Notes On Euler Angles

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The Application Of Euler Angles

Let R_x, R_y and R_z each be unit-rotors that rotate about the x, y and z axes, respectively. Then it is easy to imagine the rotation generated by the rotor $R_z R_y R_x$. A point is rotated about the x-axis, then the y, and then the z. With each rotation is about a fixed axis. We can similarly deduce the affect of the rotation generated by the rotor $R_x R_y R_z$. However, we are free to think of the rotation generated by $R_x R_y R_z$ in the order of first rotating about the x-axis, then a (not the) y-axis, and then a (not the) z-axis, if, to be more specific, we think about such a y-axis as a rotation of the y-axis. Similarly, we must think of about such a z-axis as a rotation of the z-axis. This can all be explained algebraiclyas follows. Consider a vector v to be rotated by $R_x R_y R_z$. Instead of applying R_z first, which is what we would naturally want to do to get the rotation v' of v as

$$v' = R_x R_y R_z v \tilde{R}_z \tilde{R}_y \tilde{R}_x, \tag{1}$$

we will apply R_x first, and write

$$v' = R(R_x R_y \tilde{R}_x) R_x v \tilde{R}_x (R_x \tilde{R}_y \tilde{R}_x) \tilde{R}, \tag{2}$$

where the rotor R is given by

$$R = ((R_x R_y \tilde{R}_x) R_x) R_z ((R_x R_y \tilde{R}_x) R_x)^{-1}.$$

Here we must realize that we can apply rotors to rotors to rotate the axis of rotation about which a rotor rotates. (Pardon the tongue twister.) The pattern here is seen as an application of R_x first, then the application of

R_y with what was first applied to v, (namely R_y), applied to it, then the application of R_z with what was then applied to v, (namely R), applied to it.

Visualizing the rotation of the vector v in equation (2), we see that with each rotation, we must also apply that rotation to the xyz-axis frame about which we do subsequent rotations. Indeed, this is how Euler rotations are applied. Interestingly, what we find is that when we simplify equation (2), we arrive at equation (1). It follows now that the application of the Euler angles first in the x, then y, then z-axis, is accomplished by a rotor of the form $R_x R_y R_z$.

Finding Euler Angles

In this section we concern ourselves with the extrapolation of a set of Euler angles for a given rotation axis order from a given unit-rotor R. If one were to take a purely algebraic approach to this problem, they would quickly arrive at an undetermined system of non-linear equations, which seem quite intractable. A different and perhaps better approach is to take what we know about the application of Euler angles to come up with a solution to the problem. Our approach here will be in terms of visualizing the original and final coordinate frames, and to work by using the first two Euler angles to align a single axis. The remaining Euler angle will then rotate the two frames into complete alignment. Clearly the axis we choose to align must coincide with the same axis about which the final Euler angle will rotate.