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A METHOD OF APPLYING CONFORMAL TRANSFORMATIONS USING GEOMETRIC ALGEBRA

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Abstract. It is shown that any conformal transformation may be applied to a polynomial of degree m by first converting the polynomial to an m-vector taken from a geometric algebra, applying a versor to that m-vector, and then converting the resulting m-vector back into a polynomial.

1. Preliminaries

For this paper, we assume the reader is already familiar with geometric algebra and the conformal model of geometric algebra. (See [2] for introductory material on geometric algebra. See [2, 4] for material on conformal geometric algebra.) Despite what conventions may be used in other papers on geometric algebra, here we will let the outer product take precedence over the inner product, and the geometric product take precedence over the inner and outer products.

As there are different ways of defining the inner product for different purposes, we must take a moment here to define the inner product used in this paper. It is as follows. Among vectors, the inner product is a bilinear form defining the signature of our geometric algebra which will be given in the next section. For any vector v and k-blade A, we define

$$v \cdot A = -\sum_{i=1}^{k}(-1)^i(v \cdot a_i)A_i, \tag{1.1}$$

where here, the k-blade A may be factored as $A = \bigwedge_{i=1}^{k} a_i$, and for each integer $i \in [1, k]$, we let

$$A_i = \bigwedge_{\substack{j=1 \\ j\neq i}}^{k} a_j.$$

We let v commute with A as

$$v \cdot A = -(-1)^k A \cdot v.$$

It is sometimes convenient to rewrite equation (1.1) as

$$v \cdot A = (v \cdot a_1)A_1 - (v \cdot A_1) \wedge a_1,$$

which gives a recursive version of the definition. For a k-blade A and an l-blade B, we define

$$A \cdot B = \begin{cases} A_k \cdot (a_k \cdot B) & \text{if } k \leq l, \\ (A \cdot b_1) \cdot B_1 & \text{if } k \geq l, \end{cases} \tag{1.2}$$

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where here, the l-blade B may be factored as $B = \bigwedge_{i=1}^{l} b_i$, and for each integer $i \in [1, k]$, we let

$$B_i = \bigwedge_{\substack{j=1 \\ j \neq i}}^{l} b_j.$$

In the case that $k = l$, either part of the piece-wise function of equation (1.2) may be used to evaluate $A \cdot B$.

2. The Geometric Algebra

We begin by letting $\{\mathbb{G}_i\}_{i=1}^m$ be a sequence of m Minkowski geometric algebras upon which the conformal model of n-dimensional euclidean space may be imposed. Each $\mathbb{G}_i$ is the very geometric algebra found in [4]. If $\mathbb{V}_i$ is a vector space generating $\mathbb{G}_i$, then $\{e_{i,-}, e_{i,+}\} \cup \{e_{i,j}\}_{j=1}^n$ is a set of basis vectors generating $\mathbb{V}_i$, where $\{e_{i,j}\}_{j=1}^n$ is an orthonormal basis for an n-dimensional euclidean space, and where each of $e_{i,-}$ and $e_{i,+}$, taken from [5], is given by

$$\begin{aligned} e_{i,-} &= \frac{1}{2}\infty_i + o_i, \\ e_{i,+} &= \frac{1}{2}\infty_i - o_i, \end{aligned}$$

where here, o_i and ∞_i are, for each $\mathbb{G}_i$, the familiar null-vectors representative of the points at origin and infinity, respectively. With the exception of zero, we consider the sequence of geometric algebras $\{\mathbb{G}_i\}_{i=1}^m$ to be a set of pair-wise disjoint sets.

Taking our cue from the "mother algebra" in [1], we are now interested in forming the smallest geometric algebra $\mathbb{G}$ containing each $\mathbb{G}_i$ as a geometric sub-algebra. This geometric algebra $\mathbb{G}$ is therefore generated by the vector space $\mathbb{V}$, given by

$$\mathbb{V} = \bigoplus_{i=1}^{m} \mathbb{V}_i.$$

We now introduce a function $\Psi_{i,j} : \mathbb{G} \to \mathbb{G}$ defined as

$$\Psi_{i,j}(E) = \left\{ \begin{array}{ll} S_{i,j,1} E (S_{i,j,1})^{-1} & \text{if } i \neq j, \\ E & \text{if } i = j \end{array} \right. ,$$

where $S_{i,j,k}$ is the constant given by

$$S_{i,j,k} = \left(1 - (-1)^k e_{i,-} e_{j,-}\right) \left(1 + (-1)^k e_{i,+} e_{j,+}\right) \prod_{r=1}^{n} \left(1 + (-1)^k e_{i,r} e_{j,r}\right). \tag{2.1}$$

