A Model Of Algebraic Sets Using Geometric Algebra

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Abstract. Blah.

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1. Introduction And Motivation

Letting $\mathbb{R}^n$ denote an n-dimensional Euclidean space, we are going to let $\mathbb{P}^n$ denote the set of all polynomials $f : \mathbb{R}^n \to \mathbb{R}$ of any degree. Then, letting $P \subseteq \mathbb{P}^n$ be any set of such polynomials, we define the zero set Z of P, denoted $Z(P)$, as the set given by

$$Z(P) = \{x \in \mathbb{R}^n | f(x) = 0 \text{ for all } f \in P\}. \tag{1.1}$$

Every subset S of $\mathbb{R}^n$ for which there exists a set $P \subseteq \mathbb{P}^n$ such that $S = Z(P)$ is what we refer to as an algebraic set. It is well known that for any algebraic set $S \subseteq \mathbb{R}^n$, there is always such a subset P of $\mathbb{P}^n$ of finite cardinality.

Given the definition in equation (1.1), it is easy to show that the subsets S of $\mathbb{R}^n$ that are the geometries of CGA, and other similar models of geometry based upon geometric algebra, are simply algebraic sets. The goal of this paper is to show that there exists a model of geometry, based upon geometric algebra, where every possible algebraic set has a representative in the form of an element of that geometric algebra. A desire to come up with such a model of geometric algebra is motivated by the admittedly fanciful dream of the German mathematician Leibniz, referred to in [] and claimed to have already been realized in []. In any case, it would seem that a generalization of CGA or any CGA-like model to one that hosts the set of all algebraic sets in $\mathbb{R}^n$ would bring us closer to such a goal. Work to this end has already been done in [1, 2] which has brought us, the reader and writer, to the present paper.

2. Blades As Algebraic Sets

We begin with an examination of $\mathbb{P}^n$ as a linear space, observing that it is of countably infinite dimension. A set of basis vectors for this space may be taken as the set of all unit-monomials in anywhere from 1 to n of the n variable components of an arbitrary point in $\mathbb{R}^n$.[1] Considering now any mapping from the set $\mathbb{Z}^+$ of positive integers to this said set $\{g_i\}_{i=1}^\infty$ of unit-monomials,[2] and letting $\mathbb{V}^\infty$ denote a simple Euclidean vector space of countably infinite dimension, if we define the function $p : \mathbb{R}^n \to \mathbb{V}^\infty$ as

$$p(x) = \sum_{i=1}^\infty g_i(x)e_i, \tag{2.1}$$

where $\{e_i\}_{i=1}^\infty$ is any orthonormal basis for $\mathbb{V}^\infty$, then for any polynomial $f \in \mathbb{P}^n$, there must exist a unique[3] vector $v \in \mathbb{V}^\infty$ such that

$$f(x) = p(x) \cdot v. \tag{2.2}$$

It now follows that for any subset P of $\mathbb{P}^n$, there exists a blade $B \in \mathbb{G}(\mathbb{V}^\infty)$, such that

$$Z(P) = \{x \in \mathbb{R}^n | p(x) \cdot B = 0\}. \tag{2.3}$$

Clearly, we need only consider such finite sets $P = \{f_i\}_{i=1}^k$ that are linearly independent sets of k polynomials. Then, for each polynomial f_i, there is an associated vector v_i by equation (2.2), and the set $\{v_i\}_{i=1}^k$ must clearly be a linearly independent set of k vectors. We may then take B to be the k-blade

$$B = \bigwedge_{i=1}^k v_i, \tag{2.4}$$

seeing that the equation $p(x) \cdot B = 0$ becomes

$$0 = -\sum_{i=1}^k (-1)^i (p(x) \cdot v_i) B_i, \tag{2.5}$$

where B_i denotes the product B with v_i removed. Realize that $\{B_i\}_{i=1}^k$ is a linearly independent set of $(k-1)$-blades, and therefore $p(x) \cdot B = 0$ if and only if $p(x) \cdot v_i = 0$ for all $i \in \mathbb{Z}_k$. For convenience, we will let $\mathbb{Z}_k$ denote the set of k integers in $[1, k] \cap \mathbb{Z}^+$.

Already we have fulfilled the promise of the introductory section of this paper, but we will continue now with one further development that is

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[1] The number c_k of such monomials homogeneous of a degree k is given by

$$c_k = \sum_{i=1}^n \binom{n}{i} p(k, i),$$

where $p(k, i)$ is the number of partitions of size i of the integer k. The combinatorics of the matter are not important, but they are pointed out here as an added measure of the reality of the set of monomials we are talking about.

