A Variation Of The Quadric Model Of Geometric Algebra

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Abstract. A variation of the quadric model set forth in [10] is found in which the rigid body motions are represented by versors applicable to any quadric surface. Extending this variation of the original model to include a specific form of quartic surface, we find that such surfaces are closed under the application of all conformal transformations. Results of a computer program implementing this new model are presented.

Mathematics Subject Classification (2010). Primary 14J70; Secondary 14J29.

Keywords. Quadric Surface, Quartic Surface, Geometric Algebra, Quadric Model, Conformal Model.

1. Introduction

In the original paper [10], a model for quadric surfaces was presented based upon the ideas of projective geometry. What was unfortunate about this model, however, was its lack of support for the rigid body transformations. It was predicted in the conclusion of that paper that a better model for quadric surfaces may exist that is more like the conformal model of geometric algebra, here-after abbreviated CGA. The present paper details what may be such a model. We'll find that the rigid body transformations can be incorporated into the model by using an alternative method of encoding the quadric form. An extension to this quadric form will then allow us to support the conformal transformations at the expense of expanding our model to necessarily include a specific form of quartic surfaces. The new model and its extension will both use the same geometric algebra to be given as follows.

2. The Geometric Algebra

We begin here with a description of the structure of the geometric algebra upon which our model will be imposed. This geometric algebra will contain the following vector spaces.

$$\begin{array}{ll} \text{Notation} & \text{Basis} \\ \hline \mathbb{V}^e & \{e_i\}_{i=1}^n \\ \mathbb{V}^o & \{o\} \cup \mathbb{V}^e \\ \mathbb{V}^\infty & \{\infty\} \cup \mathbb{V}^e \\ \mathbb{V} & \{o\} \cup \{\infty\} \cup \mathbb{V}^e \end{array} \tag{2.1}$$

The set of vectors $\{e_i\}_{i=1}^n$ forms an orthonormal set of basis vectors for the n-dimensional Euclidean vector space $\mathbb{V}^e$, which we'll use to represent n-dimensional Euclidean space. The vectors o and ∞ are the familiar null-vectors representing the points at origin and infinity, respectively, taken from CGA. An inner-product table for these basis vectors is given as follows, where $1 \leq i < j \leq n$.

$$\begin{array}{c|cccc} \cdot & o & e_i & e_j & \infty \\ \hline o & 0 & 0 & 0 & -1 \\ e_i & 0 & 1 & 0 & 0 \\ e_j & 0 & 0 & 1 & 0 \\ \infty & -1 & 0 & 0 & 0 \end{array} \tag{2.2}$$

We will now let $\mathbb{G}(\mathbb{V})$ denote the Minkowski geometric algebra generated by $\mathbb{V}$. For each vector space in table (2.1), we will let an over-bar above this vector space denote an identical copy of that vector space. These copies will observe the same nesting relationships given in table (2.1). That is, we have $\overline{\mathbb{V}}^e \subset \overline{\mathbb{V}}^o \subset \overline{\mathbb{V}}$ and $\overline{\mathbb{V}}^e \subset \overline{\mathbb{V}}^\infty \subset \overline{\mathbb{V}}$. The vector space $\mathbb{W}$ will denote the smallest vector space containing each of $\mathbb{V}$ and $\overline{\mathbb{V}}$ as vector subspaces. In symbols, one may write

$$\mathbb{G}(\mathbb{W}) = \mathbb{G}(\mathbb{V} \oplus \overline{\mathbb{V}}). \tag{2.3}$$

We will use over-bar notation to distinguish between vectors taken from $\mathbb{V}$ with vectors taken from $\overline{\mathbb{V}}$. For algebraic purposes, we will find it useful to see that the over-bar notation may be defined as a bijective function from $\mathbb{V}$ to $\overline{\mathbb{V}}$. Doing so, we see that for any element $E \in \mathbb{V}$, we may define $\overline{E} \in \overline{\mathbb{V}}$ as

$$\overline{E} = S_1 E S_1^{-1}, \tag{2.4}$$

where $S_k \in \mathbb{G}(\mathbb{W})$ is the element given by

$$S_k = (1 - (-1)^k e_- \overline{e}_-)(1 + (-1)^k e_+ \overline{e}_+) \prod_{i=1}^n (1 + (-1)^k e_i \overline{e}_i). \tag{2.5}$$

This definition is non-circular if we let the over-bars in equation (2.5) be purely notation. The vectors e_- and e_+, taken from [6], are defined as

$$e_- = \frac{1}{2}\infty + o, \tag{2.6}$$

$$e_+ = \frac{1}{2}\infty - o. \tag{2.7}$$

The vectors $\overline{e}_-$ and $\overline{e}_+$ are defined similarly in terms of $\overline{o}$ and $\overline{\infty}$. Defined this way, realize that, like the over-bar function defined in [10], here we have the property that for any vector $v \in \mathbb{V}$, we have $\overline{\overline{v}} = -v$. Also notice that

$$S_1^{-1} = \frac{S_0}{2^{n+2}}. \tag{2.8}$$

Letting E be any element in $\mathbb{G}(\mathbb{W})$ in equation (2.4), we extend the over-bar function to be defined on all of $\mathbb{G}(\mathbb{W})$. If it could be shown that S_k in (2.5) was a versor, we could now conclude that the over-bar function is an outermorphism. Interestingly, despite a lack of any proof that S_k is a versor, (it is likely not a versor), we will be able to show later on in this paper that the over-bar function is never-the-less an outermorphism.

3. The Form Of Quadric Surfaces In $\mathbb{G}(\mathbb{W})$

We now give a formal definition under which elements $E \in \mathbb{G}(\mathbb{W})$ are representative of n-dimensional quadric surfaces in our present variation of the original model.

Definition 3.1. Referring to an element $E \in \mathbb{G}(\mathbb{W})$ as a quadric surface, it is representative of such an n-dimensional surface as[1] the set of all points $p \in \mathbb{V}^o$ such that

$$0 = p \wedge \overline{p} \cdot E. \tag{3.1}$$

To be absolutely clear, let us briefly go over the exact definition of the inner product we are using here. The inner product between vectors is defined as being left and right distributive over addition, but not generally associative as usual. The inner product of vectors is a bilinear form. Table (2.2) then completes the definition of the inner product among vectors.

For any vector $v \in \mathbb{W}$ and any k-blade $A \in \mathbb{G}(\mathbb{W})$ with $k > 1$, we define

$$v \cdot A = (v \cdot A_k) \wedge a_k - (-1)^k (v \cdot a_k) A_k, \tag{3.2}$$

$$A \cdot v = -(-1)^k v \cdot A, \tag{3.3}$$

where A is factored as $\bigwedge_{i=1}^k a_i$, and A_i denotes the outer product A with a_i removed. Equations (3.2) and (3.3) recursively define the inner product in terms of how we have defined it among vectors.

