An Introduction To Geometric Sets

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1. Introduction

1.1. Motivation

1.2. Conventions

This paper uses capital letters A, B, C to denote blades of various grades, while using lower case letters a, b, c to denote vectors. Scalars are written using greek letters α, β, γ. Grades of blades A, B, C are denoted by lower case letters r, s, t, respectively. Lower case letters i, j, k are used as indices. Capital letters R, S, T are used to denote subsets of interest of n-dimensional euclidean space $\mathbb{R}^n$. Lower case letters x, y, z are reserved for denoting points taken from $\mathbb{R}^n$. We will use $\mathbb{G}$ to denote our geometric algebra, and $\mathbb{V}$ to denote an m-dimensional vector space generating it. $\mathbb{B}$ will denote the set of all blades taken from $\mathbb{G}$. The scalars of $\mathbb{V}$, and therefore $\mathbb{G}$, are taken from the field of real numbers $\mathbb{R}$. The capital letter I will be used to denote the unit psuedo-scalar of $\mathbb{G}$. We assume that I is invertible with respect to the geometric product.

We will let the geometric product take precedence over the inner and outer products, and the outer product take precedence over the inner product.

No particular signature of our geometric algebra $\mathbb{G}$ is assumed in this paper.

2. Preliminaries

The results of this paper will depend upon the following preliminary material. If desired, the reader is welcome to skip this material and refer back to it only as needed.

2.1. Identities

Given a vector $a \in \mathbb{V}$ and a blade $A \in \mathbb{B}$, central to all of geometric algebra is the identity

$$aA = a \cdot A + a \wedge A. \tag{2.1}$$

The inner and outer products of (2.1) may be written in terms of the geometric product as

$$a \wedge A = \frac{1}{2} \left(aA + (-1)^r Aa \right), \tag{2.2}$$

$$a \cdot A = \frac{1}{2} \left(aA - (-1)^r Aa \right), \tag{2.3}$$

where $r = \mathrm{grade}(A)$. Then, realizing that $m = \mathrm{grade}(I)$, and that by (2.1), we have $vI = v \cdot I$, we can use equation (2.3) to establish the commutativity of vectors in $\mathbb{V}$ with the unit psuedo-scalar I as

$$aI = -(-1)^m Ia. \tag{2.4}$$

Using equation (2.4) in conjunction with equation (2.3), we find that

$$(a \cdot A)I = a \wedge AI. \tag{2.5}$$

(In verifying this identity, it helps to realize that for any integer i, we have $(-1)^i = (-1)^{-i}$.) Replacing A in equation (2.5) with AI, we find that

$$(a \wedge A)I = a \cdot AI. \tag{2.6}$$

Refering to (2.3), another important formulation of the inner product between a vector and a blade is given by

$$a \cdot A = -\sum_{i=1}^{r} (-1)^i (a \cdot a_i) A_i, \tag{2.7}$$

where A is factored as $\bigwedge_{i=1}^{r} a_i$, and we define A_i as

$$A_i = \bigwedge_{j=1, j \neq i}^{r} a_i. \tag{2.8}$$

This leads to the following recursive formulation.

$$a \cdot A = (a \cdot a_1)A_1 - a_1 \wedge (a \cdot A_1) \tag{2.9}$$

If a blade $B \in \mathbb{B}$ has grade s and factorization $\bigwedge_{i=1}^{s} b_i$, then we can express the product $A \cdot B$ recursively as

$$A \cdot B = \begin{cases} A_r \cdot (a_r \cdot B) & \text{if } r \leq s, \\ (A \cdot b_1) \cdot B_1 & \text{if } r \geq s. \end{cases} \tag{2.10}$$

2.2. Lemmas

The following lemmas will help us prove the results of this paper.

Lemma 2.1. *If $A \in \mathbb{B}$ is a non-zero blade of grade $r > 0$ and $\{b_i\}_{i=1}^r$ is a set of r linearly independent vectors such that for all $b \in \{b_i\}_{i=1}^r$, we have $b \wedge A = 0$, then there exists a scalar $\beta \in \mathbb{R}$ such that*

$$A = \beta \bigwedge_{i=1}^r b_i. \tag{2.11}$$

Proof. Letting $\bigwedge_{i=1}^r a_i$ be a factorization of the r-blade A, it is clear that if $b_i \wedge A = 0$, then there exists a set of r scalars $\{\alpha_{i,j}\}_{j=1}^r$ such that

$$b_i = \sum_{j=1}^r \alpha_{i,j} a_i. \tag{2.12}$$

We then have

$$\bigwedge_{i=1}^r b_i = (\det M) \bigwedge_{i=1}^r a_i, \tag{2.13}$$

where the $r \times r$ matrix M is given by

$$M = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,r} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{r,1} & \alpha_{r,2} & \cdots & \alpha_{r,r} \end{bmatrix}. \tag{2.14}$$

