Chapters 5-8 Supplementary Exercises
Gallian's Book on Abstract Algebra

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Problem 1

A subgroup N of a group G is called a *chracteristic subgroup* if $\phi(N) = N$ for all automorphisms ϕ of G. Prove that every subgroup of a cyclic group is characteristic.

Let ϕ be an automorphism of G, a cyclic group. Let a be an element of G. Now see that

$$\phi(\langle a \rangle) = \{\phi(a^k) | k \in \mathbb{Z}\} = \{\phi^k(a) | k \in \mathbb{Z}\} = \langle \phi(a) \rangle.$$

Clearly, $|a| = |\phi(a)|$, so $|\langle a \rangle| = |\langle \phi(a) \rangle|$. We can now claim that $\langle a \rangle = \langle \phi(a) \rangle$ by the fundamental theorem of cyclic groups, because G has one and only one subgroup of each possible order.

Problem 2

Prove that the center of a group is characteristic.

Let ϕ be any automorphism of a group G. If $Z(G) = G$, then $Z(G)$ is trivially characteristic. So assume that $Z(G)$ is a proper subgroup. Letting a be an element in $\phi(Z(G))$ and g an element in G, there must exist an element $a' \in Z(G)$ and an element $g' \in G$ such that $\phi(a') = a$ and $\phi(g') = g$. It then follows that

$$ag = \phi(a')\phi(g') = \phi(a'g') = \phi(g'a') = \phi(g')\phi(a') = ag,$$

showing that $a \in Z(G)$. Thus far we have shown that $\phi(Z(G)) \subseteq Z(G)$.

At this point, if $Z(G)$ is finite, then we could argue that because ϕ is one-to-one, $\phi(Z(G))$ cannot be a proper subset of $Z(G)$, and therefore, we must have $\phi(Z(G)) = Z(G)$. But $Z(G)$ may be infinite.

In any case, let $z \in G$ such that $\phi^{-1}(z) \in G - Z(G)$. Suppose now that $z \in Z(G)$. Then for all $g \in G$, we have

$$\phi^{-1}(z)\phi^{-1}(g) = \phi^{-1}(zg) = \phi^{-1}(gz) = \phi^{-1}(g)\phi^{-1}(z),$$

showing that $\phi^{-1}(z) \in Z(G)$, (since ϕ^{-1} is onto G), which is a contradiction. Therefore, $\phi^{-1}(z) \notin Z(G) \implies z \notin Z(G)$. It follows that

$$z \in Z(G) \implies \phi^{-1}(z) \in Z(G) \implies z \in \phi(Z(G)),$$

and we have $Z(G) \subseteq \phi(Z(G))$.

Problem 4

Prove that the property of being a characterstic subgroup is transitive. That is, if N is a characteristic subgroup of K and K is a characteristic subgroup of G, then N is a characteristic subgroup of G.

Let $\phi \in \mathrm{Aut}(G)$. If $\phi(K) = K$, then ϕ, when restricted in domain to K, is an automorphism of K. It follows that $\phi(N) = N$, showing that N is a characteristic subgroup of G.

Problem 6

Let H and K be subgroups of a group G and let $HK = \{hk|h \in H, k \in K\}$ and $KH = \{kh|k \in K, h \in H\}$. Prove that HK is a group if and only if $HK = KH$.

Suppose $HK = KH$. Clearly $e \in HK$. Let $a, b \in HK$. Then there exists $h, h' \in H$ and $k, k' \in K$ such that $a = hk$ and $b = h'k'$, and we have

$$ab^{-1} = hk(h'k')^{-1} = hk(k')^{-1}(h')^{-1} = hh''k'' \in HK,$$

for some element $h'' \in H$ and another $k'' \in K$, because $HK = KH$.

Now suppose HK is a subgroup of G. If $a \in HK$, then $a^{-1} = hk$ for some $h \in H$ and $k \in K$. It follows that $a = k^{-1}h^{-1} \in KH$. If $a \in KH$, then $a = hk$ for some $h \in H$ and $k \in K$. It follows that $a^{-1} = h^{-1}k^{-1} \in HK \implies (a^{-1})^{-1} = a \in HK$.

Problem 7

Let H and K be subgroups of a finite group G. Prove that

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

Consider the group $H \oplus K$, and define an equivilance relation on it as follows. For all $a, b \in H \oplus K$, let $a \sim b$ if and only if $a_h a_k = b_h b_k$, where $a = (a_h, a_k)$ and $b = (b_h, b_k)$. It is not hard to see that this is an equivilance relation on $H \oplus K$ that partitions it into $|HK|$ distinct equivilance classes. Now consider $[(h, k)]$, the equivilance class containing (h, k). If $(h', k') \in [(h, k)]$, then $hk = h'k' \implies (h')^{-1}h = k'k^{-1} = x \in H \cap K$, showing that

$$[(h, k)] = \{(hx^{-1}, xk) | x \in H \cap K\}.$$

Furthermore, for any $x, y \in H \cap K$, if $x \neq y$, then $(hx^{-1}, xk) \neq (hy^{-1}, yk)$, showing that $|[(h, k)]| = |H \cap K|$. It now follows that

$$|H||K| = |H \oplus K| = |HK||H \cap K|.$$

Progblem 50

Suppose that H and K are subgroups of a group and that $|H|$ and $|K|$ are relatively prime. Show that $H \cap K = \{e\}$.

Let $a \in H \cap K$. Then $|a|$ divides $|H \cap K|$, but since $|H \cap K|$ divides $|H|$ and $|K|$, we must have $|a|$ dividing $|H|$ and $|K|$. But if $\gcd(|H|, |K|) = 1$, then we must have $|a| = 1 \implies a = e$.