A Variation Of The Quadric Model Of Geometric Algebra

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Abstract. A variation of the quadric model set forth in [9] is found in which the rigid body motions are represented by versors applicable to any quadric surface. Extending this variation of the original model to include a specific form of quartic surface, we find that such surfaces are closed under the application of all conformal transformations. Results of a computer program implementing this new model are presented.

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1. Introduction

In the original paper [9], a model for quadric surfaces was presented based upon the ideas of projective geometry. What was unfortunate about this model, however, was its lack of support for the rigid body transformations. It was predicted in the conclusion of that paper that a better model for quadric surfaces may exist that is more like the conformal model of geometric algebra, here-after abbreviated as CGA. The present paper details what may be such a model. We'll find that the rigid body transformations can be incorporated into the model by using an alternative method of encoding the quadric form. An extension to this quadric form will then allow us to support the conformal transformations at the expense of expanding our model to necessarily include a specific form of quartic surfaces. The new model and its extension will both use the same geometric algebra to be given as follows.

2. The Geometric Algebra

We begin here with a description of the structure of the geometric algebra upon which our model will be imposed. This geometric algebra will contain the following vector spaces.

$$\begin{array}{ll} \text{Notation} & \text{Basis} \\ \hline \mathbb{V}^e & \{e_i\}_{i=1}^n \\ \mathbb{V}^o & \{o\} \cup \{e_i\}_{i=1}^n \\ \mathbb{V}^\infty & \{e_i\}_{i=1}^n \cup \{\infty\} \\ \mathbb{V} & \{o\} \cup \{e_i\}_{i=1}^n \cup \{\infty\} \end{array} \tag{2.1}$$

The set of vectors $\{e_i\}_{i=1}^n$ forms an orthonormal set of basis vectors for the n-dimensional Euclidean vector space $\mathbb{V}^e$, which we'll use to represent n-dimensional Euclidean space. The vectors o and ∞ are the familiar null-vectors representing the points at origin and infinity, respectively, taken from CGA. An inner-product table for these basis vectors is given as follows, where $1 \leq i < j \leq n$.

$$\begin{array}{c|cccc} \cdot & o & e_i & e_j & \infty \\ \hline o & 0 & 0 & 0 & -1 \\ e_i & 0 & 1 & 0 & 0 \\ e_j & 0 & 0 & 1 & 0 \\ \infty & -1 & 0 & 0 & 0 \end{array} \tag{2.2}$$

We will now let $\mathbb{G}(\mathbb{V})$ denote the Minkowski geometric algebra generated by $\mathbb{V}$. For each vector space in table (2.1), we will let an over-bar above this vector space denote an identical copy of that vector space. The vector space $\mathbb{W}$ will denote the smallest vector space containing each of $\mathbb{V}$ and $\overline{\mathbb{V}}$ as vector subspaces. In symbols, one may write

$$\mathbb{G}(\mathbb{W}) = \mathbb{G}(\mathbb{V} \oplus \overline{\mathbb{V}}). \tag{2.3}$$

We will use over-bar notation to distinguish between vectors taken from $\mathbb{V}$ with vectors taken from $\overline{\mathbb{V}}$. For algebraic purposes, we will find it useful to see that the over-bar notation may be defined as an outermorphic function that is also an isomorphism between $\mathbb{G}(\mathbb{V})$ and $\mathbb{G}(\overline{\mathbb{V}})$. Doing so, we see that for any element $E \in \mathbb{G}(\mathbb{V})$, we may define $\overline{E} \in \mathbb{G}(\overline{\mathbb{V}})$ as

$$\overline{E} = SES^{-1}, \tag{2.4}$$

where S is the versor given by

$$S = (1 + e_-\overline{e}_-)(1 - e_+\overline{e}_+)\prod_{i=1}^n (1 - e_i\overline{e}_i). \tag{2.5}$$

This definition is non-circular if we let the over-bars in equation (2.5) be purely notation. The vectors e_- and e_+, taken from [5], are defined as

$$e_- = \frac{1}{2}\infty + o, \tag{2.6}$$

$$e_+ = \frac{1}{2}\infty - o. \tag{2.7}$$

The vectors $\overline{e}_-$ and $\overline{e}_+$ are defined similarly in terms of $\overline{o}$ and $\overline{\infty}$. Defined this way, realize that, like the over-bar function defined in [9], here we have the property that for any vector $v \in \mathbb{V}$, we have $\overline{\overline{v}} = -v$.

