tag{6.1}$$

It would then follow that a vector $v \in \mathbb{G}(\mathbb{B})$ would be representative of a surface as the set of all points $p \in \mathbb{V}^e$ such that

$$0 = \rho(p) \cdot v, \tag{6.2}$$

where we define the function ρ as

$$\rho(p) = [P(p)\overline{P(p)}]. \tag{6.3}$$

The notions of dual and direct surfaces would then emerge as they do in CGA. We could then, using the outer product, intersect dual surfaces and combine direct surfaces.[2] These features come of representing geometries as blades in a geometric algebra.

For a given versor $V \in \mathbb{G}(\mathbb{V})$, and a k-blade $B \in \mathbb{G}(\mathbb{B})$, if we could find a vector factorization $v_1 \wedge \cdots \wedge v_k$ of B, then the transformation B' of B by V would be given by

$$B' = \bigwedge_{i=1}^{k} [V\overline{V}[v_i]^{-1}(V\overline{V})^{-1}]. \tag{6.4}$$

See [2] on the problem of blade factorization. It is unfortunate that we would have to bother finding such a factorization to transform a piece of geometry.

To illustrate the use of $\mathbb{G}(\mathbb{B})$, let $s, c \in \mathbb{G}(\mathbb{B})$ be vectors dually representative of a sphere and cylinder, respectively. Then, for any point $p \in \mathbb{V}^e$, we can find the dual surface containing p and the intersection of s and c as

$$\pm(\rho(p) \wedge (s \wedge c)I)I = \rho(p) \cdot s \wedge c = (\rho(p) \cdot s)c - (\rho(p) \cdot c)s, \tag{6.5}$$

---

[1]We can, of course, easily formulate the extruded conics. That is, the surfaces of dimension $n-1$ extruded through a given dimension. To do so, we simply apply a rigid-body motion versor to any conic section that is easily formulated as being origin-centered and axis-aligned.

[2]The outer product of two direct surfaces is the direct surface containing at least the union of the surfaces taken in the product.

where I, in practice, might be the unit pseudo-scalar of the geometric algebra generated by the vector sub-space of $\mathbb{B}$ given by the set

$$\{[x\bar{y}]|x, y \in \mathbb{V}\}. \tag{6.6}$$

Even this vector space, which is of dimension $(n+2)^2$, is larger than it needs to be. We could suffice with a vector space of dimension $(n+2)(n+3)/2$. In any case, it is clear from equation (6.5) that the algebra is simply giving us the desired surface in the pencil of s and c. If all we cared about was the dual intersection $s \wedge c$, we may still need to resort to [9] to do any meaningful analysis. Contrasting this with an absense of any need to do such a thing in CGA, we see further deficiencies in our more generalized model for surfaces up to degree 2.

7. The Algebraic Set

The term used in the title of this section has already been employed in sections prior to refer to the sets of points generated by Definition 3.1, Definition 5.1, equation (6.2) and equation (4.19), to name a few. Algebraic sets come from the subject of algebraic geometry, (see [5]), and one would think that geometric algebra might play an integral role in such a subject. Though the objects of study in algebraic geometry have grown far beyond the simple idea of algebraic sets, here we try to relate the subjects of geometric algebra and algebraic geometry by showing how blades can be used to represent algebraic sets, thereby generalizing the ideas of CGA, [6] and this paper thus far, to geometries of any dimension generated by polynomials of any form.

It is really quite trivial. To begin, remove the requirement that $\mathbb{V}^e$ be embedded in $\mathbb{V}$ as we have in Table (2.1), and let $\mathbb{V}$ be an m-dimensional vector space of any geometric algebra with a Euclidean signature. The embedding of the n-dimensional Euclidean vector space $\mathbb{V}^e$ in $\mathbb{V}$ is optional, while alternative signatures of $\mathbb{V}$ are worth exploring. In any case, letting $\{b_i\}_{i=1}^m$ be an orthonormal basis for $\mathbb{V}$, we simply define a mapping $\rho : \mathbb{V}^e \to \mathbb{V}$ as

$$\rho(p) = \sum_{i=1}^{m} f_i(p)b_i, \tag{7.1}$$

where each $f_i : \mathbb{V}^e \to \mathbb{R}$ is a polynomial in the vector components of its argument that simply maps a point in our Euclidean space $\mathbb{V}^e$ to a scalar in $\mathbb{R}$. Then, for any vector $v \in \mathbb{V}$, we see that the set of all points $p \in \mathbb{V}^e$ such that

$$0 = \rho(p) \cdot v \tag{7.2}$$

is an algebraic set generated by one polynomial. The polynomial form is determined by our definition of the function ρ, while an instance of the polynomial form is determined by v. The general algebraic set, which is the zero set of one or more polynomials, is then simply given by

$$\{p \in \mathbb{V}^e | 0 = \rho(p) \cdot B\}, \tag{7.3}$$

where B is any blade taken from $\mathbb{G}(\mathbb{V})$. A factorization of the k-blade B gives us vectors containing the coeficients of k linearly independent polynomials of the form ρ.

CGA is simply an instance of this more generalized model, as is the model given in section 6 which, when used in conjunction with a bivector representation scheme, allows us to utilize the transformations of CGA. How any of this may be useful, if at all, in the study of algebraic geometry, remains to be seen.

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