Chapters 1-4 Supplementary Exercises
Gallian's Book on Abstract Algebra

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Problem 1

Let G be a group and let H be a subgroup of G. For any fixed $x \in G$, define $xHx^{-1} = \{xhx^{-1} | h \in H\}$. Prove that xHx^{-1} is a subgroup of G, that if H is cyclic, then xHx^{-1} is cyclic, and that if H is Abelian, then xHx^{-1} is Abelian.

Clearly $e \in xHx^{-1}$. Letting $a, b \in xHx^{-1}$, there exist elements $h_a, h_b \in H$ such that $a = xh_ax^{-1}$ and $b = xh_bx^{-1}$. Now since $h_ah_b^{-1} \in H$, we see that

$$ab^{-1} = xh_ax^{-1}(xh_bx^{-1})^{-1} = xh_ax^{-1}xh_b^{-1}x^{-1} = xh_ah_b^{-1}x^{-1} \in xHx^{-1}.$$

Now if H is cyclic, then there exists $h \in H$ such that $H = \langle h \rangle$. We then see that

$$xHx^{-1} = \{xh^kx^{-1} | k \in \mathbb{Z}\} = \{(xhx^{-1})^k | k \in \mathbb{Z}\} = \langle xhx^{-1} \rangle.$$

If H is Abelian, then for all $a, b \in xHx^{-1}$, we have

$$ab = xh_ax^{-1}xh_bx^{-1} = xh_ah_bx^{-1} = xh_bh_ax^{-1} = xh_bx^{-1}xh_ax^{-1} = ba.$$

Problem 2

Let G be a group and let H be a subgroup of G. Define

$$N(H) = \{x \in G | xHx^{-1} = H\}.$$

Prove that $N(H)$ (called the *normalizer* of H) is a subgroup of G.

It is clear that $e \in N(H)$. Now let $a, b \in N(H)$. Then since $aHa^{-1} = H$ and $bHb^{-1} = H$, we have

$$abH(ab)^{-1} = abHb^{-1}a^{-1} = aHa^{-1} = H,$$

showing that $ab \in N(H)$. Now notice that since $aHa^{-1} = H$, the function $\phi(a) = aha^{-1}$ is a bijection from H to H. It follows that $\phi^{-1}(a) = a^{-1}ha$ is also such a bijection, and therefore, $a^{-1}Ha = H$, showing that $a^{-1} \in N(H)$.

Problem 3

Let G be a group. For each $a \in G$, define $\mathrm{cl}(a) = \{xax^{-1} | x \in G\}$. Prove that these subsets of G partition G. [$\mathrm{cl}(a)$ is called the *conjugacy class* of a.]

For any $a, b \in G$, let $a \sim b$ if and only if there exists $x \in G$ such that $a = xbx^{-1}$. We now show that this is an equivilance relation on G.

Notice that $a \sim a$, since $a = eae^{-1}$, giving us the reflexive property. Then, letting $y = x^{-1} \in G$, we see that

$$a \sim b \implies a = xbx^{-1} \implies b = yay^{-1} \implies b \sim a,$$

giving us the symmetric property. Lastly, for $a, b, c \in G$, let $a \sim b$ and $b \sim c$ so that for some $x, y \in G$, we have $a = xbx^{-1}$ and $b = ycy^{-1}$. Then we have

$$a = xbx^{-1} = xycy^{-1}x^{-1} = xyc(xy)^{-1} \implies a \sim c,$$

since $xy \in G$, giving us the transitive property.

Seeing now that for any $a \in G$, we have

$$\begin{aligned} \mathrm{cl}(a) &= \{xax^{-1} | x \in G\} \\ &= \{b \in G | \exists x \in G \text{ s.t. } b = xax^{-1}\} \\ &= \{b \in G | b \sim a\}, \end{aligned}$$

it follows by Theorem 0.6 that the conjugacy classes of G partition G.