A Quick Guide To Geometric Algebra

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February 5, 2013

Geometric algebra (GA) is easy and you can be up and running with it quickly! The key is to simply understand the basic definitions and identities of the algebra. From these a number of simple results can be deduced. We begin with a few simple definitions from linear algebra (LA).

1 Definitions of LA

Linear algebra is the study of linear functions defined on linear spaces. We will not need to define linear functions here, but we will need to define linear spaces.

Definition 1.1. *A linear space $\mathbb{V}$ is a set of elements, where each element is called a vector. It does not stand on its own. Associated with every vector space $\mathbb{V}$ is a set of scalars, usually the set of real numbers $\mathbb{R}$, such that for every scalar $\lambda \in \mathbb{R}$ and any two vectors $a, b \in \mathbb{V}$, we have*

$$\lambda a \in \mathbb{V} \tag{1}$$
$$a + b \in \mathbb{V} \tag{2}$$

We define vector addition to be commutative and associative.

Linear spaces are more commonly referred to as vector spaces, and so we will no longer use the term linear space.

Notice that equation (1) defines the relationship between the vector space $\mathbb{V}$ and the real numbers $\mathbb{R}$ associated with it.

Definition 1.2. *A linear combination of a set of k vectors $\{v_i\}_{i=1}^{k}$, taken from $\mathbb{V}$, is a vector $v \in \mathbb{V}$, written as*

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k, \tag{3}$$

where $\{\lambda_i\}_{i=1}^k$ is a set of k scalars taken from $\mathbb{R}$.

We now introduce the idea of linear dependence and linear independence.

Definition 1.3. *A set of k vectors $\{v_i\}_{i=1}^k$, taken from $\mathbb{V}$, is said to be linearly independent if there does not exist a vector $v \in \{v_i\}_{i=1}^k$ such that v is any linear combination of the remaining $k-1$ vectors in $\{v_i\}_{i=1}^k - \{v\}$.*

In other words, no one of the k vectors in $\{v_i\}_{i=1}^k$ can be written as a linear combination of the other $k-1$ vectors. Intuitively, if we were to attach a geometric interpretation to the vectors in $\mathbb{V}$, then each vector in $\{v_i\}_{i=1}^k$, (if $\{v_i\}_{i=1}^k$ is a linearly independent set), points in a direction that is a newly unexplored dimension. If you can grasp the definition of linear independence, then the definition of linear dependence is a snap.

Definition 1.4. *A set of k vectors $\{v_i\}_{i=1}^k$ taken from $\mathbb{V}$ is said to be linearly dependent if and only if it is not linearly independent.*

This means that for such a set of k vectors, we can always find a $v \in \{v_i\}_{i=1}^k$ such that v is a linear combination of the remaining $k-1$ vectors.

Understanding the concepts of linear independence and linear dependence is critical to an understanding of geometric algebra.

Definition 1.5. *The dimension of a vector space $\mathbb{V}$, denoted $\dim \mathbb{V}$, is the largest possible set of k vectors we can take from $\mathbb{V}$ that is also a linearly independent set.*

If such a set of vectors does not exist, then the vector space has infinitely many dimensions. We will only concern ourselves here in this guide with finite-dimensional vector spaces.

Given a linearly independent set $\{v_i\}_{i=1}^k$ of k vectors, where $k = \dim \mathbb{V}$, we can describe $\mathbb{V}$ is the set of all linear combinations of the vectors in $\{v_i\}_{i=1}^k$. In this way, we refer to $\{v_i\}_{i=1}^k$ as a basis for the vector space $\mathbb{V}$. Any linearly independent set of vectors taken from $\mathbb{V}$ forms a basis for some vector sub-space of V. The set of all linear combinations of such a set is what we refer to as the span of that set, and we can say that a vector sub-space is spanned by any given basis of that space.

We are now ready to move on to definitions of geometric algebra.

2 Definitions of GA

We begin by simultaneously introducing the concept of a blade and a new kind of product, called the outer product, for vectors, which is not closed in $\mathbb{V}$. This means that, unlike what we can say about vector addition, the outer product of two vectors is not a vector.

Definition 2.1. *A non-zero blade B of grade k is the outer product of k vectors taken from a linearly independent set $\{v_i\}_{i=1}^k$, and may be written as*

$$B = v_1 \wedge v_2 \wedge \cdots \wedge v_k. \tag{4}$$

A blade is considered zero otherwise. That is, the outer product of k linearly dependent vectors is zero. The outer product is defined as anti-commutative, associative, and left and right distributive over addition.