3. The Form Of Quadric Surfaces In $\mathbb{G}(\mathbb{W})$

We now give a formal definition under which elements $E \in \mathbb{G}(\mathbb{W})$ are representative of n-dimensional quadric surfaces in our present variation of the original model.

Definition 3.1. Referring to an element $E \in \mathbb{G}(\mathbb{W})$ as a quadric surface, it is representative of such an n-dimensional surface as the set of all points $p \in \mathbb{V}^o$ such that

$$0 = p \wedge \overline{p} \cdot E. \tag{3.1}$$

From this definition it can be seen that the general form of a quadric $E \in \mathbb{G}(\mathbb{W})$ is given by

$$E = \sum_{i=1}^{k} a_i \overline{b}_i, \tag{3.2}$$

where each of $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^k$ is a sequence of k vectors taken from $\mathbb{V}^\infty$. To see why, realize that the form (3.2) can always be reduced to the form

$$E \equiv \sum_{i=1}^{n} \sum_{j=i}^{n} \lambda_{ij} e_i \overline{e}_j + \sum_{i=1}^{n} \lambda_i e_i \overline{\infty} + \lambda \infty \overline{\infty}, \tag{3.3}$$

where each of λ_{ij}, λ_i, and λ are scalars, in the sense that this reduced form represents the same surface as that in equation (3.2) under Definition 3.1. We then see that this form (3.3), when it is substituted into equation (3.1), reduces to a polynomial equation of degree 2 in the vector components of $p + (p \cdot \infty)o$. Doing so with $p = o + x$, where $x \in \mathbb{V}^e$, we get the equation

$$0 = -\sum_{i=1}^{n} \sum_{j=i}^{n} \lambda_{ij} (x \cdot e_i)(x \cdot e_j) + \sum_{i=1}^{n} \lambda_i (x \cdot e_i) - \lambda, \tag{3.4}$$

which we may recognize as the general equation of an n-dimensional quadric surface. It may be worth comparing this method of representing quadric surfaces with that done in chapter 4 of [4] in what is called a quadratic Grassmann-Caylay algebra.

In practice, a computer program might take such a bivector of the form (3.2) and extract from it the coefficients of the quadric polynomial (3.4) it represents. It could then render the surface using traditional methods, such as those used to render the traced surfaces in Figure 1 far below, or the meshed surfaces in Figure 2 yet further below.

Of course, using geometric algebra on paper, it might be undesirable and unnecessary to think of quadrics in terms of polynomial equations. A, perhaps, better way to think of quadrics is in terms of an element of a geometric algebra whose decomposition produces the parameters characterizing the quadric surface. For example, many common quadrics are the solution set in $\mathbb{V}^e$ of the equation

$$0 = -r^2 + (x - c)^2 + \lambda((x - c) \cdot v)^2 \tag{3.5}$$

in the variable x. (An explanation of the parameters r, c, v and λ was given in [9].) Then, factoring out $-p\overline{p}$, we see that the element $E \in \mathbb{G}(\mathbb{W})$, given by

$$\Omega + \lambda v\overline{v} + 2(c + \lambda(c \cdot v)v)\overline{\infty} + (c^2 + \lambda(c \cdot v)^2 - r^2)\infty\overline{\infty} \tag{3.6}$$

is representative of this very same quadric by Definition 3.1, where Ω is defined as

$$\Omega = \sum_{i=1}^{n} e_i\overline{e}_i. \tag{3.7}$$

Canonical forms similar to (3.6) can be found for specific geometries, such as planes, spheres, plane-pairs, circular cylinders, circular conical surfaces, and so on.

4. Transformations Supported By The Model

The main result of this section will depend upon the following lemma.