Take notice that

$$(S_{i,j,1})^{-1} = 2^{-(n+2)} S_{i,j,0}.$$

Our definition of $\Psi_{i,j}$ is motivated by the fact that for any vector $v_i \in \mathbb{V}_i$ and its corresponding vector $v_j \in \mathbb{V}_j$, we have

$$v_j = \Psi_{i,j}(v_i).$$

Being in correspondence, this means that for all integers $k \in [1, n]$, we have

$$v_i \cdot e_{i,k} = v_j \cdot e_{j,k},$$

as well as

$$v_i \cdot o_i = v_j \cdot o_j,$$

and

$$v_i \cdot \infty_i = v_j \cdot \infty_j.$$

Notice that $\Psi_{i,j}(v_j) = -v_i$. For any vector $v \notin \mathbb{V}_i$ and $v \notin \mathbb{V}_j$, the function $\Psi_{i,j}$ leaves v invariant.

If it could be shown that $S_{i,j,1}$ in equation (2.1) is a versor of $\mathbb{G}$, we could then conclude that $\Psi_{i,j}$ is an outermorphism. (See [3] for a definition of outermorphism.) Though no such proof will be given here, we will never-the-less be able to show that $\Psi_{i,j}$ is indeed an outermorphism. To that end, it suffices to show that for any two vectors $a, b \in \mathbb{V}$, we have

$$a \cdot b = \Psi_{i,j}(a) \cdot \Psi_{i,j}(b). \tag{2.2}$$

To see this, begin by rewriting a and b as $a = \sum_{k=1}^{m} a_k$ and $b = \sum_{k=1}^{m} b_k$, where for each pair (a_k, b_k), we have $a_k, b_k \in \mathbb{V}_k$. We then have

$$a \cdot b = \sum_{k=1}^{m} a_k \cdot b_k,$$

where we can make the observation that for any integer $k \in [1, m]$, we have

$$a_k \cdot b_k = \Psi_{i,j}(a_k) \cdot \Psi_{i,j}(b_k).$$

We now simply see that

$$\Psi_{i,j}(a) \cdot \Psi_{i,j}(b) = \sum_{k=1}^{m} \Psi_{i,j}(a_k) \cdot \Psi_{i,j}(b_k)$$

to complete the proof. For a versor $V \in \mathbb{G}$, the property equivalent to equation (2.2) is as follows.

$$a \cdot b = VaV^{-1} \cdot VbV^{-1}$$

This property of versors is imployed in the step taken from (**??**) to (**??**) in §4 below.

Having now established equation (2.2), we can apply it in a proof that $\Psi_{i,j}$ preserves the outer product.

$$\begin{aligned}
\Psi_{i,j}(a \wedge b) &= \Psi_{i,j}(ab - a \cdot b) \\
&= \Psi_{i,j}(ab) - \Psi_{i,j}(a \cdot b) \\
&= \Psi_{i,j}(a)\Psi_{i,j}(b) - a \cdot b \\
&= \Psi_{i,j}(a)\Psi_{i,j}(b) - \Psi_{i,j}(a) \cdot \Psi_{i,j}(b) \\
&= \Psi_{i,j}(a) \wedge \Psi_{i,j}(b).
\end{aligned}$$

Interestingly, it seems that although the property in equation (2.2) is important, and though we could now continue to use it in an inductive proof that Ψ preserves the outer product among blades of arbitrary grade through the use of equation (1.1), the following proof, also inductive, shows that only the properties of linearity, grade preservation and the preservation of the geometric product are required to prove such a thing.

