Chapters 9-11 Supplementary Exercises Gallian's Book on Abstract Algebra

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Exercise 36

A proper subgroup H of a group G is called *maximal* if there is no subgroup K such that $H \subset K \subset G$. Prove that Q under addition has no maximal subgroups.

Let H be any non-trivial, proper subgroup of Q. Then, if $h \in Q$ is also a member of H, then so is any integer multiple of h. In other words,

$$hZ = \{zh | z \in Z\} \subseteq H.$$

Notice that hZ is also a subgroup of H.

Now, since H is a proper subgroup of Q, there exists $r \in Q \backslash H$. If we wanted to form a subgroup of Q containing H and r, then it must contain at least H and rZ. Letting $H + rZ$ denote the set

$$H + rZ = \{h + zr | h \in H, z \in Z\},$$

it is not hard to show that $H + rZ$ is a subgroup of Q properly containing H. What remains to be shown, however, is that $H + rZ$ is a proper subgroup of Q.

To that end, suppose $Q = H + rZ$ in the hopes of reaching a contradiction. This then implies that

$$\langle r + H \rangle = Q/H,$$

which is to say that the factor group Q/H is cyclic, being generated by $r + H$. But, since $rZ \cap H$ is a non-trivial group, it follows that the order of $r + H$ is finite, and therefore $|Q/H|$ is finite.

We now must show that $|Q/H|$ is finite if and only if $H = Q$, which would give us a desired contradiction. Intuitively, it is not possible for $|Q/H|$ to be a positive integer greater than one. For example, suppose $|Q/H| = 2$. Then $Q/H = \{H, r+H\}$. But we also know that H and $r+H$ are interleaved along all of R, (the real number line), in such a way that no open interval contains members of either coset exclusively. (If such an open interval contained elements exclusively from H, then we must have $H = Q$. This conclusion is also arrived at if this open interval contained elements exclusively from $r + H$.) So, H and $r + H$ are each dense in Q. But does there exist a proper subgroup of Q that is dense in Q?