On The Problem Of Intersecting Quadric Surfaces Using Geometric Algebra

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To my dear wife Melinda.

Abstract. Progress is made on the problem of finding a conformal-like model of geometry based upon geometric algebra in which intersections of quadric surfaces may be taken.

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1. Introduction

In light of the paper [6], some encouragement has been given to the present author to develop a model of geometry, similar to the conformal model of geometric algebra, (see[4, 2, 1]), but not limited in representation to any proper subset of the set of all quadric surfaces. But for such a model to achieve adequate similarity to the conformal model, it must preserve one or more of its most desirable features; preferably, all of them. For example, in [6], the set of all conformal transformations were preserved, but intersections were not.[1]

The goal of this paper, therefore, is to preserve the intersection property. In other words, we want to find a model of geometry based upon geometric algebra giving us all quadric surfaces and the ability to intersect them as effortlessly as can be done in the conformal model. If nothing else, the attempt to do so in this paper will shed light on the feasibility and practicality of such an endeavor, and thereby bring us closer to answering the question of whether it can even be done at all.

[1] A way of representing intersections using the outer product in the model of [6] can be found, but its usefulness, if any, is highly questionable.

2. The Intersection Property

Let us begin by taking a closer look at exactly what the intersection property is. Upon initial inspection, one might suppose that this property is nothing more than the ability to represent the intersection of any two given geometries in a way consistent with the representation of any geometry of the model, but this is not enough. Such a representation has no usefulness if it does not submit to an analysis yielding the geometric characteristics of the intersection.

That having been said, we can say now that the outer product's ability to intersect geometries represented by blades in the conformal model is really not at all interesting. What is interesting is the realization that we can equate one characterization of an intersection with another, and this is the key to finding intersections in the conformal model. The reason for this is that while one such characterization is composed as the intersection we wish to take, the other characterization lends itself to analysis through decomposition.

For example, suppose we wish to take the planar intersection of a conical surface. If we know or suspect that the resulting conic section is an ellipse, then we can choose to interpret this intersection as that of a plane and an elliptical cylinder meeting the plane at right angles. This latter characterization will have an easily found decomposition yielding all features of the ellipse.[2] Having found all such features, we can then say that we've fully realized the given section, whereas before this we were only able to represent it.[3]

The quest to find our model of geometry, (the one promised in the introductory section of this paper), being quite difficult, the bringing of the example just given in the preceding paragraph to fruition will become the impetus for all choices henceforth made in finding the model. Even if our model can do nothing more than this one example, we will consider or goal achieved. Rest assured, however, that along the way, we will find results generally applicable to the problem at hand.

3. Enter The Model

So that no further delay be made, we will now let the remainder of this paper begin exactly where the first section of [6] ended, assuming all results and definitions up to that point.[4] That said, we now introduce the function

[2] Geometries having easily decomposable representations in our model will be referred to as canonical forms.

[3] Note that although non-planar intersections will be as easily represented in our model of geometry as any other type of intersection, the technique of finding intersections in this paper might not be helpful in finding non-planar intersections for the simple reason that such intersections have no obvious canonical forms. For more on non-planar intersections, see [5].

[4] The reader need read no further than the first section of [6] before preceding. In this paper, $\mathbb{R}^n$ is used to denote n-dimensional euclidean space; but we are, for the most part,

$p : \mathbb{R}^n \to \mathbb{V}$ as

$$\begin{aligned} p(x) &= e_0 + x \\ &\quad + (x\cdot e_2)(x\cdot e_3)e_4 + (x\cdot e_1)(x\cdot e_3)e_5 + (x\cdot e_1)(x\cdot e_2)e_6 \\ &\quad + (x\cdot e_1)^2 e_7 + (x\cdot e_2)^2 e_8 + (x\cdot e_3)^2 e_9, \end{aligned}$$

the vectors in $\{e_i\}_{i=0}^{9}$ forming an orthonormal basis for a 10-dimensional euclidean vector space $\mathbb{V}$.[5] This is sufficient to define the entire model as it is clear that for any quadric surface, or any intersection of two or more quadric surfaces, there exists a blade $B \in \mathbb{G}$ representative of this surface as $\dot{g}(B)$.

