Exercises and Problems of Alan Mcdonald's Vector and Geometric Calculus

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Exercise 1.4 (Genarlized)

When do two line parameterizations intersect?

$$x_1(t) = c_1 + tv_1$$
$$x_2(t) = c_2 + tv_2$$

We're interested in a scalar $t \in \mathbb{R}$ such that $x_1(t) = x_2(t)$. It is clear that zero, one or infinitely many such t may exist. Solving the equation

$$c_1 + tv_1 = c_2 + tv_2,$$

we get

$$t = (c_1 - c_2)(v_2 - v_1)^{-1}.$$

Clearly there is no solution when $v_1 = v_2$, unless $c_1 = c_2$, in which case $x_1(t) = x_2(t)$ for all t. On the other hand, t may not be a scalar by this equation, in which case we may conclude that the lines do not touch. Letting $\Delta c = c_2 - c_1$ and $\Delta v = v_2 - v_1$, we have

$$t = -\Delta c \cdot \Delta v^{-1} - \Delta c \wedge \Delta v^{-1},$$

showing that the lines intersect only when $\Delta c \wedge \Delta v = 0$.

Problem 1.1.1

Let $x(t) = a\cos(\theta i) + b\sin(\theta j)$, where $0 \leq \theta \leq 2\pi$, parameterize a curve. Show that the points on the curve are on the ellipse with equation $x^2/a^2 + y^2/b^2 = 1$.

We need only show that $x(t)$ satisfies the implicit equation for all t. Doing so, we see that

$$(a\cos\theta)^2/a^2 + (b\sin\theta)^2/b^2 = \cos^2\theta + \sin^2\theta = 1$$

by the Pythagorean Theorem.

Problem 1.3.1

Show that $\rho = 2a\sin\phi\cos\theta$ is the equation of the sphere of radius $|a|$ with center at $(a, 0, 0)$.

With $x = \rho\sin\phi\cos\theta$, notice that $\rho = 2ax/\rho$ so that we have $x^2 + y^2 + z^2 = \rho^2 = 2ax$. All that remains is to complete the square, giving us $(x-a)^2 + y^2 + z^2 = a^2$.

Exercise 2.1

Part a)

Show that the union of (arbitrariy many) open sets is open.

Let $\{U_\alpha\}$ be a set of arbitrarily many open sets, and let $U = \cup_\alpha U_\alpha$. Choosing any $x \in U$, there must exist α such that $x \in U_\alpha$. Now let N be a neighborhood of x contained in U_α, and see that $x \in N \subset U_\alpha \subseteq U$. It follows that a neighborhood N of x exists in U for all $x \in U$, and so U is open.

Part b)

Show that the intersection $O_1 \cap O_2$ of two open sets is open.

If the intersection is empty, then we're done, since the empty set is vacuously open. Supposing the intersection to be non-empty, for any $x \in O_1 \cap O_2$, we see that $x \in O_1$ and $x \in O_2$, and therefore, there exist neighborhoods N_1 and N_2 of x such that $x \in N_1 \subset O_1$ and $x \in N_2 \subset O_2$. Clearly,

$N_1 \cap N_2 \neq \emptyset$ and there exists a neighborhood N of x in $N_1 \cap N_2$. It follows that $x \in N \subset N_1 \cap N_2 \subset O_1 \cap O_2$, showing that $O_1 \cap O_2$ is open.

Part c)

The intersection of finitely many open sets is open. Show that the intersection of infinitely many open sets need not be open.

Let $\{U_\alpha\}$ be an infinite set of open sets such that $\cap_\alpha U_\alpha$ is a singleton set. For example,

$$\{0\} = \bigcap_{r>0} (r, r).$$

Clearly $\{0\}$ is closed, because there exists no neighborhood of 0 in $\{0\}$.

Exercise 2.5

Given an example of a set which is neither open nor closed.

Consider the rationals $\mathbb{Q} \subset \mathbb{R}$. It is clearly not an open set, nor is its complement, the irrationals.

Problem 2.1.3

Show that every open set U in $\mathbb{R}^n$ is a union of neighborhoods of points of U.

For any $x \in U$, let N_x denote any neighborhood of x in U. We then see that

$$U = \bigcup_{x \in U} x \subseteq \bigcup_{x \in U} N_x \subseteq U. \tag{1}$$

It follows that $U = \cup_{x \in U} N_x$.

Problem 2.3.4

Let $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$, where U is an open connected set, be continuous. Show that the range of f is connected.

For any two vectors $y_1, y_2 \in f(U)$, let $x_1, x_2 \in U$ be the vectors such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then, by virtue of U being a connected set, there exists a continuous parameterization $x(t) : [a, b] \to U$ such that $x(a) = x_1$ and $x(b) = x_2$. Letting $y(t) : [a, b] \to \mathbb{R}^n$ be defined as $y(t) = f(x(t))$, we see that y is continuous, because the composition of continuous functions is continuous. Furthermore, $y(a) = y_1$ and $y(b) = y_2$. It follows now that $f(U)$ is connected.

(Didn't use fact that U is open set?)