Versors That Give Non-Uniform Scale

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To my dear wife Melinda.

Abstract. Versors are found in a geometric algebra that, when applied to elements of that algebra that are representative of various algebraic surfaces in a constrained way, perform a non-uniform scaling of those surfaces.

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1. Motivation

Non-uniform scale is one of the outstanding problems of geometric algebra. As noted in the beginning of [3], 4×4 matrices have been a standard in computer graphics for representing affine and projective transformations, but an equivilant model for such transformations in a more modern setting has yet to emerge as a considerable replacement. This paper does not purport to provide such a setting, but it does offer a potential solution to the non-uniform scale problem. An upcoming paper by an author of [2] may provide an even better solution.

2. The Result

The result of this paper is simply a corollary to that of [4], but to see how, we must first constrain the way that we represent n-dimensional algebraic surfaces of up to degree m in the Mother Minkowski algebra of order m.[1] What we do is let $n \leq m$, and reserve certain subalgebras of our mother algebra for use in specific dimensions. To see what is meant by this, let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial in whose zero set we are interested. Now define, for any integer $k \in [1, n]$, the polynomial $f_k : \mathbb{R} \to \mathbb{R}$ as

$$f_k(\lambda) = f(e_1 + e_2 + \cdots + \lambda e_k + \cdots + e_{n-1} + e_n).$$

[1] Recall that such representations are not unique, and so we have the flexibility to choose our representations carefully.

Having done this, we will represent the surface of f in the Mother Minkowski algebra of order

$$m = \sum_{k=1}^{n} \deg f_k.$$

Now if $\mathbb{G}$ denotes our mother algebra and it is generated by m subalgebras $\mathbb{G}_i$, each generated by the vector space $\mathbb{V}_i$, then we reserve $\deg f_k$ of these subalgebras for use in dimension k of our n dimensions. (We will let $\mathbb{G}^k$, where k is an integer in $[1, n]$, denote the smallest subalgebra of $\mathbb{G}$ containing all subalgebras $\mathbb{G}_i$ reserved for dimension k, and let $[\mathbb{G}^k]$ denote the set of indices over which $\mathbb{G}_i \subseteq \mathbb{G}^k$.)

An example may be warrented at this point. Let $n = 3$ and consider the polynomial given by

$$f(x) = 3x_1^2 x_2 x_3^4 + 4x_1 x_2^5 - 7x_3^2, \tag{2.1}$$

where x_k is notation for $x_k = x \cdot e_k$. We will represent the surface that is the zero set of this polynomial using an m-vector in a Mother Minkowski algebra of order $m = 2 + 5 + 4 = 11$. The first 2 subalgebras are reserved for dimension 1, the next 5 for dimension 2, and the last 4 for dimension 3. The m-vector B representing this surface is then given by

$$\begin{aligned} B = {} & 3e_{(1,2),1} \wedge e_{3,2} \wedge \infty_{(4,5,6,7)} \wedge e_{(8,9,10),3} \wedge \infty_{11} \\ & + 4e_{1,1} \wedge \infty_2 \wedge e_{(3,4,5,6,7),2} \wedge \infty_{(8,9,10,11)} \\ & - 7\infty_{(1,2,3,4,5,6,7)} \wedge e_{(8,9),3} \wedge \infty_{(10,11)}. \end{aligned}$$

Here, notation is a challenge. The vector $e_{i,j}$ denotes the j^{th} euclidean basis vector in the i^{th} subalgebra. We then define

$$e_{(i_1,i_2,\dots,i_r),j} = e_{i_1,j} \wedge e_{i_2,j} \wedge \cdots \wedge e_{i_r,j}.$$

The notation for ∞ is similar.

We can now say that the zero set of f in equation (2.1) is given by the set of all solutions to the equation

$$\bigwedge_{k=1}^{m} p_k(x) \cdot B = 0. \tag{2.2}$$

Recall that $p_k(x) = o_k + x_k + \frac{1}{2}x^2\infty_k$. Of course, we could have represented f in a mother algebra of order $\deg f = 7$, but it will soon become clear why we needed our algebra $\mathbb{G}$ to be of order $m = 11$.

Returning from the example, suppose now we have an m-vector B representative of any polynomial $f : \mathbb{R}^n \to \mathbb{R}$ under the constraint thus illustrated. Seeing that the zero set of f is the set of solutions to equation (2.2), we make the simple observation that if D is a versor taken from a subalgebra $\mathbb{G}^k$, and further, D is the product of the same dilation versor D_i found in each subalgebra $\mathbb{G}_i$ contained in $\mathbb{G}^k$, (see [2] for an explanation of dilation versors), which is to say that $D = \prod_{i \in [\mathbb{G}^k]} D_i$, then the non-uniform scale of f in the dimension of k by the scale of each D_i is given by the set of solutions to the equation

$$\begin{aligned} &\pm \bigwedge_{i\notin[\mathbb{G}^k]} p_i(x) \wedge \bigwedge_{i\in[\mathbb{G}^k]} D_i^{-1} p_i(x) D_i \cdot B \\ = &\pm \bigwedge_{i\notin[\mathbb{G}^k]} p_i(x) \wedge D^{-1}\left(\bigwedge_{i\in[\mathbb{G}^k]} p_i(x)\right) D \cdot B = 0. \end{aligned} \tag{2.3}$$

Now realize that for all $i \notin [\mathbb{G}^k]$, D leaves $p_i(x)$ invariant. That is,

$$D^{-1} p_i(x) D = p_i(x).$$

It now follows by equations (3.2) through (3.5) of [4] that equation (2.3) may be rewritten as

$$\bigwedge_{k=1}^{m} p_k(x) \cdot DBD^{-1},$$

showing that D, when applied to B, performs a non-uniform scaling of the surface of f.

3. Closing Remarks

Though we have now shown that versors performing the non-unform scale operation exist, seeing that their application requires a great deal of cumbersome convention and notation, a question of their practicality immediately arises. It's certainly not practical on paper, but perhaps such versors may find applications on the computer.

Another immediate observation we can make about the result of this paper is that it easily generalizes to the idea of a non-uniform "X", where "X" may be replaced here by any one of the conformal transformations. Since all such transformations may be decomposed as one or more reflections and spherical inversions, the two fundamental transformations to consider here are non-uniform reflections and non-uniform inversions. Considering the former for a moment, it's possible that in some cases these are shears. Then, since reflections give us rotations, it's not surprising that shears may be a bit more fundamental than rotations as hinted at by the clever paper [1].

References

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