On The Use Of Blades As Representatives Of Geometry

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1. Introduction And Motivation

In many models of geometry that are based upon geometric algebra (see []), blades are used to represent geometries. Seeing a great deal of commonality between these models, a formal treatment of this idea deserves to be given in an abstract setting in much the same way that, for example, abstract algebra provides such a setting in which algebraic sets generated by ideals of a polynomial ring can be studied. To the best of this author's knowledge, this is the first treatment of its kind.

To keep our discussion from becoming too pedantic, the finer details upon which the major results of this paper will depend are given in the last section. Readers needing more familiarity with geometric algebra may want to read that section first.

2. Enter The Geometric Set

To lay the foundation of our work, we introduce $\mathbb{V}$ as an m-dimensional vector space generating a geometric algebra denoted by $\mathbb{G}$. We leave the signature of this geometric algebra unspecified, but in cases where a proof depends upon signature, one is given as either euclidean or non-euclidean.[1] The scalars of $\mathbb{V}$, (and therefore of $\mathbb{G}$), are taken from the field $\mathbb{R}$ of real numbers.[2] We will

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[1] The Gram-Schmidt orthogonalization process is applicable to all blades taken from and only from a geometric algebra having no null-vectors. While many proofs are simplified under the assumption that an orthogonal basis can be chosen for any given blade, no such assumption is made in this paper for the sake of generality.

[2] To be more abstract, we could have used any field with characteristic 1, but there will be no foreseable advantage to doing so in this paper.

let $\mathbb{R}^n$ denote n-dimensional euclidean space,[3] and let $\mathbb{B}$ denote the set of all blades taken from $\mathbb{G}$. Lastly, we will let $p : \mathbb{R}^n \to \mathbb{V}$ be an unspecified, yet well-defined function that we'll use in the following definition and throughout the remainder of this paper.[4]

Definition 2.1 (Direct And Dual Representation). For any blade $B \in \mathbb{B}$, we say that B directly represents the set of all points $x \in \mathbb{R}^n$ such that $p(x) \wedge B = 0$, and say that B dually represents the set of all points $x \in \mathbb{R}^n$ such that $p(x) \cdot B = 0$. For convenience, we introduction the following functions using set-builder notation.

$$\hat{g}(B) = \{x \in \mathbb{R}^n | p(x) \wedge B = 0\}$$
$$\dot{g}(B) = \{x \in \mathbb{R}^n | p(x) \cdot B = 0\}$$

From Definition 2.1, it's important to take away the realization that a given blade $B \in \mathbb{B}$ represents two geometries simultaneously; namely, $\hat{g}(B)$ and $\dot{g}(B)$. Which geometry we choose to think of B as being a representative of at any given time is completely arbitrary.

It should also be clear from Definition 2.1 that the geometry represented by a blade B, (directly or dually), remains invariant under any non-zero scaling of the blade B. Something interesting happens, however, when we take the dual of B, as Lemma 2.2 will show.

Lemma 2.2 (Dual Relationship Between Representations). *For any subset S of $\mathbb{R}^n$, if there exists $B \in \mathbb{B}$ such that $\hat{g}(B) = S$, then $\dot{g}(BI) = S$, where I is the unit psuedo-scalar of $\mathbb{G}$. Similarly, if there exists $B \in \mathbb{B}$ such that $\dot{g}(B) = S$, then $\hat{g}(BI) = S$.*

Proof. The first of these two statements is proven by

$$0 = p(x) \wedge B = -(p(x) \cdot BI)I \iff p(x) \cdot BI = 0,$$

while the second is proven by

$$p(x) \cdot B = 0 \iff 0 = (p(x) \cdot B)I = p(x) \wedge BI.$$

(See identities (5.5) and (5.6) of Section 5.) □

In other words, Lemma 2.2 is telling us that for a single given geometry, the algebraic relationship between a blade directly (dually) representative of that geometry, and a blade dually (directly) representative of that geometry, is simply that, up to scale, they are duals of one another.

Of course, there will also be a geometric relationship between the geometry that is directly represented by a single given blade $B \in \mathbb{B}$, and the geometry that is dually represented by B, but this depends upon the definition of our function p, which we choose, in this paper, to leave open to speculation.

