Section 3.4 Exercises
Herstein's Topics In Algebra

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Thoughts

It's interesting to note that, by definition, an ideal need not be a subring of a ring. But maybe this is always the case? Let U be an ideal of a ring R. U is already an additive subgroup of R. To be a subring, we must show that U is a ring. I think all that is required at this point is closure under multiplication. Let $a, b \in U$. Then for any $r \in R$, we have $rab \in U$ since $(ra)b \in U$, and $abr \in U$, since $a(br) \in U$. So we have closure.

Prove: if ϕ is a homomorphism from a ring R with unit element 1 onto a ring R' with unit element $1'$, and R' is an integral domain, then $\phi(1) = 1'$ and R is an integral domain. (Note: Problem 20 says my conditions here are more than sufficient for showing $\phi(1) = 1'$.)

Suppose there exists $1' \neq b \in R'$ such that $ba = a$ for all $0' \neq a \in R'$. Then $1'a = ba \implies (1'-b)a = 0' \implies 1'-b = 0' \implies 1' = b$; so the multiplicative identity $1'$ in R' is unique. Now note that since $\phi(1)\phi(a) = \phi(a)$ for all $a \in R$, $\phi(1)$ acts as an identity in R'; but since there is only one such element in R', we must have $\phi(1) = 1'$.

To see that R is an integral domain, notice that for all $a, b \in R$, we have

$$0 = ab \implies 0' = \phi(ab) = \phi(a)\phi(b) \implies \phi(a) = 0' \text{ or } \phi(b) = 0',$$

which, in turn, implies that $a = 0$ or $b = 0$.

Problem 2

If F is a field, prove its only ideals are $\{0\}$ and F itself.

We first note that for every homomorphism ϕ of a ring R, we find an ideal of R; namely, $\ker \phi$. And then for every ideal I of R, we find a homomorphism of R; namely, $\phi(x) = x + I$. So there is a one-to-one correspondence between ideals of R and homomorphisms of R.

By Problem 3, any homomorphism of F is trivial. So if ϕ is such a homomorphism, it is either $\phi(x) = 0$ or $\phi(x) = x$. We then find the set of all ideals of F as the kernels of these homomorphisms; which are F and $\{0\}$, respectively.

Problem 3

Prove that any homomorphism of a field is either an isomorphism or takes each element into 0.

Let ϕ be a homomorphism of a field F. If $\phi(x) = 0$, we're done; so assumes this is not the case. We can, therefore, claim that there are non-additive-identity elements in $\phi(F)$. Let $a \in F$ such that $\phi(a)$ is such an element. Now see that $\phi(a) = \phi(a \cdot 1) = \phi(a)\phi(1)$, showing that $\phi(1)$ in $\phi(F)$ acts as a multiplicative identity element in the ring that is the homomorphic image of F. We then find that for any $a \in F$,

$$\phi(1) = \phi(aa^{-1}) = \phi(a)\phi(a^{-1}) \implies \phi(a)^{-1} = \phi(a^{-1}).$$

We can now conclude that $\phi(F)$ is a division ring, and its commutativity would certainly follow from that of F. So $\phi(F)$ is a field, and therefore an integral domain. Lastly, for any pair of elements $a, b \in F$ such that $\phi(a) = \phi(b)$, we have

$$\phi(1) = \phi(a)\phi(b)^{-1} = \phi(ab^{-1}).$$

It then follows that $ab^{-1} = 1$, since $\phi(F)$ is an integral domain. We can now say that ϕ is an isomorphism.