Continuing with our example, let us now find the canonical form for an ellipse in a plane. Using the notation[6] $x_i = x \cdot e_i$, an equation for such an ellipse may be given by

$$\frac{(x_1-h)^2}{a^2} + \frac{(x_2-k)^2}{b^2} - 1 = 0, \tag{3.1}$$

provided $x_3 = 0$, which is simply an equation for the plane. Factoring $p(x)$ out of the equation $x_3 = 0$, we get $p(x)\cdot e_3 = 0$, and out of equation (3.1), we get $p(x)\cdot E = 0$, where the vector E is given by

$$E = \left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right)e_0 - 2\frac{h}{a^2}e_1 - 2\frac{k}{b^2}e_2 + \frac{1}{a^2}e_7 + \frac{1}{b^2}e_8.$$

By itself, E here represents an axis-aligned elliptical cylinder. The ellipse is given by the set of points $\dot{g}(E\wedge e_3)$. The 2-blade $E\wedge e_3$ will be the canonical form of the ellipse that we will use to find a conic intersection that is an ellipse.

Letting $\lambda \in \mathbb{R}$ be any non-zero scalar, to see that $B = \lambda E\wedge e_3$ is easily decomposable, we give the following set of equations.

$$h = (-e_{31}\cdot B)(2e_{37}\cdot B)^{-1} \tag{3.2}$$
$$k = (-e_{32}\cdot B)(2e_{38}\cdot B)^{-1} \tag{3.3}$$
$$\lambda = (h^2 e_{37} + k^2 e_{38} - e_{30})\cdot B \tag{3.4}$$
$$a = \sqrt{\lambda(e_{37}\cdot B)^{-1}} \tag{3.5}$$
$$b = \sqrt{\lambda(e_{38}\cdot B)^{-1}} \tag{3.6}$$

Using these equations, we can recover the ellipse. Note here that we are using the convenient notation $e_{ijk\ldots} = e_i\wedge e_j\wedge e_k\wedge\ldots$.

going to restrict ourselves to the case $n = 3$. The astute reader will recognize when results in this paper generalize to any positive integer n.

[5]The reader should take care to note which results of this paper depend upon this definition of the function p and which do not. Also note that signatures other than the euclidean may be worth considering, but there will be no foreseeable need to do so in this paper.

[6]This notation is overloaded. When writing x_i, this may mean the i^{th} component of x, or it may mean the i^{th} point in a sequence of points. The intended meaning will always be clear from context.

Now let us formulate the intersection we wish to take. We will intersect the $x_3 = 0$ plane with the conical surface having points satisfying the equation

$$x_1^2 + x_2^2 - (x_3 + 1)^2 \tan^2 \frac{\pi}{4} = 0, \tag{3.7}$$

where $\frac{\pi}{2}$ is the angle of aperture.[7] We have submerged it below the $x_3 = 0$ plane to get a non-trivial intersection $\dot{g}(C \wedge e_3)$, the vector C being given by

$$\begin{aligned} C &= -\left(\tan^2 \frac{\pi}{4}\right) e_0 + 2\left(\tan^2 \frac{\pi}{4}\right) e_3 + e_7 + e_8 - \left(\tan^2 \frac{\pi}{4}\right) e_9 \\ &= -e_0 + 2e_3 + e_7 + e_8 - e_9. \end{aligned}$$

4. Making Use Of The Model

Being now able to represent all quadric intersections, the task of algebraically relating them remains. For example, knowing that $C \wedge e_3$ is an ellipse, how might we decompose it as we would $E \wedge e_3$? The following lemma may be able to help.

Lemma 4.1. *Letting $B \in \mathbb{G}$ be a blade of grade k, if there exist k points $\{x_i\}_{i=1}^k \subset \mathbb{R}^n$ such that $\bigwedge_{i=1}^k p(x_i) \neq 0$, and that for all points $x \in \{x_i\}_{i=1}^k$, we have $x \in \hat{g}(B)$, then there exists a scalar $\lambda \in \mathbb{R}$ such that*

$$B = \lambda \bigwedge_{i=1}^{k} p(x_i). \tag{4.1}$$

Proof. If $x_i \in \hat{g}(B)$, then $p(x_i) \wedge B = 0$, showing that $p(x_i)$ is in the vector space spanned by any factorization of B. Then, since $\bigwedge_{i=1}^k p(x_i) \neq 0$, the set of vectors $\{p(x_i)\}_{i=1}^k$ is linearly independent and therefore a basis for this vector space. The blades B and $\bigwedge_{i=1}^k p(x_i)$ must, therefore, be equal, up to scale. □

For a blade $B \in \mathbb{G}$ having a factorization (4.1), we will refer to B as an irreducible blade for reasons that will become clear shortly.

The usefulness of Lemma 4.1 is realized in our next lemma.

