Chapter 10 Exercises
Gallian's Book on Abstract Algebra

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Lemma 1

Let H be a proper subgroup of G. Then for all $g \in G - H$ and all $h \in H$, $gh \in G - H$.

Suppose $gh = h' \in H$. Then $g = h'h^{-1} \in H$, which is a contradiction. Therefore, $gh \in G - H$.

Lemma 2

Let N be a normal subgroup of a group G. Then for any $g \in G$ and any $n \in N$, there exists $n' \in N$ such that $gn = n'g$ or such that $ng = gn'$.

Lemma 3

Let G be a group and let n be a positive integer. Then the number of elements in G of order n, if any, is divisible by $\phi(n)$, the totient of n.

Suppose G has one or more elements of order n. Let N be the set $\{x \in G||x| = n\}$. Then, for any pair of elements $a, b \in N$, let $a \sim b$ if and only if $a \in \langle b \rangle$. This defines an equivilance relation on N, since $a \in \langle a \rangle$ gives us the reflexive property, since $a \in \langle b \rangle \implies b \in \langle a \rangle$ gives us the symmetric property, and since $a \in \langle b \rangle$ and, for $c \in N$, $b \in \langle c \rangle$ implies that $a \in \langle c \rangle$, giving us the transitive property. We now note that by Theorem 4.4, the size of each equivilance class is $\phi(n)$. It follows that the number of elements of order n is G is $s\phi(n)$, where s is the number of equivilance classes.

Oh, I had already read this in the book as the Corollary to Theorem 4.4.

Lemma 4

If N is a normal subgroup of a group G and gN for some $g \in G$ is a coset in G/N, then for any $g' \in gN$, we have $gN = g'N$.

If $g' \in gN$, then there exists $n \in N$ such that $g' = gn$. Then $g'g^{-1} = gng^{-1} \in N$ by the normality of N in G, and it follows that $gN = g'N$ by Property 4 of the Lemma on cosets in Chapter 7. Thus any member of a coset can act as a representative of the coset.

Exercise 38

For each pair of positive integers m and n, we can define a homomorphism from Z to $Z_m \oplus Z_n$ by $x \to (x \bmod m, x \bmod n)$. What is the kernel when $(m, n) = (3, 4)$? What is the kernel when $(m, n) = (6, 4)$? Generalize.

Let $\phi : Z \to Z_m \oplus Z_n$ be the homomorphism. Seeing that

$$\begin{aligned}\ker \phi &= \{x \in Z | x \equiv 0 \pmod{m} \text{ and } x \equiv 0 \pmod{n}\} \\ &= \{zm | z \in Z\} \cap \{zn | z \in Z\},\end{aligned}$$

it follows that

$$\ker \phi = \{z\text{lcm}(m, n) | z \in Z\}.$$

Exercise 39

If K is a subgroup of G and N is a normal subgroup of G, prove that $K/(K \cap N)$ is isomorphic to KN/N.

Notice that the normality of the subgroup $K \cap N$ in K is proven by the problem similar to Exercise 50 in Chapter 9.

We now show that KN is a group. Let $x \in KN$. Then $x = kn$ for some $k \in K$ and $n \in N$. But then by Lemma 2 above, $x = n'k \in NK$ for some $n' \in N$. It follows that $KN \subseteq NK$. Similarly, we can show that $NK \subseteq KN$, so $NK = KN$. It then follows by Exercise 6 of the supplementary exercises for chapters 5 through 8 that NK is a group.

Is N normal in KN?

We now let $\phi : K/(K \cap N) \to KN/N$ be a function defined as

$$\phi(k(K \cap N)) = kN,$$

and show that it is a homomorphism. Let us first verify that this is a well defined function. Let $a, b \in K$ such that $a(K \cap N) = b(K \cap N)$. Then $ab^{-1} \in K \cap N \subseteq N$, showing that $aN = bN$.

We now show that ϕ is operation preserving. By the normality of N and $N \cap K$, we see that

$$\begin{aligned}
&\phi(a(K \cap N)b(K \cap N)) \\
=&\phi(ab(K \cap N)) \\
=&abN = aNbN \\
=&\phi(a(K \cap N))(\phi(b(K \cap N)),
\end{aligned}$$

showing that ϕ is operation preserving.

