The Intersection Of Rays And Algebraic Surfaces

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Abstract. It is shown that for any multi-variable polynomial defined over the real numbers that the image of a line through the domain of such a function is determined entirely by all orders of the directional derivatives of this function at any one point along the line and in a direction of the line. This result has an application in the problem of casting rays through algebraic surfaces as it shows that such a problem, in all cases, reduces to the problem of finding the roots of a single-variable polynomial having an explicit formulation in terms of the multi-variable polynomial and ray in question.

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1. Introduction

Letting $f : \mathbb{R}^n \to \mathbb{R}$ be any polynomial equation in n variables and up to degree m, it was shown in [2] that the function $\bigwedge_{i=1}^m p_i(x)$ may be factored out of this polynomial in terms of the inner product as

$$f(x) = \bigwedge_{i=1}^{m} p_i(x) \cdot B, \tag{1.1}$$

where B is an m-vector of our geometric algebra with $\infty_i \cdot B = 0$ for all integers $i \in [1, m]$, and where the function $p_i : \mathbb{R}^n \to \mathbb{V}_i$ is given by

$$p_i(x) = o_i + x_i + \frac{1}{2}x_i^2\infty_i, \tag{1.2}$$

having its origins in the paper [1]. Given a point $x \in \mathbb{R}^n$ and a direction vector $v \in \mathbb{R}^n$, we wish to find the set of all scalars $\lambda \in \mathbb{R}$ such that $f(x + \lambda v) = 0$. Utilizing equation (1.1) for this purpose, we easily find that

$$f(x + \lambda v) = \bigwedge_{i=1}^{m} (p_i(x) + \lambda v_i) \cdot B, \tag{1.3}$$

because we can ignore the $\frac{1}{2}x_i^2\infty_i$ term in equation (1.2). Looking at equation (1.3), it is immediately clear that its expansion is that of a polynomial in λ of up to degree m. What we're going to show in this paper is that an explicit formula for this polynomial can be found in terms of all orders of directional derivatives of f at x and in the direction of v.

2. The Result

We begin by rewriting equation (1.3) as

$$f(x+\lambda v) = \sum_{i=0}^{m} T_i(x), \tag{2.1}$$

where $T_i(x)$ will denote the i^{th} term involving λ^i in the series expansion of (1.3). Carefully formulating this term, we get

$$T_i(x) = \lambda^i \sum_{j=1}^{\binom{m}{i}} W_{j,i}(x) \cdot B,$$

where $W_{j,i}$ is the j^{th} way to write an outer product involving i vectors taken from $\{v_k\}_{k=1}^m$ and $m-i$ vectors taken from $\{p_k(x)\}_{k=1}^m$ in an order having ascending sub-scripts. The following examples help clarify this in the case $m=3$.

$$\begin{aligned}
W_{1,0} &= p_1 \wedge p_2 \wedge p_3 \\
W_{1,1} &= p_1 \wedge p_2 \wedge v_3 \\
W_{2,1} &= p_1 \wedge v_2 \wedge p_3 \\
W_{3,1} &= v_1 \wedge p_2 \wedge p_3 \\
W_{1,2} &= p_1 \wedge v_2 \wedge v_3 \\
W_{2,2} &= v_1 \wedge p_2 \wedge v_3 \\
W_{3,2} &= v_1 \wedge v_2 \wedge v_3 \\
W_{1,3} &= v_1 \wedge v_2 \wedge v_3
\end{aligned}$$

Having now come to terms, (no pun intended), with the general expansion of equation (1.3), we proceed now to fearlessly take the directional derivative of T_i at x and in the direction of v. Doing so, we get

$$\nabla_v T_i(x) = \lambda^i \sum_{j=1}^{\binom{m}{i}} \lim_{\delta\to 0} \frac{W_{j,i}(x+\delta v) - W_{j,i}(x)}{\delta} \cdot B,$$

knowing that each individual limit will exist. What we must realize now is that the term $W_{j,i}(x)$ will get canceled in the expansion of $W_{j,i}(x+\delta v)$, leaving only terms that are multiples of positive powers of δ. Furthermore, it is only those remaining terms that are multiples of δ itself that will survive the limit process. We are therefore left to deduce these terms in an evaluation of the limit. What we find is that all such terms are of the form $\delta W_{j,i+1}(x)$, but we need to determine just how many we have. Realizing that $\binom{m}{i}$ old terms will each contribute $m-i$ new terms of this form, of which there should be $\binom{m}{i+1}$, but that no type of term will be produced any more or less than any other, we see that

$$\frac{(m-i)\binom{m}{i}}{\binom{m}{i+1}} = i+1$$

is the number of such terms of the form $\delta W_{j,i+1}(x)$, and we may write

$$\begin{aligned}\nabla_v T_i(x) &= \lambda^i \sum_{j=1}^{\binom{m}{i+1}} \lim_{\delta\to 0} \frac{(i+1)\delta W_{j,i+1}(x)}{\delta} \cdot B \\ &= \lambda^i (i+1) \sum_{j=1}^{\binom{m}{i+1}} W_{j,i+1}(x) \cdot B \\ &= \frac{i+1}{\lambda} T_{i+1}(x). \end{aligned} \tag{2.2}$$

Returning to equation (2.1), and realizing that $T_0(x) = f(x)$, we can now finally deduce the expansion of (1.3) using the recurrence relation of equation (2.2) as

$$f(x+\lambda v) = \sum_{i=0}^{m} \frac{\lambda^i}{i!} \nabla_v^i f(x), \tag{2.3}$$

where $\nabla_v^i f(x)$ is the i^{th} order directional derivative of f at x in the direction of v with $\nabla_v^0 f(x) = f(x)$.

3. Making Use Of The Result

Admittedly, this result may be more of theoretical interest than of having any practical application to the problem of ray casting algebraic surfaces. Nevertheless, equation (2.3) has been used to formulate the coeficients of a quadric equation which can then be easily solved by the quadratic formula. A variety of quadrics were ray-taced using this method and can be seen in Figure **??**. The quadratic equation used may be written as

$$f(x) + \lambda v \cdot \nabla f(x) + \lambda^2 v_1 \wedge v_2 \cdot B.$$

As can be easily seen, each of the ceoficients are easy to compute, given f and ∇f. This is not quite the case, however, when we consider the cubic equation

$$f(x) + \lambda v \cdot \nabla f(x) + \frac{\lambda^2}{2} \nabla_v (v \cdot \nabla f(x)) + \lambda^3 v_1 \wedge v_2 \wedge v_3 \cdot B.$$

Here, all but the third coeficient is easily computed. Interestingly, we may rewrite $\nabla_v(v \cdot \nabla f(x))$ as $v \cdot \nabla_v(\nabla f(x))$, in which form, approximating the derivative may be a bit easier. In all cases, a coeficient can be computed as an approximation of the following limit.

$$\frac{T_i(x)}{\lambda^i} = \lim_{\delta \to 0} \frac{f(x + \delta v) - f(x)}{\delta^i}.$$

$$v_1 \wedge v_2 \cdot \nabla \wedge \nabla f(x)$$

References

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