Prime Formula

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Considering the prime number sieve, it is natural to ask: is there an expression in the first $n-1$ primes for the nth prime? Everything that follows is an attempt to answer that question, and is a result of Mark Tiefenbruck.

We begin with the well-known zeta function.

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_{i=1}^{\infty} \frac{1}{1 - p_i^{-s}} \tag{1}$$

Focusing on all but the first $n-1$ factors, we write

$$\zeta(s) \prod_{i=1}^{n-1} (1 - p_i^{-s}) = \prod_{i=n}^{\infty} \frac{1}{1 - p_i^{-s}} = 1 + \sum_{q \in Q} \frac{1}{q^s}, \tag{2}$$

where the set Q is given by

$$Q = \{q \geq p_n | p_i \nmid q \text{ for all } 1 \leq i < n\}.$$

We're now going to leave equation (2) for a moment and return to it later.

Let r_i be an infinite sequence of real numbers with every $r_i > 1$. We now wish to find the following limit.

$$\lim_{s \to \infty} \left[\sum_{i=1}^{\infty} \frac{1}{r_i^s} \right]^{-1/s} = R$$

By rearrangement, this becomes

$$\lim_{s \to \infty} \left[\left(\frac{R}{r_j} \right)^s + \sum_{i \neq j} \left(\frac{R}{r_i} \right)^s \right]^{-1/s} = 1.$$

From this it can be seen that $R = r_j = \min\{r_i\}$.

Returning to equation (2), we now see that

$$\lim_{s\to\infty} \left[-1 + \zeta(s) \prod_{i=1}^{n-1} (1 - p_i^{-s}) \right]^{-1/s} = \min Q = p_n.$$

We now have a formula for p_n in terms of each p_1 through p_{n-1}. One of the unsatisfactory things about this, however, is that we're still dependent upon the zeta function, which in turn depends on all the primes. Nevertheless, we can expression $\zeta(2s)$, for example, in terms of π and ther Bernoulli numbers; so it's arguable that we have still as yet found a formula of the desired form.