For blades $A, B \in \mathbb{G}(\mathbb{W})$, each of grade k and l, respectively, we define

$$A \cdot B = \begin{cases} A_k \cdot (a_k \cdot B) & \text{if } k \leq l, \\ (A \cdot b_1) \cdot B_1 & \text{if } k \geq l, \end{cases} \tag{3.4}$$

where here we have used the same notation in factoring B as we did for A earlier in equations (3.2) and (3.3). Equation (3.4) is another recursive definition. In the case that $k = l$, either part of the piece-wise definition may be used. We let the inner product be left and right distributive over addition among blades of any grade.

[1] Throughout this paper, we let the outer product take precedence over the inner product, and the geometric product take precedence over the inner and outer products.

Returning to Definition 3.1, from it can be seen that the general form of a quadric $E \in \mathbb{G}(\mathbb{W})$ is given by

$$E = \sum_{i=1}^{k} a_i \overline{b}_i, \tag{3.5}$$

where each of $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^k$ is a sequence of k vectors taken from $\mathbb{V}^\infty$. To see why, realize that the form (3.5) can always be reduced to the form

$$E \equiv \sum_{i=1}^{n} \sum_{j=i}^{n} \lambda_{ij} e_i \overline{e}_j + \sum_{i=1}^{n} \lambda_i e_i \overline{\infty} + \lambda \infty \overline{\infty}, \tag{3.6}$$

where each of λ_{ij}, λ_i, and λ are scalars, in the sense that this reduced form represents the same surface as that in equation (3.5) under Definition 3.1. We then see that this form (3.6), when it is substituted into equation (3.1), reduces to a polynomial equation of degree 2 in the vector components of $p + (p \cdot \infty)o$. Doing so with $p = o + x$, where $x \in \mathbb{V}^e$, we get the equation

$$0 = -\sum_{i=1}^{n} \sum_{j=i}^{n} \lambda_{ij} (x \cdot e_i)(x \cdot e_j) + \sum_{i=1}^{n} \lambda_i (x \cdot e_i) - \lambda, \tag{3.7}$$

which we may recognize as the general equation of an n-dimensional quadric surface. It may be worth comparing this method of representing quadric surfaces with that done in chapter 4 of [5] in what is called a quadratic Grassmann-Caylay algebra.

In practice, a computer program might take such a bivector of the form (3.5) and extract from it the coefficients of the quadric polynomial (3.7) it represents. It could then render the surface using traditional methods, such as those used to render the traced surfaces in Figure 1 far below, or the meshed surfaces in Figure 2 yet further below.

Of course, using geometric algebra on paper, it might be undesirable and unnecessary to think of quadrics in terms of polynomial equations. A perhaps better way to think of quadrics is in terms of an element of a geometric algebra whose decomposition produces the parameters characterizing the quadric surface. For example, many common quadrics, such as the cylinder, double-cone, spheroid and hyperboloid of two sheets, are the solution set in $\mathbb{V}^e$ of the equation

$$0 = -r^2 + (x - c)^2 + \lambda((x - c) \cdot v)^2 \tag{3.8}$$

in the variable $x \in \mathbb{V}^e$. Here, $v \in \mathbb{V}^e$ is a unit-length direction vector, $c \in \mathbb{V}^e$ is a center, and $r \in \mathbb{R}$ is a radius about that center. The scalar $\lambda \in \mathbb{R}$ controls the type and extremity of the surface.

Then, factoring $-p \wedge \overline{p}$ out of equation (3.8), we see that the element $E \in \mathbb{G}(\mathbb{W})$, given by

$$E = \Omega + \lambda v \overline{v} + 2(c + \lambda(c \cdot v)v)\overline{\infty} + (c^2 + \lambda(c \cdot v)^2 - r^2)\infty\overline{\infty} \tag{3.9}$$

is representative of this very same quadric by Definition 3.1, where Ω is defined as

$$\Omega = \sum_{i=1}^{n} e_i \bar{e}_i. \tag{3.10}$$

Canonical forms similar to (3.9) can be found for specific geometries, such as planes, spheres, plane-pairs, circular cylinders, circular conical surfaces, and so on. It is then possible to combine such forms with addition and subtraction to find other forms, but without as much success as we seem to have in CGA, combining its forms in the inner and outer products.

4. Transformations Supported By The Model

The main result of this section, Lemma 4.7, will depend upon Lemma 4.6 below; but first, the immediately following lemmas, although perhaps well-known, are given to add further clarification.

Lemma 4.1. *For any vector $a \in \mathbb{W}$ and any versor $V \in \mathbb{G}(\mathbb{W})$, we have*

$$VaV^{-1} \in \mathbb{W}. \tag{4.1}$$

Proof. It is left to the reader to prove that VaV^{-1} is a vector when V is a vector. Letting $V = \prod_{i=1}^{k} v_k$, where $\{v_i\}_{i=1}^{k}$ is a set of k vectors, form the inductive hypothesis that for any vector $a \in \mathbb{W}$, we have

$$(Vv_k^{-1})a(Vv_k^{-1})^{-1} \in \mathbb{W}. \tag{4.2}$$

We then see that

$$VaV^{-1} = (Vv_k^{-1}v_k)a(Vv_k^{-1}v_k)^{-1} = (Vv_k^{-1})(v_k a v_k^{-1})(Vv_k^{-1})^{-1} \in \mathbb{W}, \tag{4.3}$$

which completes the proof by induction. □

Of course, Lemma 4.1 is not unique to our geometric algebra $\mathbb{G}(\mathbb{W})$ as it applies to any geometric algebra. We will also find this to be the case with Lemmas 4.2 and 4.3 to follow.

Lemma 4.2. *Given any two vectors $a, b \in \mathbb{W}$, and a versor $V \in \mathbb{G}(\mathbb{W})$, we have*

$$a \cdot b = VaV^{-1} \cdot VbV^{-1}. \tag{4.4}$$

Proof. It suffices to show that

$$\langle V(a \wedge b)V^{-1} \rangle_0 = 0, \tag{4.5}$$

since

$$a \cdot b + \langle V(a \wedge b)V^{-1} \rangle_0 \tag{4.6}$$

$$= \langle V(a \cdot b + a \wedge b)V^{-1} \rangle_0 \tag{4.7}$$

$$= \langle VabV^{-1} \rangle_0 \tag{4.8}$$

$$= \langle VaV^{-1}VbV^{-1} \rangle_0 \tag{4.9}$$

$$= VaV^{-1} \cdot VbV^{-1} \tag{4.10}$$

by Lemma 4.1. Considering first the case $\langle v(a \wedge b)v^{-1}\rangle_0$, where $v \in \mathbb{W}$, we see by direct evaluation that

$$\langle v(a \wedge b)v^{-1}\rangle_0 = \frac{1}{v^2}\begin{vmatrix} v \cdot a & v \cdot a \\ v \cdot b & v \cdot b \end{vmatrix} = 0. \tag{4.11}$$