Lemma 4.1. *For any versor $V \in \mathbb{G}(\mathbb{W})$, and any four vectors $a, b, c, d \in \mathbb{V}$, we have*

$$V^{-1}aV \wedge \overline{V^{-1}bV} \cdot c \wedge \overline{d} = a \wedge \overline{b} \cdot V\overline{V}(c \wedge \overline{d})(V\overline{V})^{-1}. \tag{4.1}$$

Proof. We begin by first establishing that

$$\begin{aligned} & V^{-1}aV \wedge \overline{V^{-1}bV} \cdot c \wedge \overline{d} && (4.2)\\ =& -(V^{-1}aV \cdot c)(V^{-1}bV \cdot d) && (4.3)\\ =& -(a \cdot VcV^{-1})(b \cdot VdV^{-1}) && (4.4)\\ =& \; a \wedge \overline{b} \cdot VcV^{-1} \wedge \overline{VdV^{-1}}. && (4.5)\end{aligned}$$

We now notice that

$$\begin{aligned} & VcV^{-1} && (4.6)\\ =& \; V\overline{\overline{V}V^{-1}}cV^{-1} && (4.7)\\ =& \; (-1)^m V\overline{V}c\overline{V^{-1}}V^{-1} && (4.8)\\ =& \; (-1)^m V\overline{V}c(V\overline{V})^{-1}, && (4.9)\end{aligned}$$

where m is the number of vectors taken together in a geometric product to form V. We then notice that

$$\begin{aligned} & \overline{VdV^{-1}} && (4.10)\\ =& \; VV^{-1}\overline{VdV^{-1}} && (4.11)\\ =& \; (-1)^{m^2} V\overline{V}V^{-1}\overline{dV^{-1}} && (4.12)\\ =& \; (-1)^{m^2+m} V\overline{Vd}V^{-1}\overline{V^{-1}} && (4.13)\\ =& \; (-1)^{2m^2+m} V\overline{VdV^{-1}}V^{-1} && (4.14)\\ =& \; (-1)^m V\overline{Vd}(V\overline{V})^{-1}. && (4.15)\end{aligned}$$

It now follows that

$$a\wedge\overline{b}\cdot VcV^{-1}\wedge\overline{VdV^{-1}} \tag{4.16}$$

$$=a\wedge\overline{b}\cdot(-1)^{2m}V\overline{V}c(V\overline{V})^{-1}\wedge V\overline{V}\overline{d}(V\overline{V})^{-1} \tag{4.17}$$

$$=a\wedge\overline{b}\cdot V\overline{V}(c\wedge\overline{d})(V\overline{V})^{-1}, \tag{4.18}$$

which completes the proof. □

We're now ready to prove the main result as follows.

Lemma 4.2. *Letting $E\in\mathbb{G}(\mathbb{W})$ be a bivector of the form (3.2), $p,p'\in\mathbb{V}^o$ be a pair of points related by a versor $V\in\mathbb{G}(\mathbb{V})$ by the equation*

$$p'=o\cdot V^{-1}pV\wedge\infty, \tag{4.19}$$

and $E'\in\mathbb{G}(\mathbb{W})$ a bivector given by

$$E'=V\overline{V}E(V\overline{V})^{-1}, \tag{4.20}$$

the set of all points $p\in\mathbb{V}^o$ such that

$$0=p'\wedge\overline{p}'\cdot E \tag{4.21}$$

is exactly the set of all points $p\in\mathbb{V}^o$ such that

$$0=p\wedge\overline{p}\cdot E'. \tag{4.22}$$

Proof. The lemma goes through by the following chain of equalities.

$$(o\cdot V^{-1}pV\wedge\infty)\wedge\overline{(o\cdot V^{-1}pV\wedge\infty)}\cdot E \tag{4.23}$$

$$=V^{-1}pV\wedge\overline{V^{-1}pV}\cdot E \tag{4.24}$$

$$=p\wedge\overline{p}\cdot(V\overline{V})E(V\overline{V})^{-1}. \tag{4.25}$$

The first equality holds by the fact that E is of the form (3.2), while the second equality holds by Lemma 4.1. □

A corrollary to Lemma 4.2 immediately follows.

Corollary 4.3. *If $V\in\mathbb{G}(\mathbb{V})$ is a versor such that E' in equation (4.20) is of the form (3.2), then the versor $V\overline{V}$ represents a transformation closed in the set of all quadric surfaces.*

The key motivation behind Lemma 4.2 is the observation that the desired transformation of E by V is given by the algebraic set of equation (4.21), because an understanding of how V^{-1} transforms p gives us an understanding of what type of geometry we get from equation (4.21) in terms of E and V. Lemma 4.2 then shows that this is also the algebraic set of equation (4.22), thereby giving us a means of performing desired transformations on elements in $\mathbb{G}(\mathbb{W})$ representative of quadric surfaces. By Corollary 4.3, what we get from such a transformation is also a quadric surface, provided that V is a versor such that E' in (4.20), like E, is also a bivector of the form (3.2).

