CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 00, Pages 000–000 (Xxxx XX, XXXX)
S 1088-4173(XX)0000-0

A METHOD OF APPLYING CONFORMAL TRANSFORMATIONS USING GEOMETRIC ALGEBRA

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ABSTRACT. It is shown that any conformal transformation may be applied to a polynomial of degree m by first converting the polynomial to an m-vector taken from a geometric algebra, applying a versor to that m-vector, and then converting the resulting m-vector back into a polynomial.

1. Preliminaries

For this paper, we assume the reader is already familiar with geometric algebra and the conformal model of geometric algebra. (See [2] for introductory material on geometric algebra. See [2, 4] for material on conformal geometric algebra.) Despite what conventions may be used in other papers on geometric algebra, here we will let the outer product take precedence over the inner product, and the geometric product take precedence over the inner and outer products.

As there are different ways of defining the inner product for different purposes, we must take a moment here to define the inner product used in this paper. It is as follows. Among vectors, the inner product is a bilinear form defining the signature of our geometric algebra which will be given in the next section. For any vector v and k-blade A, we define

$$v \cdot A = -\sum_{i=1}^{k} (-1)^i (v \cdot a_i) A_i, \tag{1.1}$$

where here, the k-blade A may be factored as $A = \bigwedge_{i=1}^{k} a_i$, and for each integer $i \in [1, k]$, we let

$$A_i = \bigwedge_{\substack{j=1 \\ j \neq i}}^{k} a_j.$$

We let v commute with A as

$$v \cdot A = -(-1)^k A \cdot v.$$

It is sometimes convenient to rewrite equation (1.1) as

$$v \cdot A = (v \cdot a_1) A_1 - (v \cdot A_1) \wedge a_1,$$

which gives a recursive version of the definition. For the k-blade A and l-blade B, we define

$$A \cdot B = \begin{cases} A_k \cdot (a_k \cdot B) & \text{if } k \leq l, \\ (A \cdot b_1) \cdot B_1 & \text{if } k \geq l, \end{cases} \tag{1.2}$$

2010 *Mathematics Subject Classification.* Primary .

where here, the l-blade B may be factored as $B = \bigwedge_{i=1}^{l} b_i$, and for each integer $i \in [1,k]$, we let

$$B_i = \bigwedge_{\substack{j=1 \\ j\neq i}}^{l} b_j.$$

In the case that $k = l$, either part of the piece-wise function of equation (1.2) may be used to evaluate $A \cdot B$.

2. The Geometric Algebra

We begin by letting $\{\mathbb{G}_i\}_{i=1}^m$ be a sequence of m Minkowski geometric algebras upon which the conformal model of n-dimensional euclidean space may be imposed. Each $\mathbb{G}_i$ is the very geometric algebra found in [4]. If $\mathbb{V}_i$ is a vector space generating $\mathbb{G}_i$, then $\{e_{i,-}, e_{i,+}\} \cup \{e_{i,j}\}_{j=1}^n$ is a set of basis vectors generating $\mathbb{V}_i$, where $\{e_{i,j}\}_{j=1}^n$ is an orthonormal basis for an n-dimensional euclidean space, and each of $e_{i,-}$ and $e_{i,+}$, taken from [5], is given by

$$e_{i,-} = \frac{1}{2}\infty_i + o_i,$$
$$e_{i,+} = \frac{1}{2}\infty_i - o_i,$$

where here, o_i and ∞_i are, for each $\mathbb{G}_i$, the familiar null-vectors representative of the points at origin and infinity, respectively. With the exception of zero, we consider the sequence of geometric algebras $\{\mathbb{G}_i\}_{i=1}^m$ to be pair-wise disjoint.