We will talk about what anti-commutative means later on.

Notice that blades of grade 1 are simply vectors. Blades of higher grade are simply a higher-dimensional generalization of the idea of a vector. We will leave the geometric interpretation of such blades open to the reader for now, and proceed with our treatment of blades from a purely algebraic stand-point.

Recall scalar-vector multiplication in equation (1). We similarly allow such a multiplication for blades. Call it scalar-blade multiplication. Realize that a scalar distributes to one and only one vector in the product of equation (4). That is, for any scalar $\lambda \in \mathbb{R}$, we have

$$\lambda B = v_1 \wedge v_2 \wedge \cdots \wedge \lambda v_i \wedge \cdots \wedge v_k, \tag{5}$$

where $1 \leq i \leq k$, the choice of i here being complete arbitrary.

It is important to realize at this point that, by Definition 2.1, a blade may be said to represent a vector sub-space of $\mathbb{V}$. It represents a little more than that, but let us focus on this idea exclusively. Given a non-zero blade B of the form in equation (4), we can take the k vectors in the set $\{v_k\}_{i=1}^k$ as a basis for a vector sub-space of $\mathbb{V}$ of dimension k. In fact, every basis for this vector sub-space, when put in an outer product of the form (4), is, by definition, equal to B, up to scale. That is, if $\{b_i\}_{i=1}^k$ is any basis other than $\{v_i\}_{i=1}^k$ for the vector sub-space represented by B in equation (4), then there exists a scalar $\lambda \in \mathbb{R}$, such that

$$B = \lambda b_1 \wedge b_2 \wedge \cdots \wedge b_k. \tag{6}$$

The great power in this is the freedom to choose any factorization we want of a given blade. The factorization we choose will often be one of convenience that lends itself both algebraically and geometrically to the problem at hand. We'll see examples of this later on.

Given two different basis $\{v_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^k$ for a vector sub-space represented by a k-blade B, the scalar λ in equation (6), which relates their individual outer products, is addressed later on in equation (17).

Continuing with this idea of blades as being representative of vector sub-spaces of $\mathbb{V}$, we can define the following notation.

Definition 2.2 (Membership in a vector sub-space of $\mathbb{V}$). *For any vector* $v \in \mathbb{V}$ *and a non-zero blade B, we say that*

$$v \in B \text{ if and only if } v \wedge B = 0, \tag{7}$$

which is to mean that v is in the vector sub-space of $\mathbb{V}$ represented by B if and only if $v \wedge B = 0$.

It is easy to see that this definition makes sense. If $v \wedge B = 0$, and if $\{v_i\}_{i=1}^k$ is any linearly independent set such that equation (4) is satisfied, then $v \wedge B = 0$ if and only if $\{v\} \cup \{v_i\}_{i=1}^k$ is a linearly dependent set. It follows that v is a linear combination of the vectors in $\{v_i\}_{i=1}^k$.

2.1 Blade Addition

We now extend addition of vectors to addition of blades. The sum of blades of various grades are just that. They do not combine into anything new. Think of this like adding real and imaginary numbers. They don't combine into anything new, but are simply toted about as a pair. Letting scalars be what we'll call blades of grade zero, it is now clear that we can form what we'll call a multivector M, the general element of a geometric algebra, as an element of the form

$$M = B_1 + B_2 + \cdots + B_k, \tag{8}$$

where for each B_i, we have $0 \leq \text{grade}(B_i) \leq \dim \mathbb{V}$. Cleary, we must have

$$\max\{\text{grade}(B_i)\}_{i=1}^k \leq \dim \mathbb{V}, \tag{9}$$

since any blade of grade greater than k must be zero by Definition 2.1 above. To see this, realize that if $\text{grade}(B_i) = \dim \mathbb{V}$, then for all $v \in \mathbb{V}$, we have

4

$v \in B_i$, showing that B_i represents the largest possible vector sub-space of $\mathbb{V}$, which is $\mathbb{V}$ itself. The grade of B_i, therefore, is in this case the largest possible grade.

Just as vectors add as vectors, and scalars add as scalars, it is tempting to assume that blades of any grade k will add as blades of that grade. This is not necessarily true, and forces us to introduce the notion of k-vectors.

