On The Expansion Of Algebraic Expressions In Geometric Algebra

Spencer T. Parkin

Abstract. Abstract goes here...

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1. Introduction

While the expansion of algebraic expressions taken from, say, a polynomial ring, are found as a trivial matter of applying the associative and distributive properties, and combining like-terms, it is interesting to note that this is certainly not true of expressions taken from a geometric algebra.[1] In this paper, a general stratagy, or algorithm, if you will, is given for the expansion of such expressions, and it is shown that it is perhaps just as natural to write an element of a geometric algebra as a sum of "mercers" as it is to write such an element as a sum of blades. The term "mercer" is introduced in Table 1 below, along with similar, traditional terms found in geometric algebra.[2]

Table 1. Terms used in GA

Term	Definition
Blade	The outer product of zero or more linearly-independent vectors.
Versor	The geometric product of zero or more invertible vectors, not necessarily forming a linearly-independent set.
Mercer	The geometric product of zero or more vectors, not necessarily invertible and not necessarily forming a linearly-independent set.

[1] Factoring may be thought of as the problem opposite of and more interesting than that of expansion. In geometric algebra, however, expansion is not entirely trivial, and therefore a subject of interest.

[2] The term "versor" was avoided in this paper in favour of "mercer" as a matter of rigour. Not knowing a term for the algebraic form in question, and not finding one in the literature, one was made up.

From these it is clear that every versor is a mercer, but not every mercer is a versor.

Similar to the concept of grade, that of rank will be introduced in this paper with respect to mercers. As an n-blade refers to a blade of grade n, we will let an n-mercer refer to a mercer of rank n; that is, a geometric product of precisely n vectors. Note that blades of grade zero are indistinguishable from mercers of the same rank as each denotes the set of all scalars.

Unlike versors, note that mercers do not form a group over the geometric product by simple reason that not every mercer is invertible with respect to the geometric product. They are important to study, however, because they appear more often in consideration of the typical expression taken from a geometric algebra. Put a better way, versors are a special case of mercer, and we want to keep our discussion as generally applicable as possible.

2. Symmetry Between The Outer And Geometric Products

As will be shown by the results established in this section, there is perhaps a lot more in common between the outer and geometric products than one might think. Certainly the outer and inner products play a complementary role in the building up or tearing down of blades, respectively, but from a purely algebraic perspective, consider the following well-known definition of the geometric product between two vectors a and b.

$$ab = a \cdot b + a \wedge b \tag{2.1}$$

The right-hand side of equation (2.1) is a sum of blades, while the left-hand side is a sum of mercers; in this case, exactly one; namely, ab. Thus, the element ab appears naturally in a sum-of-blades and sum-of-mercers form, but what of the element $a \wedge b$? Rearranging (2.1), we simply find that

$$a \wedge b = -a \cdot b + ab, \tag{2.2}$$

showing that it too may be written as a sum of blades or that of mercers. In fine, one aim of this paper is to show that while every element has a sum-of-blades form, they too each have a sum-of-mercers form. Indeed, existance is a fundamental question in mathematics, as is the question of uniqueness. Therefore, we will also consider the uniqueness of these two forms.

2.1. From Mercer To Sum Of Blades

As the set of all elements of a geometric algebra are generated by a vector space and the outer product, it is clear that every element of a geometric algebra has a sum-of-blades form. Furthermore, for a given multivector, after sorting all blades in the sum by grade, then each blade alphabetically by vector name (accounting for the possible sign change due to the anti-commutativity of the outer product), then each subset of blades homogeneous of the same grade alphabetically accordingly as the blade reads – after doing all this, it is not hard to convince yourself that the sum-of-blades form is unique. The proof essentially lies in linear algebra where it is well-known that every vector can be written in no more than one linear combination of a given set of linearly independent basis vectors. A vector space and the outer product generate a linearly independent set of basis blades, all linear combinations of which produce all elements of the geometric algebra.

Give mercer to sum-of-blades expansion here.

2.2. From Blade To Sum Of Mercers

An interesting expansion of a blade into a sum-of-mercers form can be found in [1, p. 86] and derived by a repeated application of equation (4.6).[3] Interestingly, however, it does not take much effort to show that not all sums of mercers representing the same element of a geometric algebra need be homogeneous of mercers of the same rank; and therefore, a sum-of-mercers form, for any one given element, is not unique. To see this, consider the 3-blade $a \wedge b \wedge c$. It may be written as

$$a \wedge b \wedge c = \frac{1}{4}(abc - acb + bca - cba),$$

or written as

$$a \wedge b \wedge c = -(b \cdot c)a + (a \cdot c)b - (a \cdot b)c + abc$$

by a repeated application of equation (4.2). Since we're concerned with symmetry between the outer and geometric products, we'll focus on the latter form, and show that as a normalized sum-of-mercers form, it is unique.