Suppose now that for a fixed integer k, the versor V, being the geometric product of k vectors, satisfies equation (4.5). Then, to complete our proof by induction, we must now show that

$$\langle vV(a \wedge b)(vV)^{-1}\rangle_0 = \langle vV(a \wedge b)V^{-1}v^{-1}\rangle_0 = 0. \tag{4.12}$$

To that end, we write

$$\langle vV(a \wedge b)V^{-1}v^{-1}\rangle_0 = \left\langle v\left(\sum_{i=0}^{m} X_i\right)v^{-1}\right\rangle_0, \tag{4.13}$$

where X_i is given by

$$X_i = \langle V(a \wedge b)V^{-1}\rangle_i, \tag{4.14}$$

and where $m = n + 2$. Clearly, however, we may allow the condition $m \leq 2$, because we're only interested in the grade zero part in equation (4.13). This leaves us to consider the cases $i = 0$, $i = 1$ and $i = 2$ in the expression $\langle vX_iv^{-1}\rangle_0$. For the case $i = 0$, this expression is zero by our inductive hypothesis. For the case $i = 1$, this expression is zero by Lemma 4.1. For the case $i = 2$, this expression is zero by the same work we did to show our result in equation (4.11). □

Lemma 4.3. *For any set of k-vectors $\{a_i\}_{i=1}^k$, and a versor $V \in \mathbb{G}(\mathbb{W})$, we have*

$$VAV^{-1} = \bigwedge_{i=1}^{k} Va_iV^{-1}, \tag{4.15}$$

where here we have let $A = \bigwedge_{i=1}^k a_i$.

Proof. It is not hard to show that the set $\{a_i\}_{i=1}^k$ is linearly independent if and only if the set $\{Va_iV^{-1}\}_{i=1}^k$ is linearly independent. Clearly equation (4.15) holds in the case that we are dealing with linearly dependent sets.

Assuming now that A is a non-zero blade, we will proceed with yet another proof by induction. We first notice that our lemma trivially holds in the case $k = 1$. For the case $k = 2$, notice that

$$V(a_1 \wedge a_2)V^{-1} \tag{4.16}$$
$$= V(a_1a_2 - a_1 \cdot a_2)V^{-1} \tag{4.17}$$
$$= Va_1a_2V^{-1} - a_1 \cdot a_2 \tag{4.18}$$
$$= Va_1V^{-1}Va_2V^{-1} - Va_1V^{-1} \cdot Va_2V^{-1} \tag{4.19}$$
$$= Va_1V^{-1} \wedge Va_2V^{-1} \tag{4.20}$$

by Lemma 4.2 and Lemma 4.1. We now make the inductive hypothesis that for a fixed integer $k > 2$, that our lemma holds for the cases $k - 1$ and $k - 2$. To simplify the algebra, we will let B be the blade given by $\bigwedge_{i=1}^k b_i$, where for each integer $i \in [1,k]$, we have $b_i = Va_iV^{-1}$. Our proof by induction is then completed when we see that

$$\begin{aligned}
& VAV^{-1} && (4.21)\\
&= V(a_1A_1 - a_1 \cdot A_1)V^{-1} && (4.22)\\
&= Va_1A_1V^{-1} - V(a_1 \cdot A_1)V^{-1} && (4.23)\\
&= Va_1V^{-1}VA_1V^{-1} - V\left(\sum_{i=2}^{k}(-1)^i(a_1 \cdot a_i)(A_1)_i\right)V^{-1} && (4.24)\\
&= Va_1V^{-1}B_1 - \sum_{i=2}^{k}(-1)^i(Va_1V^{-1} \cdot Va_iV^{-1})V(A_1)_iV^{-1} && (4.25)\\
&= b_1B_1 - \sum_{i=2}^{k}(-1)^i(b_1 \cdot b_i)(B_1)_i && (4.26)\\
&= b_1B_1 - b_1 \cdot B_1 = B. && (4.27)
\end{aligned}$$

Here, for any k-blade $C = \bigwedge_{i=1}^{k} c_i$, we have re-used the notation C_i as denoting the blade C with c_i removed. The blade $(C_i)_j$ is the blade C with both c_i and c_j removed.

Our inductive hypothesis was invoked above in letting $VA_1V^{-1} = B_1$, and in letting $V(A_1)_iV^{-1} = (B_1)_i$.

□

Similar to Lemma 4.2, the following lemma will help us prove the outermorphic property of the over-bar function.

Lemma 4.4. *Given any two vectors $a, b \in \mathbb{W}$, we have*

$$a \cdot b = \overline{a} \cdot \overline{b}. \tag{4.28}$$

Proof. Let $a = a_v + a_{\overline{v}}$ with $a_v \in \mathbb{V}$ and $a_{\overline{v}} \in \overline{\mathbb{V}}$. Similarly, let $b = b_v + b_{\overline{v}}$. We then see that

$$a \cdot b = a_v \cdot b_v + a_{\overline{v}} \cdot b_{\overline{v}}, \tag{4.29}$$

since any vector in $\mathbb{V}$ is orthogonal to any vector in $\overline{\mathbb{V}}$. It is then trivial to see that $a_v \cdot b_v = \overline{a}_v \cdot \overline{b}_v$ by our knowledge of what the over-bar function in equation (2.4) does. Similarly, we see that

$$a_{\overline{v}} \cdot b_{\overline{v}} = (-\overline{a}_{\overline{v}}) \cdot (-\overline{b}_{\overline{v}}) = \overline{a}_{\overline{v}} \cdot \overline{b}_{\overline{v}}. \tag{4.30}$$

We now need only observe that

$$\overline{a} \cdot \overline{b} = \overline{a}_v \cdot \overline{b}_v + \overline{a}_{\overline{v}} \cdot \overline{b}_{\overline{v}}. \tag{4.31}$$

□

We are now ready to prove that the over-bar function is an outermorphism.

Lemma 4.5. *For any set of k-vectors $\{a_i\}_{i=1}^k$, we have*

$$\overline{A} = \bigwedge_{i=1}^{k} \overline{a}_i. \tag{4.32}$$

where here we have let $A = \bigwedge_{i=1}^k a_i$.

Proof. Revisit the proof of Lemma 4.3, replacing V with S_1 of equation (2.5), and replacing the application of Lemma 4.2 with that of Lemma 4.4. We have already established by definition that Lemma 4.1 holds when V is replaced by S_1. □

Knowing that the over-bar function is an outermorphism gives it great ease of use while performing algebraic manipulations. Admittedly, this knowledge was requisite to the paper [10], though not proved until now.

