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Abstract. Abstract...

Keywords. Key words...

Definition 0.1 (Spade). An element $M \in \mathbb{G}$ is a *spade* if it can be written as the geometric product of zero or more vectors.

It follows from Definition 0.1 that all versors are spades, but not all spades are versors. Furthermore, while the set of all spades of $\mathbb{G}$ enjoys closure under the geometric product, this set, unlike the set of versors of $\mathbb{G}$, does not form a group.

Definition 0.2 (Spade Rank/Compact Spade Form). The *rank* of a spade $M \in \mathbb{G}$, denoted $\text{rank}(M)$, is the smallest number of vectors for which M can be written as a geometric product of such. If a spade $M_r \in \mathbb{G}$ has factorization

$$M_r = \prod_{i=1}^{r} m_i,$$

we say that it is written in *compact form* if $\text{rank}(M_r) = r$.

Many identities involving a spade M_r hold whether or not it is written in compact form.

Definition 0.3 (Dull/Sharp Spade). A spade $M \in \mathbb{G}$ is *sharp* if the grade $\text{rank}(M)$ part of M is non-zero, and *dull* otherwise.

If M_r is a sharp blade written in compact form, it then follows that

$$\langle M_r \rangle_r = \bigwedge_{i=1}^{r} m_i.$$

Definition 0.4 (Ideal Spade Form). Letting $M_r^{(i)}$ denote the spade

$$M_r^{(i)} = \prod_{\substack{j=1 \\ j \neq i}}^{r} m_j,$$

a spade $M_r \in \mathbb{G}$ is written in *ideal form* when the set of r spades $\{M_r^{(i)}\}_{i=1}^r$ is a linearly independent set.

Clearly, every spade can be written in compact form. It is not clear, however, whether every spade can be written in ideal form.

Lemma 0.5. *For every non-zero r-blade $B_r \in \mathbb{G}$, with $r > 1$, and having factorization*

$$B_r = \bigwedge_{i=1}^r b_i,$$

the set of $(r-1)$-blades $\{B_r^{(i)}\}_{i=1}^r$, where

$$B_r^{(i)} = \bigwedge_{\substack{j=1 \\ j\neq i}}^r b_j,$$

is a linearly independent set.

Proof. Supposing to the contrary, and without loss of generality, let

$$B_{r-1} = B_r^{(r)} = \sum_{i=1}^{r-1} \alpha_i B_r^{(i)} = \left(\sum_{i=1}^{r-1} \alpha_i B_{r-1}^{(i)}\right) \wedge b_r.$$

Now notice that

$$0 \neq B_r = B_{r-1} \wedge b_r = B_r^{(r)} \wedge b_r = \left(\sum_{i=1}^{r-1} \alpha_i B_r^{(i)}\right) \wedge b_r = 0,$$

which is clearly a contradiction. □

Lemma 0.6. *Given any spade M_r, the set of all solution sets $\{\alpha_i\}_{i=1}^r$ of the equation*

$$0 = \sum_{i=1}^r \alpha_i M_r^{(i)}$$

is, for all integers $j \in [0, r]$, the intersection of all sets of solution sets of the equations

$$0 = \sum_{i=1}^r \alpha_i \langle M_r^{(i)} \rangle_j.$$

Proof. This is a simple consequence of there being no possibility of cancelation between elements of differing grade. □

Lemma 0.7. *Every sharp spade having rank greater than one, and written in compact form, is also written in ideal form.*

Proof. Let M_r be a sharp spade written in compact form with $r > 1$. It follows that $\{m_i\}_{i=1}^r$ is a linearly independent set of vectors, and so any one of its subsets is also linearly independent. By Lemma 0.5, it now follows that $\{\langle M_r^{(i)} \rangle_{r-1}\}_{i=1}^r$ is a linearly independent set. The linear independence of the set $\{M_r^{(i)}\}_{i=1}^r$ now follows by Lemma 0.6. □

Although it is likely that every spade, dull or sharp, written in compact form is also written in ideal form, proof of this statement will have to remain an open question as the present author was enable to resolve it. Notice that if $a, b \in \mathbb{V}$ are a pair of non-parallel, yet non-perpendicular vectors, then aba is a dull spade of rank 3, and it is easy to show that it is in ideal form, which is to say that there is no non-trivial linear combination of ab and ba that vanishes.

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