[3] Some models of geometry find affine space to be the natural space within which to work, but this will not be the case in this paper.

[4] By leaving p unspecified, we're abstracting away the definition of the function. We only care that it is a well-defined function. In some parts of this paper, we will consider the cases where p takes on some desirable properties.

With Lemma 2.2 in hand, geometric algebra's equivilant of an algebraic set may be given as follows.[5]

Definition 2.3 (Geometric Set). A subset $S \subset \mathbb{R}^n$ for which there exists a blade $B \in \mathbb{B}$ such that $\hat{g}(B) = S$ is what we'll refer to as a "geometric set."

By Lemma 2.2, it is easy to see that Definition 2.3 is equivilant to a version of itself that replaces $\hat{g}$ with $\dot{g}$. Therefore, for any geometric set S, we can claim the existence of a blade $B \in \mathbb{B}$ such that $\hat{g}(B) = S$ or $\dot{g}(B) = S$.

3. Properties Of Geometric Sets

Having now defined the notion of a geometric set, we now consider the properties of such sets.

Lemma 3.1 (The Intersection Of Two Geometric Sets Is A Geometric Set). *If $R, S \subset \mathbb{R}^n$ are a pair of geometric sets, then $R \cap S$ is also a geometric set.*

Proof. The case that $R = S$ is trivial. So we assume that $R \neq S$. Let $A, B \in \mathbb{B}$ be blades such that $\dot{g}(A) = R$ and $\dot{g}(B) = S$. Then, since $R \neq S$, we have $A \wedge B \neq 0$. (Not obvious! Prove it!) Go on... $\square$

4. Transforming Geometric Sets

5. Useful Identities And Lemmas

In this section we give a number of useful algebraic identities and results that would otherwise distract us from the flow of the paper if given in the main body.[6]

Letting $v \in \mathbb{V}$ and $B \in \mathbb{B}$, recall that

$$vB = v \cdot B + v \wedge B. \tag{5.1}$$

Also recall that

$$v \wedge B = \frac{1}{2}(vB + (-1)^s Bv), \tag{5.2}$$

$$v \cdot B = \frac{1}{2}(vB - (-1)^s Bv), \tag{5.3}$$

where $s = \text{grade}(B)$. Realizing that $\text{grade}(I) = m$, and that by (5.1), we have $vI = v \cdot I$, we can use equation (5.3) to establish the commutativity of vectors in $\mathbb{V}$ with the unit psuedo-scalar I as

$$vI = -(-1)^m Iv. \tag{5.4}$$

Using equation (5.4) in conjunection with equation (5.3), we find that

$$(v \cdot B)I = v \wedge BI. \tag{5.5}$$

[5]If p is defined appropriately, geometric sets are algebraic sets.

[6]This section is not intended as a complete or comprehensive review of geometric algebra. See [] for such a review.

(In verifying this identity, it helps to realize that for any integer k, we have $(-1)^k = (-1)^{-k}$.) Replacing B in equation (5.5) with BI, we find that

$$v \wedge B = -(v \cdot BI)I. \tag{5.6}$$

Recall that in the case that $v \wedge B$ vanishes, we can say that v is in the vector space spanned by any factorization of B. Furthermore, if B is an s-blade, at most s linearly independent vectors can be found where for each such vector v, we have $v \wedge B = 0$.

Lemma 5.1. *For any non-zero blade $C \in \mathbb{B}$, if $A, B \in \mathbb{B}$ are blades such that $C = A \wedge B$, then for all $v \in \mathbb{V}$ such that $v \wedge C = 0$, we have $v \wedge A = 0$ or $v \wedge B = 0$.*

Proof. It is clear for at least one $D \in \{A, B\}$, we have $v \wedge D = 0$. Suppose $v \wedge A = 0$ and $v \wedge B = 0$. Then there exist blades $A', B' \in \mathbb{B}$ such that $A = v \wedge A'$ and $B = v \wedge B'$. It then follows that $C = v \wedge A' \wedge v \wedge B' = 0$, which is a contradiction. We can now conclude that for exactly one $D \in \{A, B\}$, we have $v \wedge D = 0$. □

