# Chapters 9-11 Supplementary Exercises
# Gallian's Book on Abstract Algebra

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## Exercise 8

Let $k$ be a divisor of $n$. The factor gruop $(Z/\langle n\rangle)/(\langle k\rangle/\langle n\rangle)$ is isomorphic to some very familiar group. What is the group?

By Exercise 40 of Chapter 10 (The Third Isomorphism Theorem), we see that $(Z/\langle n\rangle)/(\langle k\rangle/\langle n\rangle) \approx Z/\langle k\rangle$. What more is there to say?

## Exercise 30

Let $G$ be a group and let $\phi : G \to G$ be a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1 g_2 g_3 = e = h_1 h_2 h_3$. Prove that there exists an element of $a$ in $G$ such that $\Psi(x) = a\phi(x)$ is a homomorphism.

If $a = e \in G$ and $u, v \in G$, then

$$\Psi((uv)^{-1})\Psi(u)\Psi(v) = e \implies \Psi(v^{-1}u^{-1}) = \Psi(v)^{-1}\Psi(v)^{-1}.$$

Hmmm... Can we somehow show that $\Psi$ is a homomorphism here? We cannot use homomorphic properties of $\Psi$ before we know that it's a homomorphism.

# Exercise 36

A proper subgroup $H$ of a group $G$ is called *maximal* if there is no subgroup $K$ such that $H \subset K \subset G$. Prove that $Q$ under addition has no maximal subgroups.

Let $H$ be any non-trivial, proper subgroup of $Q$. Then, if $h \in Q$ is also a member of $H$, then so is any integer multiple of $h$. In other words,

$$hZ = \{zh | z \in Z\} \subseteq H.$$

Furthermore, $hZ$ is a subgroup of $H$. Anyhow, we have shown that any non-trivial subgroup of $Q$ is infinite.

Now, since $H$ is a proper subgroup of $Q$, there exists $r \in Q \backslash H$. If we wanted to form a subgroup of $Q$ containing $H$ and $r$, then it must contain at least $H$ and $rZ$. Letting $H + rZ$ denote the set

$$H + rZ = \{h + zr | h \in H, z \in Z\},$$

it is not hard to show that $H + rZ$ is a subgroup of $Q$ properly containing $H$. What remains to be shown, however, is that $H + rZ$ is a proper subgroup of $Q$.

I am completely stumped! Here are some thoughts.

Intuitively, it should be impossible for a proper subgroup $H$ of $Q$ to complete $Q$ by throwing in a single element $r \in Q \backslash H$. What may perhaps help us realize this is a consideration of what was added to $H$ by throwing in $r$. This is precisely the set of elements

$$(H + rZ) \backslash H = \{h + s | h \in H, s \in rZ \backslash H\} = H + rZ \backslash H.$$

Notice that $H \cap rZ$ is a non-trivial subgroup of $Q$.