We can now apply Lemma 4.2 to show that the rigid body transformations are supported in our new variation of the original model. Letting $\pi \in \mathbb{V}$ be a dual plane of CGA, given by

$$\pi = v + (c \cdot v)\infty, \tag{4.26}$$

where $v \in \mathbb{V}^e$ is a unit-length vector indicating the norm of the plane, and where $c \in \mathbb{V}^e$ is a vector representing a point on the plane, we see that for any homogenized point $p \in \mathbb{V}^o$, we have

$$-\pi p \pi^{-1} = o + x - 2((x-c)\cdot v)v + \lambda\infty, \tag{4.27}$$

where $p = o+x$ with $x \in \mathbb{V}^e$, and where the scalar $\lambda \in \mathbb{R}$ is of no consequence. Letting $V = \pi$, the point $p' \in \mathbb{V}^o$ of consequence here is given by equation (4.19), from which we can recognize an orthogonal reflection about the plane π. It now follows by Lemma 4.2 that $\pi\overline{\pi}$ is a versor capable of reflecting any quadric surface about the plane π. Being able to perform planar reflections of any quadric in any plane, it now follows that we can always find a versor $V \in \mathbb{G}(\mathbb{W})$ capable of performing any rigid body motion on any quadric surface. The development of the rigid body motions, (combinations of translations and rotations), by planar reflections, is well known, and can be found in section 2.7 of [5].

In retrospect, what we have done to find the rigid body motions of quadric surfaces is similar to what was done in [6]; and according to [10], we can state more generally that what we have done is at least similar to finding an isomorphism between quadratic spaces. Section 4 of [7] shows that versors can be used to transform quadric surfaces using an entirely different approach.

5. Extending The New Model

Interestingly, if we were not content with the rigid body motions of quadrics, then we really could find what is, for example, the spherical inversion of, say, an infinitely long cylinder in a sphere. To do this, we start by changing Definition 3.1 into the following definition.

Definition 5.1. For any element $E \in \mathbb{G}(\mathbb{W})$, we may refer to it as an n-dimensional quartic surface as the set of all points $p \in \mathbb{V}^e$ such that

$$0 = P(p) \wedge \overline{P}(p) \cdot E, \tag{5.1}$$

where $P : \mathbb{V}^e \to \mathbb{V}$ is the point mapping of CGA, defined in [3] as

$$P(p) = o + p + \frac{1}{2}p^2\infty. \tag{5.2}$$

It is then helpful to introduce another definition as follows.

Definition 5.2. A versor $V \in \mathbb{G}(\mathbb{V})$ is said to be point-form preserving if for any point $x \in \mathbb{V}^e$, there exists a scalar $\lambda \in \mathbb{R}$ and a point $y \in \mathbb{V}^e$ such that $V^{-1}P(x)V = \lambda P(y)$.

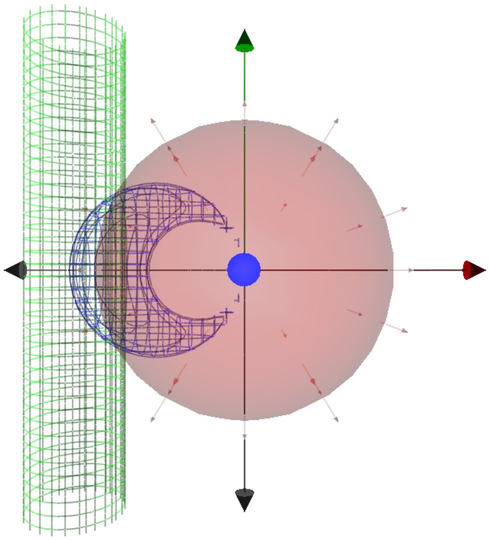

FIGURE 1. The inversion of a cylinder in a sphere. Traces in various planes were used to render the cylinder and its inversion.

We then arrive at the following upgrade of Lemma 4.2.

