

A Model Of Algebraic Sets Using Geometric Algebra

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Abstract. Blah.

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1. Introduction And Motivation

Letting \mathbb{R}^n denote an n -dimensional Euclidean space, we are going to let \mathbb{P}^n denote the set of all polynomials $f : \mathbb{R}^n \rightarrow \mathbb{R}$ of any degree. Then, letting $P \subseteq \mathbb{P}^n$ be any set of such polynomials, we define the zero set Z of P , denoted $Z(P)$, as the set given by

$$Z(P) = \{x \in \mathbb{R}^n | f(x) = 0 \text{ for all } f \in P\}. \quad (1.1)$$

Every subset S of \mathbb{R}^n for which there exists a set $P \subseteq \mathbb{P}^n$ such that $S = Z(P)$ is what we refer to as an algebraic set. It is well known that for any algebraic set $S \subseteq \mathbb{R}^n$, there is always such a subset P of \mathbb{P}^n of finite cardinality.

Given the definition in equation (1.1), it is easy to show that the subsets S of \mathbb{R}^n that are the geometries of CGA, and other similar models of geometry based upon geometric algebra, are simply algebraic sets. The goal of this paper is to show that there exists a model of geometry, based upon geometric algebra, where every possible algebraic set has a representative in the form of an element of that geometric algebra. A desire to come up with such a model of geometric algebra is motivated by the admittedly fanciful dream of the German mathematician Leibniz, referred to in [] and claimed to have already been realized in []. In any case, it would seem that a generalization of CGA or any CGA-like model to one that hosts the set of all algebraic sets in \mathbb{R}^n would bring us closer to such a goal. Work to this end has already been done in [1, 2] which has brought us, the reader and writer, to the present paper.

2. Blades As Algebraic Sets

We begin with an examination of \mathbb{P}^n as a linear space, observing that it is of countably infinite dimension. A set of basis vectors for this space may be taken as the set of all unit-monomials in anywhere from 1 to n of the n variable components of an arbitrary point in \mathbb{R}^n .¹ Considering now any mapping from the set \mathbb{Z}^+ of positive integers to this said set $\{g_i\}_{i=1}^\infty$ of unit-monomials,² and letting \mathbb{V}^∞ denote a simple Euclidean vector space of countably infinite dimension, if we define the function $p: \mathbb{R}^n \rightarrow \mathbb{V}^\infty$ as

$$p(x) = \sum_{i=1}^{\infty} g_i(x) e_i, \quad (2.1)$$

where $\{e_i\}_{i=1}^\infty$ is any orthonormal basis for \mathbb{V}^∞ , then for any polynomial $f \in \mathbb{P}^n$, there must exist a unique³ vector $v \in \mathbb{V}^\infty$ such that

$$f(x) = p(x) \cdot v. \quad (2.2)$$

It now follows that for any subset P of \mathbb{P}^n , there exists a blade $B \in \mathbb{G}(\mathbb{V}^\infty)$, such that

$$Z(P) = \{x \in \mathbb{R}^n | p(x) \cdot B = 0\}. \quad (2.3)$$

Clearly, we need only consider such finite sets $P = \{f_i\}_{i=1}^k$ that are linearly independent sets of k polynomials. Then, for each polynomial f_i , there is an associated vector v_i by equation (2.2), and the set $\{v_i\}_{i=1}^k$ must clearly be a linearly independent set of k vectors. We may then take B to be the k -blade

$$B = \bigwedge_{i=1}^k v_i, \quad (2.4)$$

seeing that the equation $p(x) \cdot B = 0$ becomes

$$0 = - \sum_{i=1}^k (-1)^i (p(x) \cdot v_i) B_i, \quad (2.5)$$

where B_i denotes the product B with v_i removed. Realize that $\{B_i\}_{i=1}^k$ is a linearly independent set of $(k-1)$ -blades, and therefore $p(x) \cdot B = 0$ if and only if $p(x) \cdot v_i = 0$ for all $i \in \mathbb{Z}_k$. For convenience, we will let \mathbb{Z}_k denote the set of k integers in $[1, k] \cap \mathbb{Z}^+$.

Already we have fulfilled the promise of the introductory section of this paper, but we will continue now with one further development that is

¹The number c_k of such monomials homogeneous of a degree k is given by

$$c_k = \sum_{i=1}^n \binom{n}{i} p(k, i),$$

where $p(k, i)$ is the number of partitions of size i of the integer k . The combinatorics of the matter are not important, but they are pointed out here as an added measure of the reality of the set of monomials we are talking about.

