Chapters 5-8 Supplementary Exercises
Gallian's Book on Abstract Algebra

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Problem 1

A subgroup N of a group G is called a *chracteristic subgroup* if $\phi(N) = N$ for all automorphisms ϕ of G. Prove that every subgroup of a cyclic group is characteristic.

Let ϕ be an automorphism of G, a cyclic group. Let a be an element of G. Now see that

$$\phi(\langle a \rangle) = \{\phi(a^k) | k \in \mathbb{Z}\} = \{\phi^k(a) | k \in \mathbb{Z}\} = \langle \phi(a) \rangle.$$

Clearly, $|a| = |\phi(a)|$, so $|\langle a \rangle| = |\langle \phi(a) \rangle|$. We can now claim that $\langle a \rangle = \langle \phi(a) \rangle$ by the fundamental theorem of cyclic groups, because G has one and only one subgroup of each possible order.

Problem 2

Prove that the center of a group is characteristic.

Let ϕ be any automorphism of a group G. Letting a be an element in $\phi(Z(G))$ and g an element in G, there must exist an element $a' \in Z(G)$ and an element $g' \in G$ such that $\phi(a') = a$ and $\phi(g') = g$. It then follows that

$$ag = \phi(a')\phi(g') = \phi(a'g') = \phi(g'a') = \phi(g')\phi(a') = ag,$$

showing that $a \in Z(G)$. Thus far we have shown that $\phi(Z(G)) \subseteq Z(G)$. But ϕ is one-to-one, so $\phi(Z(G))$ cannot be a proper subset of $Z(G)$, and therefore, we must have $\phi(Z(G)) = Z(G)$. Oops, what if $Z(G)$ is infinite?

I'm stumped...

Problem 4

Prove that the property of being a characterstic subgroup is transitive. That is, if N is a characteristic subgroup of K and K is a characteristic subgroup of G, then N is a characteristic subgroup of G.

Let $\phi \in \mathrm{Aut}(G)$. If $\phi(K) = K$, then ϕ, when restricted in domain to K, is an automorphism of K. It follows that $\phi(N) = N$, showing that N is a characteristic subgroup of G.

Problem 6

Let H and K be subgroups of a group G and let $HK = \{hk | h \in H, k \in K\}$ and $KH = \{kh | k \in K, h \in H\}$. Prove that HK is a group if and only if $HK = KH$.

Suppose $HK = KH$. Clearly $e \in HK$. Let $a, b \in HK$. Then there exists $h, h' \in H$ and $k, k' \in K$ such that $a = hk$ and $b = h'k'$, and we have

$$ab^{-1} = hk(h'k')^{-1} = hk(k')^{-1}(h')^{-1} = hh''k'' \in HK,$$

for some element $h'' \in H$ and another $k'' \in K$, because $HK = KH$.

Now suppose HK is a subgroup of G. If $a \in HK$, then $a^{-1} = hk$ for some $h \in H$ and $k \in K$. It follows that $a = k^{-1}h^{-1} \in KH$. If $a \in KH$, then $a = hk$ for some $h \in H$ and $k \in K$. It follows that $a^{-1} = h^{-1}k^{-1} \in HK \implies (a^{-1})^{-1} = a \in HK$.

Problem 7

Let H and K be subgroups of a finite group G. Prove that

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

This is easy to prove when $H \cap K = \{e\}$. Think about it.