Lemma 5.3. *Letting $E \in \mathbb{G}(\mathbb{W})$ be a bivector of any form, $p, p' \in \mathbb{V}^e$ be a pair of points related by a point-form preserving versor $V \in \mathbb{G}(\mathbb{V})$ by the equation*

$$\lambda P(p') = V^{-1} P(p) V, \tag{5.3}$$

and $E' \in \mathbb{G}(\mathbb{W})$ a bivector given by equation (4.20), *the set of all point $p \in \mathbb{V}^e$ such that*

$$0 = P(p') \wedge \overline{P}(p') \cdot E \tag{5.4}$$

is exactly the set of all point $p \in \mathbb{V}^e$ such that

$$0 = P(p) \wedge \overline{P}(p) \cdot E'. \tag{5.5}$$

Proof. The scalar λ clearly divides out of equation (5.4), so we may, without loss of generality, let $\lambda = 1$. Once again applying Lemma 4.1, we simply see that

$$V^{-1} P(p) V \wedge \overline{V^{-1} P(p) V} \cdot E = P(p) \wedge \overline{P}(p) \cdot V \overline{V} E (V \overline{V})^{-1}. \tag{5.6}$$

□

The need for a point-form preserving versor is apparent from equation (5.4), since otherwise we could not claim to understand what algebraic set we get from equation (5.4) in terms of E and V. We now see by Lemma 5.3 that if $V \in \mathbb{G}(\mathbb{W})$ is any point-form preserving versor, and if E is a surface under Definition 5.1, then the element $E' \in \mathbb{G}(\mathbb{W})$, given by equation (4.20), must, by Definition 5.1, be representative of the desired transformation of E

by the versor $V\overline{V}$. The general polynomial equation arising from the form of such elements E in Defintion 5.1 is much more involved than what we have in equation (3.4). Nevertheless, it is possible to extract a specific form of a quartic polynomial equation in the vector components of p from equation (5.1). The result being unsightly, it will not be presented here. Suffice it to say, a computerized algebra system was used to find the polynomial form. In any case, it is easy to see from equation (5.1) that the degree of the resulting polynomial will be four.

Now notice that under Definition 5.1, canonical forms such as (3.6) are still valid. This is because

$$P(p) \wedge \overline{P}(p) \cdot E = (o + p) \wedge \overline{(o + p)} \cdot E \tag{5.7}$$

in the case that E is of the form (3.2). This allows us to use what we already know about quadrics in the new model with its extension to quartic surfaces of a specific form.

An interesting side effect of Definition 5.1 is the existence of every union of any pair of geometries where either one is a circle, plane or point. We simply note that for any pair of vectors $a, b \in \mathbb{V}$, we have

$$P(p) \wedge \overline{P}(p) \cdot a \wedge \overline{b} = -(P(p) \cdot a)(P(p) \cdot b). \tag{5.8}$$

Putting theory into practice, the author wrote a piece of computer software that implements this CGA-like model for the special class of quartic surfaces of equation (5.1). Giving the program the following script as input, the output of the program is given in Figure 1 and rendered another way in Figure 2. The script is easy for anyone to read, even if they are not familiar with its language. It is given here to illustrate how one might use the model with the aide a computer system.

```
/*
 * Calculate the surface that is the
 * inversion of a cylinder in a sphere.
 */
do
(
    /* Make the cylinder. */
    v = e2, c = -7*e1, r = 2,
    cylinder = Omega - v^bar(v) + 2*c*nib + (c.c - r*r)*ni^nib,
    bind_quadric(cylinder),
    geo_color(cylinder,0,1,0),

    /* Make the sphere. */
    c = 0, r = 6,
    sphere = no + c + 0.5*(c.c - r*r)*ni,
    bind_dual_sphere(sphere),
    geo_color(sphere,1,0,0,0.2),

    /* Make the inversion of the cylinder in the sphere. */
    V = sphere*bar(sphere),
    inversion = V*cylinder*V~,
    bind_conformal_quartic(inversion),
    geo_color(inversion,0,0,1),
)
```

The functions beginning with the word "bind" create and bind an entity to the given element of the geometric algebra that is responsible for interpreting that element as a surface under Definition 5.1 or, in the case of the sphere, as a dual surface under the definition given by CGA. The computer program

