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Abstract. Abstract...

Keywords. Key words...

1. A Treatment Of Spades

In what follows we will let $\mathbb{V}$ denote a vector space generating our geometric algebra $\mathbb{G}$.

Definition 1.1 (Spade). An element $M \in \mathbb{G}$ is a *spade* if it can be written as the geometric product of zero or more vectors.

It follows from Definition 1.1 that all versors are spades, but not all spades are versors. Furthermore, while the set of all spades of $\mathbb{G}$ enjoys closure under the geometric product, this set, unlike the set of versors of $\mathbb{G}$, does not form a group.

Definition 1.2 (Spade Rank). The *rank* of a spade $M_r \in \mathbb{G}$, denoted $\text{rank}(M_r)$, is the smallest number[1] of vectors for which M_r can be written as a geometric product of such. If a spade $M_r \in \mathbb{G}$ has factorization

$$M_r = \prod_{i=1}^{r} m_i, \tag{1.1}$$

then it is not necessarily the case that $\text{rank}(M_r) = r$. However, such a factorization does exist. Clearly, it would not be unique.

Many identities involving a spade M_r hold whether or not $\text{rank}(M_r) = r$. In any case, we will become interested in precisely what we can say about the vectors in $\{m_i\}_{i=1}^r$ when $\text{rank}(M_r)$ is r.

Proposition 1.3. *For any given non-zero spade $M_r \in \mathbb{G}$ with $r > 0$,*

$$\textit{rank}(M_r) = r \textit{ if and only if } 0 \neq \textstyle\bigwedge_{i=1}^r m_i.$$

[1]This smallest number exists by the well-ordering principle. See [].

One direction of Proposition 1.3 is trivial to prove. The other is not. An outline of a proof follows.

Given a spade $M_r \in \mathbb{G}$ with factorization (1.1), the set of r vectors $\{m_i\}_{i=1}^r$ is either a linearly independent set, or a linearly dependent set. In the former case, it is clear that $\text{rank}(M_r) = r$, because $0 \neq \bigwedge_{i=1}^r m_i = \langle M_r \rangle_r$. In the latter case, we must show that $\text{rank}(M_r) < r$. To that end, let s be the largest integer with $1 \leq s < r$ such that $\{m_i\}_{i=1}^s$ is a linearly independent set, and write

$$\left\langle \prod_{i=1}^{s+1} m_i \right\rangle_{s-1} = \left\langle \left(\prod_{i=1}^{s} m_i \right) \sum_{i=1}^{s} \alpha_i m_i \right\rangle_{s-1} = \sum_{i=1}^{s} \beta_i \bigwedge_{\substack{j=1 \\ j \neq i}}^{s} m_j .$$

Just as not all scalars α_i are necessarily non-zero, we may not have all scalars β_i non-zero. Assuming for the moment that each β_i is non-zero, we may write the grade $s-1$ part as the following $(s-1)$-blade.

$$\sum_{i=1}^{s} \beta_i \bigwedge_{\substack{j=1 \\ j \neq i}}^{s} m_j = \beta_1 \bigwedge_{i=1}^{s-1} n_i,$$

where each n_i is given by

$$n_i = m_{i+1} + \frac{\beta_{i+1}}{\beta_i} m_i .$$

In any case, even if some β_i are zero, we can still come up with the $(s-1)$-blade that is the grade $s-1$ part of $\prod_{i=1}^{s+1} m_i$ in terms of vectors n_i.

With this $(s-1)$-blade in hand, we now must solve, for each integer k, the system of equations given by

$$\left\langle \prod_{i=1}^{s+1} m_i \right\rangle_{s-1-2k} = \beta_1 \left\langle \prod_{i=1}^{s-1} \left(n_i + \sum_{j=1}^{i-1} \gamma_{i,j} n_j \right) \right\rangle_{s-1-2k} . \tag{1.2}$$

If a solution in the variables $\gamma_{i,j}$ can be found, then we have shown that our geometric product of r vectors can be rewritten as a geometric product of $r-2$ vectors. We then continue this process until the set of vectors taken in the geometric product becomes a linearly independent set.

The proof outlined above is an algorithm for finding one of the smallest possible factorizations of the spade M_r; and consequently, its rank. The truthfulness of Proposition 1.3 hinges on the idea that a solution to the system of equations (1.2) can always be found.

Keeping in mind the Jewish proverb, "for example is not proof," the following examples are instructive.[2]

Give examples here...

[2]Of course, while attempting to prove a generality, a single supporting example is not a proof; but when attempting to disprove a generality, a counter-example is plenty proof. This author has failed to find a counter-example to Proposition 1.3.

Lemma 1.4. *For every non-zero r-blade $B_r \in \mathbb{G}$ with $r > 1$, and having factorization*

$$B_r = \bigwedge_{i=1}^{r} b_i,$$

the set of $(r-1)$-blades $\{B_r^{(i)}\}_{i=1}^r$, where the notation $B_r^{(i)}$ is given by

$$B_r^{(i)} = \bigwedge_{\substack{j=1 \\ j\neq i}}^{r} b_j,$$

is a linearly independent set.

Proof. Supposing to the contrary, and without loss of generality, let

$$B_{r-1} = B_r^{(r)} = \sum_{i=1}^{r-1} \alpha_i B_r^{(i)} = \left(\sum_{i=1}^{r-1} \alpha_i B_{r-1}^{(i)}\right) \wedge b_r.$$

Now notice that

$$0 \neq B_r = B_{r-1} \wedge b_r = B_r^{(r)} \wedge b_r = \left(\sum_{i=1}^{r-1} \alpha_i B_r^{(i)}\right) \wedge b_r = 0,$$

which is clearly a contradiction. □

Lemma 1.5. *Given any spade M_r, the set of all solution sets $\{\alpha_i\}_{i=1}^r$ of the equation*

$$0 = \sum_{i=1}^{r} \alpha_i M_r^{(i)}$$

is, for all integers $j \in [0, r]$, the intersection of all sets of solution sets of the equations

$$0 = \sum_{i=1}^{r} \alpha_i \langle M_r^{(i)} \rangle_j,$$

where the notation $M_r^{(i)}$ is given by

$$M_r^{(i)} = \prod_{\substack{j=1 \\ j\neq i}}^{r} m_j.$$

Proof. This is a simple consequence of there being no possibility of interference between elements of differing grade. □

Lemma 1.6. *For any given spade $M_r \in \mathbb{G}$ with $r > 1$, if rank$(M_r) = r$, then the set of spades $\{M_r^{(i)}\}_{i=1}^r$ is a linearly independent set.*

Proof. This follows easily in consideration of Proposition 1.3 with Lemma 1.4 and Lemma 1.5. □

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