Lemma 4.2. *If $A, B \in \mathbb{G}$ are blades of grade k with $\hat{g}(A) = \hat{g}(B)$, and one of these is irreducible, then there exists a scalar $\lambda \in \mathbb{R}$ such that $A = \lambda B$.*

Proof. We first establish that if one of the blades A and B is irreducible, then so is the other. Assuming, without loss of generality, that A is irreducible, let $\{x_i\}_{i=1}^k \subset \mathbb{R}^n$ be a set of k points, and $\alpha \in \mathbb{R}$ be a scalar, such that $A = \alpha \bigwedge_{i=1}^k p(x_i)$. Now since $\hat{g}(A) = \hat{g}(B)$, it is clear that for all $x \in \{x_i\}_{i=1}^k$, we have $x \in \hat{g}(B)$, and so it follows by Lemma 4.1 that there exists a scalar $\beta \in \mathbb{R}$ such that $B = \beta \bigwedge_{i=1}^k p(x_i)$.

Lastly, we simply realize that if we let $\lambda = \frac{\alpha}{\beta}$, we have $A = \lambda B$. □

[7] Yes, this will give us a circle in the $x_3 = 0$ plane and not a general ellipse, but to keep things simpler, our primary example will consider this special case of the ellipse.

In light of Lemma 4.2, the following question naturally arises. Knowing that $\hat{g}((E \wedge e_3)I) = \hat{g}((C \wedge e_3)I)$, is any one of $(E \wedge e_3)I$ and $(C \wedge e_3)I$ irreducible?[8] If so, we have found a way, by Lemma 4.2, to algebraically relate them so that an analysis by decomposition of $C \wedge e_3$ as $E \wedge e_3$ can move forward.[9] Unfortunately, it is not hard to show that neither of these is irreducible. To see why, consider the following equation which expresses the form of $p(x)$ for all points in the $x_3 = 0$ plane.

$$p(x_1 e_1 + x_2 e_2) = e_0 + x_1 e_1 + x_2 e_2 + x_1 x_2 e_6 + x_1^2 e_7 + x_2^2 e_8 \tag{4.2}$$

Now notice that an upper-bound on the dimension of a vector space that can be spanned by vectors of this form is clearly 6 as there are only 6 components on the right-hand side of equation (4.2). But each of $E \wedge e_3$ and $C \wedge e_3$ are 2-blades, making their duals blades of grade $10 - 2 = 8 > 6$. It follows that there is no set of 8 points $\{x_i\}_{i=1}^{8}$ on the ellipse such that $\bigwedge_{i=1}^{8} p(x_i) \neq 0$.

Not willing to give up just yet, we arrive at the following lemma.

Lemma 4.3. *For every k-blade $B \in \mathbb{G}$, there exists a blade $B' \in \mathbb{G}$ of grade $k' \leq k$ such that $\hat{g}(B) = \hat{g}(B')$ and B' is irreducible.*

Proof. If B is irreducible, then let $k' = k$ and $B' = B$ and we're done. If B is not irreducible, then let j be the largest possible integer for which there exists a set of j points $\{x_i\}_{i=1}^{j} \subseteq \hat{g}(B)$ with $\bigwedge_{i=1}^{j} p(x_i) \neq 0$, (clearly $j < k$), and write

$$B = B_0 \wedge \bigwedge_{i=1}^{j} p(x_i)$$

for some blade B_0 of grade $k - j$. Now realize that for any $x \in \hat{g}(B)$, if $x \in \hat{g}(B_0)$, then $p(x) \wedge \bigwedge_{i=1}^{j} p(x_i) \neq 0$ and $x \notin \{x_i\}_{i=1}^{j}$, which is a contradiction. Therefore, if $x \in \hat{g}(B)$, then, letting $k' = j$ and $B' = \bigwedge_{i=1}^{j} p(x_i)$, we have $x \in \hat{g}(B')$. Conversely, if $x \in \hat{g}(B')$, then clearly $x \in \hat{g}(B)$. It follows that $\hat{g}(B) = \hat{g}(B')$ and B' is irreducible. $\square$

Seeing that B' is potentially a reduction in grade of the blade B, but one in which the geometry represented by B is certainly not sacrificed, we will say that B is reducible in the case that $k' < k$. In the case that $k' = k$, it is clear that B is irreducible.