Definition 2.3. *A k-vector is the sum of one or more blades of grade k.*

Referring back to M in equation (8), if for all integers i, we had grade$(B_i) = k$, then M is a k-vector, and sometimes referred to as a multivector homogeneous of grade k.

From Definition 2.3, it is clear that all k-blades are simply k-vectors. The opposite, however, is not generally true. That is, not all k-vectors are k-blades. To see why, we must study the idea of adding blades of the same grade k.

Let A and B be non-zero blades of grade k, and consider the k-vector C given as

$$C = A + B, \tag{10}$$

the sum of A and B. Now, by virtue of being blades, each of A and B have a factorization in terms of a linearly independent set $\{a_i\}_{i=1}^k$ or $\{b_i\}_{i=1}^k$, respectively. That is, we may write each of A and B as

$$A = a_1 \wedge a_2 \wedge \cdots \wedge a_k, \tag{11}$$
$$B = b_1 \wedge b_2 \wedge \cdots \wedge b_k, \tag{12}$$

by applying Definition 2.1. Recall now from our earlier discussion that for blades of grade greater than 1, such factorizations are not unique! For example, there may be many such sets $\{a_i\}_{i=1}^k$ of k vectors that we could come up with such that equation (11) is satisfied, because there are many different possible sets of basis vectors we can choose that span the vector sub-space represented by A.

Now comes the critical observation. It is that, under a certain circumstance to be made clear momentarily, we can choose factorizations of A and B such that $a_k = b_k$. Then, using the distributive property of the outer product over addition, we have

$$C = (a_1 \wedge a_2 \wedge \cdots \wedge a_{k-1} + b_1 \wedge b_2 \wedge \cdots \wedge b_{k-1}) \wedge c_k, \tag{13}$$

where $c_k = a_k = b_k$. What we've now done is reduce the problem of adding k-blades to that of adding blades of grade $k - 1$. Inductively, we can say that we've found a way to add k-blades, because ultimately, if our certain circumstance persists at each step, we will eventually reduce the problem of adding the two k-blades to that of adding two 1-blades (vectors), which thing is already defined by equation (2) of Definition 1.1.

So then what is this certain circumstance? Recalling that A and B represent vector sub-spaces of $\mathbb{V}$, isn't it clear that if these vector sub-spaces non-trivially overlap, (share more than just the zero vector in common), then we can find a non-zero vector c_k that exists in both vector sub-spaces? Given such a vector c_k, we now need only write each of A and B in terms of this vector, such that $a_k = c_k$ and $b_k = c_k$.

It is now clear that two blades A and B of grade $k > 1$ fully add if and only if they share a common vector sub-space of dimension $k - 1$. Notice that two 1-blades (vectors), may represent disjoint 1-dimensional vector sub-spaces, (with the exception of the zero vector), yet we can still add them by equation (2) of Definition 1.1.

2.2   The Outer Product

We needed the outer product to get going with blades, but we now return to it here to give it a closer look and a more formal treatment.

Intuitively, the outer product simply allows us to glue vectors together to form blades. What we hope to gain in this section is some kind of geometric intuition about blades, and this will help us learn more about the outer product.

Recall in Definition 2.1 that the outer product is anti-commutative. This means that the transposition of any two adjacent vectors in equation (4), along with a negation of the entire product, leaves B unchanged. That is, we have

$$B = -v_1 \wedge v_2 \wedge \cdots \wedge v_{i+1} \wedge v_i \wedge \cdots \wedge v_k, \tag{14}$$

where $1 \leq i < k$. We'll return to the geometric significance of this momentarily.

Considering $\mathbb{V}$ to be a Euclidean vector space, all vectors $v \in \mathbb{V}$ have a Euclidean interpretation. That is, we can think of v as being a directed line segment with arbitrary position. Using this idea, we can extend the notion of vector magnitude to blade magnitude in the following definition.

Definition 2.4. *Given a non-zero k-blade B, and letting $\{v_k\}_{i=1}^k$ be an orthogonal basis for the vector sub-space of $\mathbb{V}$ represented by B, we define the magnitude of B, denoted by $|B|$, as*

$$|B| = |v_1||v_2|\ldots|v_k|, \tag{15}$$

where for each $1 \leq i \leq k$, $|v_i|$ denotes the length of v_i.

