Chapters 1-4 Supplementary Exercises
Gallian's Book on Abstract Algebra

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Problem 1

Let G be a group and let H be a subgroup of G. For any fixed $x \in G$, define $xHx^{-1} = \{xhx^{-1} | h \in H\}$. Prove that xHx^{-1} is a subgroup of G, that if H is cyclic, then xHx^{-1} is cyclic, and that if H is Abelian, then xHx^{-1} is Abelian.

Clearly $e \in xHx^{-1}$. Letting $a, b \in xHx^{-1}$, there exist elements $h_a, h_b \in H$ such that $a = xh_ax^{-1}$ and $b = xh_bx^{-1}$. Now since $h_ah_b^{-1} \in H$, we see that

$$ab^{-1} = xh_ax^{-1}(xh_bx^{-1})^{-1} = xh_ax^{-1}xh_b^{-1}x^{-1} = xh_ah_b^{-1}x^{-1} \in xHx^{-1}.$$

Now if H is cyclic, then there exists $h \in H$ such that $H = \langle h \rangle$. We then see that

$$xHx^{-1} = \{xh^kx^{-1} | k \in \mathbb{Z}\} = \{(xhx^{-1})^k | k \in \mathbb{Z}\} = \langle xhx^{-1} \rangle.$$

If H is Abelian, then for all $a, b \in xHx^{-1}$, we have

$$ab = xh_ax^{-1}xh_bx^{-1} = xh_ah_bx^{-1} = xh_bh_ax^{-1} = xh_bx^{-1}xh_ax^{-1} = ba.$$

Problem 2

Let G be a group and let H be a subgroup of G. Define

$$N(H) = \{x \in G | xHx^{-1} = H\}.$$

Prove that $N(H)$ (called the *normalizer* of H) is a subgroup of G.

It is clear that $e \in N(H)$. Now let $a, b \in N(H)$. Then since $aHa^{-1} = H$ and $bHb^{-1} = H$, we have

$$abH(ab)^{-1} = abHb^{-1}a^{-1} = aHa^{-1} = H,$$

showing that $ab \in N(H)$. Now notice that since $aHa^{-1} = H$, the function $\phi(a) = aha^{-1}$ is a bijection from H to H. It follows that $\phi^{-1}(a) = a^{-1}ha$ is also such a bijection, and therefore, $a^{-1}Ha = H$, showing that $a^{-1} \in N(H)$.

Problem 3

Let G be a group. For each $a \in G$, define $\mathrm{cl}(a) = \{xax^{-1} | x \in G\}$. Prove that these subsets of G partition G. [$\mathrm{cl}(a)$ is called the *conjugacy class* of a.]

For any $a, b \in G$, let $a \sim b$ if and only if there exists $x \in G$ such that $a = xbx^{-1}$. We now show that this is an equivilance relation on G.

Notice that $a \sim a$, since $a = eae^{-1}$, giving us the reflexive property. Then, letting $y = x^{-1} \in G$, we see that

$$a \sim b \implies a = xbx^{-1} \implies b = yay^{-1} \implies b \sim a,$$

giving us the symmetric property. Lastly, for $a, b, c \in G$, let $a \sim b$ and $b \sim c$ so that for some $x, y \in G$, we have $a = xbx^{-1}$ and $b = ycy^{-1}$. Then we have

$$a = xbx^{-1} = xycy^{-1}x^{-1} = xyc(xy)^{-1} \implies a \sim c,$$

since $xy \in G$, giving us the transitive property.

Seeing now that for any $a \in G$, we have

$$\begin{aligned}\mathrm{cl}(a) &= \{xax^{-1} | x \in G\} \\ &= \{b \in G | \exists x \in G \text{ s.t. } b = xax^{-1}\} \\ &= \{b \in G | b \sim a\},\end{aligned}$$

it follows by Theorem 0.6 that the conjugacy classes of G partition G.

Problem 5

Prove that, in any group, $|xax^{-1}| = |a|$.

