Versors That Give Non-Uniform Scale

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To my dear wife Melinda.

Abstract. It is shown that for every non-uniform scale transformation, a versor exists in a geometric algebra that can perform this transformation on any algebraic surface. Some implications and generalizations of this find are discussed, as well as the possibility of acquiring all affine transformations as versors applicable to any algebraic surface.

Keywords. Algebraic Surface, Conformal Model, Non-Uniform Scale, Geometric Algebra.

1. Motivation And Review

The question of existence of non-uniform scale versors is one of the outstanding problems of geometric algebra, and one that stands in the way of geometric algebra competing against existing and well-proven transformation models. As noted in the beginning of [7], 4×4 matrices have long-time been a standard in computer graphics for representing affine and projective transformations, but an equivalent yet hopefully more capable and universally compatible model for such transformations in a more modern setting has yet to emerge as a considerable replacement. This paper does not purport to provide such a setting, but it does offer a potential solution to the non-uniform scale problem. An upcoming paper by an author of [6] may provide an even better solution.

Being dependent upon [9], a quick review is in order before we begin. In [9], and therefore this paper, we work in a geometric algebra that is generated by a vector space that is ismorphic to an external direct product of m disjoint copies of a vector space that generates the Minkowski algebra used by the conformal model of geometric algebra. If $\mathbb{V}_i$ denotes one of these vector spaces, then our geometric algebra $\mathbb{G}$ may be thought of as being generated by the vector space $\bigoplus_{i=1}^m \mathbb{V}_i$. We will be interested in the m-vectors taken from $\bigwedge_{i=1}^m \mathbb{V}_i$ as being representative of algebraic surfaces of up to degree m by equation (2.3) below.[1]

2. The Result

The result of this paper is simply a corollary to that of [9], but to see how, we must first constrain the way in which we represent algebraic surfaces, (living in n-dimensional space), of up to degree m in the Mother Minkowski algebra of order m.[2] What we do is let $n \leq m$, and reserve certain subalgebras of our mother algebra for use in specific dimensions. To see what is meant by this, let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial in whose zero set we are interested. Now define, for each integer $k \in [1, n]$, the polynomial $f_k : \mathbb{R} \to \mathbb{R}$ as

$$f_k(\lambda) = f(e_1 + e_2 + \cdots + \lambda e_k + \cdots + e_{n-1} + e_n).$$

Having done this, we will represent the surface of f in the Mother Minkowski algebra of order

$$m = \sum_{k=1}^{n} \deg f_k.$$

Now if $\mathbb{G}$ denotes our mother algebra and it is generated by m subalgebras $\mathbb{G}_i$, each generated by the vector space $\mathbb{V}_i$, then we reserve $\deg f_k$ of these subalgebras for use in dimension k of our n dimensions. (We will let $\mathbb{G}^k$, where k is an integer in $[1, n]$, denote the smallest subalgebra of $\mathbb{G}$ containing all subalgebras $\mathbb{G}_i$ reserved for dimension k, and let $[\mathbb{G}^k]$ denote the set of indices over which $\mathbb{G}_i \subseteq \mathbb{G}^k$.)

An example may be warrented at this point. Let $n = 3$ and consider the polynomial given by

$$f(x) = 3x_1^2 x_2 x_3^4 + 4x_1 x_2^3 - 7x_3^2, \tag{2.1}$$

where x_k is notation for $x_k = x \cdot e_k$. We will represent the surface that is the zero set of this polynomial using an m-vector in a Mother Minkowski algebra of order $m = 2+3+4 = 9$. The first 2 subalgebras are reserved for dimension 1, the next 3 for dimension 2, and the last 4 for dimension 3. The m-vector B representing this surface is then given by

$$\begin{aligned} B &= 3e_{12,1} \wedge e_{3,2} \wedge \infty_{45} \wedge e_{6789,3} \\ &\quad - 4e_{1,1} \wedge \infty_2 \wedge e_{345,2} \wedge \infty_{6789} \\ &\quad + 7\infty_{12345} \wedge e_{67,3} \wedge \infty_{89}. \end{aligned} \tag{2.2}$$

Here, notation is a challenge. The vector $e_{i,j}$ denotes the j^{th} euclidean basis vector in the i^{th} subalgebra. We then define

$$e_{i_1 i_2 \ldots i_r, j} = e_{i_1,j} \wedge e_{i_2,j} \wedge \cdots \wedge e_{i_r,j}.$$

[1] The notation $\bigwedge_{i=1}^m \mathbb{V}_i$ appears to suggest a set of m-blades, but it has been used in the literature to refer to m-vectors more generally.

[2] Recall that such representations are not unique, and so we have the flexibility to choose our representations carefully.

The notation for ∞ is similar.