We now consider the kernel of ϕ. Notice that

$$\begin{aligned}
\ker \phi &= \{k(K \cap N) \in K/(K \cap N) | k \in N\}, \\
&= \{k(K \cap N) \in K/(K \cap N) | k \in K \cap N\}, \\
&= \{K \cap N\}.
\end{aligned}$$

It follows that ϕ is an isomorphism by Property 9 of Theorem 10.2.

Exercise 40

If M and N are normal subgroups of G and $N \leq M$, prove that $(G/N)/(M/N) \approx G/M$.

Notice that M/N is a subgroup of G/N. To see that M/N is normal in G/N, let $g \in G$ and let $m \in M$, and see that

$$gNmN(gN)^{-1} = gmNg^{-1}N = gmg^{-1}N \in M/N,$$

since $gmg^{-1} \in M$ by the normality of M in G.

Now consider the mapping $\phi : (G/N)/(M/N) \to G/M$, defined as

$$\phi(xN(M/N)) = yM,$$

where y is any element in the coset xN. Let us now show that this is a well defined mapping. Let $a, b \in G$ such that $aN(M/N) = bN(M/N)$. It follows that $aN(bN)^{-1} = ab^{-1}N \in M/N \implies ab^{-1} \in M$. Now let $aN(M/N)$ map to $a'M$ and $bN(M/N)$ map to $b'M$. Now if $a' \in aN \subseteq aM$, then $a'M = aM$. Similarly, if $b' \in bN \subseteq bM$, then $b'M = bM$. But now since $ab^{-1} \in M$, we see that $aM = bM$, so $a'M = b'M$.

Notice that the proof that ϕ is well defined also lets us simplify its usage. That is, for any $x \in G$, we can let $xN(M/N)$ map to xM. This will greatly ease the remainder of our proof.

We now show that ϕ is operation preserving. Letting $a, b \in G$, we have

$$\begin{aligned}
&\phi(aN(M/N)bN(M/N)) \\
&= \phi(aNbN(M/N)) \\
&= \phi(abN(M/N)) \\
&= abM = aMbM \\
&= \phi(aN(M/N))\phi(bN(M/N)).
\end{aligned}$$

We now consider the kernel of ϕ. We have

$$\begin{aligned}
\ker \phi &= \{gN(M/N) | g \in G \text{ and } \phi(gN) = M\} \\
&= \{gN(M/N) | g \in M\}.
\end{aligned}$$

Now let $a, b \in M$ and consider $aN(M/N)$ and $bN(M/N)$. Since $a, b \in M$, we have $ab^{-1}N \in M/N$, which, in turn, implies that $aN(bN)^{-1} \in M/N \implies aN(M/N) = bN(M/N)$. It follows that $|\ker \phi| = 1$, and therefore, ϕ is an isomorphism.

Exercise 47

Suppose that for each prime p, Z_p is the homomorphic image of a group G. What can we say about $|G|$? Give an example of such a group.

By Property 6 of Theorem 10.2, we see that $|\phi(G)|$ divides the order of $|G|$. So, since $\phi(G) = Z_p$, we see that p divides $|G|$.

An automorphism of Z_p may be a trivial example.

After reading the answer in the back of the book, I'm wrong, because I did not understand the problem statement. For *every* prime p, Z_p is *a* homomorphic image of *the* group G. So by Property 6 of Theorem 10.2, every prime p divides $|G|$; and since there are infinitely many primes, $|G| = \infty$.

Exercise 49

Let N be a normal subgroup of a group G. Use property 7 of Theorem 10.2 to prove that every subgroup of G/N has the form H/N, where H is a subgroup of G.

For every subgroup H of G with $N \leq H$, it is clear that N is normal in H and that $H/N \leq G/N$. Now let's consider what is somewhat the converse of this. For every subgroup K of G/N, does there exists a subgroup H of G such that $K = H/N$?

Let $\phi : G \to G/N$ be defined as $\phi(g) = gN$. This is well defined and operation preserving, so it is a homomorphism from G to G/N. Then, by property 7 of Theorem 10.2, we see that $\phi^{-1}(K)$ is a subgroup of G. Now notice that if $n \in N$, then $\phi(n) = nN = N \in K$, showing that $N \leq \phi^{-1}(K)$. It follows that N is normal in $\phi^{-1}(K)$. Letting $H = \phi^{-1}(K)$, what remains to be shown now is that $H/N = K$. Letting $g \in G$, we have

$$gN \in \phi^{-1}(K)/N \iff g \in \phi^{-1}(K) \iff \phi(g) \in K \iff gN \in K.$$

It follows that $H/N = K$.