[2] For example, if $n = 2$, then we might take $g_1(x, y) = 1$, $g_2(x, y) = x$, $g_3(x, y) = y$, $g_3(x, y) = x^2$, $g_4(x, y) = xy$, $g_5(x, y) = y^2$, and so on.

[3] Let $v_1, v_2 \in \mathbb{V}^\infty$ be two such vectors. Then $0 = f(x) - f(x) = p(x) \cdot (v_1 - v_2)$. Now notice that there does not exist $x \in \mathbb{R}^n$ such that $p(x) = 0$. It follows that $v_1 = v_2$.

motivated by a desire to pair our current ability to intersect geometries with an ability to take their union.

We start by letting $\{\mathbb{V}_i^\infty\}_{i=1}^\infty$ be a countably infinite set of vector spaces isomorphic to $\mathbb{V}^\infty$. With the exception of the zero vector, we consider these vector spaces as pair-wise disjoint, so that for any pair of non-zero vectors $v_i \in \mathbb{V}_i^\infty$ and $v_j \in \mathbb{V}_j^\infty$ with $i \neq j$, we have $v_i \cdot v_j = 0$. Let $\mathbb{V}$ simply denote the vector space spanned by the set of vectors $\cup_{i\in\mathbb{Z}^+}\{e_{ij}\}_{j=1}^\infty$, where for each $i \in \mathbb{Z}^+$, the set $\{e_{ij}\}_{j=1}^\infty$ is an orthonormal basis for the vector space $\mathbb{V}_i^\infty$. It is clear that the dimension of $\mathbb{V}$, like that of $\mathbb{V}^\infty$ above, is countably infinite as the union of countably many countable sets is countable.[4]

Definition 2.1 (The q Function). Given a blade $B \in \mathbb{G}(\mathbb{V})$, the function q is a mapping from the set of all blades in $\mathbb{G}(\mathbb{V})$ to $\mathbb{Z}^+$ so that $i \in q(B)$ if and only if there exists a vector $v \in \mathbb{V}_i^\infty$ such that $v \wedge B = 0$.

With Definition 2.1 in place, we may now proceed with the following two definitions.

Definition 2.2 (The Algebraic Set $\dot{G}$). Given a blade $B \in \mathbb{G}(\mathbb{V})$, the set $\dot{G}$ of B, denoted $\dot{G}(B)$, is the set

$$\dot{G}(B) = \left\{ x \in \mathbb{R}^n \;\middle|\; \bigwedge_{i\in q(B)} p_i(x) \cdot B = 0 \right\}, \tag{2.6}$$

where for each $i \in \mathbb{Z}^+$, we define p_i similar to equation (2.1) as

$$p_i(x) = \sum_{j=1}^\infty g_j(x) e_{ij}. \tag{2.7}$$

For any product of the form $\prod_{i\in S}$ or $\bigwedge_{i\in S}$, where S is a set of positive integers, we take the terms in the product to be in ascending order with respect to the index i.

Definition 2.3 (The Algebraic Set $\hat{G}$). Given a blade $B \in \mathbb{G}(\mathbb{V})$, the set $\hat{G}$ of B, denoted $\hat{G}(B)$, is the set

$$\hat{G}(B) = \left\{ x \in \mathbb{R}^n \;\middle|\; \bigwedge_{i\in q(B)} p_i(x) \wedge B = 0 \right\}. \tag{2.8}$$

We now proceed to investigate the consequences of Definition 2.2 and that of Definition 2.3.

[4] This may come as some comfort to the reader. If you already thought that the dimension of $\mathbb{V}^\infty$ was ridiculously huge, the dimension of the vector space $\mathbb{V}$ is no bigger. In practice, of course, a computer program, for example, would work in a finite-dimensional vector space, thereby restricting the number of possible geometries to those that are zero sets of polynomials of a specific form. This is exactly what's going on in CGA, with the additional modification of altering the signature of the geometric algebra to accommodate the conformal transformations. Non-Euclidean signatures are not considered in this paper.

3. The Algebraic Set $\dot{G}$

As promised, there now exist conditions under which we are able to take the union of any two geometries represented by blades in $\mathbb{G}(\mathbb{V})$. The result is as follows.

