On The Use Of Blades As Representatives Of Geometry

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1. Introduction And Motivation

In many models of geometry based upon geometric algebra (see []), blades are used to represent geometries. Seeing a great deal of commonality between these models, a formal treatment of this idea deserves to be given in an abstract setting.[1] To the author's knowledge, this is the first treatment of its kind.

2. Foundation

To lay the foundation of our work, we introduce $\mathbb{V}$ as denoting an m-dimensional vector space generating a geometric algebra denoted by $\mathbb{G}$. We leave the signature of this geometric algebra unspecified, but in cases where a proof depends upon signature, one is given as either euclidean or non-euclidean.[2] The scalars of $\mathbb{V}$, (and therefore $\mathbb{G}$), are taken from the field of real numbers denoted by $\mathbb{R}$.[3] We will let $\mathbb{R}^n$ denote n-dimensional euclidean space,[4] and let $\mathbb{B}$ denote the set of all blades taken from $\mathbb{G}$. Lastly, we will let $p : \mathbb{R}^n \to \mathbb{V}$ be an unspecified function we'll use in the following definition which launches us forth into a general theory of models of geometry based upon blades in

[1]For example, the use of ideals of polynomial rings as representatives of algebraic sets is studied in the setting of abstract algebra.

[2]The Gram-Schmidt orthogonalization process is applicable to all blades taken from and only from geometric algebras having euclidean signatures.

[3]To be more abstract, we could have used any field with characteristic 1, but there will be no foreseable advantage to doing so in this paper.

[4]Some models of geometry find affine space to be the natural space within which to work, but this will not be the case in this paper.

a geometric algebra. In our abstract setting, the definition of this function does not matter. All that matters is that it is a function.

Definition 2.1 (Direct And Dual Representation). For any blade $B \in \mathbb{B}$, we say that B directly represents the set of all points $x \in \mathbb{R}^n$ such that $p(x) \wedge B = 0$, and say that B dually represents the set of all points $x \in \mathbb{R}^n$ such that $p(x) \cdot B = 0$. For convenience, we introduction the following functions using set-builder notation.

$$\hat{g}(B) = \{x \in \mathbb{R}^n | p(x) \wedge B = 0\}$$
$$\dot{g}(B) = \{x \in \mathbb{R}^n | p(x) \cdot B = 0\}$$

From Definition 2.1, it's important to take away the realization that a given blade $B \in \mathbb{B}$ represents two geometries simultaneously; namely, $\hat{g}(B)$ and $\dot{g}(B)$. Which geometry we choose to think of B as being a representative of at any given time is completely arbitrary.[5]

It should also be clear from Definition 2.1 that the geometry represented by a blade B, (directly or dually), remains invariant under any non-zero scaling of the blade B. Something interesting happens, however, when we take the dual of B, as our first lemma shows.

Lemma 2.2 (Dual Something). *For any subset S of $\mathbb{R}^n$, if there exists $B \in \mathbb{B}$ such that $\hat{g}(B) = S$, then $\dot{g}(BI) = S$, where I is the unit psuedo-scalar of $\mathbb{G}$. Similarly, if there exists $B \in \mathbb{B}$ such that $\dot{g}(B) = S$, then $\hat{g}(BI) = S$.*

Proof. The first of these two statements is proven by

$$0 = p(x) \wedge B = -(p(x) \cdot BI)I \iff p(x) \cdot BI = 0,$$

while the second is proven by

$$p(x) \cdot B = 0 \iff 0 = (p(x) \cdot B)I = p(x) \wedge BI.$$

See identities (3.5) and (3.6) of Section 3. □

In words, Lemma 2.2 is telling us that for a single given geometry, the algebraic relationship between a blade directly (dually) representative of that geometry, and a blade dually (directly) representative of that geometry, is simply that, up to scale, they are duals of one another.

Of course, there will also be a geometric relationship between the geometry that is directly represented by a single given blade $B \in \mathbb{B}$, and the geometry that is dually represented by B, but this depends upon the definition of our function p, which we choose, in this paper, to leave open to speculation.

[5]In some literature on geometric algebra, a blade B intended to represent some peice of geometry directly or dually is referred to as a "geometry" or a "dual geometry," respectively. This is confusing and not practiced in this paper. A blade is a blade; and when we refer to geometry, we will use proper language in identifying what represents it and how it does so. In this paper, a geometry is a subset of $\mathbb{R}^n$ that can be represented dually or directly by some blade $B \in \mathbb{B}$ under Definition 2.1.

3. Useful Identities

In this section we give a number of useful algebraic identities that would otherwise distract us from the flow of the paper if given in the main body. This section is not intended as a complete review of geometric algebra. See [] for such a review.

Letting $v \in \mathbb{V}$ and $B \in \mathbb{B}$, recall that

$$vB = v \cdot B + v \wedge B. \tag{3.1}$$

Also recall that

$$v \wedge B = \frac{1}{2}(vB + (-1)^{\text{grade}(B)}Bv), \tag{3.2}$$

$$v \cdot B = \frac{1}{2}(vB - (-1)^{\text{grade}(B)}Bv). \tag{3.3}$$

Realizing that grade$(I) = m$, and that by (3.1), we have $vI = v \cdot I$, we can use equation (3.3) to establish the commutativity of vectors in $\mathbb{V}$ with the unit psuedo-scalar I as

$$vI = -(-1)^m Iv. \tag{3.4}$$

Using equation (3.4) in conjunection with equation (3.3), we find that

$$(v \cdot B)I = v \wedge BI. \tag{3.5}$$

(In verifying this identity, it helps to realize that for any integer k, $(-1)^k = (-1)^{-k}$.) Replacing B in equation (3.5) with BI, we find that

$$v \wedge B = -(v \cdot BI)I. \tag{3.6}$$

References

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