Having now set forth lemmas 4.1 through 4.5, we are ready to tackle our main result. For brevity, Lemmas 4.1 through 4.5 are invoked as needed without making any explicit reference to them. For example, Lemma 4.2 is employed in the step taken from equation (4.35) to equation (4.36) below. Lemma 4.3 is used in the step taken from equation (4.49) to (4.50). We will often make use of the fact that the over-bar function preserves the geometric product, and in the case that we might require its preservation of the outer product, Lemma 4.5 justifies our use of this property.

Lemma 4.6. *For any versor $V \in \mathbb{G}(\mathbb{W})$, and any four vectors $a, b, c, d \in \mathbb{W}$, we have*

$$V^{-1}aV \wedge \overline{V^{-1}bV} \cdot c \wedge \overline{d} = a \wedge \overline{b} \cdot V\overline{V}(c \wedge \overline{d})(V\overline{V})^{-1}. \tag{4.33}$$

Proof. We begin by first establishing that

$$\begin{aligned}
& V^{-1}aV \wedge \overline{V^{-1}bV} \cdot c \wedge \overline{d} && (4.34)\\
&= -(V^{-1}aV \cdot c)(V^{-1}bV \cdot d) && (4.35)\\
&= -(a \cdot VcV^{-1})(b \cdot VdV^{-1}) && (4.36)\\
&= a \wedge \overline{b} \cdot VcV^{-1} \wedge \overline{VdV^{-1}}. && (4.37)
\end{aligned}$$

We now notice that

$$\begin{aligned}
& VcV^{-1} && (4.38)\\
&= V\overline{\overline{V}V^{-1}}cV^{-1} && (4.39)\\
&= (-1)^m V\overline{V}c\overline{V^{-1}}V^{-1} && (4.40)\\
&= (-1)^m V\overline{V}c(V\overline{V})^{-1}, && (4.41)
\end{aligned}$$

where m is the number of vectors taken together in a geometric product to form V. We then notice that

$$\overline{VdV^{-1}} \tag{4.42}$$

$$= VV^{-1}\overline{VdV^{-1}} \tag{4.43}$$

$$= (-1)^{m^2} V\overline{V}V^{-1}\overline{dV^{-1}} \tag{4.44}$$

$$= (-1)^{m^2+m} V\overline{Vd}V^{-1}\overline{V^{-1}} \tag{4.45}$$

$$= (-1)^{2m^2+m} V\overline{VdV^{-1}}V^{-1} \tag{4.46}$$

$$= (-1)^{m} V\overline{V}\overline{d}(V\overline{V})^{-1}. \tag{4.47}$$

It now follows that

$$a \wedge \overline{b} \cdot VcV^{-1} \wedge \overline{VdV^{-1}} \tag{4.48}$$

$$= a \wedge \overline{b} \cdot (-1)^{2m} V\overline{V}c(V\overline{V})^{-1} \wedge V\overline{V}\overline{d}(V\overline{V})^{-1} \tag{4.49}$$

$$= a \wedge \overline{b} \cdot V\overline{V}(c \wedge \overline{d})(V\overline{V})^{-1}, \tag{4.50}$$

which completes the proof. □

We're now ready to prove the main result as follows.

Lemma 4.7. *Letting $E \in \mathbb{G}(\mathbb{W})$ be a bivector of the form* (3.5)*, $p, p' \in \mathbb{V}^o$ be a pair of points related by a versor $V \in \mathbb{G}(\mathbb{V})$ by the equation*

$$p' = o \cdot V^{-1}pV \wedge \infty, \tag{4.51}$$

and $E' \in \mathbb{G}(\mathbb{W})$ a bivector given by

$$E' = V\overline{V}E(V\overline{V})^{-1}, \tag{4.52}$$

the set of all points $p \in \mathbb{V}^o$ such that

$$0 = p' \wedge \overline{p}' \cdot E \tag{4.53}$$

is exactly the set of all points $p \in \mathbb{V}^o$ such that

$$0 = p \wedge \overline{p} \cdot E'. \tag{4.54}$$

Proof. The lemma goes through by the following chain of equalities.

$$(o \cdot V^{-1}pV \wedge \infty) \wedge \overline{(o \cdot V^{-1}pV \wedge \infty)} \cdot E \tag{4.55}$$

$$= V^{-1}pV \wedge \overline{V^{-1}pV} \cdot E \tag{4.56}$$

$$= p \wedge \overline{p} \cdot (V\overline{V})E(V\overline{V})^{-1}. \tag{4.57}$$

The first equality holds by the fact that E is of the form (3.5), while the second equality holds by Lemma 4.6. □

A corollary to Lemma 4.7 immediately follows.

Corollary 4.8. *If $V \in \mathbb{G}(\mathbb{V})$ is a versor such that E' in equation* (4.52) *is of the form* (3.5)*, then the versor $V\overline{V}$ represents a transformation closed in the set of all quadric surfaces.*

To see the step from (4.55) to (4.56), the reader must first realize that while $V^{-1}pV$ is not necessarily in $\mathbb{V}^o$, the vector $o \cdot V^{-1}pV \wedge \infty$ is always in $\mathbb{V}^o$. Now realize that it doesn't matter that $V^{-1}pV$ is not in $\mathbb{V}^o$ if E is of the form (3.5), because this form masks off, if you will, the parts of $V^{-1}pV$ that would otherwise contribute to the result of the inner product.

The key motivation behind Lemma 4.7 is the observation that the desired transformation of E by V is given by the algebraic set of equation (4.53), because an understanding of how V^{-1} transforms p gives us an understanding of what type of geometry we get from equation (4.53) in terms of E and V. Lemma 4.7 then shows that this is also the algebraic set of equation (4.54), thereby giving us a means of performing desired transformations directly on elements in $\mathbb{G}(\mathbb{W})$ representative of quadric surfaces. By Corollary 4.8, what we get from such a transformation is also a quadric surface, provided that V is a versor such that E' in (4.52), like E, is also a bivector of the form (3.5).

We can now apply Lemma 4.7 to show that the rigid body transformations are supported in our new variation of the original model. Letting $\pi \in \mathbb{V}$ be a dual plane of CGA, (see section 4.2 of [2]), given by

$$\pi = v + (c \cdot v)\infty, \tag{4.58}$$

where $v \in \mathbb{V}^e$ is a unit-length vector indicating the norm of the plane, and where $c \in \mathbb{V}^e$ is a vector representing a point on the plane, we see that for any homogenized point $p \in \mathbb{V}^o$, we have

$$-\pi p \pi^{-1} = o + x - 2((x - c) \cdot v)v + \lambda\infty, \tag{4.59}$$

where $p = o + x$ with $x \in \mathbb{V}^e$, and where the scalar $\lambda \in \mathbb{R}$ is of no consequence. Letting $V = \pi$, the point $p' \in \mathbb{V}^o$ of consequence here is given by the additive inverse of equation (4.51), from which we can recognize an orthogonal reflection about the plane π. It now follows by Lemma 4.7 that $\pi\overline{\pi}$ is a versor capable of reflecting any quadric surface about the plane π. Being able to perform planar reflections of any quadric in any plane, it now follows that we can always find a versor $V \in \mathbb{G}(\mathbb{W})$ capable of performing any rigid body motion on any quadric surface. The development of the rigid body motions, (combinations of translations and rotations), by planar reflections, is well known, and can be found in section 2.7 of [6].