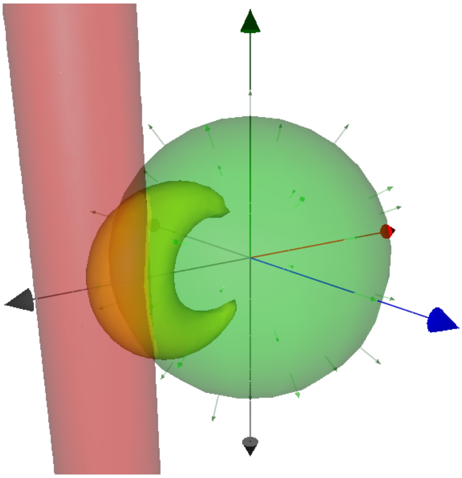

FIGURE 2. The inversion of a cylinder in a sphere. A surface mesh generation algorithm was used to skin the cylindrical and inverted surfaces. The inverted mesh suffers where the curvature becomes extreme.

can then use traditional methods to render the surface from the extracted polynomial equation. For example, the polynomial equation in x, y and z for the inverted surface presented in Figure 1 is given by

$$\begin{aligned} 0 = 28.8x^2 + 11.2x^3 + x^4 + 11.2xy^2 + 2x^2y^2 + \\ 11.2xz^2 + 2x^2z^2 + y^4 + 2y^2z^2 + 28.8z^2 + z^4. \end{aligned} \tag{5.9}$$

It is interesting how a bit of reasoning in geometric algebra has given us such a simple means to obtaining this polynomial equation. Of course, while such equations lend themselves to computer algorithms, they are not practical on paper. This is where the canonical forms of elements might become useful; although, admittedly, even these forms have proven to be unwieldy and impractical for the author, unlike their CGA counterparts.

6. Dual And Direct Surfaces

The goal from the beginning has been to find a model, similar to CGA, for the general set of surfaces up to degree 2, not just the specific class of surfaces, up to degree 2, that are just the spheres and planes of CGA. While this has been accomplished to some extent, one of the greatest deficiencies remaining appears to be the inability for the model to represent surfaces of up to the desired degree for all dimensions from zero to n in the same manner that this is possible in CGA.[1] One possible solution to this is that of utilizing the

[1]We can, of course, easily formulate the extruded conics. That is, the surfaces of dimension $n-1$ extruded through a dimension orthogonal to the conic section. To do so, we simply apply a rigid-body motion versor to any conic section that is easily formulated as being

geometric algebra that is generated by the linear space of bivectors in $\mathbb{G}(\mathbb{W})$. We could define a linear function on this space that maps it to a vector space $\mathbb{B}$. If $[]$ was such a function, then for any pair of 2-blades $A, B \in \mathbb{G}(\mathbb{W})$, we could define

$$[A] \cdot [B] = A \cdot B. \tag{6.1}$$

It would then follow that a vector $v \in \mathbb{G}(\mathbb{B})$ would be representative of a surface as the set of all points $p \in \mathbb{V}^e$ such that

$$0 = \rho(p) \cdot v, \tag{6.2}$$

where we define[2] the function ρ as

$$\rho(p) = [P(p)\overline{P(p)}]. \tag{6.3}$$

The notions of dual and direct surfaces would then emerge as they do in CGA. A blade $B \in \mathbb{G}(\mathbb{B})$ is dually representative of a surface as the set $\dot{G}(B)$, defined as

$$\dot{G}(B) = \{p \in \mathbb{V}^e | 0 = \rho(p) \cdot B\}. \tag{6.4}$$

A blade $B \in \mathbb{G}(\mathbb{B})$ is directly representative of a surface as the set $\hat{G}(B)$, defined as

$$\hat{G}(B) = \{p \in \mathbb{V}^e | 0 = \rho(p) \wedge B\}. \tag{6.5}$$

Using the outer product, we can now intersect dual surfaces and combine direct surfaces.[3] These features arise as a consequence of representing geometries as blades in a geometric algebra.

To illustrate the use of $\mathbb{G}(\mathbb{B})$, let $s, c \in \mathbb{G}(\mathbb{B})$ be vectors dually representative of a sphere and cylinder, respectively. Then, for any point $p \in \mathbb{V}^e$, we can find the dual surface containing p and the intersection of s and c as

$$\pm(\rho(p) \wedge (s \wedge c)I)I = \rho(p) \cdot s \wedge c = (\rho(p) \cdot s)c - (\rho(p) \cdot c)s, \tag{6.6}$$

where I, in practice, might be the unit pseudo-scalar of the geometric algebra generated by the vector sub-space of $\mathbb{B}$ given by the set

$$\{[x\overline{y}] | x, y \in \mathbb{V}\}. \tag{6.7}$$

Even this vector space, which is of dimension $(n+2)^2$, is larger than it needs to be. We could suffice with a vector space of dimension $(n+2)(n+3)/2$. In

origin-centered and axis-aligned. For example, $0 = -r^2 + x^2 + y^2$ is a circle in the plane, but also an extruded circle (a cylinder) in 3-dimensional space.

