Chapter 9 Exercises
Gallian's Book on Abstract Algebra

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Exercise 1

Let $H = \{(1), (12)\}$. Is H normal in S_3.

No, $(123)H \neq H(123)$, because $(123)(12) = (13) \neq (23) = (12)(123)$.

Exercise 2

Prove that A_n is normal in S_n.

Let $\alpha \in A_n$ and let $\beta \in S_n$. Now notice that $\beta\alpha\beta^{-1} \in A_n$, in the case that β is an even permutation, or an odd permuation. It then follows by Theorem 9.1 that A_n is a normal subgroup of S_n.

Exercise 3

Show that if G is the internal direct product of $H_1, H_2, \dots, H_n$ and $i \neq j$ with $1 \leq i \leq n$, $1 \leq j \leq n$, then $H_i \cap H_j = \{e\}$.

Without loss of generality, let $i < j$. Now notice that

$$H_i \subseteq H_1 H_2 \dots H_i \dots H_{j-2} H_{j-1}$$

and that $H_1 H_2 \dots H_i \dots H_{j-2} H_{j-1} \cap H_j = \{e\}$. It follows that $H_i \cap H_j = \{e\}$.

Finishing Theorem 9.6

We are given $\phi(h_1h_2\dots h_n) = (h_1, h_2, \dots, h_n)$. It is immediately clear that ϕ is onto $H_1 \oplus H_2 \oplus \cdots \oplus H_n$. By the uniqueness of representation of elements in $H_1H_2\dots H_n$ already proven, it follows that ϕ is one-to-one. That ϕ is operation preserving follows from the commutativity among disjoint subgroups. For all integers $i \in [1, n]$, for all $a_i, b_i \in H_i$, we have

$$\begin{aligned}
&\phi(a_1a_2\dots a_nb_1b_2\dots b_n)\\
=&\,\phi(a_1b_1a_2b_2\dots a_nb_n)\\
=&\,(a_1b_1, a_2b_2, \dots, a_nb_n)\\
=&\,(a_1, a_2, \dots, a_n)(b_1, b_2, \dots, b_n)\\
=&\,\phi(a_1a_2\dots a_n)\phi(b_1b_2\dots b_n).
\end{aligned}$$

Exercise 61

Suppose that H is a normal subgroup of a finite group G. If G/H has an element of order n, show that H has an element of order n. Show, by example, that the assumption that G is finite is necessary.

The case $n = 1$ is trivial, so let $n > 1$. Let $a \in G$ such that $|aH| = n$. Clearly $a \neq e$. It follows that the mapping $\phi : H \to H$, given by $\phi(h) = a^nh$ is a non-trivial permutation of the elements of H and so ϕ is a member of the group of permutations of H. We then see that $a^{|\phi|n} = e$. But it is easy to see that for all integers $i \in [1, |\phi|n - 1]$, we have $a^i \neq e$. So $|a^{|\phi|}| = n$.