We can now say that the zero set of f in equation (2.1) is given by the set of all solutions to the equation

$$\bigwedge_{k=1}^{m} p_k(x) \cdot B = 0. \tag{2.3}$$

Recall that $p_k(x) = o_k + x_k + \frac{1}{2}x^2\infty_k$, where here, x_k denotes the embedding of x in $\mathbb{V}_k$. Of course, we could have represented f in a mother algebra of order $\deg f = 7$, but it will soon become clear why we needed our algebra $\mathbb{G}$ to be of order $m = 9$.

Before moving on, notice that $[\mathbb{G}^1] = \{1, 2\}$, $[\mathbb{G}^2] = \{3, 4, 5\}$ and $[\mathbb{G}^3] = \{6, 7, 8, 9\}$.

Returning from the example, suppose now we have an m-vector B representing any polynomial $f : \mathbb{R}^n \to \mathbb{R}$ under the constraint thus illustrated. Seeing that the zero set of f is the set of solutions to equation (2.3), we make the simple observation that if D is a versor taken from a subalgebra $\mathbb{G}^k$, and further, D is the product of the same origin-centered dilation versor D_i found in each subalgebra $\mathbb{G}_i$ contained in $\mathbb{G}^k$, (see [6] for an explanation of dilation versors), which is to say that $D = \prod_{i\in[\mathbb{G}^k]} D_i$, then the non-uniform scale of f in the dimension of k by the scale of each D_i is given by the set of solutions to the equation

$$\begin{aligned} &\bigwedge_{i\notin[\mathbb{G}^k]} p_i(x) \wedge \bigwedge_{i\in[\mathbb{G}^k]} D_i^{-1} p_i(x) D_i \cdot B \\ &= \bigwedge_{i\notin[\mathbb{G}^k]} p_i(x) \wedge D^{-1} \left(\bigwedge_{i\in[\mathbb{G}^k]} p_i(x) \right) D \cdot B = 0, \end{aligned} \tag{2.4}$$

since for any $i \neq j$, we have

$$D_j^{-1} p_i(x) D_j = p_i(x).$$

Similarly, for all $i \notin [\mathbb{G}^k]$, D leaves $p_i(x)$ invariant. That is,

$$D^{-1} p_i(x) D = p_i(x).$$

It now follows by equations (3.2) through (3.5) of [9] that equation (2.4) may be rewritten as

$$\bigwedge_{k=1}^{m} p_k(x) \cdot DBD^{-1} = 0,$$

showing that D, when applied to B, performs a non-uniform scaling of the surface of f.

Putting this result into practice, let us apply a non-uniform scale transformation to the polynomial in equation (2.1). Suppose we wish to scale the surface represented by this polynomial by a factor of 2 in the e_2 dimension. Letting $D_k = (o_k - \infty_k)\left(o_k - \frac{1}{2}\infty_k\right)$, the non-uniform scale versor D we want is $D = D_3D_4D_5$. Then, using B in equation (2.2), we find that

$$\begin{aligned}\frac{1}{2^3}DBD^{-1} &= \frac{3}{2}e_{12,1} \wedge e_{3,2} \wedge \infty_{45} \wedge e_{6789,3} \\ &- \frac{4}{2^3}e_{1,1} \wedge \infty_2 \wedge e_{345,2} \wedge \infty_{6789} \\ &+ 7\infty_{12345} \wedge e_{67,3} \wedge \infty_{89},\end{aligned} \tag{2.5}$$

which is just what we would hope to get when checking this against the polynomial $f\left(x_1e_1 + \frac{1}{2}x_2e_2 + x_3e_3\right)$.[3] Notice that the $1/2^3$ factor on the left-hand side homogenizes DBD^{-1}.

3. Discussion

Though we have now shown that versors performing non-unform scaling exist, seeing that their application requires a great deal of cumbersome convention and notation, a question of their practicality immediately arises. It's certainly not practical on paper, but perhaps such versors may find applications on the computer.

In any case, it might now be possible to show that any affine transformation has an associated versor in our mother algebra that performs this transformation on an algebraic surface. This would be interesting, because the set of all algebraic surfaces of a certain degree are classified by defining an equivalence relation on this set which states that two surfaces, (in our case, m-vectors), are equivalent if and only if there exists an affine transformation, (in our case, an inner automorphism of the versor group of our mother algebra), that takes one of these surfaces to the other. The versor that would take any algebraic surface to the principle representative of its equivalence class would represent an important transformation.[4] It is not at all clear, however, whether geometric algebra is the right tool for studying such equivalence classes. As far as the mother Minkowski algebra is concerned, the only transformation we yet lack is that of shear.

In an initial search for shear, an immediate and possibly helpful observation we can make about the result of this paper is that it easily generalizes to the idea of a non-uniform "X", where "X" may be replaced here by any one of the conformal transformations. Since all such transformations may be decomposed as one or more planar reflections and spherical inversions, the two fundamental transformations to consider here are non-uniform reflections and non-uniform inversions. We first note that, without loss of generality, we need only consider reflections and inversions about the origin, since any problem can be translated into and out of this situation. Secondly, we can rule

[3]The present author used symbolic computation software to make the calculation in equation (2.5). This software can be found at `https://github.com/spencerparkin/GAVisTool`, though it is not recommended for general use.

