On The Problem Of Intersecting Quadric Surfaces Using Geometric Algebra

Spencer T. Parkin

To my dear wife Melinda.

Abstract. Progress is made on the problem of finding a conformal-like model of geometry based upon geometric algebra in which intersections of quadric surfaces may be taken.

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1. Introduction

In light of the paper [], some encouragement has been given to the present author to develop a model of geometry, similar to the conformal model of geometric algebra, but not limited in representation to any proper subset of the set of all quadric surfaces. But for such a model to achieve adequate similarity to the conformal model, it must preserve one of more of its most desirable features; preferably, all of them. For example, in [], the set of all conformal transformations were preserved, but intersections were not.[1]

The goal of this paper, therefore, is to preserve the intersection property. In other words, we want to find a model of geometry based upon geometric algebra giving us all quadric surfaces and the ability to intersect them as effortlessly as can be done in the conformal model. If nothing else, the attempt to do so in this paper will shed light on the feasibility of such an endeavor, and thereby bring us closer to answering the question of whether it can even be done.

[1] A way of representing intersections using the outer product in the model of [] can be found, but its usefulness, if any, is highly questionable.

2. The Intersection Property

Let us begin by taking a closer look at exactly what the intersection property is. Upon initial inspection, one might suppose that this property is nothing more than the ability to represent the intersection of any two given geometries in a way consistent with the representation of any geometry of the model, but this is not enough. Such a representation has no usefulness if it does not submit to an analysis yielding the geometric characteristics of the intersection.

That having been said, we can say now that the outer product's ability to intersect geometries represented by blades in the conformal model is really not at all interesting. What is interesting is the realization that we can equate one characterization of an intersection with another, and this is the key to finding intersections in the conformal model. The reason for this is that while one such characterization is composed as the intersection we wish to take, the other characterization lends itself to analysis through decomposition.

For example, suppose we wish to take the planar intersection of a conical surface. If we know that the resulting conic section is an ellipse, then we can choose to interpret this intersection as that of a plane and an elliptical cylinder meeting the plane at right angles. This latter characterization will have an easily found decomposition yielding all features of the ellipse.[2] Having found all such features, we can then say that we've fully realized the given section, whereas before this we were only able to represent it.[3]

The quest to find our model of geometry, (the one promised in the introductory section of this paper), being quite difficult, the bringing of the example just given in the preceding paragraph to fruition will become the impetus for all choices henceforth made in finding the model. Even if our model can do nothing more than this one example, we will consider or goal achieved. Rest assured, however, that along the way, we will find results generally applicable to the problem at hand.

3. Enter The Model

So that no further delay be made, we will now let the remainder of this paper begin exactly where the first section of [] ended, assuming all results and definitions up to that point. That said, we now introduce the function

[2]Geometries having easily decomposable representations in our model will be referred to as canonical forms.

[3]Note that although non-planar intersections will be as easily represented in our model of geometry as any other type of intersection, the technique of finding intersections in this paper may not be helpful in finding non-planar intersections for the simple reason that such intersections have no obvious canonical form.

$p : \mathbb{R}^n \to \mathbb{V}$ as

$$\begin{aligned} p(x) &= e_0 + x \\ &\quad + (x \cdot e_2)(x \cdot e_3)e_4 + (x \cdot e_1)(x \cdot e_3)e_5 + (x \cdot e_1)(x \cdot e_2)e_6 \\ &\quad + (x \cdot e_1)^2 e_7 + (x \cdot e_2)^2 e_8 + (x \cdot e_3)^2 e_9, \end{aligned}$$

the vectors in $\{e_i\}_{i=0}^9$ forming an orthonormal basis for a 10-dimensional euclidean vector space.[4] This is sufficient to define the entire model as it is clear that for any quadric surface, or any intersection of two or more quadric surfaces, there exists a blade $B \in \mathbb{G}$ representative of this surface as $\dot{g}(B)$.

