Outermorphisms

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Letting $\mathbb{G}$ be a geometric algebra generated by a vector space $\mathbb{V}$, and letting $f : \mathbb{V} \to \mathbb{V}$ be a linear transform defined on that vector space, there exists an extension $\underline{f}$ of f to all of $\mathbb{G}$ given by $\underline{f}(a) = f(a)$ whenever $a \in \mathbb{V}$, and whenever $\lambda \in \mathbb{R}$ and $A, B \in G$, we have

$$\underline{f}(A + B) = \underline{f}(A) + \underline{f}(B), \tag{1}$$

$$\underline{f}(\lambda A) = \lambda \underline{f}(A), \tag{2}$$

$$\underline{f}(\lambda) = \lambda, \tag{3}$$

$$\underline{f}(A \wedge B) = \underline{f}(A) \wedge \underline{f}(B). \tag{4}$$

Clearly $\underline{f}$ is grade perserving. That is, $\underline{f}(\langle A \rangle_i) = \langle \underline{f}(A) \rangle_i$. It follows that $\underline{f}(I) = \lambda I$, where I is the unit pseudo-scalar of $\mathbb{G}$. In this case, we define $\det f = \lambda$. That is, we define

$$\det \underline{f} = I^{-1} \underline{f}(I). \tag{5}$$

(Notice that $\det \underline{f}$ is an eigen-value for the eigen-blade I.) Hestenes then defines the adjoint or transpose $\overline{f}$ of $\underline{f}$ as being implicitly given by

$$\langle \overline{f}(A) B \rangle_0 = \langle A \underline{f}(B) \rangle_0. \tag{6}$$

From this it is not at all obvious to me that $\overline{f}$ is an outermorphism, or even a linear transform that can be extended to an outermorphism. Assuming it is, however, it is easy to show that it has the same determinant as $\underline{f}$.

$$\det \underline{f} = \langle I^{-1} \underline{f}(I) \rangle_0 = \langle \overline{f}(I^{-1}) I \rangle_0 = \langle \overline{f}(I) I^{-1} \rangle_0 = \langle I^{-1} \overline{f}(I) \rangle_0 = \det \overline{f} \tag{7}$$

Then, citing a references I don't have access to, Hestenes claims that from all this he can derive the following identity, provided grade$(A) \leq$ grade(B).

$$\underline{f}(\overline{f}(A) \cdot B) = A \cdot \underline{f}(B) \tag{8}$$

Notice that this is clearly consistent with equation (6) as far as scalars go.

Now, realizing that the inner and geometric products are interchangable when one operand is a pseudo-scalar, we can use equation (8) to find that

$$A = \frac{AI^{-1}\underline{f}(I)}{\det f} = \frac{\underline{f}(\overline{f}(AI^{-1})I)}{\det f} = \frac{\underline{f}(\overline{f}(AI)I^{-1})}{\det f}, \tag{9}$$

provided, of course, that $\det f \neq 0$. Finally, we see from this that

$$\underline{f}^{-1}(A) = \underline{f}^{-1}\left(\frac{\underline{f}(\overline{f}(AI)I^{-1})}{\det f}\right) = \frac{\overline{f}(AI)I^{-1}}{\det f}. \tag{10}$$

This is really interesting to me, although I still have no idea how to use it in practice to, say, calculate the inverse of a matrix. This is certainly a more elegant formulation of the inverse of a linear transformation than the one presented in my linear algebra textbook.

Returning to (8), a special case of this is easy to prove.

$$\underline{f}\left(\overline{f}(a) \cdot \bigwedge_{i=1}^{n} b_i\right) \tag{11}$$

$$= \underline{f}\left(-\sum_{i=1}^{n}(-1)^i(\overline{f}(a) \cdot b_i) \bigwedge_{j=1, j\neq i}^{n} b_j\right) \tag{12}$$

$$= -\sum_{i=1}^{n}(-1)^i(\overline{f}(a) \cdot b_i) \bigwedge_{j=1, j\neq i}^{n} \underline{f}(b_j) \tag{13}$$

$$= -\sum_{i=1}^{n}(-1)^i(a \cdot \underline{f}(b_i)) \bigwedge_{j=1, j\neq i}^{n} \underline{f}(b_j) \tag{14}$$

$$= a \cdot \bigwedge_{i=1}^{n} \underline{f}(b_i) \tag{15}$$

$$= a \cdot \underline{f}\left(\bigwedge_{i=1}^{n} b_i\right) \tag{16}$$

Then, if grade$(A) \leq$ grade(B) and $A = \bigwedge_{i=1}^{m} a_i$ and $B = \bigwedge_{i=1}^{n} b_j$, we can use

$$A \cdot B = \bigwedge_{i=1}^{m-1} a_i \cdot \left(a_m \cdot \bigwedge_{i=1}^{n} b_i \right) \tag{17}$$

to show, by induction, that equation (8) holds in the case of blades. We have

$$= \underline{f} \left(\overline{f} \left(\bigwedge_{i=1}^{m} a_i \right) \cdot \bigwedge_{i=1}^{n} b_i \right) \tag{18}$$

$$= \underline{f} \left(\overline{f} \left(\bigwedge_{i=1}^{m-1} a_i \right) \cdot \left(\overline{f}(a_m) \cdot \bigwedge_{i=1}^{n} b_i \right) \right) \tag{19}$$

$$= \bigwedge_{i=1}^{m-1} a_i \cdot \underline{f} \left(\overline{f}(a_m) \cdot \bigwedge_{i=1}^{n} b_i \right) \tag{20}$$

$$= \bigwedge_{i=1}^{m-1} a_i \cdot \left(a_m \cdot \underline{f} \left(\bigwedge_{i=1}^{n} b_i \right) \right) \tag{21}$$

$$= \left(\bigwedge_{i=1}^{m} a_i \right) \cdot \underline{f} \left(\bigwedge_{i=1}^{n} b_i \right). \tag{22}$$

Notice that the inductive hypothesis was used going from equation (19) to (20).

To finish a proof of equation (8) for multivectors in general, I believe all we need to do is cite the distributivity of outermorphisms over addition.

Returning to equation (10), calculating the inverse of a matrix may involving calculating $\underline{f}^{-1}(e_i)$ for each basis vectors e_i as all linear transformations are determined by how they transform a basis of the vector space. Hestenes found an explicit form for the adjoint $\overline{f}$ using some calculus. Perhaps I should look there.