In retrospect, what we have done to find the rigid body motions of quadric surfaces is similar to what was done in [7]; and according to [11], we can state more generally that what we have done is at least similar to finding an isomorphism between quadratic spaces. Section 4 of [8] shows that versors can be used to transform quadric surfaces using an entirely different approach.

5. Extending The New Model

Interestingly, if we were not content with the rigid body motions of quadrics, then we really could find what is, for example, the spherical inversion of,

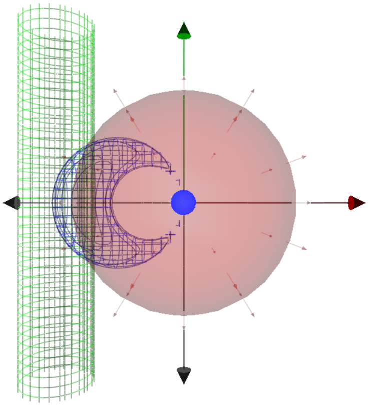

Figure 1. The inversion of a cylinder in a sphere. Traces in various planes were used to render the cylinder and its inversion.

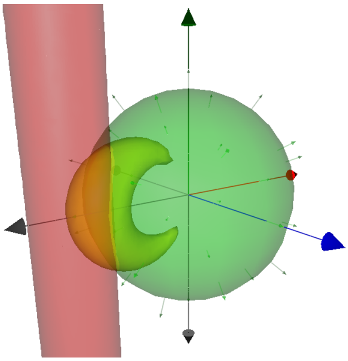

Figure 2. The inversion of a cylinder in a sphere. A surface mesh generation algorithm was used to skin the cylindrical and inverted surfaces. The inverted mesh suffers where the curvature becomes extreme.

say, an infinitely long cylinder in a sphere. To do this, we start by changing Definition 3.1 into the following definition.

Definition 5.1. For any element $E \in \mathbb{G}(\mathbb{W})$, we may refer to it as an n-dimensional quartic surface as the set of all points $p \in \mathbb{V}^e$ such that

$$0 = P(p) \wedge \overline{P}(p) \cdot E, \tag{5.1}$$

where $P : \mathbb{V}^e \to \mathbb{V}$ is the point mapping of CGA, defined in [4] as

$$P(p) = o + p + \frac{1}{2}p^2\infty. \tag{5.2}$$

It is then helpful to introduce another definition as follows.

Definition 5.2. Given a mapping $f : \mathbb{V}^e \to \mathbb{V}$, a versor $V \in \mathbb{G}(\mathbb{W})$ is said to be f-form preserving if for any point $x \in \mathbb{V}^e$, there exists a non-zero scalar $\lambda \in \mathbb{R}$ and a point $y \in \mathbb{V}^e$ such that $Vf(x)V^{-1} = \lambda f(y)$.

We then arrive at the following upgrade of Lemma 4.7.

Lemma 5.3. *Letting $E \in \mathbb{G}(\mathbb{W})$ be a bivector of any form, $p, p' \in \mathbb{V}^e$ be a pair of points related by a P-form preserving versor $V \in \mathbb{G}(\mathbb{V})$ by the equation*

$$\lambda P(p') = V^{-1}P(p)V, \tag{5.3}$$

and $E' \in \mathbb{G}(\mathbb{W})$ a bivector given by equation (4.52)*, the set of all points $p \in \mathbb{V}^e$ such that*

$$0 = P(p') \wedge \overline{P}(p') \cdot E \tag{5.4}$$

is exactly the set of all point $p \in \mathbb{V}^e$ such that

$$0 = P(p) \wedge \overline{P}(p) \cdot E'. \tag{5.5}$$

Proof. We first note that the function P is a stable mapping. (See Definition 7.1 far below.) This insures that p' may be correctly considered a well defined function of p.

Now notice that the scalar λ clearly cancels out of equation (5.4) so that we may, without loss of generality, let $\lambda = 1$. Once again applying Lemma 4.6, we simply see that

$$V^{-1}P(p)V \wedge \overline{V^{-1}P(p)V} \cdot E = P(p) \wedge \overline{P}(p) \cdot V\overline{V}E(V\overline{V})^{-1}. \tag{5.6}$$

□

The need for a P-form preserving versor is apparent from equation (5.4), since otherwise we could not claim to understand what algebraic set we get from equation (5.4) in terms of E and V. Our claim is justified by an understanding of how points transform in CGA.

We now see by Lemma 5.3 that if $V \in \mathbb{G}(\mathbb{W})$ is any P-form preserving versor, and if E is a surface under Definition 5.1, then the element $E' \in \mathbb{G}(\mathbb{W})$, given by equation (4.52), must, by Definition 5.1, be representative of the desired transformation of E by the versor $V\overline{V}$. Realizing that all of the conformal transformations of CGA are P-form preserving, and that P is a stable mapping, we now see that our extended model supports the entire set of conformal transformations.

The general polynomial equation arising from the form of such elements E in Definition 5.1 is much more involved than what we have in equation (3.7). Nevertheless, it is possible to extract a specific form of a quartic polynomial equation in the vector components of p from equation (5.1). The result being unsightly, it will not be presented here. Suffice it to say, a computerized algebra system was used to find the polynomial form. In any case, it is easy to see from equation (5.1) that the degree of the resulting polynomial will be four.

Now notice that under Definition 5.1, canonical forms such as (3.9) are still valid. This is because

$$P(p) \wedge \overline{P}(p) \cdot E = (o + p) \wedge \overline{(o + p)} \cdot E \tag{5.7}$$

in the case that E is of the form (3.5). This allows us to use what we already know about quadrics in the new model with its extension to quartic surfaces of a specific form.