[4]The principle representatives are the origin-centered, axis-aligned surfaces having unit characteristics where possible without loss of generality.

out non-uniform inversions right away, since they can't be any more helpful to us than non-uniform dilations. This leaves non-uniform reflections.

A quick analysis of the 2-dimensional case shows that a point undergoing this operation in the horizontal dimension with a plane having a unit-normal determined by an angle θ undergoes the same transformation illustrated by the following matrix equation.

$$\begin{bmatrix} -\cos 2\theta & -\sin 2\theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -(\cos 2\theta)x - (\sin 2\theta)y \\ y \end{bmatrix}$$

This bears some semblance of shear but clearly misses the mark.

4. Comparison And Possible Reconciliation

A review of an early draft of this paper rightly pointed out the work of [1] as providing all affine transformations for points in affine spaces. In writing [9], however, the author, in his weakness, was only able to take away from [1] the idea of "gluing" m geometric algebras together to form a mother algebra with added structure making the representation of surfaces by m-vectors in that algebra a natural consequence.[5] Once done, one can then leverage the known transformations available in the geometric algebra that was replicated to form the mother algebra. In the case of [9], it was the algebra of the conformal model; so, naturally, all surfaces became subject to the conformal transformations.

The soon-to-be-published paper [8], however, makes many of the ideas of [1] more accessible to the average reader. It even features a method of representing quadric surfaces in a way similar to that found in [9], with the only drawback being that it does not generally preserve the sandwich operation of versors on quadrics. To solve this problem, extend the representation scheme to the set of all algebraic surfaces of degree m, and gain all of the affine transformations, the following proposition is made, even if at the cost of the bloat created by [9] and exacerbated by the present paper.[6]

The paper [8] uses a geometric algebra generated by a vector space which may be denoted by $W \oplus W^*$, (with W a vector space isomorphic to $\mathbb{R}^4$, and W^* its dual), and shows that versors in this algebra give the affine transformations on homogeneous points in the vector space W. Leveraging this algebra as was done with the conformal model in [9], we can represent surfaces of degree m as m-vectors in the geometric algebra generated by $\bigoplus_{i=1}^m W_i$, which algebra is a subalgebra of the mother algebra generated by

$$\bigoplus_{i=1}^m W_i \oplus W_i^* = \bigoplus_{i=1}^m W_i \oplus \bigoplus_{i=1}^m W_i^*.$$

Specifically, we would be interested in the m-vectors taken from $\bigwedge_{i=1}^m W_i$.

[5] Through personal communication with David Hestenes, (an author of [1]), I suggested the representation scheme of [9], after which he pointed me to the paper [1] as having already explored similar propositions.

[6] Here, "bloat" refers to the excessive use of dimension.

Seeing that any further details of this idea would warrant a paper of its own, let us leave it at that. Seeing that it would not at all be hard for anyone familiar with [9] and [8] to fill in such details in the absence of such a paper, it may not at all be warranted.

5. Closing Remarks

Lastly, the apparent "coordinitis," (see [3]), suffered by this paper and [9] should be addressed. Polynomials are inherently non-coordinate-free; and so, naturally, a study of their zero sets in the framework of geometric algebra is not always free of coordinates. Despite this, if one begins with a coordinate-free formulation of a surface, $\bigwedge p_k(x)$ can often be factored out of it in terms of the inner product in a coordinate-free manner. For example, many quadrics characterized by a scalar λ, center c, radius r and unit-length vector v, are solutions in x to the coordinate-free equation

$$(x-c)^2 + \lambda((x-c)\cdot v)^2 - r^2 = 0,$$

which, when expanded, becomes

$$x^2 + \lambda(x\cdot v)^2 - 2x\cdot(c + \lambda(c\cdot v)v) + c^2 + \lambda(c\cdot v)^2 - r^2 = 0,$$

out of which may be factored $p_1(x)\wedge p_2(x)$ to get

$$p_1p_2\cdot(-\Omega - \lambda v_1v_2 - 2(c+\lambda(c\cdot v)v)_1\infty_2 - (c^2+\lambda(c\cdot v)^2 - r^2)\infty_{12}) = 0,$$

where the constant Ω is given by $\sum_{i=1}^n e_{12,i}$. (Recall that the subscripts can act as outermorphisms, and so have algebraic properties. See the admittedly poor paper [10].) The usefulness, if any, of these forms remains to be seen.

The study of algebraic surfaces using polynomial rings in abstract algebra, (algebraic geometry), does not suffer from coordinitis, because it is so abstract. There is a certain appeal to geometric algebra, however, because its language and results, as greatly illustrated by the conformal model, are just down-right fun.

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