²For example, if $n = 2$, then we might take $g_1(x, y) = 1$, $g_2(x, y) = x$, $g_3(x, y) = y$, $g_4(x, y) = x^2$, $g_5(x, y) = xy$, $g_6(x, y) = y^2$, and so on.

³Let $v_1, v_2 \in \mathbb{V}^\infty$ be two such vectors. Then $0 = f(x) - f(x) = p(x) \cdot (v_1 - v_2)$. Now notice that there does not exist $x \in \mathbb{R}^n$ such that $p(x) = 0$. It follows that $v_1 = v_2$.

motivated by a desire to pair our current ability to intersect geometries with an ability to take their union.

We start by letting $\{\mathbb{V}_i^\infty\}_{i=1}^\infty$ be a countably infinite set of vector spaces isomorphic to \mathbb{V}^∞ . With the exception of the zero vector, we consider these vector spaces as pair-wise disjoint, so that for any pair of non-zero vectors $v_i \in \mathbb{V}_i^\infty$ and $v_j \in \mathbb{V}_j^\infty$ with $i \neq j$, we have $v_i \cdot v_j = 0$. Let \mathbb{V} simply denote the vector space spanned by the set of vectors $\cup_{i \in \mathbb{Z}^+} \{e_{ij}\}_{j=1}^\infty$, where for each $i \in \mathbb{Z}^+$, the set $\{e_{ij}\}_{j=1}^\infty$ is an orthonormal basis for the vector space \mathbb{V}_i^∞ . It is clear that the dimension of \mathbb{V} , like that of \mathbb{V}^∞ above, is countably infinite as the union of countably many countable sets is countable.⁴

Given a blade $B \in \mathbb{G}(\mathbb{V})$, we now define a function q as a mapping from the set of all blades in $\mathbb{G}(\mathbb{V})$ to \mathbb{Z}^+ , so that $i \in q(B)$ if and only if there exists a vector $v \in \mathbb{V}_i^\infty$ such that $v \wedge B = 0$. With this in place, we define the set \dot{G} of B , denoted $\dot{G}(B)$, as the set

$$\dot{G}(B) = \left\{ x \in \mathbb{R}^n \left| \bigwedge_{i \in q(B)} p_i(x) \cdot B = 0 \right. \right\}, \quad (2.6)$$

where for each $i \in \mathbb{Z}^+$, we define p_i similar to equation (2.1) as

$$p_i(x) = \sum_{j=1}^{\infty} g_j(x) e_{ij}. \quad (2.7)$$

For any product of the form $\prod_{i \in S}$ or $\bigwedge_{i \in S}$, where S is a set of positive integers, we take the terms in the product to be in ascending order.

Similarly, given a blade $B \in \mathbb{G}(\mathbb{V})$, we will define the set $\hat{G}(B)$ as

$$\hat{G}(B) = \left\{ x \in \mathbb{R}^n \left| \bigwedge_{i \in q(B)} p_i(x) \wedge B = 0 \right. \right\}. \quad (2.8)$$

We now proceed to investigate the consequences of these two definitions.

⁴This may come as some comfort, since if you already thought that the dimension of \mathbb{V}^∞ was ridiculously huge, the dimension of the vector space \mathbb{V} is no bigger. In practice, of course, a computer program, for example, would work in a finite-dimensional vector space, thereby restricting the number of possible geometries to those that are zero sets of polynomials of a specific form. This is exactly what's going on in CGA, with the additional modification of altering the signature of the geometric algebra to accommodate the conformal transformations. Non-Euclidean signatures are not considered in this paper.

3. The Algebraic Set \dot{G}

4. The Algebraic Set \hat{G}

5. Relating \dot{G} with \hat{G}

Given any two blades $A, B \in \mathbb{G}(\mathbb{V})$, we can now show that there exists a rotor $R \in \mathbb{G}(\mathbb{V})$, such that

$$G(A) \cup G(B) = G(A \wedge B'), \quad (5.1)$$

where $B' = RBR^{-1}$. Letting R be the rotor given by

$$R = \prod_{i \in q(A)} \prod_{j=1}^{\infty} (1 - e_{ij} e_{(i+k)j}), \quad (5.2)$$