Continuing with our example, let us now find the canonical form for an ellipse in a plane. Using the notation[5] $x_i = x \cdot e_i$, an equation for such an ellipse may be given by

$$\frac{(x_1 - h)^2}{a^2} + \frac{(x_2 - k)^2}{b^2} - 1 = 0, \tag{3.1}$$

provided $x_3 = 0$, which is simply an equation for the plane. Factoring $p(x)$ out of the equation $x_3 = 0$, we get $p(x) \cdot e_3 = 0$, and out of equation (3.1), we get $p(x) \cdot E = 0$, where the vector E is given by

$$E = \left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right) e_0 - 2\frac{h}{a^2} e_1 - 2\frac{k}{b^2} e_2 + \frac{1}{a^2} e_7 + \frac{1}{b^2} e_8.$$

By itself, E here represents an axis-aligned elliptical cylinder. The ellipse is given by the set of points $\dot{g}(E \wedge e_3)$. The 2-blade $E \wedge e_3$ will be the canonical form of the ellipse that we will use to find a conic intersection that is an ellipse.

Letting $\lambda \in \mathbb{R}$ be any non-zero scalar, to see that $B = \lambda E \wedge e_3$ is easily decomposable, we give the following set of equations.

$$h = \frac{-e_3 \wedge e_1 \cdot B}{2 e_3 \wedge e_7 \cdot B} \tag{3.2}$$

$$k = \frac{-e_3 \wedge e_2 \cdot B}{2 e_3 \wedge e_8 \cdot B} \tag{3.3}$$

$$\lambda = (e_3 \wedge (h^2 e_7 + k^2 e_8 - e_0)) \cdot B \tag{3.4}$$

$$a = \sqrt{\lambda (e_3 \wedge e_7 \cdot B)^{-1}} \tag{3.5}$$

$$b = \sqrt{\lambda (e_3 \wedge e_8 \cdot B)^{-1}} \tag{3.6}$$

Using these equations, we can recover the ellipse.

[4]The reader should take care to note which results of this paper depend upon this definition of the function p and which do not. Also note that signatures other than the euclidean may be worth considering, but there will be no foreseeable need in this paper.

[5]This notation is overloaded. When writing x_i, this may mean the i^{th} component of x, or it may mean the i^{th} point in a sequence of points. The intended meaning will always be clear from context.

Now let us formulate the intersection we wish to take. We will intersect the $x_3 = 0$ plane with the conical surface having points satisfying the equation

$$x_1^2 + x_2^2 - (x_3 + 1)^2 \tan^2 \frac{\pi}{4} = 0, \tag{3.7}$$

where $\frac{\pi}{2}$ is the angle of aperture.[6] We have submerged it below the $x_3 = 0$ plane to get a non-trivial intersection $\dot{g}(C \wedge e_3)$, the vector C being given by

$$\begin{aligned} C &= -\left(\tan^2 \frac{\pi}{4}\right) e_0 + 2\left(\tan^2 \frac{\pi}{4}\right) e_3 + e_7 + e_8 - \left(\tan^2 \frac{\pi}{4}\right) e_9 \\ &= -e_0 + 2e_3 + e_7 + e_8 - e_9. \end{aligned}$$

4. Making Use Of The Model

Being now able to represent all quadric intersections, the task of algebraically relating them remains. For example, knowing that $C \wedge e_3$ is an ellipse, how might we decompose it as we would $E \wedge e_3$? The following lemma may be able to help.

Lemma 4.1. *Letting $B \in \mathbb{G}$ be a blade of grade k, if there exist k points $\{x_i\}_{i=1}^k \subset \mathbb{R}^n$ such that $\bigwedge_{i=1}^k p(x_i) \neq 0$, and that for all points $x \in \{x_i\}_{i=1}^k$, we have $x \in \hat{g}(B)$, then there exists a scalar $\lambda \in \mathbb{R}$ such that*

$$B = \lambda \bigwedge_{i=1}^{k} p(x_i). \tag{4.1}$$

Proof. If $x_i \in \hat{g}(B)$, then $p(x_i) \wedge B = 0$, showing that $p(x_i)$ is in the vector space spanned by any factorization of B. Then, since $\bigwedge_{i=1}^k p(x_i) \neq 0$, the set of vectors $\{p(x_i)\}_{i=1}^k$ is linearly independent and therefore a basis for this vector space. The blades B and $\bigwedge_{i=1}^k p(x_i)$ must, therefore, be equal, up to scale. □

For a blade $B \in \mathbb{G}$ having a factorization (4.1), we will refer to B as an irreducible blade for reasons that will become clear shortly.