Let $a_i = (xax^{-1})^i = xa^ix^{-1}$. Suppose $a_i = e$ for $0 \leq i < |a|$. Then $xa^ix^{-1} = e \implies a^i = e$, which is a contradiction. Therefore, since $a_{|a|} = e$, we see that $|a_i| = e$.

Problem 15

Let G be an Abelian group and let n be a fixed positive integer. Let $G^n = \{g^n | g \in G\}$. Prove that G^n is a subgroup of G. Give an example showing that G^n need not be a subgroup of G when G is non-Abelian.

Clearly $e \in G^n$, since $e^n = e$. Then, for any $a, b \in G^n$, there exists $g_a, g_b \in G$ such that $a = g_a^n$ and $b = g_b^n$, and we see that

$$ab^{-1} = g_a^n (g_b^n)^{-1} = g_a^n (g_b^{-1})^n = (g_a g_b^{-1})^n \in G^n,$$

by the Abelian property of G.

I'm failing to come up with an example.

Problem 18

Prove that the subset of elements of finite order in an Abelian group forms a subgroup. Is the same thing true for non-Abelian groups?

Let H be the said subset of an Abelian group G. Clearly $e \in H$, since $|e| = 1$. For all $a, b \in H$, we see that

$$(ab)^{|a||b|} = a^{|a|} b^{|b|} = e,$$

by the Abelian property of G, showing that $|ab|$ divides $|a||b|$ and therefore $|ab| \leq |a||b|$, so $ab \in H$. Lastly, notice that for all $a \in H$, $|a^{-1}| = |a|$, so $a^{-1} \in H$.

We need to find an example of two elements having finite order whose product has infinite order...

Problem 26

Let H be a subgroup of a group G and let $|g| = n$. If g^m belongs to H and m and n are relatively prime, prove that g belongs to H.

Since $g^m \in H$, we see that $\langle g^m \rangle \leq H$. Then, by Theorem 4.2, notice that $\langle g^m \rangle = \langle g^{\gcd(n,m)} \rangle = \langle g \rangle$, so that clearly $g \in H$ also.

Problem 34

Suppose that G is a group that has exactly one nontrivial proper subgroup. Prove that G is cyclic and $|G| = p^2$, where p is prime.

Let $\{e\} < H < G$. Then, for any non-identity $h \in H$, $\langle h \rangle$ is a subgroup of H, but it cannot be a proper subgroup. Therefore, $\langle h \rangle = H$. Furthermore, since for all non-identity $h \in H$, we have $\langle h \rangle = H$, we see that H has one and only one non-trivial cyclic subgroup; namely, itself. Therefore, H being cyclic, and non-trivial, we see that $|H|$ must be prime by Theorem 4.3. Let $|H| = p$.

Now choose $x \in G - H$. Clearly $x \neq e$. Consider $\langle x \rangle$. This must be H or G. But if $\langle x \rangle = H$, then $x \in H$, which is a contradiction. Therefore, $\langle x \rangle = G$. Now since H is a proper subgroup of G, $|H|$ is a non-trivial divisor of $|G|$. So $|G| = pk$ for some integer $k > 1$. But H is the only proper subgroup of G, and so $|H|$ is the only non-trivial divisor of $|G|$. Therefore, $k = p$. (1 and $|G|$ are the trivial divisors of $|G|$.)

Problem 45

Let G be a cyclic group of order n and let H be the subgroup of order d. Show that $H = \{x \in G || x| \text{ divides } d\}$.

For an $x \in G$ such that $|x|$ divides d, consider the subgroup $\langle x \rangle$. G being cyclic, there is one and only one subgroup of G of order $|x|$, namely $\langle x \rangle$. Now, seeing that $|x|$ is a divisor of d, H must have one and only one subgroup of order $|x|$, call it K. But then K is also a subgroup of G, and therefore, we must have $K = \langle x \rangle$, showing that $x \in H$.