Chapters 9-11 Supplementary Exercises
Gallian's Book on Abstract Algebra

Spencer T. Parkin

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Exercise 8

Let k be a divisor of n. The factor gruop $(Z/\langle n\rangle)/(\langle k\rangle/\langle n\rangle)$ is isomorphic to some very familiar group. What is the group?

By Exercise 40 of Chapter 10 (The Third Isomorphism Theorem), we see that $(Z/\langle n\rangle)/(\langle k\rangle/\langle n\rangle) \approx Z/\langle k\rangle$. What more is there to say?

Exercise 30

Let G be a group and let $\phi : G \to G$ be a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1g_2g_3 = e = h_1h_2h_3$. Prove that there exists an element of a in G such that $\Psi(x) = a\phi(x)$ is a homomorphism.

If $a = e \in G$ and $u, v \in G$, then

$$\Psi((uv)^{-1})\Psi(u)\Psi(v) = e \implies \Psi(v^{-1}u^{-1}) = \Psi(v)^{-1}\Psi(v)^{-1}.$$

Hmmm... Can we somehow show that Ψ is a homomorphism here? We cannot use homomorphic properties of Ψ before we know that it's a homomorphism.

Exercise 36

A proper subgroup H of a group G is called *maximal* if there is no subgroup K such that $H \subset K \subset G$. Prove that Q under addition has no maximal subgroups.

For any $q \in Q$, let $Z(q)$ denote the set $\{zq | z \in \mathbb{Z}\}$. Then for any proper subgroup H of the rationals Q under addition, we will assume that there exists $q \in Q - H$ such that $Z(q) \cap H$ is the trivial subgroup of Q. (How might I prove that this is true, if it's true? It is easy to show that H has no upper bound on the set of its non-members in Q. We can then find a finite sequence of any length where the elements are evenly spaced and the sequence misses H altogether. But none of these has the form $Z(q)$ for some $q \in Q - H$.)

Now if $q \in Q - H$, it is easy to show that $H + Z(q)$ properly contains H and is a subgroup of Q. Let $q \in Q - H$ be an element of Q such that $Z(q) \cap H$ is the trivial group. This can be done by our assumption above. What remains to be shown is that $H + Z(q)$ is a proper subgroup of Q. Suppose $Q = H + Z(q)$. Notice that $q/2 \notin H$, (since $q/2 \in H$ would imply that $q \in H$), and $q/2 \notin Z(q)$. Yet we must have $q/2 = zq + h$ for some $h \in H$ and $z \in \mathbb{Z}$. Rearranging, we have $2h = (1 - 2z)q$. Now since $2h \in H$ and $(1-2z)q \in Z(q)$, we must have $2h = (1-2z)q \in Z(q) \cap H$ which implies that $h = 0$ and $q = 0$. But this contradicts the facts that $q \notin H$ and $q/2 \notin Z(q)$. So $H + Z(q)$ is a proper subgroup of Q.

I have found a major problem with the assumption made in this proof. Notice that all subgroups H of Q have principle subgroups, (subgroups of the form $Z(q)$ for $q \in H$.) And yet, for any two rationals $q, p \in Q$, the intersection $Z(q) \cap Z(p)$ is never trivial! In fact, it's another group of the form $Z(r)$ for some $r \in Q$.