Now since $\{b_i\}_{i=1}^r$ is a linearly independent set of vectors, $\det M \neq 0$, and we may choose $\beta = (\det M)^{-1}$ and equation (2.11) now holds. □

Lemma 2.2. *Generalize the previous lemma to the case when the number of linearly independent vectors is less than the grade of the blade in question.*

Proof. □

Lemma 2.3. *For a non-zero r-blade A factored as $\bigwedge_{i=1}^r a_i$, the set of $(r-1)$-blades $\{A_i\}_{i=1}^r$ is linearly independent.*

Proof. Suppose there exists a non-trivial set of r scalars $\{\alpha_i\}_{i=1}^r$ such that $0 = \sum_{i=1}^r \alpha_i A_i$. Then, without loss of generality, suppose that $\alpha_r \neq 0$, and rearrange our equation as $-\alpha_r A_r = \sum_{i=1}^{r-1} \alpha_i A_i$. Now notice that while $a_r \wedge -\alpha_r A_r \neq 0$, we have $a_r \wedge \sum_{i=1}^{r-1} \alpha_i A_i = 0$, which is a contradiction. □

Note that a perhaps more elegant proof of Lemma 2.3 could have been given under the assumption of a euclidean signature; in which case, the Gram-Schmidt orthogonalization process would have allowed us to choose, without loss of generality, an orthogonal factorization of the blade A. Doing so, A becomes the versor $\prod_{i=1}^r a_i$, and we may write

$$0 = \sum_{i=1}^r \alpha_i A_i \iff 0 = \left(\sum_{i=1}^r \alpha_i A_i \right) A = -\sum_{i=1}^r (-1)^{r-i} A_i^2 \alpha_i a_i. \tag{2.15}$$

Lemma 2.4 (The Zero Product Property). *For any two non-zero blades* $A, B \in \mathbb{B}$ *of grades r and s, respectively, if $AB = 0$ and at least one of A and B is invertible, then at least one of A and B is zero.*

Proof. Without loss of generality, suppose B^{-1} exists. We then see that

$$A = ABB^{-1} = 0B^{-1} = 0. \tag{2.16}$$

$\square$

Notice the requirment here of Lemma 2.4 that at least one of A and B be invertible. This requirement comes about in consideration of the square of a non-zero null-vector.

3. Results

3.1. Foundation

Our discussion begins with a non-zero, well-defined, yet undefined function $p : \mathbb{R}^n \to \mathbb{V}$. By leaving this function undefined, the results to follow generalize to the homogeneous model, conformal model, and any other model of geometry that is based upon the use of blades as representatives of geometry.

Definition 3.1 (Direct And Dual Representation). For the two functions $\hat{g} : \mathbb{B} \to \mathbb{R}^n$ and $\dot{g} : \mathbb{B} \to \mathbb{R}^n$, given by

$$\hat{g}(A) = \{x \in \mathbb{R}^n | p(x) \wedge A = 0\}, \tag{3.1}$$

$$\dot{g}(A) = \{x \in \mathbb{R}^n | p(x) \cdot A = 0\}, \tag{3.2}$$

we say that A directly represents the set of points $\hat{g}(A)$ and dually represents the set of points $\dot{g}(A)$.

From Definition 3.1, it is important to take away the realization that a given blade $A \in \mathbb{B}$ represents two subsets of $\mathbb{R}^n$ simultaneously; namely, $\hat{g}(A)$ and $\dot{g}(A)$. Which we choose to think of A as being a representative of at any given time is completely arbitrary.

It should also be clear from Definition 3.1 that the subset of $\mathbb{R}^n$ represented by a blade A, (directly or dually), remains invariant under any non-zero scaling of the blade A.

Finally, now enters this paper's object of study: the *geometric set*.

Definition 3.2 (Geometric Set). A subset $R \subset \mathbb{R}^n$ for which there exists a blade $A \in \mathbb{B}$ such that $\hat{g}(A) = R$ is what we'll refer to as a *geometric set*.

In the course of our study, we will find the concept of *irreducibility* important.

Definition 3.3 (Irreducible/Reducible Blade). Given an r-blade $A \in \mathbb{B}$, if there does not exist an s-blade $B \in \mathbb{B}$ with $s < r$ such that $\hat{g}(A) = \hat{g}(B)$, then A is what we'll refer to as an *irreducible* blade. A blade that is not irreducible is referred to as *reducible*.

3.2. Developments

Having given the definitions in the previous section, we may now focus on the results that follow from these definitions.