If $\{v_k\}_{k=1}^k$ is an orthogonal basis, then for any pair of vectors $a, b \in \{v_i\}_{i=1}^k$, we have $a \cdot b = 0$, which is to say that a and b are perpendicular to one another. Clearly, $|B|$ is a k-dimensional hyper-volume. That is, length if $k = 1$, area if $k = 2$, volume if $k = 3$, and so on.

But how do we know that such a basis as that given in Definition 2.4 always exists? The answer is the Gram-Schmidt orthogonalization process. Notice that for any scalar $\lambda \in \mathbb{R}$, we can rewrite B in equation (4) as

$$B = v_1 \wedge v_2 \wedge \cdots \wedge (v_i + \lambda v_j) \wedge \cdots \wedge v_k, \tag{16}$$

where $1 \leq i \leq k$, $1 \leq j \leq k$ and $i \neq j$. Every step of the Gram-Schmidt orthogonalization process, which process, for brevity, we'll leave as a research project for the reader, is of the form of equation (16), and in the end, produces a factorization of B in terms of an orthogonal basis.

The geometric take-home point of all this is that blades, under a geometric interpretation, have arbitrary shape. While a vector is always a line segment, we may visualize a blade of grade greater than one as an area in a plane or a volume in a hyper-plane, the shape of that area or volume being completely arbitrary. The defining characteristics of a k-blade are simply the orientation of the k-dimensional hyper-volume, the amount of that hyper-volume, and another property known as handedness. Handedness is a property that manifests itself only in the comparison of two blades A and B of the same grade with the same orientation and magnitude. In this case, if $\{a_i\}_{i=1}^k$ is a sequence of vectors such that equation (11) is satisfied, then B is given by equation (12), where $\{b_i\}_{i=1}^k$ is simply an even or odd permutation of the vectors in $\{a_i\}_{i=1}^k$. That is, there exists a bijective map between $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^k$. Then, seeing that any permutation $\{b_i\}_{i=1}^k$ of $\{a_i\}_{i=1}^k$ is reached by performing a finite sequence of adjacent transpositions, this means that by equation (14), A and B differ algebraicly at most by a sign. That is, $A = \pm B$. In the case that $A = B$, each have the same handedness, while in the case that $A = -B$, they are said to have opposite handedness.

A last look at the outer product that summarizes all of the geometric characteristics of a k-blade is given by

$$B = \begin{vmatrix} b_1 \cdot v_1 & b_1 \cdot v_2 & \ldots & b_1 \cdot v_k \\ b_2 \cdot v_1 & b_2 \cdot v_2 & \ldots & b_2 \cdot v_k \\ \vdots & \vdots & \ddots & \vdots \\ b_k \cdot v_1 & b_k \cdot v_2 & \ldots & b_k \cdot v_k \end{vmatrix} b_1 \wedge b_2 \wedge \cdots \wedge b_k, \qquad (17)$$

where here, B was originally written as it is in equation (4), and for each $1 \leq i \leq k$, we have here rewritten v_i in terms of the alternative basis $\{b_i\}_{i=1}^k$, to get equation (17). It is not too hard to prove equation (17) by induction.

If $\{b_i\}_{i=1}^k$ is an orthonormal basis, then we can take the absolute value of the determinant in equation (17) to be $|B|$, and its sign as an indication of handedness relative to what we usually refer to as a right-handed system. (The determinant as a whole is the scalar λ we talked about earlier in equation (6).) The orientation of B is that of the unit k-blade $b_1 \wedge b_2 \wedge \cdots \wedge b_k$ in the same sense that we might use unit-vectors to indicate pure directions without magnitude.

2.3   The Inner Product

Thus far we have already used the reader's knowledge of the inner product between vectors taken from $\mathbb{V}$. As a function, the inner product of vectors is what we call a bilinear form. How we define this bilinear form on $\mathbb{V}$ will determine what is called the signature of our geometric algebra, because, as we'll see, the inner product among blades of various grades is ultimately defined in terms of the inner product among vectors.

Like the outer product, we define the inner product to be left and right distributive over addition. Unlike the outer product, however, the inner product commutes vectors and is not generally associative.

The inner product between vectors already understood, we begin by defining the inner product between vectors and blades.

