An Interesting Question About The Rationals $\mathbb{Q}$

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Axiom

For any $q \in \mathbb{Q}$, let $Z(q)$ denote the set $\{zq|z \in \mathbb{Z}\}$. Then for any proper subgroup H of the rationals $\mathbb{Q}$ under addition, there exists $q \in \mathbb{Q} - H$ such that $Z(q) \cap H$ is the trivial subgroup of $\mathbb{Q}$.

The Problem

Show that no proper subgroup H of $\mathbb{Q}$ is maximal.

If $q \in \mathbb{Q} - H$, it is easy to show that $H + Z(q)$ properly contains H and is a subgroup of $\mathbb{Q}$. Let $q \in \mathbb{Q} - H$ be an element of $\mathbb{Q}$ such that $Z(q) \cap H$ is the trivial group. This can be done by the axiom above.

What remains to be shown is that $H + Z(q)$ is a proper subgroup of $\mathbb{Q}$. Suppose $\mathbb{Q} = H + Z(q)$. Notice that $q/2 \notin H$, (since this would imply that $q \in H$), and $q/2 \notin Z(q)$. Yet we must have $q/2 = zq + h$ for some $h \in H$ and $z \in \mathbb{Z}$. Rearranging, we have $2h = (1 - 2z)q$. Now since $2h \in H$ and $(1-2z)q \in Z(q)$, we must have $2h = (1-2z)q \in Z(q) \cap H$ which implies that $h = 0$ and $q = 0$. But this contradicts the facts that $q \notin H$ and $q/2 \notin Z(q)$. So $H + Z(q)$ is a proper subgroup of $\mathbb{Q}$.