Lemma 3.1 (The Union Of Geometries). *For any two blades $A, B \in \mathbb{G}(\mathbb{V})$ with the property that $q(A) \cap q(B)$ is empty, we have*

$$\dot{G}(A) \cup \dot{G}(B) = \dot{G}(A \wedge B). \tag{3.1}$$

Proof. Without loss of generality, we may write $C = A \wedge B$ as $C = \bigwedge_{i=1}^{k} C_k$, where for each $i \in \mathbb{Z}_k$, we have $C_i \in \mathbb{G}(V_i^\infty)$. (Notice that $q(C) = \mathbb{Z}_k$.) It is now clear that

$$\bigwedge_{i\in\mathbb{Z}_k} p_i(x) \cdot C = \pm \bigwedge_{i\in\mathbb{Z}_{k-1}} p_i(x) \cdot (p_k(x) \cdot C_k) \wedge \bigwedge_{i\in\mathbb{Z}_{k-1}} C_i \tag{3.2}$$

$$= \pm \bigwedge_{i\in\mathbb{Z}_k} p_i(x) \cdot C_i. \tag{3.3}$$

(Notice that the outer product in (3.3) becomes a scalar product in the case that every C_i is a vector.) It is now clear that $\dot{G}(C)$ is indeed the union of $\dot{G}(A)$ and $\dot{G}(B)$. □

It is not hard to show that for any blade $B \in \mathbb{G}(\mathbb{V})$, there exists a rotor R, such that B', given by $B' = RBR^{-1}$, has the property $\dot{G}(B') = \dot{G}(B)$ while $q(B')$ is mapped to any other set of integers of size $|q(B)|$. This fact can be used to adjust any given pair of blades $A, B \in \mathbb{G}(\mathbb{V})$, if needed, so that $q(A) \cap q(B)$ is empty. Admittedly, the need for any such adjustment prior to a union operation seems to detract from our dream of an algebra where the geometric elements would combine effortlessly in products that perform desired geometric operations. Unfortunately, things don't get any better as the next lemma shows.

Lemma 3.2 (The Intersection Of Geometries). *For any two blades $A, B \in \mathbb{G}(\mathbb{V})$ such that $q(A) = q(B)$ with $|q(A)| = |q(B)| = 1$, we have*

$$\dot{G}(A) \cap \dot{G}(B) = \dot{G}(A \wedge B). \tag{3.4}$$

Proof. Revisit the conversation of this paper surrounding equation (2.5). □

In this case, a simple rotor adjustment to one of A and B will only help in the case that $|q(A)| = |q(B)| = 1$. If either one of $|q(A)|$ or $|q(B)|$ is greater than one, however, more adjustments are needed before an intersection can be taken. The following equation illustrates why.

$$\bigcup_{i\in q(A)} \dot{G}(A_i) \cap \bigcup_{i\in q(B)} \dot{G}(B_i) = \bigcup_{\substack{i\in q(A)\\ j\in q(B)}} \dot{G}(A_i) \cap \dot{G}(B_j), \tag{3.5}$$

Here we have considered A as the blade $\bigwedge_{i\in q(A)} A_i$, where for each $i \in q(A)$, we have $A_i \in \mathbb{G}(\mathbb{V}_i^\infty)$. We have similiarly considered B in terms of the blades

$\{B_i\}_{i\in q(B)}$. It is now easy to see by the right-hand side of equation (3.5) that there exists a blade C with the property that $\dot{G}(C) = \dot{G}(A) \cap \dot{G}(B)$, but we do not necessarily have $\text{grade}(C) = \text{grade}(A \wedge B)$, showing that a rotor adjustment, which is grade preserving, to one or both of A and B cannot be of general help. We need the factorizations of A and B to formulate C. One possible saving grace taunts us with the following lemma.

Lemma 3.3. *For any blade $B \in \mathbb{G}(\mathbb{V})$, there exists a blade $B' \in \mathbb{G}(\mathbb{V})$ with $|q(B')| = 1$ such that $\dot{G}(B') = \dot{G}(B)$.*

Proof. It is clear that $\dot{G}(B)$ is an algebraic set as finite unions and arbitrary intersections of such sets are algebraic. Now simply see that the set of all algebraic sets is covered by the set

$$\{\dot{G}(B') | B' \in \mathbb{G}(\mathbb{V}) \text{ and } |q(B')| = 1\}. \tag{3.6}$$

□

Lemma 3.3 proves the existance of a blade with a desired property, but does not give us any clue to a means of calculating it. It would be a great achievement of geometric algebra if it provided such a means.

4. The Algebraic Set $\hat{G}$

5. Relating $\dot{G}$ with $\hat{G}$

6. Transforming The Geometries of $\mathbb{G}(\mathbb{V})$

References

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