Lemma 3.4 (Dual Relationship Between Representations). *For any geometric set $R \subset \mathbb{R}^n$, if $A \in \mathbb{B}$ is a blade such that $\hat{g}(A) = R$, then $\dot{g}(AI) = R$, and similarly, if $A \in \mathbb{B}$ is a blade such that $\dot{g}(A) = R$, then $\hat{g}(AI) = R$.*

Proof. By the identity of equation (2.6), and Lemma 2.4, the first of these two latter statements is proven by

$$0 = p(x) \wedge A = -(p(x) \cdot AI)I \iff p(x) \cdot AI = 0, \tag{3.3}$$

while the second, by the identity of equation (2.5), and again Lemma 2.4, is proven by

$$p(x) \cdot A = 0 \iff 0 = (p(x) \cdot A)I = p(x) \wedge AI. \tag{3.4}$$

$\square$

In other words, Lemma 3.4 is telling us that for a single given geometric set, the algebraic relationship between a blade directly (dually) representative of that set, and a blade dually (directly) representative of that set, is simply that, up to scale, they are duals of one another.

Of course, there will also be a geometric relationship between the geometry that is directly represented by a single given blade $A \in \mathbb{B}$, and the geometry that is dually represented by A, but this would depend upon how we choose to define the function $p : \mathbb{R}^n \to \mathbb{V}$.

Lemma 3.5. *For any geometric set $R \subset \mathbb{R}^n$, there exists a blade $A \in \mathbb{B}$ such that $\dot{g}(A) = R$.*

Lemma 3.6. *For all non-zero blades $A \in \mathbb{B}$, we have $\hat{g}(A) \cap \dot{g}(A) = \emptyset$.*

Lemma 3.7. *If $A \in \mathbb{B}$ is an irreducible r-blade with $\hat{g}(A) \neq \emptyset$, then there exists a set of r points $\{x_i\}_{i=1}^r \subset \mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$ such that*

$$A = \alpha \bigwedge_{i=1}^r p(x_i). \tag{3.5}$$

Proof. Let s be the largest integer for which there exists a set of s points $\{x_i\}_{i=1}^s \subset \hat{g}(A)$ such that $\bigwedge_{i=1}^s p(x_i) \neq 0$. Clearly $s \geq 1$, because $\hat{g}(A)$ is non-empty; and clearly $s \leq r$ because of the requirement that for each $x \in \{x_i\}_{i=1}^s$, we have $p(x) \wedge A = 0$. Now if $s = r$, we're done by Lemma 2.1. Therefore, suppose $s < r$. In this case, there must exist, by Lemma **??**, a factorization of A of the form $A = B \wedge C$, where B is a blade of grade $r - s$, and C is an s-blade given by

$$C = \bigwedge_{i=1}^s p(x_i), \tag{3.6}$$

Now realize that $\hat{g}(A) \subseteq \hat{g}(C)$ or else s is not the largest of its kind, and that $\hat{g}(C) \subseteq \hat{g}(A)$, because C is a subspace of A. It now follows that $\hat{g}(A) = \hat{g}(C)$, which contradicts the irreducibility of the blade A. $\square$

Notice that in the proof of Lemma **??** that $\hat{g}(B) = \emptyset$. It is important to realize, however, that although there is no point $x \in \hat{g}(A)$ such that $x \in \hat{g}(B)$, this does not imply that $x \in \hat{g}(C)$, despite the correctness of this conclusion.

It was noted earlier that if $A, B \in \mathbb{B}$ are blades such that for a scalar $\beta \in \mathbb{R}$, we have $A = \beta B$, then $\hat{g}(A) = \hat{g}(B)$. The converse of this statement, however, is not generally true, but leads us to an important and fundamental theorem.

Theorem 3.8 (The Fundamental Theorem Of Geometric Set Representation). *For every geometric set $R \subset \mathbb{R}^n$, there exists, up to scale, a unique, irreducible blade $A \in \mathbb{B}$ such that $\hat{g}(A) = R$. Moreover, this blade A is a subspace of every blade B directly representative of R.*

Proof. First show that we can always reduce a reducible blade to an irreducible blade.

Use Lemma 3.7 to prove that two found irreducible blades directly representative of R share each other's factorization, up to scale. $\qquad\square$

The importance of Theorem 3.7 can be realized in the utility of the conformal model. By Theorem 3.7, we are justified in algebraically relating two independently made formulations of a given geometric set. For example, we may equate the intersection of two spheres as some scalar multiple of the canonical intersection of a sphere centered on a plane. The former formulation is what we may wish to calculate, while the latter formulation lends itself to analysis through decomposition.

References

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