# Section 2.6 Exercises
# Hertein's Topics In Algebra

Spencer T. Parkin

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## Thoughts

Remembering Gallian's book, he shows that the operation Herstein introduces here is well-defined. Let $N$ be a normal subgroup of $G$ and define, for any $a, b \in G$,

$$(Na)(Nb) = N(ab).$$

Is this a well-defined operation? Well, $Na = Nb$ if and only if $ab^{-1} \in N$. So let $a', b' \in G$ such that $Na = Na'$ and $Nb = Nb'$ and write

$$(Na')(Nb') = N(a'b').$$

Can we show that

$$ab(b')^{-1}(a')^{-1} = ab(a'b')^{-1} \in N?$$

Well, clearly $n_b = b(b')^{-1} \in N$. Now since $n_a = a(a')^{-1} \in N$, we have

$$ab(b')^{-1}(a')^{-1} = an_b(a')^{-1} = n_a[(a')n_b(a')^{-1}] \in N,$$

since $N$ is normal.

## Problem 9

Suppse $H$ is the only subgroup of order $|H|$ in the finite group $G$. Prove that $H$ is a normal subgroup of $G$.

This would follow from proving the following statement. If $\{H_i\}$ is a finite set of subgroups of $G$, each of order $n$, then for all $g \in G$, and every integer $i$, there exists an integer $j$, such that

$$gH_i = H_j g.$$

Interestingly, this presents the idea of two subgroups of $G$ being co-normal. Neither is necessarily normal by themselves, but together, they're co-normal.

Here's an idea. Let $H$ be a subroup of $G$. Then, for any $g \in G$, let $K$ be the set given by

$$K = \{g^{-1}hg | h \in H\}.$$

It is clear that $gK = Hg$. We now show, whether or not $H$ is a normal subgroup of $G$, that $K$ is a subgroup of $G$ having the same order as $H$.

Clearly $e \in K$, so $K$ is non-empty. Closure is trivial, for

$$(g^{-1}h_1g)(g^{-1}h_2g) = g^{-1}h_1h_2g \in K$$

since $h_1h_2 \in H$. And then

$$(g^{-1}hg)^{-1} = g^{-1}h^{-1}g \in K$$

since $h^{-1} \in H$.

To show now that $|K| = |H|$, let $\phi_g(h) = g^{-1}hg$ and write

$$g^{-1}h_1g = g^1h_2g \implies h_1 = h_2,$$

showing that $\phi_g$ is one-to-one. Then since $H$ is finite, $\phi_g$ is also onto $K$. It follows that $|K| = |H|$.

Returning to the original problem, we see that $H$ must be normal in $G$, because $H$ is the only subgroup of $G$ of its order. (That is, we must have $K = H$.)

Now, if two subgroups of co-normal, can we make a group out of the set of cosets shared between the two subgroups? I don't see how. There are two identity elements.