The usefulness of Lemma 4.1 is realized in our next lemma.

Lemma 4.2. *If $A, B \in \mathbb{G}$ are blades of grade k with $\hat{g}(A) = \hat{g}(B)$, and one of these is irreducible, then there exists a scalar $\lambda \in \mathbb{R}$ such that $A = \lambda B$.*

Proof. We first establish that if one of the blades A and B is irreducible, then so is the other. Assuming, without loss of generality, that A is irreducible, let $\{x_i\}_{i=1}^k \subset \mathbb{R}^n$ be a set of k points, and $\alpha \in \mathbb{R}$ be a scalar, such that $A = \alpha \bigwedge_{i=1}^k p(x_i)$. Now since $\hat{g}(A) = \hat{g}(B)$, it is clear that for all $x \in \{x_i\}_{i=1}^k$, we have $x \in \hat{g}(B)$, and so it follows by Lemma 4.1 that there exists a scalar $\beta \in \mathbb{R}$ such that $B = \beta \bigwedge_{i=1}^k p(x_i)$.

Lastly, we simply realize that if we let $\lambda = \frac{\alpha}{\beta}$, we have $A = \lambda B$. □

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[6] Yes, this will give us a circle in the $x_3 = 0$ plane and not a general ellipse, but to keep things simpler, our primary example will consider this special case of the ellipse.

In light of Lemma 4.2, the following question naturally arises. Knowing that $\dot{g}(E\wedge e_3) = \dot{g}(C\wedge e_3)$, is any one of $(E\wedge e_3)I$ and $(C\wedge e_3)I$ irreducible?[7] If so, we have found a way to algebraically relate them so that an analysis by decomposition, up to scale, of $C \wedge e_3$ as $E \wedge e_3$ can move forward.[8] Unfortunately, it is not hard to show that neither of these is irreducible. To see why, consider the following equation which expresses the form of $p(x)$ for all points in the $x_3 = 0$ plane.

$$p(x_1e_1 + x_2e_2) = e_0 + x_1e_1 + x_2e_2 + x_1x_2e_6 + x_1^2e_7 + x_2^2e_8 \tag{4.2}$$

Now notice that an upper-bound on the dimension of a vector space that can be spanned by vectors of this form is clearly 6 as there are only 6 components on the right-hand side of equation (4.2). But each of $E \wedge e_3$ and $C \wedge e_3$ are 2-blades, making their duals blades of grade $10 - 2 = 8 > 6$. It follows that there is no set of 8 points $\{x_i\}_{i=1}^8$ on the ellipse such that $\bigwedge_{i=1}^8 p(x_i) \neq 0$.

Not willing to give up just yet, we arrive at the following lemma.

Lemma 4.3. *For every k-blade $B \in \mathbb{G}$, there exists a blade $B' \in \mathbb{G}$ of grade $k' \leq k$ such that $\hat{g}(B) = \hat{g}(B')$ and B' is irreducible.*

Proof. If B is irreducible, then let $k' = k$ and $B' = B$ and we're done. If B is not irreducible, then let j be the largest possible integer for which there exists a set of j points $\{x_i\}_{i=1}^j \subseteq \hat{g}(B)$ with $\bigwedge_{i=1}^j p(x_i) \neq 0$, (clearly $j < k$), and write

$$B = B_0 \wedge \bigwedge_{i=1}^{j} p(x_i)$$

for some blade B_0 of grade $k - j$. Now realize that for any $x \in \hat{g}(B)$, if $x \in \hat{g}(B_0)$, then $p(x) \wedge \bigwedge_{i=1}^j p(x_i) \neq 0$ and $x \notin \{x_i\}_{i=1}^j$, which is a contradiction. Therefore, if $x \in \hat{g}(B)$, then, letting $k' = j$ and $B' = \bigwedge_{i=1}^j p(x_i)$, we have $x \in \hat{g}(B')$. Conversely, if $x \in \hat{g}(B')$, then clearly $x \in \hat{g}(B)$. It follows that $\hat{g}(B) = \hat{g}(B')$ and B' is irreducible. □

Seeing that B' is potentially a reduction in grade of the blade B, but that it certainly does not sacrifice the geometry represented by B, we will say that B is reducible in the case that $k' < k$. In the case that $k' = k$, it is clear that B is irreducible.

