Versors That Give Non-Uniform Scale

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To my dear wife Melinda.

Abstract. It is shown that for every non-uniform scale transformation, a versor exists in a geometric algebra that can perform this transformation on any algebraic surface. Some implications and generalizations of this find are discussed.

Keywords. Algebraic Surface, Conformal Model, Non-Uniform Scale, Geometric Algebra.

1. Motivation

The question of existence of non-uniform scale versors is one of the outstanding problems of geometric algebra, and one that stands in the way of geometric algebra competing against existing and well-proven transformation models. As noted in the beginning of [5], 4×4 matrices have long-time been a standard in computer graphics for representing affine and projective transformations, but an equivalent yet hopefully more capable and universally compatible model for such transformations in a more modern setting has yet to emerge as a considerable replacement. This paper does not purport to provide such a setting, but it does offer a potential solution to the non-uniform scale problem. An upcoming paper by an author of [4] may provide an even better solution.

2. The Result

The result[1] of this paper is simply a corollary to that of [6], but to see how, we must first constrain the way in which we represent n-dimensional algebraic surfaces of up to degree m in the Mother Minkowski algebra of order m.[2] What we do is let $n \leq m$, and reserve certain subalgebras of our

[1] To understand this paper, the reader must be familiar with [6].

[2] Recall that such representations are not unique, and so we have the flexibility to choose our representations carefully.

mother algebra for use in specific dimensions. To see what is meant by this, let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial in whose zero set we are interested. Now define, for any integer $k \in [1, n]$, the polynomial $f_k : \mathbb{R} \to \mathbb{R}$ as

$$f_k(\lambda) = f(e_1 + e_2 + \cdots + \lambda e_k + \cdots + e_{n-1} + e_n).$$

Having done this, we will represent the surface of f in the Mother Minkowski algebra of order

$$m = \sum_{k=1}^{n} \deg f_k.$$

Now if $\mathbb{G}$ denotes our mother algebra and it is generated by m subalgebras $\mathbb{G}_i$, each generated by the vector space $\mathbb{V}_i$, then we reserve $\deg f_k$ of these subalgebras for use in dimension k of our n dimensions. (We will let $\mathbb{G}^k$, where k is an integer in $[1, n]$, denote the smallest subalgebra of $\mathbb{G}$ containing all subalgebras $\mathbb{G}_i$ reserved for dimension k, and let $[\mathbb{G}^k]$ denote the set of indices over which $\mathbb{G}_i \subseteq \mathbb{G}^k$.)

An example may be warrented at this point. Let $n = 3$ and consider the polynomial given by

$$f(x) = 3x_1^2 x_2 x_3^4 + 4x_1 x_2^3 - 7x_3^2, \tag{2.1}$$

where x_k is notation for $x_k = x \cdot e_k$. We will represent the surface that is the zero set of this polynomial using an m-vector in a Mother Minkowski algebra of order $m = 2+3+4 = 9$. The first 2 subalgebras are reserved for dimension 1, the next 3 for dimension 2, and the last 4 for dimension 3. The m-vector B representing this surface is then given by

$$\begin{aligned} B &= 3e_{12,1} \wedge e_{3,2} \wedge \infty_{45} \wedge e_{6789,3} \\ &\quad - 4e_{1,1} \wedge \infty_2 \wedge e_{345,2} \wedge \infty_{6789} \\ &\quad + 7\infty_{12345} \wedge e_{67,3} \wedge \infty_{89}. \end{aligned} \tag{2.2}$$

Here, notation is a challenge. The vector $e_{i,j}$ denotes the j^{th} euclidean basis vector in the i^{th} subalgebra. We then define

$$e_{i_1 i_2 \ldots i_r, j} = e_{i_1,j} \wedge e_{i_2,j} \wedge \cdots \wedge e_{i_r,j}.$$

The notation for ∞ is similar.

We can now say that the zero set of f in equation (2.1) is given by the set of all solutions to the equation

$$\bigwedge_{k=1}^{m} p_k(x) \cdot B = 0. \tag{2.3}$$

Recall that $p_k(x) = o_k + x_k + \frac{1}{2}x^2\infty_k$. Of course, we could have represented f in a mother algebra of order $\deg f = 7$, but it will soon become clear why we needed our algebra $\mathbb{G}$ to be of order $m = 9$.

Before moving on, notice that $[\mathbb{G}^1] = \{1, 2\}$, $[\mathbb{G}^2] = \{3, 4, 5\}$ and $[\mathbb{G}^3] = \{6, 7, 8, 9\}$.