Lemma 5.2. *If $C \in \mathbb{B}$ is a non-zero blade of grade $t > 0$ and $\{c_i\}_{i=1}^t$ is a linearly independent set of t vectors such that for all $v \in \{c_i\}_{i=1}^t$, we have $v \wedge C = 0$, then there exists $\lambda \in \mathbb{R}$ such that*

$$C = \lambda \bigwedge_{i=1}^{t} c_i.$$

Proof. The case of $t = 1$ is trivial. We now proceed by strong induction on $t > 1$. Let $A, B \in \mathbb{B}$ be blades of positive grades r and s, respectively, such that $C = A \wedge B$. Then, by Lemma **??**, we can partition the set $\{c_i\}_{i=1}^t$ into two subsets $\{a_i\}_{i=1}^r$ and $\{b_i\}_{i=1}^s$, where for all integers $1 \leq i \leq r$, we have $a_i \wedge A = 0$, and for all integers $1 \leq i \leq s$, we have $b_i \wedge B = 0$. Then, by our inductive hypothesis, there exist scalars $\alpha, \beta \in \mathbb{R}$ such that $A = \alpha \bigwedge_{i=1}^r a_i$ and $B = \beta \bigwedge_{i=1}^s b_i$. Letting $\lambda = \alpha\beta$, we now have

$$C = \lambda A \wedge B = \lambda \bigwedge_{i=1}^{t} c_i,$$

which completes the proof by induction. □

Returning to the product $v \cdot B$, an alternative expansion is given by

$$v \cdot B = -\sum_{i=1}^{s} (-1)^i (v \cdot b_i) B_i,$$

where B is factored as $\bigwedge_{i=1}^s b_i$, and we define B_i as

$$B_i = \bigwedge_{j=1, j \neq i}^{s} b_i.$$

This leads to the following recursive formulation.

$$v \cdot B = (v \cdot b_1)B_1 - b_1 \wedge (v \cdot B_1)$$

If a blade $A \in \mathbb{B}$ has grade r and factorization $\bigwedge_{i=1}^{r} a_i$, then we can express the product $A \cdot B$ recursively as

$$A \cdot B = \begin{cases} A_r \cdot (a_r \cdot B) & \text{if } r \leq s, \\ (A \cdot b_1) \cdot B_1 & \text{if } r \geq s. \end{cases}$$

Interestingly, though it is not at all obvious from equation (**??**), the product $v \cdot B$ is a blade. It is clearly homogeneous of grade $s-1$, but it is not immediately clear that it is a blade. To see that it is a blade, let $\beta = \prod_{i=1}^{s} v \cdot b_i$, and let β_i be given by

$$\beta_i = \prod_{j=1, j \neq i} v \cdot b_j.$$

(If $\beta = 0$, the following argument can be reduced to a smaller, equivilant case; so we assume $\beta \neq 0$.) Then, letting $c_i = \beta_1 b_1 - (-1)^i \beta_i b_i$, notice that for all integers $1 < i \leq s$, we have

$$c_i \wedge (v \cdot B) = \beta B - \beta B = 0.$$

Seeing now that the linear independence of the set of $s-1$ vectors $\{c_i\}_{i=2}^{s}$ follows from that of the set of s vectors $\{b_i\}_{i=1}^{s}$, we can invoke Lemma **??** in claiming that for some non-zero scalar $\lambda \in \mathbb{R}$, we have

$$v \cdot B = \lambda \bigwedge_{i=2}^{s} c_i,$$

showing that $v \cdot B$ is indeed a blade of grade $s-1$.

Lemma 5.3. *For a non-zero s-blade B factored as $\bigwedge_{i=1}^{s} b_i$, the set of $(s-1)$-blades $\{B_i\}_{i=1}^{s}$ is linearly independent.*

Proof. It is clear that the set of 1-blades $\{b_i\}_{i=1}^{s}$ is linearly independent. Go on... □

Note that a much simpler proof of Lemma **??** could have been given under the assumption of a euclidean signature, in which case, the Gram-Schmidt orthogonalization process would have allowed us to assume, without loss of generality, an orthogonal factorization of the blade B.

References

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