On The Use Of Blades As Representatives Of Geometry

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1. Introduction And Motivation

In many models of geometry that are based upon geometric algebra (see []), blades are used to represent geometries. Seeing a great deal of commonality between these models, a formal treatment of this idea deserves to be given in an abstract setting in much the same way that, for example, abstract algebra provides such a setting in which algebraic sets generated by ideals of a polynomial ring can be studied. To the author's knowledge, this is the first treatment of its kind.

2. Foundation

To lay the foundation of our work, we introduce $\mathbb{V}$ as an m-dimensional vector space generating a geometric algebra denoted by $\mathbb{G}$. We leave the signature of this geometric algebra unspecified, but in cases where a proof depends upon signature, one is given as either euclidean or non-euclidean.[1] The scalars of $\mathbb{V}$, (and therefore of $\mathbb{G}$), are taken from the field $\mathbb{R}$ of real numbers.[2] We will let $\mathbb{R}^n$ denote n-dimensional euclidean space,[3] and let $\mathbb{B}$ denote the set of all blades taken from $\mathbb{G}$. Lastly, we will let $p : \mathbb{R}^n \to \mathbb{V}$ be an unspecified, yet

[1]The Gram-Schmidt orthogonalization process is applicable to all blades taken from and only from a geometric algebra having a euclidean signature, anti-euclidean signature or perhaps some combination of the two as long as null-vectors are not possible.
[2]To be more abstract, we could have used any field with characteristic 1, but there will be no foreseeable advantage to doing so in this paper.
[3]Some models of geometry find affine space to be the natural space within which to work, but this will not be the case in this paper.

well-defined function that we'll use in the following definition and throughout the remainder of this paper.[4]

Definition 2.1 (Direct And Dual Representation). For any blade $B \in \mathbb{B}$, we say that B directly represents the set of all points $x \in \mathbb{R}^n$ such that $p(x) \wedge B = 0$, and say that B dually represents the set of all points $x \in \mathbb{R}^n$ such that $p(x) \cdot B = 0$. For convenience, we introduction the following functions using set-builder notation.

$$\hat{g}(B) = \{x \in \mathbb{R}^n | p(x) \wedge B = 0\}$$
$$\dot{g}(B) = \{x \in \mathbb{R}^n | p(x) \cdot B = 0\}$$

From Definition 2.1, it's important to take away the realization that a given blade $B \in \mathbb{B}$ represents two geometries simultaneously; namely, $\hat{g}(B)$ and $\dot{g}(B)$. Which geometry we choose to think of B as being a representative of at any given time is completely arbitrary.[5]

It should also be clear from Definition 2.1 that the geometry represented by a blade B, (directly or dually), remains invariant under any non-zero scaling of the blade B. Something interesting happens, however, when we take the dual of B, as Lemma 2.2 will show.

Lemma 2.2 (Dual Relationship Between Representations). *For any subset S of $\mathbb{R}^n$, if there exists $B \in \mathbb{B}$ such that $\hat{g}(B) = S$, then $\dot{g}(BI) = S$, where I is the unit psuedo-scalar of $\mathbb{G}$. Similarly, if there exists $B \in \mathbb{B}$ such that $\dot{g}(B) = S$, then $\hat{g}(BI) = S$.*

Proof. The first of these two statements is proven by

$$0 = p(x) \wedge B = -(p(x) \cdot BI)I \iff p(x) \cdot BI = 0,$$

while the second is proven by

$$p(x) \cdot B = 0 \iff 0 = (p(x) \cdot B)I = p(x) \wedge BI.$$

See identities (3.5) and (3.6) of Section 3. □

In words, Lemma 2.2 is telling us that for a single given geometry, the algebraic relationship between a blade directly (dually) representative of that geometry, and a blade dually (directly) representative of that geometry, is simply that, up to scale, they are duals of one another.

Of course, there will also be a geometric relationship between the geometry that is directly represented by a single given blade $B \in \mathbb{B}$, and the

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[4]By leaving p unspecified, we're abstracting away the definition of the function. We only care that it is a well-defined function. In some parts of this paper, we will consider the cases where p takes on some desirable properties.

[5]In some literature on geometric algebra, a blade B intended to represent some peice of geometry directly or dually is referred to as a "geometry" or a "dual geometry," respectively. This is confusing and not practiced in this paper. A blade is a blade; and when we refer to geometry, we will use proper language in identifying what represents it and how it does so. In this paper, a geometry is a subset of $\mathbb{R}^n$ that can be represented dually or directly by some blade $B \in \mathbb{B}$ under Definition 2.1. See Defintion 2.3.

geometry that is dually represented by B, but this depends upon the definition of our function p, which we choose, in this paper, to leave open to speculation.

With Lemma 2.2 in hand, geometric algebra's equivilant of an algebraic set may be given as follows.[6]

Definition 2.3 (Geometric Set). A subset $S \subset \mathbb{R}^n$ for which there exists a blade $B \in \mathbb{B}$ such that $\hat{g}(B) = S$ is what we'll refer to as a "geometric set."

By Lemma 2.2, it is easy to see that Definition 2.3 is equivilant to a version of itself that replaces $\hat{g}$ with $\dot{g}$. Therefore, for any geometric set S, we can claim the existence of a blade $B \in \mathbb{B}$ such that $\hat{g}(B) = S$ or $\dot{g}(B) = S$.

3. Useful Identities

In this section we give a number of useful algebraic identities that would otherwise distract us from the flow of the paper if given in the main body. This section is not intended as a complete review of geometric algebra. See [] for such a review.

Letting $v \in \mathbb{V}$ and $B \in \mathbb{B}$, recall that

$$vB = v \cdot B + v \wedge B. \tag{3.1}$$

Also recall that

$$v \wedge B = \frac{1}{2}(vB + (-1)^{\text{grade}(B)} Bv), \tag{3.2}$$

$$v \cdot B = \frac{1}{2}(vB - (-1)^{\text{grade}(B)} Bv). \tag{3.3}$$

Realizing that $\text{grade}(I) = m$, and that by (3.1), we have $vI = v \cdot I$, we can use equation (3.3) to establish the commutativity of vectors in $\mathbb{V}$ with the unit psuedo-scalar I as

$$vI = -(-1)^m Iv. \tag{3.4}$$

Using equation (3.4) in conjunection with equation (3.3), we find that

$$(v \cdot B)I = v \wedge BI. \tag{3.5}$$

(In verifying this identity, it helps to realize that for any integer k, $(-1)^k = (-1)^{-k}$.) Replacing B in equation (3.5) with BI, we find that

$$v \wedge B = -(v \cdot BI)I. \tag{3.6}$$

References

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[6] If p is defined appropriately, geometric sets are algebraic sets.