Taking our cue from the "mother algebra" in [1], we are now interested in forming the smallest geometric algebra $\mathbb{G}$ containing each $\mathbb{G}_i$ as a geometric sub-algebra. This geometric algebra $\mathbb{G}$ is therefore generated by the vector space $\mathbb{V}$, given by

$$\mathbb{V} = \bigoplus_{i=1}^{m} \mathbb{V}_i.$$

We now introduce a function $\Psi_{i,j} : \mathbb{G} \to \mathbb{G}$ defined as

$$\Psi_{i,j}(E) = \left\{ \begin{array}{ll} S_{i,j,1} E (S_{i,j,1})^{-1} & \text{if } i \neq j, \\ E & \text{if } i = j \end{array} \right. ,$$

where $S_{i,j,k}$ is the constant given by

$$S_{i,j,k} = \left(1 - (-1)^k e_{i,-} e_{j,-}\right)\left(1 + (-1)^k e_{i,+} e_{j,+}\right) \prod_{r=1}^{n} \left(1 - (-1)^k e_{i,r} e_{j,r}\right). \tag{2.1}$$

Take notice that

$$(S_{i,j,1})^{-1} = 2^{-(n+2)} S_{i,j,0}.$$

Our definition of $\Psi_{i,j}$ is motivated by the fact that for any vector $v_i \in \mathbb{V}_i$ and its corresponding vector $v_j \in \mathbb{V}_j$, we have

$$v_j = \Psi_{i,j}(v_i).$$

Being in correspondence, this means that for all integers $k \in [1,n]$, we have

$$v_i \cdot e_{i,k} = v_j \cdot e_{j,k},$$

as well as

$$v_i \cdot o_i = v_j \cdot o_j,$$
$$v_i \cdot \infty_i = v_j \cdot \infty_j.$$

Notice that $\Psi_{i,j}(v_j) = -v_i$. For any vector $v \notin \mathbb{V}_i$ and $v \notin \mathbb{V}_j$, the function $\Psi_{i,j}$ leaves v invariant.

Though $S_{i,j,1}$ in equation (2.1) is not a versor of $\mathbb{G}$, we will now show that $\Psi_{i,j}$ is an outermorphism. (See [3] for a definition of outermorphism.) It suffices to show that for any two vectors $a, b \in \mathbb{V}$, we have

$$a \cdot b = \Psi_{i,j}(a) \cdot \Psi_{i,j}(b).$$

To see this, begin by rewriting a and b as

$$a = \sum_{k=1}^{m} a_k,$$
$$b = \sum_{k=1}^{m} b_k,$$

where for each pair (a_k, b_k), we have $a_k, b_k \in \mathbb{V}_k$. We then have

$$a \cdot b = \sum_{k=1}^{m} a_k \cdot b_k,$$

where we can make the observation that for any $k \in [1, m]$, we have

$$a_k \cdot b_k = \Psi_{i,j}(a_k) \cdot \Psi_{i,j}(b_k).$$

We now simply see that

$$\Psi_{i,j}(a) \cdot \Psi_{i,j}(b) = \sum_{k=1}^{m} \Psi_{i,j}(a_k) \cdot \Psi_{i,j}(b_k)$$

to complete the proof.

3. Geometric Representation

Having now set forth our geometric algebra $\mathbb{G}$, we're ready to discuss geometric representation. Letting $\mathbb{R}^n$ be the n-dimensional euclidean vector sub-space of $\mathbb{V}_1$, our geometric representation scheme is to let a geometry be the set of all points $x \in \mathbb{R}^n$, such that

$$\bigwedge_{j=1}^{m} p_j(x) \cdot A = 0,$$

where A is an m-vector (not necessarily an m-blade) of $\mathbb{G}$, and where the function $p_j : \mathbb{R}^n \to \mathbb{V}_1$, reminding us of the conformal mapping found in [4], is defined as

$$p_j(x) = \Psi_{1,j}\left(o_1 + x + \frac{1}{2}x^2\infty_1\right).$$

Here, it is the m-vector A that serves as the representative of our geometry. It is not hard to see that the point-set generated by A through equation (**??**) is simply the zero set of a polynomial of degree at most $2m$.[1]

