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THE MOTHER MINKOWSKI ALGEBRA OF ORDER m

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ABSTRACT. Put abstract here.

1. MOTIVATION

Before presenting the Mother Minkowski algebra of order m, we lead up to it here with some background and motivation. We begin by recalling that an algebraic set is any subset of an n-dimensional euclidean space $\mathbb{R}^n$ that is also the zero set of one or more polynomials. Given a geometric algebra $\mathbb{G}$, we can represents such sets using blades $B \in \mathbb{G}$ as the set of all points $x \in \mathbb{R}^n$ such that

$$p(x) \cdot B = 0,$$

where $p : \mathbb{R}^n \to \mathbb{V}$ maps points in $\mathbb{R}^n$ to a vector space $\mathbb{V}$ generating our geometric algebra $\mathbb{G}$. Though not necessary, $\mathbb{R}^n$ is often embedded in $\mathbb{V}$; but regardless of this, the function p is necessarily defined in such a way that the expression $p(x) \cdot B$ is a polynomial in the vector components of x when $B \in \mathbb{V}$.

Letting $\mathbb{B}$ denote the set of all blades found in $\mathbb{G}$, and letting $P(\mathbb{R}^n)$ denote the power set of $\mathbb{R}^n$, we will find it useful to define the mapping $g : \mathbb{B} \to P(\mathbb{R}^n)$ as

$$\dot{g}(B) = \{x \in \mathbb{R}^n | p(x) \cdot B = 0\}.$$

To see that $\dot{g}(B)$ is an algebraic set, we first observe that when $B \in \mathbb{V}$, $\dot{g}(B)$ is the zero set of a polynomial in the vector components of x. Secondly, we observe that if $\bigwedge_{i=1}^k b_i$ is a factorization of the k-blade B, each b_i being in $\mathbb{V}$, then

$$p(x) \cdot B = -\sum_{i=1}^k (-1)^i (p(x) \cdot b_i) B_i,$$

where B_i is given by

$$B_i = \bigwedge_{j=1, j \neq i} b_j,$$

and therefore, since $\{B_i\}_{i=1}^k$ is a linearly independent set, we have

$$\dot{g}(B) = \bigcap_{i=1}^k \dot{g}(b_i).$$

This model of representing algebraic sets using blades of a geometric algebra presents some interesting properties. To begin, if $A, B \in \mathbb{B}$ are blades with $A \wedge B \neq 0$, then

$$\dot{g}(A) \cap \dot{g}(B) = \dot{g}(A \wedge B).$$

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In this way, the outer product serves to take the intersection of two surfaces. But we can also look at the outer product in a different light as an operator that takes at least the union of its two given surfaces. To see this, we must consider an alternative interpretation of blades $B \in \mathbb{B}$ as algebraic sets. Defining $\hat{g} : \mathbb{B} \to P(\mathbb{R}^n)$ as

$$\hat{g}(B) = \{x \in \mathbb{R}^n | p(x) \wedge B = 0\},$$

we see that $\hat{g}(B) = \dot{g}(BI)$, where I is the unit psuedo-scalar of $\mathbb{G}$, showing that the image of $\hat{g}$, like $\dot{g}$, consists of algebraic sets. Under this new interpretation, we find that for blades $A, B \in \mathbb{B}$, we have

$$\hat{g}(A) \cup \hat{g}(B) \subseteq \hat{g}(A \wedge B).$$

Exactly what surface we get from $A \wedge B$ in terms of $\hat{g}$ can be deduced by considering the surface $(A \wedge B)I$ in terms of $\dot{g}$.

What's further a benefit of using blades to represent surfaces are the transformations performable on such geometries through the use of outermorphisms; in particular, outermorphisms $f : \mathbb{B} \to \mathbb{B}$ of the form

$$f(B) = VBV^{-1},$$

where V is a versor of $\mathbb{G}$. Given such a function, we wish to compare $\dot{g}(B)$ with $\dot{g}(f(B))$. Interestingly, to understand the latter in terms of the former, we need only understand the mapping from $\mathbb{R}^n \to \mathbb{R}^n$, if any, induced by V through p as being each point $x \in \mathbb{R}^n$ mapped to a point $y \in \mathbb{R}^n$, where

$$Vp(x)V^{-1} = \lambda p(y),$$

λ being some scalar in $\mathbb{R}$. This is, of course, only a well defined mapping, provided that for every point $x \in \mathbb{R}^n$, there exists such a point $y \in \mathbb{R}^n$, and that it is unique. Assuming that V and p meet these requirements, and so do indeed induce such a mapping $h : \mathbb{R}^n \to \mathbb{R}^n$, we can now show that

$$\dot{g}(f(B)) = h(\dot{g}(B)).$$

We need only show that

$$\dot{g}(VBV^{-1}) = \{x \in \mathbb{R}^n | V^{-1}p(x)V \cdot B = 0\}.$$

Do that here...

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