[2]Notice that we need not define ρ here in terms of P and $[]$. We could simply forget all of that machinary and provide an equivilant function in terms a basis for the vector space $\mathbb{B}$. In fact, in how ρ is defined on a given vector space, we can find a model capable of representing almost any subset of the set of all algebraic sets.

[3]The outer product of two dual surfaces is the dual surface that is the intersection, if any, of the two dual surfaces taken in the product. That is, for any two blades $A, B \in \mathbb{G}(\mathbb{B})$ with $A \wedge B \neq 0$, we have

$$\dot{G}(A \wedge B) = \dot{G}(A) \cap \dot{G}(B).$$

Similarly, the outer product of two direct surfaces is the direct surface containing at least the union of the surfaces taken in the product. That is, for any two blades $A, B \in \mathbb{G}(\mathbb{B})$, we have

$$\hat{G}(A \wedge B) \supseteq \hat{G}(A) \cup \hat{G}(B).$$

Imaginary dual intersections may be reinterpreted as real direct surfaces.

any case, it is clear from equation (6.6) that the algebra is simply giving us the desired surface in the pencil of s and c.

If all we cared about was the dual intersection $s \wedge c$, we may still need to resort to [12] to do any meaningful analysis. Contrasting this with an absence of any need to do such a thing in CGA, we see further deficiencies in our more generalized model for surfaces up to degree 2. To further illustrate the point, consider the intersection of any quadratic dual surface with a line. It is much easier to setup and solve a quadratic equation than it is to take the outer product of the dual surface with a dual line and then make sense of the result. For example, given a quadric $E \in \mathbb{G}(\mathbb{W})$ of the form (3.2), and letting $f : \mathbb{V}^o \to \mathbb{R}$ be defined as

$$f(p) = p \wedge \overline{p} \cdot E, \tag{6.8}$$

we have

$$0 = f(p + \lambda v) = f(p) + \lambda \nabla_v f(p) + \lambda^2 f(v), \tag{6.9}$$

where $p \in \mathbb{V}^o$ is a point and $v \in \mathbb{V}^e$ is a direction vector, where $\nabla_v f(p)$ is the directional derivative of f at p in the direction v, and from which we easily recognize a quadratic equation in the scalar variable λ. In CGA, point-pairs are easily decomposable. Section 4 of [7] offers a solution to the intersection problem using a different method of utilizing CGA to represent conic and quadric surfaces. See also section 4.2 of [13].

7. Transformations Of Dual And Direct Surfaces

For a given versor $V \in \mathbb{G}(\mathbb{V})$, and a k-blade $B \in \mathbb{G}(\mathbb{B})$, if we could find a vector factorization $v_1 \wedge \cdots \wedge v_k$ of B, then the transformation B' of B by V would be given by

$$B' = \bigwedge_{i=1}^{k} [V\overline{V}[v_i]^{-1}(V\overline{V})^{-1}]. \tag{7.1}$$

(See [2] on the problem of factoring blades.) It is unfortunate that we would have to bother finding such a factorization in order to apply a given transformation.

Leaving the versors of $\mathbb{G}(\mathbb{V})$ behind, we are left to consider the versors of $\mathbb{G}(\mathbb{B})$. Following the line of thinking that led to Lemma 4.2 and emma 5.3, we begin by considering the set of all versors $V \in \mathbb{G}(\mathbb{B})$ preserving the form $\rho(p)$ as the point-form preserving versors $V \in \mathbb{G}(\mathbb{V})$ preserve the form $P(p)$ under Definition 5.2. We will also refer to such versors $V \in \mathbb{G}(\mathbb{B})$ as point-form preserving.

If we are able to develop any point-form preserving versor $V \in \mathbb{G}(\mathbb{B})$, and understand what kind of transformation a point undergoes by an application of this transformation, then it is not hard to show that an application of such a versor to any blade $B \in \mathbb{G}(\mathbb{B})$, dually or directly representative of a given surface, can be well understood. We begin with the following definition.

