Conjugates and Commutators

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This note gives some insight into why useful sequences for twisty puzzles so often take the form of conjugates and commutators. We begin with a permutation group $G \leq S(\Omega)$, and for all $g \in G$, let

$$\phi(g) = \{\omega \in \Omega | \omega^g \neq \omega\}.$$

We then establish, for all $a, b \in G$, the following relations.

$$\begin{aligned} |\phi(bab^{-1})| &= |\phi(a)| \\ |\phi(aba^{-1}b^{-1})| &= |\phi(a) \cap \phi(b)| \\ &\leq \min(|\phi(a)|, |\phi(b)|) \end{aligned}$$

The first of these is a consequence of the fact that for any $\omega \in \Omega$, we have $\omega^a = \omega$ if and only if $\omega^{bab^{-1}} = \omega$. For the second of these, the relation clearly holds when a and b commute. If they do not commute, then for any $\omega \in \phi(a) \cup \phi(b)$, we consider the following three cases.

$$\begin{aligned} &\text{case 1: } \omega \in \phi(a) - \phi(b) \\ &\text{case 2: } \omega \in \phi(b) - \phi(a) \\ &\text{case 3: } \omega \in \phi(a) \cap \phi(b) \end{aligned}$$

Clearly, $\omega^{aba^{-1}b^{-1}} = \omega$ for cases 1 and 2, but not case 3.

What we learn here is that conjugates translate a useful sequence into another useful sequence acting on a different subset of Ω of the same size in a similar way. We also learn that the total action performed by a commutator is less than the minimum action performed by either permutation taken in the commutator product. This is essential to descending a stabilizer chain, where getting from one subgroup to a smaller subgroup requires stabilizing more points of Ω.