Chapter 14 Exercises
Gallian's Book on Abstract Algebra

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March 4, 2014

Understanding Example 7

Let R be the ring of all real-valued functions of a real variable. The subset S of all differentiable functions is a subgring of R but not an ideal of R.

Let f be any real-valued function of a real variable that is not differential and let g be such a function that is differentiable. Now notice that the function $h(x) = f(x)g(x)$ is not differentiable.

Understanding Example 15

If it can be shown that A contains a non-zero real number c, then, by virtue of being an ideal, it absorbs all elements of $R[x]$, so all $f(x)c$ with $f(x) \in R[x]$ is all of $R[x]$, showing that $A = R[x]$.

Exercise 3

Verify that the set I in Example 5 is an ideal and that if J is any ideal of R that contains $a_1, a_2, \dots, a_n$, then $I \subseteq J$. (Hence, $\langle a_1, a_2, \dots, a_n \rangle$ is the smallest ideal of R that contains $a_1, a_2, \dots, a_n$.)

For reference, note that

$$I = \langle a_1, a_2, \dots, a_n \rangle = \{r_1 a_1 + r_2 a_2 + \cdots + r_n a_n | r_i \in R\}.$$

Clearly $0 \in I$. If $x, y \in I$, then for elements $x_1, x_2, \ldots, x_n \in R$ and elements $y_1, y_2, \ldots, y_n \in R$, we have

$$\begin{aligned} x - y &= x_1a_1 + x_2a_2 + \cdots + x_na_n - (y_1a_1 + y_2a_2 + \cdots + y_na_n) \\ &= (x_1 - y_1)a_1 + (x_2 - y_2)a_2 + \cdots + (x_n - y_n)a_n \in I, \end{aligned}$$

since each $x_i - y_i \in R$. Letting $r \in R$, we have

$$rx = rx_1a_1 + rx_2a_2 + \cdots + rx_na_n \in I,$$

since each $rx_i \in R$.

Now let J be an ideal of R containing $a_1, a_2, \ldots, a_n$, and let x be any element of I. As before, let $x = x_1a_1 + x_2a_2 + \cdots + x_na_n$. Now by the definition of what an ideal is, it is clear that each $x_ia_i \in J$, because $x_i \in R$ and $a_i \in J$. Furthermore, $x \in J$, because each $x_ia_i \in J$ and J is a group.

Exercise 7

Let a belong to a commutative ring R. Show that $aR = \{ar | r \in R\}$ is an ideal of R. If R is the ring of even integers, list the elements of $4R$.

Clearly the additive identity is in aR, since $0 = a \cdot 0$. Let $x, y \in aR$. Then there exist elements $r_x, r_y \in R$ such that $x = ar_x$ and $y = ar_y$. We then have $x - y = ar_x - ar_y = a(r_x - r_y) \in aR$ since $r_x - r_y \in R$. Now let $r \in R$ and see that $rx = rar_x = arr_x \in aR$ since $rr_x \in R$. It follows that aR is an ideal of R.

In that case, $4R = \{0, \pm 8, \pm 16, \pm 24, \pm 32, \ldots\}$, I think.

Exercise 9

If n is an integer greater than 1, show that $\langle n \rangle = nZ$ is a prime ideal of Z if and only if n is prime.

Notice that nZ is an ideal of Z by Exercise 7.

Suppose n is prime. Let $a, b \in Z$ such that $ab \in nZ$. Then there exists $z \in Z$ such that $ab = nz$. It follows that $n | ab$ which implies that $n | a$ or $n | b$ by Euclid's Lemma. So there exists $z' \in Z$ such that $a = nz' \in nZ$ or $b = nz' \in nZ$, showing that nZ is a prime ideal of Z.

Now suppose nZ is a prime ideal of Z. Then if $a, b \in Z$ such that $ab = nz$ for some $z \in Z$, we must have, for some $z' \in Z$, $a = nz'$ or $b = nz'$. In other words, if $n|ab$, we must have $n|a$ or $n|b$ in every case. There is no composite number that can do this, so n must be prime. (We can also conclude n is prime by continually factoring what n divides, and then know that n divides one of the factors. Repeating, we're eventually left with only one prime factor.)

Exercise 15

If A is an ideal of a ring R and 1 belongs to A, prove that $A = R$.

Since A is an ideal of R and $1 \in A$, we have, for all $r \in R$, $r = 1r \in A$, showing that $A = R$.

Exercise 21

Verify the claim made in Example 10 about the size of R/I.

For reference,

$$R = \left\{ \left[\begin{array}{cc} a_1 & a_2 \\ a_3 & a_4 \end{array} \right] \middle| a_i \in Z \right\}$$

and I is the subset of R consisting of matrices with even entries.

The example in the text helps make the verification easy. Let $M \in R$. Then the coset $M+I = B+I$, where B is a matrix consisting of just ones and zeros. Since the matrices have 4 possible entries, there are $2^4 = 16$ possible elements in R/I.

Exercise 23

Show that the set B in the latter half of the proof of Theorem 14.4 is an ideal of R.

For reference, $B = \{br + c | r \in R, a \in A\}$ with $b \in R - A$. The subset A is an ideal of R, and R is a commutative ring with unity.

Letting $x \in R$, we must show that for any $y \in B$, that $xy \in B$ and $yx \in B$. Let $y = br + a$ for elements $r \in R$ and $a \in A$. Then $xy = bxr + xa \in B$ since $xr \in R$ and $xa \in A$. (Remember that A is an ideal of R.) And we have $yx = bry + ay \in B$ since $ry \in R$ and $ay \in A$.