Spades
A New Way To Represent Geometric Sets

Spencer T. Parkin

Abstract. Abstract...

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1. Introduction And Motivation

Traditionally, *blades* are used, in places such as the conformal model, to represent geometric sets in geometric algebra.[1] Doing so, the meet and join operations become the principle means by which geometries are combined or intersected into new geometries. In this paper we show that *spades* may also be used to represent geometric sets; and in so doing, the geometric product becomes the principle means by which geometries are combined or intersected to make new geometries.

The term "spade" requires some explanation. See Table 1 below.

TABLE 1. A few terms used in GA

Term	Definition
Blade	An outer product of zero or more linearly-independent vectors.
Versor	A geometric product of zero or more *invertible* vectors, not necessarily forming a linearly-independent set.
Spade	A geometric product of zero or more vectors, not necessarily forming a linearly-independent set.
Null Versor	A geometric product of one or more vectors where at least one of them is null.

From these definitions it is clear that every versor is a spade, but not every spade is a versor. The definition of a versor, which can be found in [2,

[1] A formal treatment of geometric sets in an abstract setting is given in [1]. Geometric sets are a generalization of algebraic sets.

p. 90], is well established and engrained in the literature just as it is written in Table 1. Not requiring that each vector in the factorization of a spade be invertible, this justifies the new term.

2. Geometric Sets

We must begin with a review of geometric sets. Given an n-dimensional space $\mathbb{F}^n$, we let $p : \mathbb{F}^n \to \mathbb{V}$ be a vector-valued function mapping points in $\mathbb{F}^n$ to vectors in a vector space $\mathbb{V}$ generating our geometric algebra $\mathbb{G}$. With this in hand, we are ready for the following definition.

Definition 2.1 (Geometric Set). A subset S of $\mathbb{F}^n$ is a *geometric set* if and only if there exists a set of r vectors $\{v_i\}_{i=1}^r \subset \mathbb{V}$, such that

$$S = \{x \in \mathbb{F}^n | p(x) \cdot v_i \text{ for all } i \in [1, r]\} = \bigcap_{i=1}^{r} \{x \in \mathbb{F}^n | p(x) \cdot v_i = 0\}.$$

If each expression $p(x) \cdot v_i$ is a polynomial in the components of x, then it is easy to see that, in this case, geometric sets are algebraic. By the Hilbert Basis Theorem (see [3, p. 204]), r needs not be infinite. This is good news for us, as we'll see, since we'll eventually want to show that every algebraic set can be represented by a blade of grade r.

Lemma 2.2. *If $\{E_i\}_{i=1}^r$ is any linearly independent set of elements taken from $\mathbb{G}$, then the set of all solutions in $\mathbb{F}^n$ to the equation*

$$0 = \sum_{i=1}^{r} (p(x) \cdot v_i) E_i \tag{2.1}$$

is a geometric set.

Proof. Being a linearly independent set of elements, the only linear combination of these elements that vanishes is the trivial linear combination. It follows that for each integer $i \in [1, r]$, we must have $p(x) \cdot v_i = 0$. □

Lemma 2.3. *If $\{E_i\}_{i=1}^r$ is any sequence of elements taken from $\mathbb{G}$, then the set of all solutions in $\mathbb{F}^n$ to equation (2.1) is a geometric set.*

Proof. If $\{E_i\}_{i=1}^r$ is a linearly independent set, then we're done by Lemma 2.2. Supposing to the contrary, and without loss of generality, we can let s be an integer with $1 \leq s < r$ such that $\{E_i\}_{i=1}^s$ is a linearly independent set, and

$$\text{span}\{E_i\}_{i=1}^r = \text{span}\{E_i\}_{i=1}^s.$$

Now for each integer $i \in [s + 1, r]$, write E_i as a linear combination of the elements in $\{E_i\}_{i=1}^s$ as

$$E_i = \sum_{j=1}^{s} \alpha_{i,j} E_j.$$

Having done so, we see now that equation (2.3) becomes

$$\begin{aligned}
0 &= \sum_{i=1}^{r}(p(x)\cdot v_i)E_i \\
&= \sum_{i=1}^{s}(p(x)\cdot v_i)E_i + \sum_{i=s+1}^{r}(p(x)\cdot v_i)\sum_{j=1}^{s}\alpha_{i,j}E_j \\
&= \sum_{i=1}^{s}\left[p(x)\cdot v_i - \sum_{j=s+1}^{r}\alpha_{j,i}(p(x)\cdot v_j)\right]E_i \\
&= \sum_{i=1}^{s}\left[p(x)\cdot\left(v_i - \sum_{j=s+1}^{r}\alpha_{j,i}v_i\right)\right]E_i.
\end{aligned}$$

We see now that the set of all solutions to equation (2.3) is given by

$$\bigcap_{i=1}^{s}\left\{x\in\mathbb{F}^n \,\middle|\, p(x)\cdot\left(v_i - \sum_{j=s+1}^{r}\alpha_{j,i}v_i\right) = 0\right\},$$

which is clearly a geometric set by Definition 2.1. □

3. Perliminary Material

Before we can show how blades and spades can represent geometric sets, we need to lay some ground work with the following definitions, lemmas, and identities.