For any polynomial of degree r with $m < r \leq 2m$, there does not necessarily exist an m-vector A representative of its zero set. For all polynomials of degree r with $0 \leq r \leq m$, however, there does exist such an m-vector A. Taking advantage of the linearity of the inner product, it suffices to show that this is the case for every monomial of such a degree. Indeed, the m-vector A, (in this case an m-blade), is given by

$$A = \lambda \bigwedge_{j=1}^{r} a_j \wedge \bigwedge_{j=r+1}^{m} \infty_j,$$

where $\lambda \in \mathbb{R}$ is a scalar, $\{a_j\}_{j=1}^{r}$ is a set of r vectors with each $a_j \in \mathbb{V}_j$, and the first product is one in the case that $r = 0$. To see this, let $x_j = \Psi_{1,j}(x)$ and write

$$\begin{aligned}\bigwedge_{j=1}^{m} p_j(x) \cdot A &= \bigwedge_{j=1}^{r} x_j \wedge \bigwedge_{j=r+1}^{m} o_j \cdot A \\ &= \lambda(-1)^{m-r} \bigwedge_{i=1}^{r} x_j \cdot \bigwedge_{i=1}^{r} a_j \\ &= \lambda(-1)^{m-r} \prod_{i=1}^{r} x_j \cdot a_j,\end{aligned}$$

which shows that for an appropriate choice of the vectors in $\{a_j\}_{j=1}^{r}$, and that of λ, we can formulate A as being representative of any monomial in n independent variables $\{x \cdot e_{1,j}\}_{j=1}^{n}$.

It is not difficult to convert between polynomial functions and m-vectors A of our geometric algebra $\mathbb{G}$. That is, not difficult if we are using a computer algebra system. Never-the-less, once implemented, we have a nice subroutine allowing us to convert between the two forms.

4. Applying Conformal Transformations To Geometries

We now come to the main result of this papers, which is to show that the conformal transformations are easily applicable to any geometry that may be represented as an m-vector by equation (**??**).[2] We simply observe that if $V \in \mathbb{G}_1$ is a versor representative of a conformal transformation, such as those that can be found in [2], then the desired geometry is given by the set of all points $x \in \mathbb{R}^n$, such that

$$\bigwedge_{j=1}^{m} V^{-1} p_j V \cdot A = 0.$$

It will not be hard to show that the element A' representative of this very set of points by equation (**??**) is given by

$$A' = WAW^{-1},$$

[1] An algebraic set is the zero set of one or more polynomial functions. It can be shown that such sets may be represented by the blades of a geometric algebra, but the geometric representation method of this paper is restricted to those sets that are the zero set of one and only one polynomial function.

[2] Again, by this we mean easy for a computer algebra system.

where the versor W is given by

$$W = \prod_{j=1}^{m} V_j,$$

where $V_j = \Psi_{1,j}(V)$. Taking advantage of the linearity of the inner product once again, we need only prove our main result in the case that A is an m-blade. Let $\{a_j\}_{j=1}^{m}$ be a set m vectors, such that

$$A = \bigwedge_{j=1}^{m} a_j.$$

We then have

$$\begin{aligned}
\bigwedge_{j=1}^{m} V^{-1} p_j V \cdot \bigwedge_{j=1}^{m} a_j &= \prod_{j=1}^{m} V^{-1} p_j V \cdot a_j \\
&= \prod_{j=1}^{m} p_j \cdot V a_j V^{-1} \\
&= \bigwedge_{j=1}^{m} p_j \cdot \bigwedge_{j=1}^{m} V a_j V^{-1} \\
&= \bigwedge_{j=1}^{m} p_j \cdot \bigwedge_{j=1}^{m} W a_j W^{-1} \\
&= \bigwedge_{j=1}^{m} p_j \cdot W A W^{-1}.
\end{aligned}$$

This completes the proof.

5. A Trial Implementation

Putting theory into practice, a computer algebra system with visualization capabilities was implemented and used to generate Figure **??**, Figure **??** and Figure **??**.

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