What we see now is that $(C\wedge e_3)I$ and $(E\wedge e_3)I$ are reducible blades, and that equating them is possible if they can both be reduced. Unfortunately, an irreducible canonical form of our ellipse is not obvious, nor is a solution to the problem of either fully reducing a given blade, or showing that it is already irreducible. Each of these problems may be about as hard as finding, not just any, but a specific type of factorization of the blade in question, and there may be no way of getting around that. That is, if we're bent on equating one blade with another.

[7]Here we're letting I be the unit psuedo-scalar of our 10-dimensional geometric algebra $\mathbb{G}$.
[8]Think of this as replacing B with $\lambda C \wedge e_3$ in equations (3.2) through (3.6).

5. Taking a Different Approach to the Problem

Fortunately, what may be an alternative to our original plan is that of using $C \wedge e_3$ and $E \wedge e_3$ to generate a system of non-linear equations. The idea is simple. If we know that we have a point $x \in \dot{g}(E \wedge e_3)$, (our canonical form), then we also have $x \in \dot{g}(C \wedge e_3)$, (our desired intersection), and this generates for us one or more equations in the components x_i of x. For example, if we let $x = (h + a)e_1 + ke_2$, then $p(x) \cdot E \wedge e_3$ is clearly zero, and therefore, $p(x) \cdot C \wedge e_3 = (p(x) \cdot C)e_3$ must be zero too.[9]

A lemma is now in order. For the following lemma, the asterisk symbol "$*$" may be replaced, if done in a consistent manner, by either the inner product symbol "$\cdot$" or the outer product symbol "$\wedge$."

Lemma 5.1. *Given a blade $B \in \mathbb{G}$, if $\{x_i\}_{i=1}^j \subseteq g^*(B)$ is a set of j points with $\bigwedge_{i=1}^{j-1} p(x_i) \neq 0$ and $\bigwedge_{i=1}^{j} p(x_i) = 0$, then the system of equations generated by, for all integers $i \in [1, j]$, $0 = p(x_i) * B$, is not any more determined than the system of equations generated by, for all integers $i \in [1, j-1]$, $0 = p(x_i) * B$.*[10]

Proof. For an appropriate set of $j - 1$ scalars $\{\lambda_i\}_{i=1}^{j-1} \subset \mathbb{R}$, letting $p(x_j) = \sum_{i=1}^{j-1} \lambda_i p(x_i)$, the equation $0 = p(x_j) * B$ contributes to the system all equations generated by

$$0 = \sum_{i=1}^{j-1} \lambda_i p(x_i) * B,$$

which is not any new information. (Generating new equations by adding scalar multiples of existing equations together does not make a system of equations any more determined.) □

What Lemma 5.1 is telling us is that if we want any hope of generating a sufficiently determined system, then we must only consider sets of points $\{x_i\}_{i=1}^j \subset \dot{g}(E \wedge e_3)$ for which $\bigwedge_{i=1}^{j} p(x_i) \neq 0$. Realizing this, it is worth taking a moment to consider the circumstances under which such a set of j points produces a linearly independent set of vectors $\{p(x_i)\}_{i=1}^j$.

Doing so, it is immediately clear that if $\{x_i\}_{i=1}^j$ is a linearly independent set of points in $\mathbb{R}^n$, then so is the set $\{p(x_i)\}_{i=1}^j$, but we can do a little better than this with the following lemma.

Lemma 5.2. *If a given set of $j > 2$ points $\{x_i\}_{i=1}^j$ are non-co-planar for a plane of dimension $j - 2$, then the set of vectors $\{p(x_i)\}_{i=1}^j$ is linearly independent.*

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[9]This idea calls into question the use of geometric algebra as a framework for solving the problem at hand. Could we not just as easily take this approach using nothing more than our original polynomial equations (3.1) and (3.7)? Yes; however, the use of some framework, whether it be geometric, linear or abstract algebra, may help us make useful generalizations about the problem as the lemmas of this paper attempt to do.