Definition 7.1. For any k-blade $B \in \mathbb{G}(\mathbb{B})$, we refer to it as point-fit-able, if there exists a set of k points $\{p_i\}_{i=1}^k \subset \mathbb{V}^e$, such that

$$B = \bigwedge_{i=1}^{k} \rho(p_i). \tag{7.2}$$

It is clear that the surface directly represented by a point-fit-able k-blade fits any k points that can be used to formulate a factorization of that blade by equation (7.2). Given a set $\{p_i\}_{i=1}^k$ of such points, and a point-form preserving versor $V \in \mathbb{G}(\mathbb{B})$, if we know which surface must fit those points, then it is also clear that we'll know which surface is fit by the set of points $\{p_i'\}_{i=1}^k$, where for each integer i, we have $V\rho(p_i)V^{-1} = \lambda\rho(p_i')$. Given the surface B of equation (7.2), this is simply the surface VBV^{-1}.

There are, however, at least two problems with this. First, assuming that the set $\{\rho(p_i)\}_{i=1}^\infty$ is linearly independent, or that we understand under what circumstances of the set $\{p_i\}_{i=1}^k$ that the set $\{\rho(p_i)\}_{i=1}^\infty$ will be linearly independent, determining the surface that must fit such a given set of points is non-trivial. Secondly, it is not clear whether all direct surfaces are point-fit-able. Fortunately for us, the property of being point-fit-able plays no part in the following lemma.

Lemma 7.2. *Given a k-blade $B \in \mathbb{G}(\mathbb{W})$ and a point-form preserving versor $V \in \mathbb{G}(\mathbb{B})$, if for any piont $p \in \mathbb{V}^e$, we understand the action of V on $\rho(p)$ as $V^{-1}\rho(p)V$, then we understand the action of V on the dual surface B as VBV^{-1}.*

Proof. Writing B in terms of the k vectors in $\{b_i\}_{i=1}^k$ as $B = \bigwedge_{i=1}^k b_i$, we have $0 = V^{-1}\rho(p)V \cdot B$ if and only if for all integers $i \in [1, k]$, we have $0 = V^{-1}\rho(p)V \cdot b_i = \rho(p) \cdot Vb_iV^{-1}$, since the set $\{B_i\}_{i=1}^k$, where B_i denotes the product B with b_i removed, is a linearly independent set. Then, for all integers $i \in [1, k]$, we have $0 = \rho(p) \cdot Vb_iV^{-1}$, if and only if $0 = \rho(p) \cdot VBV^{-1}$, since the set $\{VB_iV^{-1}\}_{i=1}^k$ is also linearly independent.

It follows now that $0 = V^{-1}\rho(p)V \cdot B$ if and only if $0 = \rho(p) \cdot VBV^{-1}$. Then, in terms of understanding the transformation of $\rho(p)$ by V as $V^{-1}\rho(p)V$, we also understand the transformation of B by V as VBV^{-1}. □

All that remains now is to show that an understanding of how dual surfaces transform gives us an understanding of how direct surfaces transform. To see this, realize that $\hat{G}(B) = \dot{G}(BI)$, and then that

$$\hat{G}(VBV^{-1}) = \hat{G}(VBI^2V^{-1}) = \hat{G}(VBIV^{-1}I) = \dot{G}(VBIV^{-1}), \tag{7.3}$$

where I is the unit psuedo-scalar of $\mathbb{G}(\mathbb{B})$. This shows that direct surfaces are affected by versors in the same way that dual surface are.

The challenge now is to find a point-form preserving versor $V \in \mathbb{G}(\mathbb{B})$ and understand its action on $\rho(p)$. We will have to leave this as an open question for now.

8. Closing Remarks

With a background in abstract algebra and topology, an accessible introduction to the subject of algebraic geometry is given in [8]. Not surprisingly, and perhaps ironically, there is no mention of geometric algebra; although one might presume at first glance that the similarly named subjects would have a great deal to do with one another. From the present paper, a method for using blades of a geometric algebra to represent algebraic sets generated by polynomials of any form, though not specifically stated, can now be inferred, and it is quite trivial. Algebraic geometry, however, has grown far beyond algebraic sets as the central objects of study. Not being competent in geometric calculus, much less the vast and arcane subject of algebraic geometry, the author cannot pursue a reformulation of any part of one subject with the other, but surely others can and will. Until then, geometric algebra continues to offer fun and interesting ways to do geometry with models such as CGA and perhaps the newly established model of this paper.

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