Chapter 5 Exercises
Gallian's Book on Abstract Algebra

Spencer T. Parkin

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Problem 10

Show that a function from a finite set S to itself is one-to-one if and only if it is onto. Is this true when S is infinite?

If we show that a function from a finite set S_0 to another S_1 of the same cardinality is one-to-one if and only if it is onto, then we have shown the desired result if we let $S_0 = S_1$.

That said, let S_0 be a set of n pigeons, and S_1 be a set of n pigeon holes. If $f : S_0 \to S_1$ is not onto, then at least one pigeon hole must be in use by more than one pigeon, and therefore, f is not one-to-one. Conversely, if $f : S_0 \to S_1$ is not one-to-one, then at least one pigeon hole is in use by more than one pigeon. But this is impossible without leaving at least one pigeon hole out of use, so f is not onto.

Problem 12

If α is even, prove that α^{-1} is even. If α is odd, prove that α^{-1} is odd.

Write α as a product of disjoint cycles. Notice that each n-cycle can be written as a product of $n-1$ 2-cycles. Now notice that α^{-1} can be written as a product of disjoint cycles by taking each n-cycles for α and reversing the winding order of the cycle to produce that cycle's inverse, which is also an n-cycle. It follows that α^{-1} can be written as the same number of disjoint 2-cycles as can be done for α.

Problem 13

Prove Theorem 5.6 – The set of even permutations in S_n forms a subgroup of S_n.

Note that the identity permutation is even. Note that the product of any two even permutations is even. Lastly, notice that by Problem 12 above, the inverse of any even permutation is also in the set of even permutations.

Problem 19

Show that if H is a subgroup of S_n, then either every member of H is an even permutation or exactly half of the members are even.

If all members of H are even, we're done. So assume that not all members of H are even. Let $\beta \in H$ be odd and consider the function $\phi(\alpha) = \alpha\beta$. Notice that ϕ maps the set of all even permutation of S_n into the set of all odd permutations of S_n. Being a well defined function, it follows that there are at least as many even permutations as there are odd permutations. On the other hand, notice that ϕ maps the set of all odd permutations of S_n into the set of all even permutations of S_n. Being a well defined function, it follows that there are at least as many odd permutations as there are even permutations.

Problem 21

Do the odd permutations of S_n form a group? Why?

No. We don't have closure. And the identity is not odd.

Problem 22

Let α and β belong to S_n. Prove that $\alpha^{-1}\beta^{-1}\alpha\beta$ is an even permutation.

By Problem 12, α^{-1} has the same parity as α. The same can be said of β^{-1} and β. Now simply realize that for all 4 cases of parity between α and β, each of the two parities appears twice in the expression, so that any odd parity gets canceled, letting the net parity always be even.

Problem 31

Let G be a group of permutations on a set X. Let $a \in X$ and define $\text{stab}(a) = \{a \in G | \alpha(a) = a\}$. We call $\text{stab}(a)$ the *stabilizer of a in G* (since it consists of all members of G that leave a fixed). Prove that $\text{stab}(a)$ is a subgroup of G.

It is clear that the identity permutation is in $\text{stab}(a)$, since it leaves all elements of X invariant. Let $\alpha \in \text{stab}(a)$. Then, since $\alpha(a) = a$, we have $\alpha^{-1}(a) = a$, showing that $\alpha^{-1} \in \text{stab}(a)$. Now let $\alpha, \beta \in \text{stab}(a)$. Then, since $(\alpha\beta)(a) = \alpha(\beta(a)) = \alpha(a) = a$, we have $\alpha\beta \in \text{stab}(a)$.

Problem 34

Let $H = \{\beta \in S_5 | \beta(1) = 1 \text{ and } \beta(3) = 3\}$. Prove that H is a subgroup of S_5. Is your argument valid when 5 is replaced by any $n \geq 3$?

Let i and j be distinct positive integers and let

$$H = \{\beta \in S_n | \beta(i) = i \text{ and } \beta(j) = j\},$$

where $n \geq \max(i, j)$. Now notice that

$$H = \text{stab}(i) \cap \text{stab}(j).$$

It is then clear that H is a subgroup of S_n, because the intersection of any two subgroups is a subgroup.