

Notes On Euler Angles

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The Application Of Euler Angles

Let R_x , R_y and R_z each be unit-rotors that rotate about the x , y and z axes, respectively. Then it is easy to imagine the rotation generated by the rotor $R_z R_y R_x$. A point is rotated about the x -axis, then the y , and then the z . Each rotation is about an axis from the same fixed coordinate frame. We can similarly deduce the effect of the rotation generated by the rotor $R_x R_y R_z$. However, like $R_z R_y R_x$, we are also free to think of the rotation generated by $R_x R_y R_z$ in the order of first rotating about the x -axis, then a (not the) y -axis, and then a (not the) z -axis, if, to be a bit more specific, we think about such a y -axis as a rotation of the y -axis. Similarly, we must think about such a z -axis as a rotation of the z -axis. This can all be explained algebraically as follows. Consider a vector v to be rotated by $R_x R_y R_z$. Instead of applying R_z first, which is what we would naturally want to do to get the rotation v' of v as

$$v' = R_x R_y R_z v \tilde{R}_z \tilde{R}_y \tilde{R}_x, \quad (1)$$

we will apply R_x first, and write

$$v' = R(R_x R_y \tilde{R}_x) R_x v \tilde{R}_x (R_x \tilde{R}_y \tilde{R}_x) \tilde{R}, \quad (2)$$

where the rotor R is given by

$$R = ((R_x R_y \tilde{R}_x) R_x) R_z ((R_x R_y \tilde{R}_x) R_x)^{-1}.$$

Here we must realize that we can apply rotors to rotors to rotate the axis of rotation about which a rotor rotates. (Pardon the tongue twister.) The pattern here is seen as an application of R_x first, then the application of

R_y with what was first applied to v , (namely R_y), applied to it, then the application of R_z with what was then applied to v , (namely R), applied to it.

Visualizing the rotation of the vector v in equation (2), we see that with each rotation, we must also apply that rotation to the xyz -axis frame about which we do subsequent rotations. Indeed, this is how Euler rotations are applied. Interestingly, what we find is that when we simplify equation (2), we arrive at equation (1). It follows now that the application of the Euler angles first in the x , then y , then z -axis, is accomplished by a rotor of the form $R_x R_y R_z$.

Euler Angle Decomposition

In this section we concern ourselves with the extrapolation of a set of Euler angles for a given rotation axis order from a given unit-rotor R . If one were to take a purely algebraic approach to this problem, they would quickly arrive at an undetermined system of non-linear equations, which seems quite intractable. A different and perhaps better approach is to take what we know about the application of Euler angles to come up with a solution to the problem. Our approach here will be in terms of visualizing the original and final coordinate frames, and to work by using the first two Euler angles to align a single axis. The remaining Euler angle will then rotate the two frames into complete alignment. Clearly the axis we choose to align must coincide with the same axis about which the final Euler angle will rotate.

We begin by making sure we have a well defined idea about what it means to rotate about an axis by an angle. Given an axis a (a unit-length vector) and an angle θ , we will define the unit-rotor $R(a, \theta)$ as

$$R(a, \theta) = \exp\left(-\frac{\theta}{2}aI\right) = \cos\left(\frac{\theta}{2}\right) - aI \sin\left(\frac{\theta}{2}\right),$$

and its application to a vector v to get v' as

$$v' = Rv\tilde{R},$$

where I is the unit-pseudo scalar. Defined this way, if the axis vector a is pointing towards us, we can imagine positive angles of rotation as moving points counter-clock-wise. It is important to realize these fine details if we are to succeed in the Euler decomposition of a rotor.

Having now a handle on rotations about arbitrary axes, let R be the unit-rotor we wish to decompose into a set of Euler angles θ_x , θ_y and θ_z , applied in that order. It then becomes clear that our first order of business is to find the angle θ_x such that e_2 (the y -axis) is rotated by an amount sufficient to align it orthogonal to $e'_1 = Re_1\tilde{R}$. This is in preparation for the application of the angle θ_y , which will align the rotation of e_3 by θ_x about e_1 into e_3 . (We will let $e'_2 = Re_2\tilde{R}$ and $e'_3 = Re_3\tilde{R}$.) Setting up our equation, we have

$$R(e_1, \theta_x)e_2\tilde{R}(e_1, \theta_x) \cdot e'_1 = 0.$$

(No, we need to get e_2 orthogonal to e'_3 . This is wrong.)

Simplifying this with the eventual use of the double-angle formulas, we get

$$(e'_1 \cdot e_2) \cos \theta_x = -(e'_1 \cdot e_3) \sin \theta_x.$$

Once a solution to this can be found, we then realize that

$$\theta_y = \pm \cos^{-1}(e'_3 \cdot e_3),$$

the sign of which is determined by the sign of the pseudo-scalar $e'_3 \wedge e_3 \wedge e_2$. We then find that

$$\theta_z = \pm \cos^{-1}(e'_2 \cdot e_2),$$

the sign of which is determined by the sign of the pseudo-scalar $e'_2 \wedge e_2 \wedge e_3$.

Now, there are quite a few cases still to consider in all of these steps, but this is the basic path of reasoning. It should also be noted that Euler angles are not unique, even if we constrain all angles to be within the range $[0, 2\pi]$.