# Untitled

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**Abstract.** Abstract...

**Keywords.** Key words...

**Definition 0.1 (Spade).** An element $M \in \mathbb{G}$ is a *spade* if it can be written as the geometric product of zero or more vectors.

It follows from Definition 0.1 that all versors are spades, but not all spades are versors. Furthermore, while the set of all spades of $\mathbb{G}$ enjoys closure under the geometric product, this set, unlike the set of versors of $\mathbb{G}$, does not form a group.

**Definition 0.2 (Spade Rank/Compact Spade Form).** The *rank* of a spade $M \in \mathbb{G}$, denoted $\text{rank}(M)$, is the smallest number of vectors for which $M$ can be written as a geometric product of such. If a spade $M_r \in \mathbb{G}$ has factorization

$$M_r = \prod_{i=1}^{r} m_i,$$

we say that it is written in *compact form* if $\text{rank}(M_r) = r$.

Many identities involving a spade $M_r$ hold whether or not it is written in compact form.

**Definition 0.3 (Dull/Sharp Spade).** A spade $M \in \mathbb{G}$ is *sharp* if the grade $\text{rank}(M)$ part of $M$ is non-zero, and *dull* otherwise.

If $M_r$ is a sharp blade written in compact form, it then follows that

$$\langle M_r \rangle_r = \bigwedge_{i=1}^{r} m_i.$$

**Definition 0.4 (Ideal Spade Form).** Letting $M_r^{(i)}$ denote the spade

$$M_r^{(i)} = \prod_{\substack{j=1 \\ j \neq i}}^{r} m_j,$$

a spade $M_r \in \mathbb{G}$ is written in *ideal form* when the set of $r$ spades $\{M_r^{(i)}\}_{i=1}^r$ is a linearly independent set.

Clearly, every spade can be written in compact form. At this point, however, it is not clear whether every spade can be written in ideal form. We will address this in Lemma **??**.

**Lemma 0.5.** *For every non-zero $r$-blade $B_r \in \mathbb{G}$, with $r > 1$, and having factorization*

$$B_r = \bigwedge_{i=1}^r b_i,$$

*the set of $(r-1)$-blades $\{B_r^{(i)}\}_{i=1}^r$, where*

$$B_r^{(i)} = \bigwedge_{\substack{j=1\\j\neq i}}^r b_j,$$

*is a linearly independent set.*

*Proof.* Supposing to the contrary, and without loss of generality, let

$$B_{r-1} = B_r^{(r)} = \sum_{i=1}^{r-1} \alpha_i B_r^{(i)} = \left(\sum_{i=1}^{r-1} \alpha_i B_{r-1}^{(i)}\right) \wedge b_r.$$

Now notice that

$$0 \neq B_r = B_{r-1} \wedge b_r = B_r^{(r)} \wedge b_r = \left(\sum_{i=1}^{r-1} \alpha_i B_r^{(i)}\right) \wedge b_r = 0,$$

which is clearly a contradiction. □

**Lemma 0.6.** *Given any spade $M_r$, the set of all solution sets $\{\alpha_i\}_{i=1}^r$ of the equation*

$$0 = \sum_{i=1}^r \alpha_i M_r^{(i)}$$

*is, for all integers $j \in [0, r]$, the intersection of all sets of solution sets of the equations*

$$0 = \sum_{i=1}^r \alpha_i \langle M_r^{(i)} \rangle_j.$$

*Proof.* This is a simple consequence of there being no possibility of cancelation between elements of differing grade. □

**Lemma 0.7.** *Every spade, sharp or dull, of rank $r > 1$, that is written in compact form is also written in ideal form.*

*Proof.* Supposing to the contrary, and without loss of generlaity, let

$$M_{r-1} = M_r^{(r)} = \sum_{i=1}^{r-1} \alpha_i M_r^{(i)}.$$

Now, by Lemma 0.6, we need only show that for some integer $j \in [0, r-1]$, there is no solution in each $\alpha_i$ to the equation

$$\langle M_{r-1} \rangle_j = \langle M_r^{(r)} \rangle_j = \sum_{i=1}^{r-1} \alpha_i \langle M_r^{(i)} \rangle_j.$$

In the case that $M_r$ is sharp, it follows that for each $i \in [1, r]$ that $M_r^{(i)}$ is sharp, and we reach the same contradiction here for $j = r-1$ as we did in Lemma 0.5.

Suppose now that $M_r$ is dull, but $M_{r-1}$ is sharp. This is very hard...

$\square$

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