Chapter 7 Exercises
Gallian's Book on Abstract Algebra

Spencer T. Parkin

February 10, 2014

Exercise 29

Let G be a group of permutations of a set S. Prove that the orbits of the members of S consitute a partition of S.

For any $a, b \in S$, let $a \sim b$ if and only if $a \in \text{orb}_G(b)$. To see that this forms an equivilance relation on the set S, we begin by noting that for all $a \in S$, we have $a \in \text{orb}_G(a)$, giving us the reflexive property. For all $a, b \in S$, if $a \sim b$, then $a \in \text{orb}_G(b)$ and therefore there exists $\phi \in G$ such that $\phi(b) = a$. Seeing that $\phi^{-1}(a) = b$ and $\phi^{-1} \in G$, it is clear that $b \in \text{orb}_g(a)$ and therefore $b \sim a$, giving us the symmetric property. Lastly, for all $a, b, c \in S$, if $a \sim b$ and $b \sim c$, there exist $\phi_0, \phi_1 \in G$ such that $\phi_0(b) = a$ and $\phi_1(c) = b$. Then, letting $\phi = \phi_0\phi_1 \in G$, it is clear that $\phi(c) = a$ and therefore $a \in \text{orb}_G(c)$. It follows that $a \sim c$, and we have the transitive property.

Now notice that the equivilance class containing a is given by

$$[a] = \{x \in S | x \sim a\} = \{x \in S | x \in \text{orb}_G(a)\} = \text{orb}_G(a),$$

showing that the orbits of elements in S are the equivilance classes that partition S by Theorem 0.6.