Section 2.7 Exercises
Hertein's Topics In Algebra

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Problem 17

Let G be the group of real numbers under addition and let N be the subgroup of G consisting of all the integers. Prove that G/N is isomorphic to the group of all complex numbers of absolute vlaue 1 under multiplication.

Let $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be defined by

$$\phi(\mathbb{Z} + r) = \exp(2\pi ri).$$

Now for any $a, b \in \mathbb{R}$, we have

$$\mathbb{Z} + a = \mathbb{Z} + b \iff a = b + z,$$

for some integer $z \in \mathbb{Z}$. We then have

$$\exp(2\pi ai) = \exp(2\pi(b+z)i) = \exp(2\pi bi)\exp(2\pi zi) = \exp(2\pi bi).$$

Thus far we have shown that ϕ is well-defined and onto-to-one. Clearly ϕ is onto $\mathbb{C}$. Is ϕ operation preserving?

$$\begin{aligned}\phi(\mathbb{Z} + a + \mathbb{Z} + b) &= \phi(\mathbb{Z} + a + b) = \exp(2\pi(a+b)i) \\ &= \exp(2\pi ai)\exp(2\pi bi) = \phi(\mathbb{Z} + a)\phi(\mathbb{Z} + b)\end{aligned}$$