Let A be a k-blade. It is clear that Ψ trivially and vacuously preserves the outer product in the case $k = 1$. Assuming now that $k > 1$, we make the inductive hypothesis that Ψ preserves the outer product for blades A of grade $k-1$. We then see that

$$\begin{aligned}
\Psi_{i,j}(A) &= \Psi_{i,j}(a_1 \wedge A_1) \\
&= \Psi_{i,j}\left(\frac{1}{2}(a_1 A_1 + (-1)^{k-1} A_1 a_1\right) \\
&= \frac{1}{2}\left(\Psi_{i,j}(a_1)\Psi_{i,j}(A_1) + (-1)^{k-1}\Psi_{i,j}(A_1)\Psi_{i,j}(a_1)\right) \\
&= \Psi_{i,j}(a_1) \wedge \Psi_{i,j}(A_1) \\
&= \bigwedge_{l=1}^{k} \Psi_{i,j}(a_l).
\end{aligned}$$

As you can see, our inductive hypothesis was invoked here in the last step, but this was not our first use of it. In the second-to-last step, the property of grade preservation is required. We know that Ψ preserves grade in the case $k=1$. By our inductive hypothesis, we know that Ψ preserves grade in the case $k-1$.

Establishing the outermorphic property of Ψ was not actually needed to prove the main result of this paper. It is, however, a property worth showing, because it allows for greater ease of use. That is, we don't have to carefully avoid any temptation to employ an outermorphic property in the case that a given blade is not also a versor by virtue of having a factorization in terms of pair-wise orthogonal vectors. With versors we can employ our property of preserving the geometric product. With blades that are not versors, we must have the property of preserving the outer product.

3. Geometric Representation

Having now set forth our geometric algebra $\mathbb{G}$, we're ready to discuss geometric representation. Letting $\mathbb{R}^n_j$ denote the n-dimensional euclidean vector sub-space of $\mathbb{V}_j$, our geometric representation scheme is to let a geometry be the set of all points $x_1 \in \mathbb{R}^n_1$, such that

$$\bigwedge_{j=1}^{m} p_j(x_1) \cdot A = 0, \tag{3.1}$$

where A is an m-vector (not necessarily an m-blade) of $\mathbb{G}$, and where the function $p_j : \mathbb{R}^n_1 \to \mathbb{V}_j$, reminding us of the principle mapping found in [4], is defined as

$$\begin{aligned}
p_j(x_1) &= \Psi_{1,j}\left(o_1 + x_1 + \frac{1}{2}x_1^2\infty_1\right) \\
&= o_j + x_j + \frac{1}{2}x_j^2\infty_j,
\end{aligned}$$

where $x_j \in \mathbb{R}^n_j$. Here, it is the m-vector A that serves as the representative of our geometry. It is not hard to see that the point-set generated by A through equation (3.1) is simply the zero set of a polynomial of degree at most $2m$.[1]

[1] An algebraic set is the zero set of one or more polynomial functions. It can be shown that such sets may be represented by the blades of a geometric algebra, but the geometric representation scheme of this paper is restricted to those sets that are the zero set of one and only one polynomial function.

For any polynomial of degree r with $m < r \leq 2m$, there does not necessarily exist an m-vector A representative of its zero set in terms of the definition just given by equation (3.1). For all polynomials of degree r with $0 \leq r \leq m$, however, there does exist such an m-vector A. Taking advantage of the linearity of the inner product, it suffices to show that this is the case for every monomial of such a degree. Indeed, the m-vector A (in this case an m-blade) is given by

$$A = \lambda \bigwedge_{j=1}^{r} a_j \wedge \bigwedge_{j=r+1}^{m} \infty_j,$$

where $\lambda \in \mathbb{R}$ is a scalar, $\{a_j\}_{j=1}^{r}$ is a set of r vectors with each $a_j \in \mathbb{R}_j^n$, and the product $\bigwedge_{j=1}^{r} a_j$ is one in the case that $r = 0$. (Notice that A is not just an m-blade here, but also a versor.) To see this, let $x_j = \Psi_{1,j}(x_1)$ and write

$$\begin{aligned}
\bigwedge_{j=1}^{m} p_j(x_1) \cdot A &= \bigwedge_{j=1}^{m} \left(o_j + x_j + \frac{1}{2} x_j^2 \infty_j \right) \cdot A \\
&= \bigwedge_{j=1}^{r} x_j \wedge \bigwedge_{j=r+1}^{m} o_j \cdot A \\
&= \lambda(-1)^k \bigwedge_{i=1}^{r} x_j \cdot \bigwedge_{i=1}^{r} a_j \\
&= \lambda(-1)^k \prod_{i=1}^{r} x_j \cdot a_j,
\end{aligned}$$