where $k = \max q(A)$, observe that $q(A) \cap q(B')$ is empty. If it was already the case that $q(A) \cap q(B)$ is empty, then we may have taken R as simply the identity rotor. In any case, a rotor $R \in \mathbb{G}(\mathbb{V})$ always exists such that for any pair of blades A and B , the sets $q(A)$ and $q(B')$ are disjoint. Furthermore, it is not hard to show that for the rotor R given in equation (5.2), we have $G(B) = G(B')$. Then, without loss of generality, we may rewrite $C = A \wedge B'$ as $C = \bigwedge_{i=1}^k C_k$, where for each $i \in \mathbb{Z}_k$, we have $C_i \in \mathbb{G}(V_i^\infty)$. (Notice that $q(C) = \mathbb{Z}_k$.) It is now clear that

$$\bigwedge_{i \in \mathbb{Z}_k} p_i(x) \cdot C = \pm \bigwedge_{i \in \mathbb{Z}_{k-1}} p_i(x) \cdot (p_k(x) \cdot C_k) \wedge \bigwedge_{i \in \mathbb{Z}_{k-1}} C_i \quad (5.3)$$

$$= \pm \bigwedge_{i \in \mathbb{Z}_k} p_i(x) \cdot C_i. \quad (5.4)$$

(Notice that the outer product in (5.4) becomes a scalar product in the case that every C_i is a vector.) It is now clear that $G(C)$ is indeed the union of $G(A)$ and $G(B)$.

Having established the union operation, it is not clear at this point how the intersection operation is preserved. We begin with the observation that for all blades $A, B \in \mathbb{V}$ such that $q(A) = q(B)$ with $|q(A)| = |q(B)| = 1$, we have $G(A) \cap G(B) = G(A \wedge B)$. On the other hand, if $q(A) \neq q(B)$ with the persisting requirement that $|q(A)| = |q(B)| = 1$, then there clearly exists a rotor R such that $G(A) \cap G(B) = G(A \wedge B')$, where $B' = RBR^{-1}$. Performing an intersection in the case that one of $|q(A)|$ and $|q(B)|$ is not one requires a little more effort. What we'll show is that, in general, for any two blades $A, B \in \mathbb{G}(\mathbb{V})$, there exist a blade $C \in \mathbb{G}(\mathbb{V})$ such that

$$G(A) \cap G(B) = G(C). \quad (5.5)$$

We begin by making the observation that

$$\bigcup_{i \in q(A)} G(A_i) \cap \bigcup_{i \in q(B)} G(B_i) = \bigcup_{\substack{i \in q(A) \\ j \in q(B)}} G(A_i) \cap G(B_j), \quad (5.6)$$

where for each $i \in q(A)$, we have $A = \bigwedge_{i \in q(A)} A_i$ with each $A_i \in \mathbb{G}(\mathbb{V}_i^\infty)$. The same can be said for each B_i in terms of B . We now carefully select a set of rotors $\{R_{ij}\}$ and $\{S_{ij}\}$, each of size $|q(A)||q(B)|$, such that

$$C = \bigwedge_{\substack{i \in q(A) \\ j \in q(B)}} A'_i \wedge B'_j, \quad (5.7)$$

where for each $(i, j) \in q(A) \times q(B)$, we have

$$A'_i = R_{ij} A_i R_{ij}^{-1}, \quad (5.8)$$

$$B'_j = S_{ij} B_j S_{ij}^{-1}, \quad (5.9)$$

such that

$$G(A_i) = G(A'_i), \quad (5.10)$$

$$G(B_j) = G(B'_j), \quad (5.11)$$

$$q(A'_i) = q(B'_j), \quad (5.12)$$

and such that for all $(i, j) \neq (k, l)$, we have $q(A'_i \wedge B'_j) \cap q(A'_k \wedge B'_l)$ an empty set. Admittedly, this is quite terrible.

Notice that, assuming $A \wedge B \neq 0$, the grade of $A \wedge B$ is not necessarily the grade of $A' \wedge B'$. This means that we cannot always apply a rotor to one or both of A and B before taking them in an outer product to perform an intersection operation.

6. Comments And Criticisms

A valid complaint, which the reader may have had by this point, is that in most circumstances, a rotor needs to be applied to one or both of the two blades A and B before they may be taken in the outer product in the performance of a union or intersection operation. Understandably, something like this has to happen if, borrowing a computer science term, we are able to mathematically overload the outer product. Owing to the definition of the q function, what's further difficult about this is that, prior to a union or intersection operation, one would, in practice, need to perform some sort of analysis on A and B .

References

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