Spades As Representatives Of Geometric Sets

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Abstract. Abstract...

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1. Introduction And Motivation

In [1] it was shown how *blades* taken from a geometric algebra are representative of geometric sets. In this paper we show how *spades* taken from a geometric algebra may serve as alternative means of representing of such sets. We'll find that this has some interesting applications in the conformal model.

So what is a spade? Table 1 below gives a definition of this new term among a few of its traditional counter-parts.

TABLE 1. A few terms used in GA

Term	Definition
Blade	An outer product of zero or more linearly-independent vectors.
Versor	A geometric product of zero or more invertible vectors, not necessarily forming a linearly-independent set.
Spade	A geometric product of zero or more vectors, not necessarily forming a linearly-independent set.

The difference between "versor" and "spade" is a subtle one, but important. We will not require that each vector in the product be invertible. It is the invertibility of the geometric product, however, that provides us with perhaps the greatest motivation to use spades, as apposed to blades, as representatives of geometric sets.

Similar to the concept of grade, that of rank will be introduced in this paper with respect to spades. As an r-blade refers to a blade of grade r, we will let an r-spade refer to a spade of rank r. If an element of a geomtric algebra can be written as any geometric product of vectors, then it is a spade.

The rank of that spade is then the smallest possible number of vectors for which it can be written as such a product.[1] Note that blades of grade zero are indistinguishable from spades of the same rank as each denotes the set of all scalars. Similarly, blades of grade one and spades of rank one each refer to the set of all vectors.

2. Representation By Blades And Spades

Letting a denote a vector, and B_r a blade of grade r having factorization $\bigwedge_{i=1}^r b_i$, recall the following identity, albeit in what may be a slightly unfamiliar form.

$$a \cdot B_r = \langle B_r \rangle_0 a - \sum_{i=1}^r (-1)^i (a \cdot b_i) \bigwedge_{\substack{j=1 \\ j \neq i}}^r b_i \tag{2.1}$$

Fascinatingly, replacing every instance of the outer product in equation (2.1) with a geometric product gives us a new identity, equation (2.2), which does indeed hold. Letting M_r denote a spade of at most rank r, having a factorization of $\prod_{i=1}^r m_i$, we have

$$a \cdot M_r = \langle M_r \rangle_0 a - \sum_{i=1}^r (-1)^i (a \cdot m_i) \prod_{\substack{j=1 \\ j \neq i}}^r m_j. \tag{2.2}$$

At this point it is immediately clear by a comparison of equations (2.1) and (2.2) that a representation of geometric sets can be accomplished by spades as well as by blades. We now go on to formalize this notion.

Letting $p : \mathbb{R}^n \to \mathbb{V}$ be a non-zero, vector-valued function from an n-dimensional space[2] to the vector space $\mathbb{V}$ generating our geometric algebra $\mathbb{G}$, recall from [1] the characterization of a geometric set as being the largest sub-set of $\mathbb{R}^n$ over which p vanishes in the inner product with any given blade $B_r \in \mathbb{G}$ of grade $r > 0$. In set-builder notation, we would write

$$\dot{g}(B_r) = \{x \in \mathbb{R}^n | p(x) \cdot B_r = 0\}.$$

Now, letting each $m_i = b_i$, and defining

$$\dot{G}(M_r) = \{x \in \mathbb{R}^n | p(x) \cdot M_r = \langle M_r \rangle_0 p(x)\}, \tag{2.3}$$

we see, by equations (2.1) and (2.2), that

$$\dot{G}(M_r) = \bigcap_{i=1}^r \dot{G}(m_i) = \bigcap_{i=1}^r \dot{g}(b_i) = \dot{g}(B_r).$$

This shows that, under the definition given by equation (2.3), every geometric set has a spade representative as well as that of a blade.

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[1] While the correctness of many identities of this paper do not require a spade to be written in the most compact form, the concept of rank would be ill-defined without its consideration.

[2] The field of real numbers $\mathbb{R}$ is most typically used. However, there are often advantages to using the complex numbers $\mathbb{C}$ instead as they form an algebraicly closed field.

Recalling now that

$$\hat{g}(B_r) = \{x \in \mathbb{R}^n | p(x) \wedge B_r = 0\}, \tag{2.4}$$

it is not hard to show that $\hat{g}(B_r) = \dot{g}(B_r I)$, where I is the unit-psuedo-scalar of our geometric algebra $\mathbb{G}$. In the case that $M_r I$ is also a spade, this prompts in us an investigation of $\dot{G}(M_r I)$. If I is a spade, then clearly $M_r I$ is a spade, and one of rank at most $q + r$, where $q = \text{grade}(I)$. Using equation (**??**), we then find that $\dot{G}(M_r I) = \hat{G}(M_r)$, where

$$\hat{G}(M_r) = \{x \in \mathbb{R}^n | p(x) \wedge M_r = (-1)^q \langle M_r \rangle_q p(x)\}. \tag{2.5}$$

In [1] the concept of irreducibility was established, and it was shown that for every geometric set, there existed, up to scale, a unique irreducible blade representative of that set. This irreducible blade is also a sub-space of every other blade representative of the same set. Having now a direct (equation (2.5)) and indirect (equation (2.3)) way of representing geometric sets with spades, it stands to reason whether there exists a similar result concerning them.

3. Applications In The Conformal Model

After all that's been said, why might we care to use spades in place of blades? Well, here, while investigating the use of spades in the conformal model, each example will illustrate the potential advantage that doing so may have.

4. Proof Of Identities

$$a \cdot B_r I = (a \wedge B_r) I \tag{4.1}$$

$$a \cdot M_r I = (a \wedge M_r) I \tag{4.2}$$

References

[1] S. Parkin, *An Introduction To Geometric Sets*. Advances in Applied Clifford Algebras, Volume 25, Issue Unknown, pp. 639-655, 2015.

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