Go on...

2.3. The Inner Product And Sums Of Blades

Letting a denote a vector and B_r a blade of grade r having the factorization given in equation (4.1), we wish here to express the inner product $a \cdot B_r$ as a sum of blades. Since the case $r = 1$ is trivial, we begin by writing, for all $r > 1$,

$$\begin{aligned}
a \cdot B_r &= a \cdot (B_{r-1} \wedge b_r) \\
&= (-1)^{r-1} a \cdot (b_r \wedge B_{r-1}) && (2.3) \\
&= -(-1)^r \left(-b_r \wedge (a \cdot B_{r-1}) + (a \cdot b_r) B_{r-1}\right) && (2.4) \\
&= -(-1)^r \left(-(-1)^r (a \cdot B_{r-1}) \wedge b_r + (a \cdot b_r) B_{r-1}\right) \\
&= (a \cdot B_{r-1}) \wedge b_r - (-1)^r (a \cdot b_r) B_{r-1}. && (2.5)
\end{aligned}$$

Here, we've gone from equation (2.3) to that of (2.4) by applying the identity given in equation (4.7).

Applied recursively, it is easy to see here from equation (2.5) that the expansion of $a \cdot B_r$ as a sum of blades is given by

$$a \cdot B_r = \langle B_r \rangle_0 a - \sum_{i=1}^{r} (-1)^i (a \cdot b_i) \bigwedge_{j=1, j \neq i}^{r} b_j. \qquad (2.6)$$

[3] Note the errata...

One might also simply use equation (2.5) to give an inductive argument of equation (2.6).

Notice that for all $r > 0$, the term $\langle B_r\rangle_0 a$ vanishes in equation (2.6), yet its presence allows us the case $r = 0$ if we define the summation to be zero in the vacuous case.

2.4. The Inner Product And Sums Of Mercers

Letting a denote a vector and M_r a mercer of rank r having the factorization given in equation (4.9), we wish here to express the inner product $a \cdot M_r$ as a sum of mercers. Since the case $r = 1$ is trivial, we begin by writing, for all $r > 1$,

$$\begin{aligned} a \cdot M_r &= a \cdot (M_{r-1} m_r) \\ &= a \cdot ((\langle M_{r-1}\rangle_0 + \langle M_{r-1}\rangle_1 + \langle M_{r-1}\rangle_2^r) m_r) \\ &= \langle M_{r-1}\rangle_0 a \cdot m_r + (\langle M_{r-1}\rangle_1 \cdot m_r) a \\ &+ (a \cdot \langle M_{r-1}\rangle_1) m_r - (a \cdot m_r)\langle M_{r-1}\rangle_1 + a \cdot (\langle M_{r-1}\rangle_2^r m_r). \end{aligned} \tag{2.7}$$

We will return to this equation momentarily. Until then, to ease notation, let us write $M = \langle M_{r-1}\rangle_2^r$ and see that

$$a \cdot (M m_r) = a \cdot (M \cdot m_r + M \wedge m_r)$$
$$= -(-1)^{r-1} a \cdot (m_r \cdot M) + (-1)^{r-1} a \cdot (m_r \wedge M) \tag{2.8}$$
$$= (-1)^r m_r \cdot (a \cdot M) - (-1)^r \left[-m_r \wedge (a \cdot M) + (a \cdot m_r) M\right] \tag{2.9}$$
$$= (a \cdot M) \cdot m_r + (a \cdot M) \wedge m_r - (-1)^r (a \cdot m_r) M$$
$$= (a \cdot M) m_r - (-1)^r (a \cdot m_r) M. \tag{2.10}$$

Note here our use of equations (4.8) and (4.7) to arrive at equation (2.9) from (2.8).

Returning now to equation (2.7), if we plug equation (2.10) into it under the assumption that r is odd, we get

$$a \cdot M_r = (a \cdot M_{r-1}) m_r + (a \cdot m_r) M_{r-1} - \langle M_{r-1}\rangle_0 a m_r. \tag{2.11}$$

And if we plug equation (2.10) into equation (2.7) under the assumption that r is even, we get

$$a \cdot M_r = (a \cdot M_{r-1}) m_r - (a \cdot m_r) M_{r-1} + (\langle M_{r-1}\rangle_1 \cdot m_r) a. \tag{2.12}$$

It then follows, despite the parity of r, that

$$\begin{aligned} a \cdot M_r = (a \cdot M_{r-1}) m_r - (-1)^r (a \cdot m_r) M_{r-1} \\ - \langle M_{r-1}\rangle_0 a m_r + \langle M_r\rangle_0 a. \end{aligned} \tag{2.13}$$

Note the use of equation (4.15) here in our arrival at equation (2.13).

