On The Problem Of Intersecting Algebraic Surfaces Using Geometric Algebra

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Abstract. Abstract goes here...

Keywords. Keywords go here...

1. Introduction

The intersection of two algebraic surfaces of the same dimension may be realized in different ways. A natural realization of such an intersection would simply be that of an algebraic surface, real or imaginary,[1] of dimension one less than those taken in the intersection. In [], the intersection of quadratic surfaces are found as parametric curves through space. In this paper, however, we will attempt to realize the intersection of two algebraic surfaces as a reinterpretation of a given characterization of the intersection with what may be considered its canonical characterization.

For example, suppose we wish to realize a conic section given as the intersection of a plane and a conical surface. If we know enough about this intersection to begin with, then we may be justified in reinterpreting this intersection as that of a plane and an elliptical cylinder whose axis meets the plane at a right angle. The benefit of doing so is in noticing that the latter characterization of the given intersection lends itself to analysis through decomposition. That is, we can easily break the canonical form of the intersection down into its characteristic parts; which parts, collectively, help us realize the intersection as, in this case, the foci and orientation of the ellipse that is the intersection of the elliptical cylinder and the plane. By this analysis we have effectively found the intersection originally given to us.

[1] The surfaces in question may have no real intersection. Cases of tangency should also be considered.

The rest of this paper will now begin exactly where the first section[2] of [] ended, and so it is assumed that, before continuing on, the reader has become familiar with all terms and definitions given in that section.

2. Further Developments Of Our Model

What has been given thus far in [] is a general model of geometry based upon a geometric algebra $\mathbb{G}$. Continuing with this model, we wish here to develop it further for the purpose of finding intersections as described in the introductory section.

To that end, we begin by letting $B \in \mathbb{G}$ be a k-blade, and then consider the surface $\hat{g}(B)$. If now there exist k points $\{x_i\}_{i=1}^k \subseteq \hat{g}(B)$ such that $\bigwedge_{i=1}^k p(x_i) \neq 0$, then it is clear that there must exist a scalar $\beta \in \mathbb{R}$ such that

$$B = \beta \bigwedge_{i=1}^{k} p(x_i). \tag{2.1}$$

Being able to find such a factorization of B is perhaps desirable for many reasons, but the reason it is sought after in this paper is explained as follows.

Given two k-blades $A, B \in \mathbb{G}$, if there exists a scalar $\lambda \in \mathbb{R}$ such that $A = \lambda B$, it is clear that $\hat{g}(A) = \hat{g}(B)$. If, however, we know that the blades A and B each characterize the same geometry, (even though they may do so in different ways), which is to say that $\hat{g}(A) = \hat{g}(B)$, then it does not necessarily follow that there exists a scalar $\lambda \in \mathbb{R}$ such that $A = \lambda B$. If such a scalar did exist, then we would have a convenient algebraic means of relating the two characterizations so that an analysis[3] by decomposition of either A or B could move forward.

Returning now to equation (2.1), notice that if B has such a factorization, then clearly so must A, and we may write $A = \alpha \bigwedge_{i=1}^k p(x_i)$ for some scalar $\alpha \in \mathbb{R}$. Then, letting

$$\lambda = \frac{\alpha}{\beta},$$

we have our convenient relation $A = \lambda B$.

Seeing now why such factorizations as that in equation (2.1) are desireable for our purposes, let us consider when such factorizations exist. It should already be clear that such factorizations must depend upon how we define the function p.

We may first observe that if p is a linear function, then $\{p(x_i)\}_{i=1}^k$ is linearly independent if and only if $\{x_i\}_{i=1}^k$ is linearly independent. Requiring p be linear, however, we could never represent non-linear surfaces within our model.

More to be written here...

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[2]The reader need not read all of []; just the first section.

[3]If A was composed as some intersection we had wished to take, then B may be the canonical form that we wish to decompose.

Unfortunately, what we're going to find is that a given k-blade B representing a surface as $\hat{g}(B)$ does not always have a factorization as that given in equation (2.1). Interestingly, however, in all a cases, we can show that the blade B can always be reduced to a blade B' of lesser or equal grade such that $\hat{g}(B) = \hat{g}(B')$ and where B' does have such a factorization.

Let $\{x_i\}_{i=1}^j \subseteq \hat{g}(B)$ be a set of the largest possible cardinality such that $\bigwedge_{i=1}^j p(x_i) \neq 0$. If $j = k$, then $B' = B$, and we're done. If $j < k$, then write

$$B = B_0 \wedge \bigwedge_{i=1}^j p(x_i),$$

where B_0 is a blade of grade $k - j$. Now realize that for any $x \in \hat{g}(B)$, if $x \in \hat{g}(B_0)$, then $x \notin \{x_i\}_{i=1}^j$ and $p(x) \wedge \bigwedge_{i=1}^j p(x_i) \neq 0$, which is a contradiction. Therefore, letting $B' = \bigwedge_{i=1}^j p(x_i)$, we must have $x \in \hat{g}(B')$. Lastly, noticing that if $x \in \hat{g}(B')$, then $x \in \hat{g}(B)$, we can conclude that $\hat{g}(B) = \hat{g}(B')$.

The question that now remains is how do we relate a formulated intersection with its canonical form in all cases?

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