Versors That Give Non-Uniform Scale

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To my dear wife Melinda.

Abstract. Versors are found in a geometric algebra that, when applied to elements of that algebra that are representative of various algebraic surfaces in a constrained way, perform a non-uniform scaling of those surfaces.

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1. Motivation

Non-uniform scale is one of the last remaining problems of geometric algebra.

2. The Result

The result of this paper is simply a corollary to that of [1], but to see how, we must first constrain the way that we represent n-dimensional algebraic surfaces of up to degree m in the Mother Minkowski algebra of order m.[1] What we do is let $n \leq m$, and reserve certain sub-algebras of our mother algebra for use in specific dimensions. To see what is meant by this, let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial in whose zero set we are interested. Now define, for any integer $k \in [1, n]$, the polynomial $f_k : \mathbb{R} \to \mathbb{R}$ as

$$f_k(\lambda) = f(e_1 + e_2 + \cdots + \lambda e_k + \cdots + e_{n-1} + e_n).$$

Having done this, we will represent the surface of f in the Mother Minkowski algebra of order

$$m = \sum_{k=1}^{n} \deg f_k.$$

Now if $\mathbb{G}$ denotes our mother algebra and it is generated by m sub-algebras $\mathbb{G}_i$, each generated by the vector space $\mathbb{V}_i$, then we reserve $\deg f_k$ of these

[1] Recall that such representations are not unique, and so we have the flexibility to choose our representations carefully.

sub-algebras for use in dimension k of our n dimensions. (We will let $\mathbb{G}^k$, where k is an integer in $[1, n]$, denote the largest sub-algebra of $\mathbb{G}$ containing all sub-algebras $\mathbb{G}_i$ reserved for dimension k.)

An example may be warrented at this point. Let $n = 3$ and consider the polynomial given by

$$f(x) = 3x_1^2 x_2 x_3^4 + 4x_1 x_2^5 - 7x_3^2, \tag{2.1}$$

where x_k is notation for $x_k = x \cdot e_k$. We will represent the surface that is the zero set of this polynomial using an m-vector in a Mother Minkowski algebra of order $m = 2 + 5 + 4 = 11$. The first 2 sub-algebras are reserved for dimension 1, the next 5 for dimension 2, and the last 4 for dimension 3. The m-vector B representing this surface is then given by

$$\begin{aligned} B = {} & 3e_{(1,2),1} \wedge e_{3,2} \wedge \infty_{(4,5,6,7)} \wedge e_{(8,9,10),3} \wedge \infty_{11} \\ & + 4e_{1,1} \wedge \infty_2 \wedge e_{(3,4,5,6,7),2} \wedge \infty_{(8,9,10,11)} \\ & - 7\infty_{(1,2,3,4,5,6,7)} \wedge e_{(8,9),3} \wedge \infty_{(10,11)}. \end{aligned}$$

Here, notation is a challenge. The vector $e_{i,j}$ denotes the j^{th} euclidean basis vector in the i^{th} sub-algebra. We then define

$$e_{(i_1,i_2,\dots,i_r),j} = e_{i_1,j} \wedge e_{i_2,j} \wedge \cdots \wedge e_{i_r,j}.$$

The notation for ∞ is similar.

We can now say that the zero set of f in equation (2.1) is given by the set of all solutions to the equation

$$\bigwedge_{k=1}^{m} p_k(x) \cdot B = 0. \tag{2.2}$$

Recall that $p_k(x) = o_k + x_k + \frac{1}{2}x^2\infty_k$. Of course, we could have represented f in a mother algebra of order $\deg f = 7$, but it will soon become clear why we needed our algebra $\mathbb{G}$ to be of order $m = 11$.

Returning from the example, suppose now we have an m-vector B representative of any polynomial $f : \mathbb{R}^n \to \mathbb{R}$ under the constraint thus illustrated. Seeing that the zero set of f is the set of solutions to equation (2.2), we make the simple observation that if D is a versor taken from a sub-algebra $\mathbb{G}^k$, and further, D is the product of the same dilation versor D_i found in each sub-algebra $\mathbb{G}_i$ contained in $\mathbb{G}^k$, then the non-uniform scale of f in the dimension of k by the scale of each D_i is given by the set of solutions to the equation

$$p_1(x) \wedge p_2(x) \wedge \cdots \wedge (D^{-1}p_k(x)D) \wedge \cdots \wedge p_{n-1}(x) \wedge p_n(x) \cdot B = 0. \tag{2.3}$$

Now realize that for all $j \neq k$, D leaves $p_j(x)$ invariant. That is,

$$D^{-1}p_j(x)D = p_j(x).$$

It now follows by equations (3.2) through (3.5) of [1] that equation (2.3) may be rewritten as

$$\bigwedge_{k=1}^{n} p_k(x) \cdot DBD^{-1},$$

showing that D, when applied to B, performs a non-uniform scaling of the surface of f.

3. Closing Remarks

Though we have now shown that versors performing the non-unform scale operation exist, seeing that their application requires a great deal of cumbersome convention and notation, a question of their practicality immediately arises. It's certainly not practical on paper, but perhaps such versors may find applications on the computer.

References

[1] S. Parkin, *Mother Minkowski Algebra Of Order M*. Advances in Applied Clifford Algebras (2013).

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