Chapters 9-11 Supplementary Exercises
Gallian's Book on Abstract Algebra

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Exercise 8

Let k be a divisor of n. The factor gruop $(Z/\langle n\rangle)/(\langle k\rangle/\langle n\rangle)$ is isomorphic to some very familiar group. What is the group?

By Exercise 40 of Chapter 10 (The Third Isomorphism Theorem), we see that $(Z/\langle n\rangle)/(\langle k\rangle/\langle n\rangle) \approx Z/\langle k\rangle$. What more is there to say?

Exercise 30

Let G be a group and let $\phi : G \to G$ be a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1g_2g_3 = e = h_1h_2h_3$. Prove that there exists an element of a in G such that $\Psi(x) = a\phi(x)$ is a homomorphism.

If $a = e \in G$ and $u, v \in G$, then

$$\Psi((uv)^{-1})\Psi(u)\Psi(v) = e \implies \Psi(v^{-1}u^{-1}) = \Psi(v)^{-1}\Psi(v)^{-1}.$$

Hmmm... Can we somehow show that Ψ is a homomorphism here? We cannot use homomorphic properties of Ψ before we know that it's a homomorphism.

Exercise 36

A proper subgroup H of a group G is called *maximal* if there is no subgroup K such that $H \subset K \subset G$. Prove that Q under addition has no maximal subgroups.

This is a very difficult problem, and after giving it a great deal of thought, the best I can come up with so far is the following hypothesis.

Let H be any subgroup of Q and let B be a subset of H of smallest possible cardinality such that

$$H = \{z_1 b_1 + \cdots + z_k b_k | z_i \in Z, b_i \in B, k \in Z^+\}.$$

Call such a subset B of H a basis for H. If B is of finite cardinality k, then we may write

$$H = \langle b_1, \ldots, b_k \rangle.$$

It is easy to show that if B is of finite cardinality, then H is a proper subgroup of Q. Also, if B is a basis for Q, then B is infinite. The converse of either of these statements, however, is not obvious. Let's suppose for the moment, however, that H is a proper subgroup of Q if and only if B is of finite cardinality. If then H is a proper subgroup of Q and we let $q \in Q - H$, it is clear that

$$\langle b_1, \ldots, b_k \rangle + \langle q \rangle = \langle b_1, \ldots, b_k, q \rangle$$

is a subgroup of Q that properly contains H. That it is a proper subgroup of Q follows from our assumption above and a realization that a basis for $H + \langle q \rangle$ is $B \cup \{q\}$, which is clearly finite.

Notice that all subgroups of Q of the form $\langle q \rangle$ for some non-zero $q \in Q$ are minimal subgroups of Q isomoprhic to Z.

Can an example be found that disproves the assumption? Can the assumption be proved? One approach to proving it is to take a subgroup H of Q having an infinite basis B and showing that for any $q \in Q$, we have $q \in H$. This would show that $H = Q$. It may be easy to show that q is always a limit point of H, but this does not imply membership in H.

After some thought, I see that this approach doesn't work at all. I can think of many proper subgroups of Q for which no basis can exist. Consider, for example,

$$\langle q \rangle + \langle q/2 \rangle + \langle q/2^2 \rangle + \langle q/2^3 \rangle + \ldots,$$

where $q \in Q$. There is no basis for this subgroup of Q.