An interesting side effect of Definition 5.1 is the existence of every union of any pair of geometries where either one is a circle, plane or point. We simply note that for any pair of vectors $a, b \in \mathbb{V}$, we have

$$P(p) \wedge \overline{P}(p) \cdot a \wedge \bar{b} = -(P(p) \cdot a)(P(p) \cdot b). \tag{5.8}$$

Putting theory into practice, the author wrote a piece of computer software[2] that implements this CGA-like model for the special class of quartic surfaces of equation (5.1). Giving the program the following script as input, the output of the program is given in Figure 1 and rendered another way in Figure 2. The script is easy for anyone to read, even if they are not familiar with its language. It is given here to illustrate how one might use the model with the aide a computer system.

```
/*
 * Calculate the surface that is the
 * inversion of a cylinder in a sphere.
 */
do
(
    /* Make the cylinder. */
    v = e2, c = -7*e1, r = 2,
    cylinder = Omega - v^bar(v) + 2*c*nib + (c.c - r*r)*ni^nib,
    bind_quadric(cylinder),
    geo_color(cylinder,0,1,0),

    /* Make the sphere. */
    c = 0, r = 6,
    sphere = no + c + 0.5*(c.c - r*r)*ni,
    bind_dual_sphere(sphere),
    geo_color(sphere,1,0,0,0.2),

    /* Make the inversion of the cylinder in the sphere. */
    V = sphere*bar(sphere),
    inversion = V*cylinder*V~,
    bind_conformal_quartic(inversion),
    geo_color(inversion,0,0,1),
)
```

The language used here was invented for the software and has no name. The functions beginning with the word "bind" create and bind an entity to the given element of the geometric algebra that is responsible for interpreting that element as a surface under Definition 5.1 or, in the case of the sphere, as a dual surface under the definition given by CGA. The computer program can then use traditional methods to render the surface from the extracted polynomial equation. For example, the polynomial equation in x, y and z for the inverted surface presented in Figure 1 is given by

$$\begin{aligned} 0 = 28.8x^2 + 11.2x^3 + x^4 + 11.2xy^2 + 2x^2y^2 + \\ 11.2xz^2 + 2x^2z^2 + y^4 + 2y^2z^2 + 28.8z^2 + z^4. \end{aligned} \tag{5.9}$$

[2]The software is available at `https://github.com/spencerparkin/GAVisTool`. The author does not recommend making use of this software for research purposes. Instead, the reader is referred to tools such as GAViewer and CLUCalc.

It is interesting how a bit of reasoning in geometric algebra has given us such a simple means to obtaining this polynomial equation. Of course, while such equations lend themselves to computer algorithms, they are not practical on paper. This is where the canonical forms of elements might become useful; although, admittedly, even these forms have proven to be unwieldy and impractical for the author, unlike their CGA counterparts.

6. Dual And Direct Surfaces

The goal from the beginning has been to find a model, similar to CGA, for the general set of surfaces up to degree 2, not just the specific class of surfaces, (up to degree 2), that are just the spheres and planes of CGA. While this has been accomplished to some extent, one of the greatest deficiencies remaining appears to be the inability for the model to represent surfaces of up to the desired degree for all dimensions from zero to n in the same manner that this is possible in CGA. One possible solution to this is that of utilizing the geometric algebra that is generated by the linear space of bivectors in $\mathbb{G}(\mathbb{W})$. Pursuing this idea, we could define a linear function on this space that maps it to a vector space $\mathbb{B}$. If $[]$ was such a function, then for any pair of 2-blades $A, B \in \mathbb{G}(\mathbb{W})$, we could define

$$[A] \cdot [B] = A \cdot B. \tag{6.1}$$

It would then follow that a vector $v \in \mathbb{G}(\mathbb{B})$ would be representative of a surface as the set of all points $p \in \mathbb{V}^e$ such that

$$0 = \rho(p) \cdot v, \tag{6.2}$$

where we define[3] the function ρ as

$$\rho(p) = [P(p)\overline{P(p)}]. \tag{6.3}$$

The notions of dual and direct surfaces would then emerge as they do in CGA. A blade $B \in \mathbb{G}(\mathbb{B})$ is dually representative of a surface as the set $\dot{G}(B)$, defined as

$$\dot{G}(B) = \{p \in \mathbb{V}^e | 0 = \rho(p) \cdot B\}. \tag{6.4}$$

A blade $B \in \mathbb{G}(\mathbb{B})$ is directly representative of a surface as the set $\hat{G}(B)$, defined as

$$\hat{G}(B) = \{p \in \mathbb{V}^e | 0 = \rho(p) \wedge B\}. \tag{6.5}$$

Using the outer product, we can now intersect dual surfaces and combine direct surfaces. Specifically, the outer product of two dual surfaces is the dual surface that is the intersection, if any, of the two dual surfaces taken in

[3]The reader should be made aware of the fact that we do not need the machinery of the over-bar function, nor that of the functions P or $[]$, to define the function ρ. Though not as easily, we could have defined ρ in terms of any basis of $\mathbb{G}(\mathbb{B})$ using elementary functions (addition and multiplication.) Furthermore, taking the liberty of defining ρ in other ways, we can easily come up with models of GA capable of representing almost any subset of the set of all algebraic sets. This is clarified in the closing remarks.

the product. That is, for any two blades $A, B \in \mathbb{G}(\mathbb{B})$ with $A \wedge B \neq 0$, we have

$$\dot{G}(A \wedge B) = \dot{G}(A) \cap \dot{G}(B). \tag{6.6}$$

Similarly, the outer product of two direct surfaces is the direct surface containing at least the union of the surfaces taken in the product. That is, for any two blades $A, B \in \mathbb{G}(\mathbb{B})$, we have

$$\hat{G}(A \wedge B) \supseteq \hat{G}(A) \cup \hat{G}(B). \tag{6.7}$$

Imaginary dual intersections may often be reinterpreted as real direct surfaces. That is, for any blade $C \in \mathbb{G}(\mathbb{B})$, if $\dot{G}(C)$ is empty, consider $\hat{G}(C)$. These features arise as a consequence of representing geometries as blades in a geometric algebra.

To illustrate the use of $\mathbb{G}(\mathbb{B})$, let $s, c \in \mathbb{G}(\mathbb{B})$ be vectors dually representative of a sphere and cylinder, respectively. Then, for any point $p \in \mathbb{V}^e$, we can find the dual surface containing p and the intersection of s and c as

$$\pm(\rho(p) \wedge (s \wedge c)I)I = \rho(p) \cdot s \wedge c = (\rho(p) \cdot s)c - (\rho(p) \cdot c)s, \tag{6.8}$$

where I, in practice, might be the unit pseudo-scalar of the geometric algebra generated by the vector sub-space of $\mathbb{B}$ given by the set

$$\{[x\overline{y}]|x, y \in \mathbb{V}\}. \tag{6.9}$$

Even this vector space, which is of dimension $(n+2)^2$, is larger than it needs to be. We could suffice with a vector space of dimension $(n+2)(n+3)/2$. In any case, it is clear from equation (6.8) that the algebra is simply giving us the desired surface in the pencil of s and c.