Applied recursively, it is now easy to see from equation (2.13) that the expansion of $a \cdot M_r$ as a sum of mercers is given by

$$a \cdot M_r = \langle M_r\rangle_0 a - \sum_{i=1}^{r} (-1)^r (a \cdot m_i) \prod_{j=1, j\neq i}^{r} m_j. \tag{2.14}$$

To see this, consider an inductive argument. The cases $r = 0$ and $r = 1$ follow trivially by inspection. Now make the inductive hypothesis that equation (2.14) holds for a fixed case $r-1$. Then, applying the recursive formula (2.13) to the equation in (2.14), adjusted for the case $a \cdot M_{r-1}$, we get equation (2.14), thereby completing our proof by induction.

It is very interesting now to compare this equation (2.14) with that of (2.6). Indeed, the outer and geometric products do appear to have some natural symmetry between them. But perhaps this really shouldn't be too supprising after one compares equation (2.1) with that of (2.2).

3. The Expansion Algorithm

4. Appendix Of Identities

Identities used in this paper are thrown into this appendix so as not to encumber the main body of the paper.

4.1. Identities Involving Blades

Letting a denote a vector, and B_r a blade of grade r having factorization

$$B_r = \bigwedge_{i=1}^{r} b_i, \tag{4.1}$$

recall that

$$aB_r = a \cdot B_r + a \wedge B_r. \tag{4.2}$$

Recalling also the commutativities of a with B_r in the inner and outer products as

$$a \cdot B_r = -(-1)^r B_r \cdot a, \tag{4.3}$$

$$a \wedge B_r = (-1)^r B_r \wedge a, \tag{4.4}$$

we find that

$$\begin{aligned} a \cdot B_r &= \frac{1}{2} a \cdot B_r - \frac{1}{2}(-1)^r B_r \cdot a \\ &= \frac{1}{2}(aB_r - a \wedge B_r - (-1)^r (B_r a - B_r \wedge a)) \\ &= \frac{1}{2}(aB_r - (-1)^r B_r a), \end{aligned} \tag{4.5}$$

and that

$$\begin{aligned} a \wedge B_r &= \frac{1}{2} a \wedge B_r + \frac{1}{2}(-1)^r B_r \wedge a \\ &= \frac{1}{2}(aB_r - a \cdot B_r + (-1)^r (B_r a - B_r \cdot a)) \\ &= \frac{1}{2}(aB_r + (-1)^r B_r a). \end{aligned} \tag{4.6}$$

Now letting a and b each denote a vector, it is not hard to show that for all $r \geq 1$, we have

$$a \cdot (b \wedge B_r) + b \wedge (a \cdot B_r) = (a \cdot b)B_r. \tag{4.7}$$

To that end, we apply equations (4.5) and (4.6) in writing

$$\begin{aligned} a \cdot (b \wedge B_r) &= \frac{1}{2}\left(a\frac{1}{2}\left(bB_r + (-1)^r B_r b\right) - (-1)^{r+1}\frac{1}{2}\left(bB_r + (-1)^r B_r b\right)a\right) \\ &= \frac{1}{4}\left(baB_r + (-1)^r aB_r b + (-1)^r bB_r a + B_r ba\right), \\ b \wedge (a \cdot B_r) &= \frac{1}{2}\left(b\frac{1}{2}\left(aB_r - (-1)^r B_r a\right) + (-1)^{r-1}\frac{1}{2}\left(aB_r - (-1)^r B_r a\right)b\right) \\ &= \frac{1}{4}\left(baB_r - (-1)^r bB_r a - (-1)^r aB_r b + B_r ab\right), \end{aligned}$$

from which it is easy to see that

$$\begin{aligned} a \cdot (b \wedge B_r) + b \wedge (a \cdot B_r) &= \frac{1}{4}(ab + ba)B_r + \frac{1}{4}B_r(ba + ab) \\ &= \frac{1}{2}(a \cdot b)B_r + \frac{1}{2}B_r(b \cdot a) = (a \cdot b)B_r. \end{aligned}$$