What we see now is that $(C \wedge e_3)I$ and $(E \wedge e_3)I$ are reducible blades, and that finding an algebraic relation between them may be possible if either one or both can be reduced. Since finding an irreducible canonical form of the ellipse $(E \wedge e_3)I$ does not seem in the least bit trivial, let us take a moment now to consider reducing the intersection $(C \wedge e_3)I$ we wish to find.

To do this, we set out to find the largest irreducible factor of $(C \wedge e_3)I$. Clearly, if $x \in \dot{g}(C \wedge e_3)$, then $p(x)$ is a factor of $(C \wedge e_3)I$; so the question of when a set of points produces a linearly independent set of vectors naturally

[8]Here we're letting $I = e_{0123456789}$ be the unit pseudo-scalar of our 10-dimensional geometric algebra $\mathbb{G}$.
[9]Think of this as replacing B with $\lambda C \wedge e_3$ in equations (3.2) through (3.6).

arises. It is immediately clear that if the set of j points $\{x_i\}_{i=1}^j$ is linearly independent, then so is the set of j vectors $\{p(x_i)\}_{i=1}^j$, but the following lemma helps us do a little better than this.

Lemma 4.4. *If a given set of $j > 2$ points $\{x_i\}_{i=1}^j$ are non-co-planar for a plane of dimension $j-2$, then the set of vectors $\{p(x_i)\}_{i=1}^j$ is linearly independent.*

Proof. Proving the contrapositive of the lemma, let $\{\lambda_i\}_{i=1}^j$ be a set of scalars in $\mathbb{R}$, not all zero, such that $0 = \sum_{i=1}^j \lambda_i p(x_i)$. It follows that $0 = \sum_{i=1}^j \lambda_i(e_0 + x_i)$ and therefore $0 = \sum_{i=1}^j \lambda_i$ and $0 = \sum_{i=1}^j \lambda_i x_i$. Now realize that if there exists an integer $a \in [1, j]$ such that $\lambda_a \neq 0$, then there must exist an integer $b \in [1, j] - \{a\}$ such that $\lambda_b \neq 0$. Without loss of generality, let $a = j$ so that $1 \leq b \leq j-1$, and write

$$0 = \sum_{i=1}^{j} \lambda_i x_i = \sum_{i=1}^{j-1} \lambda_i x_i - \left(\sum_{i=1}^{j-1} \lambda_i\right) x_j = -\sum_{i=1}^{j-1} \lambda_i (x_j - x_i),$$

which shows that the set of vectors $\{x_j - x_i\}_{i=1}^{j-1}$ is linearly dependent. It now follows that the $(j-1)$-dimensional simplex determined by the points in $\{x_i\}_{i=1}^j$ has no $(j-1)$-dimensional hyper-volume. That is,

$$0 = \frac{1}{(j-1)!} \bigwedge_{i=1}^{j-1} (x_j - x_i).$$

But this can only be if the j points are co-planar for a hyper-plane of dimension $j-2$. □

Lemma 4.4 is a good start, but there are certainly more conditions on $\{x_i\}_{i=1}^j$ to be found upon which $\bigwedge_{i=1}^j p(x_i) \neq 0$. The non-linearity of our function p makes these conditions difficult to find, to say the least. Nevertheless, Lemma 4.4 can help guide our initial choice of probing vectors in the naïve blade factorization algorithm;[10] and we, as Lemma 4.6 below will show, do not have to complete this factorization. We need go only so far as to know that we have found the irreducible factor we're trying to find.

Lemma 4.5. *If $B \in \mathbb{G}$ is an irreducible k-blade with $k > 2$ and $B_0, B_1 \in \mathbb{G}$ is any factorization of B in terms of two blades as $B = B_0 \wedge B_1$, each having non-zero grade, then both B_0 and B_1 are irreducible blades.*

Proof. Let $\{x_i\}_{i=1}^k \subseteq \hat{g}(B)$ be a set of k points on the geometry represented by B such that $\bigwedge_{i=1}^k p(x_i) \neq 0$. Then, seeing that $B = B_0 \wedge B_1$, it is clear that there must exist a partition of $\{x_i\}_{i=1}^k$ into two non-empty sets, one of size $j = \text{grade}(B_0)$ with $0 < j < k$, and the other of size $k - j = \text{grade}(B_1)$,

[10] A better blade factorization algorithm can been found in [3], but remember, we're not looking for just any factorization; we're looking for one having a factor of the form (4.1) of largest possible grade. What further complicates the matter is that, in finding our desired factorization, we need points on the intersection we are in the process of trying to find. Calculus methods may be in order.