Returning from the example, suppose now we have an m-vector B representative of any polynomial $f : \mathbb{R}^n \to \mathbb{R}$ under the constraint thus illustrated.

Seeing that the zero set of f is the set of solutions to equation (2.3), we make the simple observation that if D is a versor taken from a subalgebra $\mathbb{G}^k$, and further, D is the product of the same origin-centered dilation versor D_i found in each subalgebra $\mathbb{G}_i$ contained in $\mathbb{G}^k$, (see [4] for an explanation of dilation versors), which is to say that $D = \prod_{i\in[\mathbb{G}^k]} D_i$, then the non-uniform scale of f in the dimension of k by the scale of each D_i is given by the set of solutions to the equation

$$\begin{aligned}&\bigwedge_{i\notin[\mathbb{G}^k]} p_i(x) \wedge \bigwedge_{i\in[\mathbb{G}^k]} D_i^{-1} p_i(x) D_i \cdot B \\ =& \bigwedge_{i\notin[\mathbb{G}^k]} p_i(x) \wedge D^{-1}\left(\bigwedge_{i\in[\mathbb{G}^k]} p_i(x)\right) D \cdot B = 0. \end{aligned} \tag{2.4}$$

Now realize that for all $i \notin [\mathbb{G}^k]$, D leaves $p_i(x)$ invariant. That is,

$$D^{-1} p_i(x) D = p_i(x).$$

It now follows by equations (3.2) through (3.5) of [6] that equation (2.4) may be rewritten as

$$\bigwedge_{k=1}^{m} p_k(x) \cdot DBD^{-1},$$

showing that D, when applied to B, performs a non-uniform scaling of the surface of f.

Putting this result into practice, let us apply a non-uniform scale transformation to the polynomial in equation (2.1). Suppose we wish to scale the surface represented by this polynomial by a factor of 2 in the e_2 dimension. Letting $D_k = (o_k - \infty_k)\left(o_k - \frac{1}{2}\infty_k\right)$, the non-uniform scale versor D we want is $D = D_3 D_4 D_5$. Then, using B in equation (2.2), we find that

$$\begin{aligned}\frac{1}{2^3} DBD^{-1} &= \frac{3}{2} e_{12,1} \wedge \infty_{45} \wedge e_{6789,3} \\ &- \frac{4}{2^3} e_{1,1} \wedge \infty_2 \wedge e_{345,2} \wedge \infty_{6789} \\ &+ 7\infty_{12345} \wedge e_{67,3} \wedge \infty_{89}, \end{aligned} \tag{2.5}$$

which is just what we would hope to get when checking this against the polynomial $f\left(x_1 e_1 + \frac{1}{2} x_2 e_2 + x_3 e_3\right)$.[3] Notice that the $1/2^3$ factor on the left-hand side homogenizes DBD^{-1}.

3. Closing Remarks

Though we have now shown that versors performing the non-unform scale operation exist, seeing that their application requires a great deal of cumbersome convention and notation, a question of their practicality immediately

[3]The present author used symbolic computation software to make the calculatin in equation (2.5). This software can be found at `https://github.com/spencerparkin/GAVisTool`, though it is not recommended for general use.

arises. It's certainly not practical on paper, but perhaps such versors may find applications on the computer.

Another immediate observation we can make about the result of this paper is that it easily generalizes to the idea of a non-uniform "X", where "X" may be replaced here by any one of the conformal transformations. Since all such transformations may be decomposed as one or more reflections and spherical inversions, the two fundamental transformations to consider here are non-uniform reflections and non-uniform inversions. Considering the former for a moment, it's possible that in some cases these are shears. Then, since reflections give us rotations, it's not surprising that shears may be a bit more fundamental than rotations as hinted at by the clever paper [1].

In any case, it might now be possible to show that any affine transformation has an associated versor in our mother algebra that performs this transformation on an algebraic surface. This would be interesting, because the set of all algebraic surfaces of a certain degree are classified by defining an equivalence relation on this set which states that two surfaces, (in our case, m-vectors), are equivalent if and only if there exists an affine transformation, (in our case, an inner automorphism of the versor group of our mother algebra), that takes one of these surfaces to the other. The versor that would take any algebraic surface to the principle representative of its equivalence class would represent an important transformation, the decomposition of which would probably reveal the defining characteristics of the surface. In any event, it is not at all clear whether geometric algebra is the right tool for studying such equivalence classes. Showing any of this is well beyond the present author's capabilities.

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