[10]This is also to say that the system of equations generated by all j points has the same solution set as that generated by all $j - 1$ points.

Proof. Proving the contrapositive of the lemma, let $\{\lambda_i\}_{i=1}^j$ be a set of scalars in $\mathbb{R}$, not all zero, such that $0 = \sum_{i=1}^j \lambda_i p(x_i)$. It follows that $0 = \sum_{i=1}^j \lambda_i(e_0 + x_i)$ and therefore $0 = \sum_{i=1}^j \lambda_i$ and $0 = \sum_{i=1}^j \lambda_i x_i$. Now realize that if there exists an integer $a \in [1, j]$ such that $\lambda_a \neq 0$, then there must exist an integer $b \in [1, j] - \{a\}$ such that $\lambda_b \neq 0$. Without loss of generality, let $a = j$ so that $1 \leq b \leq j - 1$, and write

$$0 = \sum_{i=1}^{j} \lambda_i x_i = \sum_{i=1}^{j-1} \lambda_i x_i - \left(\sum_{i=1}^{j-1} \lambda_i\right) x_j = -\sum_{i=1}^{j-1} \lambda_i (x_j - x_i),$$

which shows that the set of vectors $\{x_j - x_i\}_{i=1}^{j-1}$ is linearly dependent. It now follows that the $(j-1)$-dimensional simplex determined by the points in $\{x_i\}_{i=1}^j$ has no $(j-1)$-dimensional hyper-volume. That is,

$$0 = \frac{1}{(j-1)!} \bigwedge_{i=1}^{j-1} (x_j - x_i).$$

But this can only be if the j points are co-planar for a hyper-plane of dimension $j - 2$. □

Lemma 5.2 is a good start, but there are certainly more conditions on $\{x_i\}_{i=1}^j$ to be found upon which $\bigwedge_{i=1}^j p(x_i) \neq 0$. The non-linearity of our function p makes these conditions difficult to find, to say the least.

At this point we would wish to return to our example, but lemmas 5.1 and 5.2 do not allow us to do so. Lemma 5.1 offers a necessary, but insufficient condition for obtaining a completely determined system, and Lemma 5.2 is not even enough to help us satisfy this condition.

6. Returning To Our Original Approach

Still not ready to give up, let us introduce yet another lemma to help us in our quest. While Lemma 4.3 dealt with existence, this one deals with uniqueness.

Lemma 6.1. *Given any blade $B \in \mathbb{G}$, if k points $\{x_i\}_{i=1}^k \subset \mathbb{R}^n$ can be found such that $\hat{g}(B) = \hat{g}\left(\bigwedge_{i=1}^k p(x_i)\right)$, then there is no irreducible form of B of a grade less than k, nor greater than k.*

Proof. □

Armed with Lemma 6.1, we are justified in claiming that, up to scale, the irreducible form of our desired intersection $\dot{g}(C \wedge e3)$ is given by $\hat{g}(R)$, where

$$R = e_{01267} - e_{01268} + e_{12678}, \tag{6.1}$$

which is a blade of grade 5 in our algebra.[11] Admittedly, knowledge of the intersection was used to obtain this form; but, assuming that there is an

[11] Choosing any 5 points $\{x_i\}_{i=1}^5$ on the unit circle in the plane for which $\bigwedge_{i=1}^5 p(x_i) \neq 0$, you'll find that $\bigwedge_{i=1}^5 p(x_i)$ is a scalar multiple of R in equation (6.1).

algorithm for finding it that does not depend on such knowledge, let's move forward unashamed and not the least bit discouraged.

Having now fully reduced the 8-blade $(C \wedge e_3)I$ down to the 5-blade R, we know that there must exist a 3-blade R_0 such that the equation

$$(R_0 \wedge R)I = E \wedge e_3$$

has real solutions in a, b, h and k. Indeed, by examination, it is not hard to see that if we let $R_0 = e_{459}$, then this is the case. The 8-blade $R_0 \wedge R$ may be decomposed using equations (3.2) through (3.6).

7. Closing Remarks

Spencer T. Parkin
102 W. 500 S.,
Salt Lake City, UT 84101
e-mail: spencerparkin@outlook.com