Models Of Geometry In Geometric Algebra

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Abstract. Abstract goes here...

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1. Introduction And Motivation

Many of the pieces of geometry we study can be described as the roots of a given polynomial. Consequently, the geometries generated by many different polynomials have been studied; and, naturally, this has led to a general theory of geometries that can be described as the zero set of one or more polynomials; namely, algebraic geometry. Ideals of polynomial rings being the generators of algebraic sets, abstract algebra became the basic framework of the theory.[1]

Similarly, various models of geometry have been developed in the framework of geometric algebra, but they all have one thing in common: they're based upon the idea of the algebraic set. It stands to reason, then, that it may be worth trying to find a unified theory of such models in geometric algebra. In other words, it may be worth studying such models in a more abstract setting. That is the focus of this paper.

2. From Polynomials To Blades

Let F be a field of characteristic 1, and let $f \in F[x_1, \dots, x_n]$ be a polynomial of arbitrary degree in n variables. Being interested in the set of all $x \in F(x_1, \dots, x_n)$ such that $f(x) = 0$, how might we translate the description of this set into the language of geometric algebra? Letting $\mathbb{G}$ denote a geometric algebra generated by an infinitely-dimensional vector space $\mathbb{V}$ whose scalars are taken from F, (the set of euclidean vectors $\{e_i\}_{i=1}^{\infty}$ generate $\mathbb{V}$),

[1] Modern algebraic geometry has grown far beyond algebraic sets as the primary object of study; but originally, this is what algebraic geometry was about.

we introduce an appropriately defined function $p : \mathbb{V} \to F$ and simply factor it out of the equation $f(x) = 0$ to obtain

$$p(x) \cdot v = 0,$$

where $v \in \mathbb{V}$. Defined appropriately, for every $f \in F[x_1, \dots, x_n]$, there would exist a unique $v \in \mathbb{V}$ such that $p(x) \cdot v = 0$ if and only if $f(x) = 0$. The existence of such a function p, and the establishment of the ensuing claim, therefore, must constitute our first result.

Lemma 2.1. *Letting $p : \mathbb{V} \to F$ be defined as*

$$p(x) = \sum_{i=1}^{\infty} m_i(x) e_i, \tag{2.1}$$

where the polynomial sequence $\{m_i\}_{i=1}^{\infty} \subset F[x_1, \dots, x_n]$ enumerates all possible unit monomials in the variables $x_1, \dots, x_n$, there exists, for every polynomial $f \in F[x_1, \dots, x_n]$, a unique vector $v \in \mathbb{V}$, such that for all $x \in F$, we have $f(x) = p(x) \cdot v$.

Proof. It is clear that there must exist a sequence of scalars $\{\alpha_i\}_{i=1}^{\infty} \subset F$ such that

$$f(x) = \sum_{i=1}^{\infty} \alpha_i m_i(x).$$

Letting $v = \sum_{i=1}^{\infty} \alpha_i e_i$, we find that v factors out of this equation as $f(x) = p(x) \cdot v$, as desired. To show uniqueness, suppose $v \neq w \in \mathbb{V}$ satisfies the equation $f(x) = p(x) \cdot w$ with $w = \sum_{i=1}^{\infty} \beta_i e_i$. Then, since $v \neq w$, there exists a positive integer i such that $\alpha_i = v \cdot e_i \neq w \cdot e_i = \beta_i$, and therefore, $\alpha_i m_i(x) \neq \beta_i m_i(x)$, which is a contradiction. □

At this point it is important to say that we should not get caught up in the way that p is or may be defined. It really doesn't matter. What does matter is that p is defined in such a way as to satisfy the property of Lemma 2.1 (i.e., that there is a one-to-one correspondence between polynomials in $F[x_1, \dots, x_n]$ and vectors in $\mathbb{V}$.) We therefore shall not make any further use of equation (2.1) in the remainder of this paper.

Our interest, however, does not stop at the zero set of a single polynomial. For a set of r polynomials $\{f_j\}_{j=1}^{r} \subset F[x_1, \dots, x_n]$, we want the set of all $x \in F(x_1, \dots, x_n)$ such that for all $f \in \{f_j\}_{i=1}^{r}$, we have $f(x) = 0$. Interestingly, geometric algebra provides a convenient description of such a set.

References

[1] S. Parkin, *Mother Minkowski Algebra Of Order M*. Advances in Applied Clifford Algebras (2013).

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