Generalizing the Dot Product

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Hello, Paul. Consider $a \cdot B$ where a is a vector and B is an n-dimensional blade (or n-blade) with $n \geq 1$. We are very familiar with the case of $n = 1$, and we really just take it as a definition to understand the meaning of $a \cdot B$ when B is a vector. But how do we generalize this idea to higher dimensions? To do that, I like to take a step back and consider $a \wedge B$. The outer product is used to build up blades. You can think of the vector a in this case as being used to extrude B into a new dimension, provided a really does go into a dimension B does not. If this is not the case, then $a \wedge B$ collapses to zero. So to intuitively grasp $a \cdot B$, we simply think of it as doing the opposite of $a \wedge B$. That is, $a \cdot B$ is taking B and collapsing it down by one dimension. The part of a that is perpendicular to B, if any, is the only part of a that extrudes B to get $a \wedge B$ non-zero. Similarly, the only part of a that is parallel to B, if any, is the only part of a that collapses B to get $a \cdot B$ non-zero.

So let's look at $a \cdot B$ when B is a 2-blade. Like you said, we can write $a \cdot B = a \cdot (b \wedge c)$ with b perpendicular to c, without loss of generality, but we don't need to introduce the geometric product here. Furthermore, if we let $a_\perp$ be the part of a perpendicular to B, and $a_\parallel$ be the part of a parallel to B, then we can choose b such that $b = a_\parallel$, and write...

$$a \cdot B = (a_\perp + a_\parallel) \cdot (b \wedge c) = a_\parallel.(b \wedge c) \equiv (a_\parallel \cdot b)c = |a_\parallel|^2 c$$

...where here I have used the symbol $\equiv$ to denote an equality that follows by definition. (Again, instead of extruding a blade into a new dimension, if we collapse B down into a lower dimensional blade using the direction of $a_\parallel$, we get a vector in the direction of c.)

Note that up to now, getting $a \cdot B$ into pure vector form ($|a_\parallel|^2 c$) was all about our clever choices for how to factor B into vectors b and c. But if we chose any vector b and c such that $B = b \wedge c$, then how do we arrive at...

$$a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b$$

Well, going back to the vectors b and c we had chosen before, let's play with what we had. We can write...

$$a \cdot B = (a_\parallel \cdot b)c = (a_\parallel \cdot b)c - (a_\parallel \cdot c)b$$

...because $a_\parallel \cdot c = 0$. Now replace $a_\parallel$ with $a - a_\perp$ to get...

$$a \cdot B = (a \cdot b)c - (a \cdot c)b$$

But this is still working under the assumption of our original choice of vectors for b and c. To generalize here, I think we have to look at...

$$b \wedge c = (b + \gamma c) \wedge c = b \wedge (c + \beta b)$$

...for any scalars β and γ, and then try to convince ourselves that the original formula will continue to hold. For example, let's replace b with $b + \gamma c$, and then see what happens to our formula. We get...

$$\begin{aligned} a \cdot B &= (a \cdot b)c - (a \cdot c)b \\ &= (a \cdot (b + \gamma c))c - (a \cdot c)(b + \gamma c) \\ &= (a \cdot b)c + \gamma(a \cdot c)c - (a \cdot c)b - \gamma(a \cdot c)c \\ &= (a \cdot b)c - (a \cdot c)b \end{aligned}$$

As we can see, the formula was not modified by the replacement. Similiarly, replacing c with $c + \beta b$ should give the same result. This shows that the formula holds for any factorization of B, because any one factorization can be reached from any other factorization by a sequence of such replacements.

Anyhow, that's my attempt to generalize the dot product after years of not doing any GA. I can't promise I didn't make a mistake, but it seems sound. A further generalization, by the way, would be to consider $A \cdot B$ where A is an m-blade and B is an n-blade. The result will be of grade $|m - n|$.