Chapter 2 Exercises
Gallian's Book on Abstract Algebra

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Problem 1

Give two reasons why the set of odd integers under addition is not a group.
We have no closure and no identity element.

Problem 2

Referring to Example 13, verify the assertion that subtraction is not associative.

$$(a-b)-c = a-b-c \neq a-b+c = a-(b-c)$$

Problem 3

Show that $\{1,2,3\}$ under multilication modulo 4 is not a group but that $\{1,2,3,4\}$ under multilication modulo 5 is a group.
Notice that $2^2 \equiv 0 \pmod 4$ and $0 \notin \{1,2,3\}$.
Notice that $\{1,2,3,4\} = U(5)$ which has already been established as a group.

Problem 6

Given an example of group elements a and b with the property that $a^{-1}ba \neq b$. In D_4, we have

$$R_{270}HR_{90} = V.$$

Problem 8

Show that the set $\{5, 15, 25, 35\}$ is a group under multiplication modulo 40. What is the identity element of this group? Can you see any relationship between this group and $U(8)$.

The Caylay table shows that it is a group with identity element 25. The Caylay table for $\{5, 15, 25, 35\}$ can be found by multiplying all entries in the Caylay table for $U(8)$ by 5. The groups are isomorphic.

Problem 11

Prove that the set of all 2×2 matrices with entries from $\mathbb{R}$ and determinant $+1$ is a group under matrix multiplication.

We have closure by the property of determinants. For matrices A and B, this is given by

$$1 = (\det A)(\det B) = \det AB.$$

Matrix mulltiplication is associative. The identity matrix is the group identity. All matrices with non-zero determinants have inverses.

Problem 12

For any integers $n > 2$, show that there are at least two elements in $U(n)$ that satisfy $x^2 = 1$.

Clearly $1 \in U(n)$ satisfies this. Now consider $n - 1 \in U(n)$.

$$(n-1)^2 = n^2 - 2n + 1 \equiv 1 \pmod{n}$$

Problem 14

Let G be a group with the following property: Whenever a, b and c belong to G and $ab = ca$, then $b = c$. Prove that G is Abelian. ("Cross" cancellation implies commutativity.)

Let $x = ba$ for two elements $a, b \in G$. It follows that $ax = aba$. Then, by "cross" cancelation, we get $x = ab$. We now see that $ba = x = ab$.

Problem 15

(Law of Exponents for Abelian Groups) Let a and b be elements of an Abelian group and let n be any integer. show that $(ab)^n = a^n b^n$. Is this also true for non-Abelian groups?

Clearly this holds for the case $n = 1$, even in non-Abelian groups. Assume it holds for the case $n - 1 \geq 1$. Then $(ab)^n = (ab)^{n-1}ab = a^{n-1}b^{n-1}ab$ by our inductive hypothesis. Then, since our group is Abelian, we have $a^{n-1}b^{n-1}ab = a^n b^n$.

In a non-Abelian group, consider the case $n = 2$, and let a and b be non-commuting members of the group. Then if $abab = aabb$, it follows by the cancelation property that $ab = ba$, which is a contradiction. Therefore, $abab \neq aabb$. It follows that the equation $(ab)^n = a^n b^n$ does not generaly hold for non-Abelian groups.

Problem 16

(Socks-Shoes Property) In a group, prove that $(ab)^{-1} = b^{-1}a^{-1}$. Find an example that shows that it is possible to have $(ab)^{-2} \neq b^{-2}a^{-2}$. Find distinct nonidentity elements a and b from a non-Abelian group with the property that $(ab)^{-1} = a^{-1}b^{-1}$. Draw an analogy between the statement $(ab)^{-1} = b^{-1}a^{-1}$ and the act of putting on and taking off your socks and shoes.

Notice that, by associativity, we have $ab(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = e$.

Now notice, letting $x = a^{-1}$ and $y = b^{-1}$, that while $(ab)^{-2} = ((ab)^{-1})^2 = (b^{-1}a^{-1})^2 = b^{-1}a^{-1}b^{-1}a^{-1} = yxyx$, we have $b^{-2}a^{-2} = (b^{-1})^2(a^{-1})^2 = b^{-1}b^{-1}a^{-1}a^{-1} = yyxx$. It then follows, by Problem 15, that distinct nonidentity and non-commuting elements taken from a non-Abelian group can be found such that $(ab)^{-2} \neq b^{-2}a^{-2}$. Knowing that it exists, I don't feel a need to find any one particular example.

I fail to see the analogy.

Problem 17

Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

Letting $x = a^{-1}$ and $y = b^{-1}$, suppose $(ab)^{-1} = xy$. Then $xy = (ab)^{-1} = b^{-1}a^{-1} = yx$.

Now suppose G is Abelian. Then $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$.

Problem 18

In a group, prove that $(a^{-1})^{-1} = a$ for all a.

Let $x = a^{-1}$. We must show that $x^{-1} = a$. To that end, we have

$$xa = e \implies x^{-1}xa = x^{-1} \implies a = x^{-1}.$$