The Meet, Join, And Geometric Sets

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Abstract. A follow-up to [], this paper shows that the intersection of any two goemetric sets is geometric, then goes on to consider other results about geometric sets in terms of the *meet* and *join* operations.

1. Introduction

In [] the notion of a geometric set was introduced, and it was shown that under some circumstances, the intersection of two geometric sets is geometric. In this paper we show that this is the case under all circumstances.

All conventions and notation used in [] are carried forward here, and so for brevity, will not be needlessly repeated.

2. The Meet And Join Of Vector Sub-Spaces

An excellent treatment of *meet* and *join* is given in []. Unlike [], however, here we will not, as they said, abuse language by referring to any blade as a vector sub-space.[1] In an attempt to be as rigorous as possible, we will introduce the following definition.

Definition 2.1. For any blade $B \in \mathbb{B}$, we let

$$\hat{v}(B) = \{v \in \mathbb{V} | v \wedge B = 0\}, \tag{2.1}$$

$$\dot{v}(B) = \{v \in \mathbb{V} | v \cdot B = 0\}. \tag{2.2}$$

Realize that for any two blades $A, B \in \mathbb{B}$, we have $\hat{v}(A) = \hat{v}(B)$ if and only if there exists a scalar $\lambda \in \mathbb{R}$ such that $A = \lambda B$.

As operations (as well defined functions), *meet* and *join* operate on vector spaces, not blades. We therefore do not speak of taking the *meet* or *join* of two blades. Rather, the blades of a geometric algebra will help us calculate the *meet* and *join* of vector sub-spaces represented by those blades.

[1]The present author has abused language in this way in other writings, but now repents of this, at least in the present paper.

Definition 2.2 (The *meet* of two vector sub-spaces). For any two blades $A, B \in \mathbb{B}$, the *meet* $\mathbb{M}$ of $\hat{v}(A)$ and $\hat{v}(B)$ is the vector space given by

$$\mathbb{M} = \hat{v}(A) \cap \hat{v}(B) = \{x \in \mathbb{V} | x \in \hat{v}(A) \text{ and } x \in \hat{v}(B)\}. \tag{2.3}$$

From Defintion 2.2 we see that the *meet* of two vector subspaces is the largest common sub-space. Notice that *meet* is clearly communtative.

Definition 2.3 (The *join* of two vector sub-spaces). For any two blades $A, B \in \mathbb{B}$, the *join* $\mathbb{J}$ of $\hat{v}(A)$ and $\hat{v}(B)$ is the vector space given by

$$\mathbb{J} = \hat{v}(A) + \hat{v}(B) = \{x_1 + x_2 \in \mathbb{V} | x_1 \in \hat{v}(A) \text{ and } x_2 \in \hat{v}(B)\}. \tag{2.4}$$

From Definition 2.3 we see that the *join* of two vector spaces is the smallest common super-space. Notice here too the commutativity of *join* .

Some care should be taken with Definition 2.2 and Definition 2.3 by easily verifying that $\mathbb{M}$ and $\mathbb{J}$ each satisfy the necessary properties of a vector space.

At this point a natural question arrises. Given two blades $A, B \in \mathbb{B}$, how do we find blades $M, J \in \mathbb{B}$ such that $\hat{v}(M) = \hat{v}(A) \cap \hat{v}(B)$ and $\hat{v}(J) = \hat{v}(A) + \hat{v}(B)$? While it is not immediately clear if a closed-form formula exists for an M or a J in terms of A and B, [] shows that there are many identities that relate all four blades, and [] gives an algorithm for calculating the *join* of any two given blades. Once the *join* is known, it is then easy to calculate the *meet* , and vice-versa. In any case, this paper is merely concerned with the realization that there must exist blades $A', B' \in \mathbb{B}$ such that

$$\hat{v}(J) = \hat{v}(A' \wedge M \wedge B'), \tag{2.5}$$

where $A' \wedge M = A$ and $M \wedge B' = B$.

3. The Intersection Of Geometric Sets

We now come to the main result of this paper, stated with the following theorem.

Theorem 3.1. *The intersection of any two geometric sets is geometric.*

Proof. For any pair of geometric sets R and S, let $A, B \in \mathbb{B}$ be a pair of blades such that $\dot{g}(A) = R$ and $\dot{g}(B) = S$. In the case that $A \wedge B \neq 0$, we have

$$\dot{g}(A \wedge B) = R \cap S. \tag{3.1}$$

On the other hand, if $A \wedge B = 0$, consider a blade J given by

$$J = A' \wedge M \wedge B', \tag{3.2}$$

where $A' \wedge M = A$ and $M \wedge B' = B$. In this case, we have

$$\begin{aligned}
\dot{g}(J) &= \dot{g}(A') \cap \dot{g}(M) \cap \dot{g}(B') && (3.3)\\
&= (\dot{g}(A') \cap \dot{g}(M)) \cap (\dot{g}(M) \cap \dot{g}(B')). && (3.4)\\
&= R \cap S. && (3.5)
\end{aligned}$$

In any case, notice that a blade directly representative of the *join* is also a blade dually representative of the intersection. □

Theorem **??** then begs the question, if the *join* gives us intersections, what does the *meet* give us? It is not hard to see that if blades A and B are directly representative of R and S, respectively, then $\hat{g}(M) = R \cap S$. But what does the *meet* give us if A and B are dually representative of R and S? This may be the same as asking: what does the *join* give us if A and B are directly representative of R and S? In that case, one way to proceed is to consider irreducible representatives.

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