Exercise 50

Show that a homomorphism defined on a cyclic group is completely determined by its action on a generator of the group.

Let $\phi : G \to \overline{G}$ be a homomorphism of the cyclic group $G = \langle g \rangle$. Then $\phi(g^k) = \phi(g)^k$.

Exercise 51

Use the First Isomorphism Theorem to prove Theorem 9.4.

We want to show that for any group G, $G/Z(G) \approx \mathrm{Inn}(G)$. To that end, let $\phi_g(x) = gxg^{-1} \in \mathrm{Inn}(G)$, and define $\psi(g) = \phi_g$. It is clear that ψ is a homomorphism from G onto $\mathrm{Inn}(G)$. Now realize that $\psi(g) = \phi_e$ if and only if $gxg^{-1} = e$, showing that $\ker \psi = Z(G)$. It now follows from Theorem 9.4 that $G/G(Z) = G/\ker\psi \approx \psi(G) = \mathrm{Inn}(G)$.

Exercise 52

Let α and β be group homomorphisms from G to $\overline{G}$ and let $H = \{g \in G|\alpha(g) = \beta(g)\}$. Prove or disprove that H is a subgroup of G.

Clearly $e \in H$ by Property 1 of Theorem 10.1. Now let $a, b \in H$. We then have

$$\alpha(ab^{-1}) = \alpha(a)\alpha(b)^{-1} = \beta(a)\beta(b)^{-1} = \beta(ab^{-1}),$$

showing that $ab^{-1} \in H$. So I think it's a subgroup of G.

Exercise 54

If H and K are normal subgroups of G and $H \cap K = \{e\}$, prove that G is isomorphic to a subgroup of $G/H \oplus G/K$.

Let $\phi(g) = (gH, gK)$. It is clear that ϕ is a homomorphism from G to $G/H \oplus G/K$. Then, by property 1 of Theorem 10.2, we see that $\phi(G)$ is a subgroup of $G/H \oplus G/K$. We will now show that $G \approx \phi(G)$, and do so by showing that ϕ is an isomorphism between G and $\phi(G)$. Notice that ϕ is clearly onto. We already know it is operation preserving. All that remains to be shown then is that ϕ is one-to-one. So, let $a, b \in G$ such that $(aH, aK) = (bH, bK)$. Then $aH = bH$ and $aK = bK$, so $ab^{-1} \in H$ and $ab^{-1} \in K$. It follows that $ab^{-1} \in H \cap K \implies ab^{-1} = e \implies a = b$, showing that ϕ is one-to-one.

Exercise 55

Suppose that H and K are distinct subgroups of G of index 2. Prove that $H \cap K$ is a normal subgroup of G of index 4 and that $G/(H \cap K)$ is not cyclic.

It is easy to show that both H and K are normal in G. By symmetry of the problem, we need only show that H is normal in G. (The proof for K is similar.) Let $g \in G - H$. Then $gH \neq H$ and $Hg \neq H$. But gH is the only remaining of the 2 cosets of H in G, so $gH = Hg$, and therefore H is normal in G.

We then see that $H \cap K$ is normal in G by Exercise 50 of chapter 9.

At this point I had to peak at the back of the book to get some hints. By Exercise 39 in this chapter, we see that $K/(K \cap H) \approx KH/H$, so $|K|/|K \cap$

$H| = |KH|/|H|$. But $|H| = |K| = |G|/2$, so we see that $|G|^2/4 = |KH||K \cap H|$. Now realize that $|KH| = |G|$, because $K \neq H \implies |HK| > |G|/2$, and $KH = HK$ implies that HK is a subgroup of G, and so $|HK|$ divides $|G|$.

Lastly, the back of the book says that $G/(H \cap K)$ has two subgroups of order 2, which clearly means that it can't by cyclic by the Fundamental Theorem of Cyclic Groups. To which two subgroups of $G/(H \cap K)$ is Gallian refering? I'm not entirely sure, and need to go to bed.