Similarly, we must note that for all $r > 1$, we have

$$a \cdot (b \cdot B_r) = -b \cdot (a \cdot B_r). \tag{4.8}$$

To see this, we apply equation (4.5) in writing

$$\begin{aligned} a \cdot (b \cdot B_r) &= \frac{1}{2}\left(a\frac{1}{2}\left(bB_r - (-1)^r B_r b\right) - (-1)^{r-1}\frac{1}{2}\left(bB_r - (-1)^r B_r b\right)a\right) \\ &= \frac{1}{4}\left(abB_r - (-1)^r aB_r b + (-1)^r bB_r a - B_r ba\right), \end{aligned}$$

Then, by substitution, we can immediately write

$$b \cdot (a \cdot B_r) = \frac{1}{4}\left(baB_r - (-1)^r bB_r a + (-1)^r aB_r b - B_r ab\right).$$

Adding these, we then see that

$$\begin{aligned} a \cdot (b \cdot B) + b \cdot (a \cdot B) &= \frac{1}{4}\left(abB_r + baB_r\right) - \frac{1}{4}\left(B_r ba + B_r ab\right) \\ &= \frac{1}{4}\left(ab + ba\right)B_r - \frac{1}{4}B_r\left(ba + ab\right) \\ &= \frac{1}{2}(a \cdot b)B_r - \frac{1}{2}B_r(b \cdot a) = 0. \end{aligned}$$

Note that we may have arrived at this conclusion sooner had we written

$$a \cdot (b \cdot B_r) = (a \wedge b) \cdot B_r = -(b \wedge a) \cdot B_r = -b \cdot (a \cdot B_r),$$

but the justification for some intermediate steps is not immediately clear.

4.2. Identities Involving Mercers

Letting M_r denote a mercer of rank r having factorization

$$M_r = \prod_{i=1}^{r} m_i, \tag{4.9}$$

recall that

$$M_r = \sum_{i=1}^{r} \langle M_r \rangle_i ,$$

where here we're making use of the angled-brackets notation $\langle \cdot \rangle_i$ which takes the grade i part of what it encloses. (Note that this requires us to visualize the expansion of the enclosure as a sum of blades.) To be more precise, if M_r is a mercer of even rank, (if r is even), then

$$M_r = \sum_{i=0}^{r/2} \langle M_r \rangle_{2i} , \tag{4.10}$$

while if M_r is a mercer of odd rank, we have

$$M_r = \sum_{i=1}^{(r+1)/2} \langle M_r \rangle_{2i-1} . \tag{4.11}$$

To see this, consider first the trivial case of $r = 0$; then, for any $r > 0$, the equation

$$M_r = M_{r-1} m_r = \langle M_{r-1} \rangle_1^r \cdot m_r + \langle M_{r-1} \rangle_1^r \wedge m_r + \langle M_{r-1} \rangle_0 m_r. \tag{4.12}$$

Here we have extended our notation $\langle \cdot \rangle_i^j$ to mean a culling of all enclosed blades not of a grade falling in the interval $[i, j]$.

An inductive hypothesis can now be stated that equations (4.10) and (4.11) hold for $r - 1$. If r is even, then, by our inductive hypothesis, M_{r-1}, when expanded as a sum of blades, consists only of blades of odd grade, and it is clear that equation (4.12) becomes (4.10). If r is odd, then, by our inductive hypothesis, M_{r-1}, when expanded as a sum of blades, consists only of blades of even grade, and it is clear that equation (4.12) becomes (4.11).

Now let a be a vector, and convince yourself that

$$a \cdot M_r = -(-1)^r M_r \cdot a, \tag{4.13}$$

$$a \wedge M_r = (-1)^r M_r \wedge a. \tag{4.14}$$

Refer to equations (4.3) and (4.4) to see this.

We now turn our attention to the following identity.

$$\langle M_r \rangle_0 = \langle M_{r-1} \rangle_1 \cdot m_r \tag{4.15}$$

Note that this is trivial in the case that r is odd, since neither M_r nor M_{r-1} have parts of grade zero nor one, respectively. Letting r be even, we write

$$M_r = M_{r-1} m_r = M_{r-1} \cdot m_r + M_{r-1} \wedge m_r - \langle M_r \rangle_0 m_r.$$

Now taking the grade zero part of both sides, we get

$$\langle M_r \rangle_0 = \langle M_{r-1} \cdot m_r \rangle_0 = \langle M_{r-1} \rangle_1 \cdot m_r.$$

References

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Spencer T. Parkin
e-mail: `spencerparkin@outlook.com`