If, however, all we cared about was the dual intersection $s \wedge c$, we may still need to resort to [13] to do any meaningful analysis. Contrasting this with an absence of any need to do such a thing in CGA, we see further deficiencies in our more generalized model for surfaces up to degree 2. To further illustrate the point, consider the intersection of any quadratic dual surface with a line. It is much easier to setup and solve a quadratic equation than it is to take the outer product of the dual surface with a dual line and then make sense of the result. For example, given a quadric $E \in \mathbb{G}(\mathbb{W})$ of the form (3.5), and letting $f : \mathbb{V}^o \to \mathbb{R}$ be defined as

$$f(p) = p \wedge \overline{p} \cdot E, \tag{6.10}$$

we have

$$0 = f(p + \lambda v) = f(p) + \lambda \nabla_v f(p) + \lambda^2 f(v), \tag{6.11}$$

where $p \in \mathbb{V}^o$ is a point and $v \in \mathbb{V}^e$ is a direction vector, where $\nabla_v f(p)$ is the directional derivative of f at p in the direction v, and from which we easily recognize a quadratic equation in the scalar variable λ. In CGA, point-pairs are easily decomposable; the point-pair equivalent in our current model, however, is not. Section 4 of [8] offers a solution to the intersection problem using a different method of utilizing CGA to represent conic and quadric surfaces. See also section 4.2 of [14].

7. Transformations Of Dual And Direct Surfaces

As we've been able to see, using blades, not bivectors, to represent the geometries of our model allows us to recover one of the benefits enjoyed by CGA in its ability to perform operations on geometry that result as a consequence of the operations we can perform on sub-spaces. Switching to blades, what we would hope to be able to do at this point is preserve the transformations we have already developed. The question is as follows. For a given quartic $E \in \mathbb{G}(\mathbb{W})$ and a versor $V \in \mathbb{G}(\mathbb{V})$, does there exist a versor $Q \in \mathbb{G}(\mathbb{B})$ such that

$$Q[E]Q^{-1} = [V\overline{V}E(V\overline{V})^{-1}]? \tag{7.1}$$

Unfortunately, considering V to be a vector, there appears only to exist such a versor Q in the special case that E is a 2-blade $a \wedge \overline{b}$ with $a, b \in \mathbb{V}$, and that $V \cdot a = 0$ or $V \cdot b = 0$. In this case, Q is the rotor given by

$$Q = [\overline{V} \wedge a^{-1}][\overline{b}^{-1} \wedge V], \tag{7.2}$$

which, sadly, does not depend upon V alone.

Alternatively, we might consider allowing our model to utilize the geometric algebras $\mathbb{G}(\mathbb{W})$ and $\mathbb{G}(\mathbb{B})$ in concert with one another. For a given versor $V \in \mathbb{G}(\mathbb{V})$, and a k-blade $B \in \mathbb{G}(\mathbb{B})$, if we could find a vector factorization $b_1 \wedge \cdots \wedge b_k$ of B, then the transformation B' of B by V would be given by

$$B' = \bigwedge_{i=1}^{k} [V\overline{V}[b_i]^{-1}(V\overline{V})^{-1}]. \tag{7.3}$$

(See [3] on the problem of factoring blades.) It is unfortunate, however, that we would have to bother finding such a factorization in order to apply a given transformation. Not wanting our model to be spread across two geometric algebras working in conjunction with one another, we now consider $\mathbb{G}(\mathbb{B})$ alone.

Leaving the versors of $\mathbb{G}(\mathbb{V})$ behind, we are left to consider the versors of $\mathbb{G}(\mathbb{B})$. Following the line of thinking that led to Lemma 4.7 and Lemma 5.3, we begin by considering the set of all versors $V \in \mathbb{G}(\mathbb{B})$ preserving the form $\rho(p)$. That is, the set of all ρ-form preserving versors in $\mathbb{G}(\mathbb{B})$ by Definition 5.2. We then need the following definition.

Definition 7.1. For any mapping $f : \mathbb{V}^e \to \mathbb{V}$, we refer to it as a stable mapping if there do not exist distinct scalars $\alpha, \beta \in \mathbb{R}$ and distinct points $x, y \in \mathbb{V}^e$ such that

$$\alpha f(x) = \beta f(y). \tag{7.4}$$

We can now make the observation that if ρ is a stable mapping, then ρ-form preserving versors $V \in \mathbb{G}(\mathbb{B})$ induce a map from $\mathbb{V}^e$ to $\mathbb{V}^e$ by taking $p \in \mathbb{V}^e$ to $p' \in \mathbb{V}^e$, where p' is the point satisfying $V\rho(p)V^{-1} = \lambda\rho(p')$ for some scalar $\lambda \in \mathbb{R}$. The need for ρ to be a stable mapping simply insures that this induced mapping is well-defined.

Lemma 7.2. *The mapping $\rho : \mathbb{V}^e \to \mathbb{V}$, as it is defined in equation* (6.3)*, is a stable mapping.*

Proof. Assume that ρ is unstable. We can therefore find distinct $\alpha, \beta \in \mathbb{R}$ and distinct $x, y \in \mathbb{V}^e$ such that $\alpha\rho(x) = \beta\rho(y)$. This then implies that

$$\alpha P(x) \wedge \overline{P}(x) = \beta P(y) \wedge \overline{P}(y), \tag{7.5}$$

but there does not exist a non-zero scalar $\lambda \in \mathbb{R}$ such that

$$\alpha P(x) = \beta P(y) + \lambda \overline{P}(y). \tag{7.6}$$

Therefore, we must have $\alpha P(x) = \beta P(y)$, yet we know from CGA that P is a stable mapping. It follows that ρ is stable by contradiction. □

Now, if we are able to develop any ρ-form preserving versor V, and understand what kind of transformation a point in $\mathbb{V}^e$ undergoes by an application of this transformation to ρ, then it is not hard to show that an application of such a versor to any blade $B \in \mathbb{G}(\mathbb{B})$, dually or directly representative of a given surface, can be well understood. We begin with the following definition.

Definition 7.3. For any k-blade $B \in \mathbb{G}(\mathbb{B})$, we refer to it as point-fit-able, if there exists a set of k points $\{p_i\}_{i=1}^k \subset \mathbb{V}^e$, such that

$$B = \bigwedge_{i=1}^{k} \rho(p_i). \tag{7.7}$$

It is clear that the surface directly represented by a point-fit-able k-blade fits any k points that can be used to formulate a factorization of that blade by equation (7.7). Given any set $\{p_i\}_{i=1}^k$ of such points, and a ρ-form preserving versor $V \in \mathbb{G}(\mathbb{B})$, if we know which surface must fit those points, then it is also clear that we'll know which surface is fit by the set of points $\{p_i'\}_{i=1}^k$, where for each integer i, we have $V\rho(p_i)V^{-1} = \lambda\rho(p_i')$. Given the direct surface B of equation (7.7), this is simply the direct surface VBV^{-1}.

There are, however, at least two problems with this. First, assuming that the set $\{\rho(p_i)\}_{i=1}^\infty$ is linearly independent, or that we understand under what circumstances of the set $\{p_i\}_{i=1}^k$ that the set $\{\rho(p_i)\}_{i=1}^\infty$ will be linearly independent, determining the surface that must fit such a given set of points is non-trivial. Secondly, it is not clear whether all direct surfaces are point-fit-able. Fortunately for us, the property of being point-fit-able plays no part in the following lemma.