Though already given in Table 1, the term spade deserves its own formal definition as follows.

Definition 3.1 (Spade). An element $M_r \in \mathbb{G}$ is called a *spade* if and only if there exists a set of r vectors $\{m_i\}_{i=1}^r$ such that it may be written as

$$M_r = \prod_{i=1}^{r} m_i.$$

It is easy to show that spades, like blades, to not have unique factorizations. Unlike blades, however, the size of a spade's factorization can very. This leads us to the following definition.

Definition 3.2 (Spade Rank). Given any spade $M_r \in \mathbb{G}$, the rank of the spade M_r, denoted $\operatorname{rank}(M_r)$ is the smallest integer $s \leq r$ such that M_r may be rewritten as a geometric product of s vectors.[2]

It is clear that if $\{m_i\}_{i=1}^r$ is a linearly independent set of vectors, then $\operatorname{rank}(M_r) = r$. The converse of this statement, however, is not so easily proven (or disproved) and of such considerable importance to the theory of spades in

[2]This smallest integer s exists by the well-ordering principle.

geometric algebra, that, if proven, would be elevated to the level of theorem. For now, however, the following lemma is offered.

Lemma 3.3. *For any given invertible spade $M_r \in \mathbb{G}$, if there exist integers $1 \leq i < j \leq r$ such that $m_i = m_j$, and m_i is invertible, then rank$(M_r) < r$.*

Proof. This is trivial in the case that $j = i + 1$. In the case that $j = i + 2$, simply notice that

$$m_i m_{i+1} m_j = m_i m_{i+1} m_i = 2(m_i \cdot m_{i+1}) m_i - m_i^2 m_{i+1}.$$

In the case that $j > i + 2$, we see that

$$m_i \left(\prod_{k=i+1}^{j-1} m_k \right) m_j = m_i^2 \prod_{k=i+1}^{j-1} m_i m_k m_i^{-1}.$$

□

For completeness, we now give a formal definition of a blade.

Definition 3.4 (Blade). An element $B_r \in \mathbb{G}$ is called an *r-blade* if and only if there exists a linearly independent set of r vectors $\{b_i\}_{i=1}^r$ such that

$$B_r = \bigwedge_{i=1}^{r} b_i.$$

Lemma 3.5. *Letting $B_r^{(i)}$ denote the $(r-1)$-blade*

$$B_r^{(i)} = \bigwedge_{\substack{j=1 \\ j \neq i}}^{r} b_i,$$

the set of r blades $\{B_r^{(i)}\}_{i=1}^r$ is linearly independent.

Proof. Supposing to the contrary, and without loss of generality, let

$$B_{r-1} = B_r^{(r)} = \sum_{i=1}^{r-1} \alpha_i B_r^{(i)} = \left(\sum_{i=1}^{r-1} \alpha_i B_{r-1}^{(i)} \right) \wedge b_r.$$

Now notice that

$$0 \neq B_r = B_{r-1} \wedge b_r = B_r^{(r)} \wedge b_r = \left(\sum_{i=1}^{r-1} \alpha_i B_r^{(i)} \right) \wedge b_r = 0,$$

which is clearly a contradiction. □

We will need a result similar to Lemma 3.5 as concerning spades. It is as follows.

Lemma 3.6. *Letting $M_r^{(i)}$ denote the spade*

$$M_r^{(i)} = \prod_{\substack{j=1 \\ j \neq i}}^{r} m_i,$$

if $0 \neq \bigwedge_{i=1}^r m_i$, then the set $\{M_r^{(i)}\}_{i=1}^r$ is a linearly independent set.

Proof. We first consider, for all $j \in [0, r]$, the j equations given by

$$0 = \sum_{i=1}^{r} \alpha_i \langle M_r^{(i)} \rangle_j.$$

Letting A_k denote the set of all solutions in each α_i to equation j, it is clear that the set of all solutions A in each α_i to the equation

$$0 = \sum_{i=1}^{r} \alpha_i M_r^{(i)}$$

is given by

$$A = \bigcap_{i=1}^{k} A_k.$$

Thus, it suffices to show that the set $\{\langle M_r^{(i)} \rangle_{r-1}\}_{i=1}^r$ is a linearly independent set. Now since $0 \neq \bigwedge_{i=1}^r m_i$, it is clear that

$$\langle M_r^{(i)} \rangle_{r-1} = \bigwedge_{\substack{j=1 \\ j \neq i}}^{r} m_i.$$

Seeing this, the linear independence of the set $\{\langle M_r^{(i)} \rangle_{r-1}\}_{i=1}^r$ follows immediately from Lemma 3.5. $\square$

4. Blades As Representatives Of Geometric Sets

5. Spades As Representatives Of Geometric Sets

References

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Spencer T. Parkin
e-mail: `spencerparkin@outlook.com`