Definition 2.5. *For any vector v and blade B of grade k, we define*

$$v \cdot B = (v \cdot b_1)B_1 - (v \cdot b_2)B_2 + \cdots - (-1)^k(v \cdot b_k)B_k, \qquad (18)$$

where here, B is factored into the k vectors in $\{b_i\}_{i=1}^k$ as it is in equation (12), and we have let B_i denote the $(k-1)$-blade given by

$$B_i = b_1 \wedge b_2 \wedge \cdots \wedge b_{i-1} \wedge b_{i+1} \wedge \cdots \wedge b_k, \qquad (19)$$

which is simply the product B with b_i removed. Similarly, we define

$$B \cdot v = (v \cdot b_k)B_k - (v \cdot b_{k-1})B_{k-1} + \cdots - (-1)^k(v \cdot b_1)B_1. \qquad (20)$$

From this definition it is easy to deduce the commutativity of vectors and blades in the inner product. We have

$$v \cdot B = (-1)^k B \cdot v. \qquad (21)$$

Compare this to the commutativity of vectors and blades in the outer product.

$$v \wedge B = -(-1)^k B \wedge v. \qquad (22)$$

It is also easy to deduce from Definition 2.5 the following useful identities.

$$v \cdot B = (v \cdot B_k) \wedge b_k - (-1)^k(v \cdot b_k)B_k \qquad (23)$$
$$B \cdot v = (v \cdot b_k)B_k - (v \cdot B_k) \wedge b_k \qquad (24)$$

We are now ready to define the inner product among blades in general.

Definition 2.6. *For any two blades A and B of grades k and l, respectively, we define their inner product as*

$$A \cdot B = \begin{cases} (a_1 \wedge a_2 \wedge \cdots \wedge a_{k-1}) \cdot (a_k \cdot B) & \text{if } k \leq l, \\ (A \cdot b_1) \cdot (b_2 \wedge b_3 \wedge \cdots \wedge b_{l-1}) & \text{if } k \geq l, \end{cases} \qquad (25)$$

where here, A and B have vector factorizations in terms of the vectors in $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^l$, respectively, as given in equations (11) and (12). In the case that $k = l$, either case may be chosen.

Definition 2.6 defines the inner product between two blades in terms of Definition 2.5. Realizing that we have let the inner product be left and right distributive over addition, we now know how to take the inner product between any two multivectors.

2.4   Comparing the Inner and Outer Products

While motivations for our definition of the outer product are readily available, such motivations are not entirely clear at this point for our recent definition of the inner product. In this section we explain the definition of the inner product by revealing its relationship with the outer product.

For a given vector v and a blade B, notice that we have

$$v \wedge B = v_\perp \wedge B, \tag{26}$$
$$v \cdot B = v_\parallel \cdot B, \tag{27}$$

where we have let $v = v_\parallel + v_\perp$, the sum of the vector components of v parallel and perpendicular to the blade B, respectively. To be more precise, the vector $v_\parallel$ is the orthogonal projection of v down onto the k-blade B, while $v_\perp$ is simply $v - v_\parallel$, sometimes referred to as the orthogonal rejection of v from B.

Equation (26) is easily verified by realizing that $v_\parallel \in B$. Equation (27) can be verified by an application of equation (18), seeing that for all vectors $b \in B$, we have $b \cdot v_\perp = 0$.

Intuitively, equation (26) is showing us that the only contribution of v in the product $v \wedge B$ is the part of v entirely out of the blade, which is also indicating the new dimension through which we are extruding B to form the higher-dimensional volume element $v \wedge B$. On the other hand, equation (27) is showing us that the only contribution of v in the product $v \cdot B$ is the part of v entirely in the blade, which is also indicating the old dimension through which we are collapsing B to form the lower-dimensional volume element $v \cdot B$. So you see, the inner and outer product perform complementary operations. The outer product can be used to build blades up, while the inner product can be used to tare blades down. The grade lowering and grade raising properties of the inner and outer products, respectively, should now be summarized as follows.

$$\begin{aligned} \mathrm{grade}(v \wedge B) &= \mathrm{grade}(B) + 1 \quad \text{if } v \notin B, \\ \mathrm{grade}(v \cdot B) &= \mathrm{grade}(B) - 1 \quad \text{if } v_\parallel \neq 0. \end{aligned} \tag{28}$$

The keen observer at this point, however, might point out that it is not entirely obvious that equation (18) gives us a blade. To see that it does, ...

2.5 The Geometric Product