such that for all x in the first partition, we have $x \in \hat{g}(B_0)$; and for all x in the second partitoin, we have $x \in \hat{g}(B_1)$. Without loss of generality, let the first partition be $\{x_i\}_{i=1}^{j}$, and the second $\{x_i\}_{i=j+1}^{k}$. Then, since $\bigwedge_{i=1}^{j} p(x_i)$ and $\bigwedge_{i=j+1}^{k} p(x_i)$ are each non-zero, it follows, by Lemma 4.1, that there exist scalars $\lambda_0, \lambda_1 \in \mathbb{R}$ such that $B_0 = \lambda_0 \bigwedge_{i=1}^{j} p(x_i)$, and $B_1 = \lambda_1 \bigwedge_{i=j+1}^{k} p(x_i)$. □

Lemma 4.6. *Given any blade $B \in \mathbb{G}$, if k points $\{x_i\}_{i=1}^{k} \subset \mathbb{R}^n$ can be found such that $\hat{g}(B) = \hat{g}(R)$ with $R = \bigwedge_{i=1}^{k} p(x_i)$, then there is no irreducible blade B' of a grade greater than k, or less than k, such that $\hat{g}(B) = \hat{g}(B')$.*

Proof. Suppose, contrary to the lemma, that $B' \in \mathbb{G}$ is an irreducible blade of grade greater than k such that $\hat{g}(B) = \hat{g}(B')$. Letting $B_0 \in \mathbb{G}$ be a blade such that $B' = R \wedge B_0$, we see that B_0 is irreducible by Lemma 4.5. But if $\hat{g}(R) = \hat{g}(B)$, then we must have $R \wedge B_0 = 0$, which is a contradiction, because $B' \neq 0$.

Suppose now, also contrary to the lemma, that $B' \in \mathbb{G}$ is an irreducible blade of grade less than k such that $\hat{g}(B) = \hat{g}(B')$. Letting $B_0 \in \mathbb{G}$ be a blade such that $R = B' \wedge B_0$, we see that B_0 is irreducible by Lemma 4.5. But if $\hat{g}(B') = \hat{g}(B)$, then we must have $B' \wedge B_0 = 0$, which is a contradiction, because $R \neq 0$. □

Concerning irreducible blades, we see that while Lemma 4.3 has dealt with the question of existence, Lemma 4.6 has dealt with the question of uniqueness. Lemmas 4.3, 4.6 and 4.2, all taken together, we can say now that every blade representative of a non-empty geometry has an irreducible form that is unique, up to scale. Lemma 4.5 shows that irreducible blades always factor as irreducible blades.

By Lemma 4.6, we are justified in claiming that, up to scale, the irreducible form of our desired intersection $\dot{g}(C \wedge e_3)$ is given by $\hat{g}(R)$, where

$$R = e_{012} \wedge (e_{67} + e_{78} + e_{86}) \tag{4.3}$$

is a blade of grade 5 in our algebra.[11] Admittedly, knowledge of the intersection was used to obtain this form; but, assuming that there is an algorithm for finding it that does not depend upon such knowledge, let's move forward unashamed and undiscouraged.

Having now fully reduced the 8-blade $(C \wedge e_3)I$ down to the 5-blade R, we know that there must exist a 3-blade R_0 such that the equation

$$(R_0 \wedge R)I = E \wedge e_3$$

has real solutions in a, b, h and k. Indeed, by examination, it is not hard to see that if we let $R_0 = e_{459}$, then this is the case. The 2-blade $(R_0 \wedge R)I$ may now be decomposed using equations (3.2) through (3.6). As the reader can check, we get the unit circle at origin.

[11] Choosing any 5 points $\{x_i\}_{i=1}^{5}$ on the unit circle in the plane for which $\bigwedge_{i=1}^{5} p(x_i) \neq 0$, you'll find that $\bigwedge_{i=1}^{5} p(x_i)$ is a scalar multiple of R in equation (4.3). It is also not hard to show that $\hat{g}(R) = \dot{g}(C \wedge e_3)$.

5. Closing Remarks

To see any efficacy in this, admittedly, anticlimactic paper, we have to look at what we've learned from our attempt at finding the model of geometry we had hoped for in the introductory section, and then try to ask the right questions moving forward. To the thoughtful reader, a few of these are given as follows.

Do that here...

Lastly, it must be said here that geometric algebra may not be the right tool for studying the intersections of algebraic surfaces. For those readers possessing a savy for algebraic geometry, the article [7], and similar publications by the same author, give the modern approach to intersection theory.

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