Chapters 9-11 Supplementary Exercises
Gallian's Book on Abstract Algebra

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Exercise 36

A proper subgroup H of a group G is called *maximal* if there is no subgroup K such that $H \subset K \subset G$. Prove that Q under addition has no maximal subgroups.

Let H be any non-trivial, proper subgroup of Q. Then, if $h \in Q$ is also a member of H, then so is any integer multiple of h. In other words,

$$hZ = \{zh | z \in Z\} \subseteq H.$$

Notice that hZ is also a subgroup of H.

Now, since H is a proper subgroup of Q, there exists $r \in Q \backslash H$. If we wanted to form a subgroup of Q containing H and r, then it must contain at least H and rZ. Letting $H + rZ$ denote the set

$$H + rZ = \{h + zr | h \in H, z \in Z\},$$

it is not hard to show that $H + rZ$ is a subgroup of Q properly containing H. What remains to be shown, however, is that $H + rZ$ is a proper subgroup of Q.

To that end, suppose $Q = H + rZ$ in the hopes of reaching a contradiction. This then implies that

$$\langle r + H \rangle = Q/H,$$

which is to say that the factor group Q/H is cyclic, being generated by $r + H$. But, since $rZ \cap H$ is a non-trivial group, it follows that the order of $r + H$

is finite. (Then, interestingly, since $Q = H + rZ$ is the smallest subgroup of Q containing H properly, we're also assuming here that H is maximal; and since H is maximal, $|Q/H|$ must be a cyclic group of prime order, it having no non-trivial and proper subgroups.) In any case, let $n = |r + H|$. (We do not care that n is prime.) Now realize that for the rational $r/n \in Q$, there must exist $h \in H$ and $z \in Z$ such that $r/n = h + zr$. But then this implies that

$$r = nh + nzr \in H,$$

(since $nzr \in H$ by the order of $r + H$), which is a contradiction. Our assumption, therefore, that $Q = H + rZ$, is false, and we must have $H + rZ$ a proper subgroup of Q. This completes the proof!