Lemma 7.4. *Given a k-blade $B \in \mathbb{G}(\mathbb{B})$ and a ρ-form preserving versor $V \in \mathbb{G}(\mathbb{B})$, we have*

$$\{p \in \mathbb{V}^e | 0 = V^{-1}\rho(p)V \cdot B\} = \dot{G}(VBV^{-1}). \tag{7.8}$$

Proof. Writing B in terms of the k vectors in $\{b_i\}_{i=1}^k$ as $B = \bigwedge_{i=1}^k b_i$, we have $0 = V^{-1}\rho(p)V \cdot B$ if and only if for all integers $i \in [1, k]$, we have $0 = V^{-1}\rho(p)V \cdot b_i = \rho(p) \cdot V b_i V^{-1}$, since the set $\{B_i\}_{i=1}^k$, where B_i denotes the product B with b_i removed, is a linearly independent set. (See equation (8.1) below.) Then, for all integers $i \in [1, k]$, we have $0 = \rho(p) \cdot V b_i V^{-1}$ if and only if $0 = \rho(p) \cdot VBV^{-1}$, since the set $\{VB_iV^{-1}\}_{i=1}^k$ is also linearly independent. It follows now that $0 = V^{-1}\rho(p)V \cdot B$ if and only if $0 = \rho(p) \cdot VBV^{-1}$. $\square$

Intuitively, Lemma 7.4 is telling us that if we understand the mapping from $\mathbb{V}^e$ to $\mathbb{V}^e$ induced by V on $\rho(p)$ as $V^{-1}\rho(p)V = \lambda\rho(p')$, then we understand the geometric transformation B undergoes by V as VBV^{-1}.

All that remains now is to show that an understanding of how dual surfaces transform gives us an understanding of how direct surfaces transform. To see this, realize that $\hat{G}(B) = \dot{G}(BI)$, and then that

$$\hat{G}(VBV^{-1}) = \hat{G}(VBI^2V^{-1}) = \hat{G}(VBIV^{-1}I) = \dot{G}(VBIV^{-1}), \tag{7.9}$$

where I is the unit pseudo-scalar of $\mathbb{G}(\mathbb{B})$. This shows that direct surfaces are affected by versors in the same way that dual surface are.

The challenge now is to find a ρ-form preserving versor V and then understand the mapping from $\mathbb{V}^e$ to $\mathbb{V}^e$ it induces through the use of ρ. Not yet being able find such a mapping, we will have to leave this as an open question for now.

8. Closing Remarks

With a background in abstract algebra and topology, an accessible introduction to the subject of algebraic geometry is given in [9]. Not surprisingly, and perhaps ironically, there is no mention of geometric algebra; although one might presume at first glance that the similarly named subjects would have a great deal to do with one another. From the present paper, a method for using blades of a geometric algebra to represent algebraic sets generated by polynomials of any form, though not specifically stated, can now be inferred, and it is quite trivial. Simply take a second glance at equation (6.4). The subset of the set of all algebraic sets you are able to represent this way using blades B of your geometric algebra is simply determined by how you choose to define the vector-valued function ρ. A realization that this is an algebraic set, (the zero set of one or more polynomial equations), is had by a revisitation of equation (3.2). Utilizing and expanding this equation, we see that

$$0 = \rho(p) \cdot B = -\sum_{i=1}^{k} (-1)^i (\rho(p) \cdot b_i) B_i. \tag{8.1}$$

It is now clear that

$$\dot{G}(B) = \bigcap_{i=1}^{k} \dot{G}(b_i), \tag{8.2}$$

since $\{B_i\}_{i=1}^k$ is a linearly independent set. We may think of ρ as defining the form of each polynomial equation while thinking of each b_i as an instantiation of that form, it containing the desired coefficients of the polynomial.

The results of this paper then show that in searching for such a function ρ, the properties of stability and form preservation are desirable in one possible method of coming to an understanding of how versors transform the geometries (algebraic sets) represented by blades B of the geometric algebra.

Geometric algebra may give us a new way to study algebraic sets. Algebraic geometry, however, has grown far beyond algebraic sets as the central objects of study. Not being competent in geometric calculus, much less the vast and arcane subject of algebraic geometry, the present author cannot hope to pursue a reformulation of even the tiniest part of one subject with the other. Assuming it is even possible or practical to do so, (an assumption built upon the hype that geometric algebra is, as many claim, the ultimate and universal mathematical language), perhaps someone will make roads in that direction. Until then, geometric algebra continues to offer fun and interesting ways to do geometry with models such as CGA and perhaps the newly established model of this paper.

References

1. L. Dorst, D. Fontijne, and S. Mann, *Geometric algebra for computer science*, Morgan Kaufmann, 2007.
2. L. Falcón-Morales, *Design of algorithms of robot vision using conformal geometric algebra*, **no. 20** (2007), 981–1005.
3. D. Fontijne and L. Dorst, *Efficient algorithms for factorization and join of blades*, Geometric Algebra Computing **Springer-Verlag London Limited** (2010), 457.
4. D. Hestenes, *Old wine in new bottles: A new algebraic framework for computational geometry*, Advances in Geometric Algebra with Applications in Science and Engineering (2001), 1–14.
5. L. Hongbo, *Invariant algebras and geometric reasoning*, World Scientific Publishing, 2008.
6. L. Hongbo, D. Hestenes, and A. Rockwood, *Generalized homogeneous coordinates for computational geometry*, Geometric Computing with Clifford Algebra **Volume 24., Berlin Heidelberg, Springer-Verlag** (2001), 27–60.
7. M. Langer, `http://www.cim.mcgill.ca/~langer/557/lecture8.pdf`.
8. A. Lasenby, *Recent applications of conformal geometric algebra*, IWMM 2004, LNCS 3519 Springer-Verlag (2005).
9. J. Milne, *Algebraic geometry (v5.22)*, 2012, Available at www.jmilne.org/math/, p. 260.
10. S. Parkin, *A model for quadric surfaces using geometric algebra*, (to appear in) Advances in Applied Clifford Algebras (2012).
11. A. Pfister, *Quadratic forms with applications to algebraic geometry and topology*, Cambridge University Press, 1995.
12. G. Sobczyk, *Conformal mappings in geometric algebra*, AMS Notices **Volume 59** (2012), 264–5.
13. W. Wang, R. Goldman, and C. Tu, *Enhancing Levin's method for computing quadric-surface intersections*, Computer Aided Geometric Design **Volume 20** (2003), 401–22.

14. R. Wareham, J. Cameron, and J. Lasenby, *Applications of conformal geometric algebra in computer vision and graphics*, IWMM 2004, LNCS 3519, pp. 329-349, 2005. Springer-Verlag (2005).

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