Chapters 5-8 Supplementary Exercises
Gallian's Book on Abstract Algebra

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Problem 1

A subgroup N of a group G is called a *chracteristic subgroup* if $\phi(N) = N$ for all automorphisms ϕ of G. Prove that every subgroup of a cyclic group is characteristic.

Let ϕ be an automorphism of G, a cyclic group. Let a be an element of G. Now see that

$$\phi(\langle a\rangle) = \{\phi(a^k)|k \in \mathbb{Z}\} = \{\phi^k(a)|k \in \mathbb{Z}\} = \langle\phi(a)\rangle.$$

Clearly, $|a| = |\phi(a)|$, so $|\langle a\rangle| = |\langle\phi(a)\rangle|$. We can now claim that $\langle a\rangle = \langle\phi(a)\rangle$ by the fundamental theorem of cyclic groups, because G has one and only one subgroup of each possible order.

Problem 2

Prove that the center of a group is characteristic.

Let ϕ be any automorphism of a group G. Letting a be an element in $\phi(Z(G))$ and g an element in G, there must exist an element $a' \in Z(G)$ and an element $g' \in G$ such that $\phi(a') = a$ and $\phi(g') = g$. It then follows that

$$ag = \phi(a')\phi(g') = \phi(a'g') = \phi(g'a') = \phi(g')\phi(a') = ag,$$

showing that $a \in Z(G)$. Thus far we have shown that $\phi(Z(G)) \subseteq Z(G)$. But ϕ is one-to-one, so $\phi(Z(G))$ cannot be a proper subset of $Z(G)$, and therefore, we must have $\phi(Z(G)) = Z(G)$. Oops, what if G is infinite?

I'm stumped...