which shows that for an appropriate choice of the vectors in $\{a_j\}_{j=1}^{r}$, and that of λ, we can formulate A as being representative of any monomial in n independent variables $\{x_1 \cdot e_{1,j}\}_{j=1}^{n}$. The integer k is given by

$$k =$$

It is not difficult to convert between polynomial functions and m-vectors A of our geometric algebra $\mathbb{G}$. That is, not difficult if we are using a computer algebra system. We'll no doubt want to make further use of such a system in the section to follow. It should also be mentioned that there are often nice conversions between m-vectors and vector-based equations, (as apposed to the perhaps less intuitive form of polynomial equations), that can be easily performed by hand by simply factoring $\bigwedge_{j=1}^{m} p_j(x_1)$ out of the equation with respect to the inner product. See, for example, the vector-based equations for the quadric surfaces found in [6].

4. Applying Conformal Transformations To Geometries

We now come to the main result of this paper, which is to show that the conformal transformations are easily applicable to any geometry that may be represented as an m-vector by equation (3.1). We simply observe that if $V_1 \in \mathbb{G}_1$ is a versor representative of a conformal transformation, (those that can be found in [2, 5]), then, letting $V_j = \Psi_{1,j}(V_1)$, the desired geometry is given by the set of all points $x_1 \in \mathbb{R}_1^n$, such that

$$\bigwedge_{j=1}^{m} V_j^{-1} p_j(x_1) V_j \cdot A = 0. \tag{4.1}$$

There are two properties of such versors V_1 in relation to the function p_1 that make this possible. The first is form preservation, up to scale. It is well known in the conformal model of geometric algebra that for any versor V_1 of that model, there exists a scalar $\lambda \in \mathbb{R}$ and a point $y_1 \in \mathbb{R}^n_1$ such that

$$V_1^{-1} p_1(x_1) V_1 = \lambda p_1(y_1). \tag{4.2}$$

Here, the sandwitch product of V_1 with $p_1(x_1)$ has preserved the form of the function p_1, up to scale. The second property is uniqueness. It is also well known in the conformal model that while such a scalar $\lambda \in \mathbb{R}$ and a point $y_1 \in \mathbb{R}^n_1$ in equation (4.2) exist, this pair is also unique. It follows that that V_1 and p_1 together induce a well-defined mapping from $\mathbb{R}^n_1$ to $\mathbb{R}^n_1$, the point y_1 being a function of the point x_1.

Returning to the geometry that is the solution set in $\mathbb{R}^n_1$ of equation (4.1), it will not be hard to show that the m-vector A' representative of this very set of points by equation (3.1) is given by

$$A' = WAW^{-1},$$

where the versor W is given by

$$W = \prod_{j=1}^{m} V_j.$$

Taking advantage of the linearity of the inner product once again, we need only prove our main result in the case that A is an m-blade. Let $\{a_j\}_{j=1}^m \subset \mathbb{V}$ be a set m vectors, such that

$$A = \bigwedge_{j=1}^{m} a_j.$$

We then have

$$\begin{aligned}
\bigwedge_{j=1}^{m} V_j^{-1} p_j(x_1) V_j \cdot \bigwedge_{j=1}^{m} a_j &= \prod_{j=1}^{m} V_j^{-1} p_j(x_1) V_j \cdot a_j \\
&= \prod_{j=1}^{m} p_j(x_1) \cdot V_j a_j V_j^{-1} \\
&= \bigwedge_{j=1}^{m} p_j(x_1) \cdot \bigwedge_{j=1}^{m} V_j a_j V_j^{-1} \\
&= \bigwedge_{j=1}^{m} p_j(x_1) \cdot \bigwedge_{j=1}^{m} W a_j W^{-1} \\
&= \bigwedge_{j=1}^{m} p_j(x_1) \cdot WAW^{-1}.
\end{aligned}$$

This completes the proof. The see the second-to-last step here, one must realize that if $i \neq j$, then V_i leaves a_j invariant, up to sign, as $V_i a_j V_i^{-1} = \pm a_j$. Furthermore, a lot of sign cancellation goes on that has not been typeset here. (Revisit. Make sure signs all go away.)

We can now convert A' into a polynomial equation, which may be a more usable form than that of an m-vector.

5. Closing Remarks

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