# Section 3.5 Exercises
# Herstein's Topics In Algebra

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## Thoughts

Let $R$ be a commutative ring and let $a \in R$. Is there such a thing as "the ideal $I$ generated by $a$?" I suppose this would be the smallest possible ideal $I$ of $R$ containing $a$. Consider

$$I = \{ra | r \in R\}.$$

It is not hard to show that this is an ideal of $R$. Is it the smallest? Or could there be other ideals of $R$ of the same size and containing $a$?

## Problem 4

Let $R$ be the ring of all real-valued continuous functions on the closed unit interval. If $M$ is a maximal ideal of $R$, prove that there exists a real number $\gamma$, $0 \leq \gamma \leq 1$, such that $M = M_\gamma = \{f(x) \in R | f(\gamma) = 0\}$.

Let $f \in R$ be a non-zero-valued continuous function on $[0,1]$, and suppose $I$, an ideal of $R$, contains it. Now let $g \in R$ be any member of $R$. Does there exist a function $h \in R$ such that $fh = g$? Clearly there must, since $f$, being non-zero on $[0,1]$, allows us to write $h = g/f$. We can now conclude that $I = R$, and that for every properly contained ideal $I$ of $R$, if $f \in I$, then there exists $\gamma \in [0,1]$ such that $f(\gamma) = 0$. Now if $I$ is a maximal ideal of $R$, it is properly contained in $R$, so such an $f$ exists in $I$. We could further argue that such an $f$ exists in $I$ that has *exactly one* zero at $\gamma$. (Make argument here.) Now if $I$ contains such an $f$, what other functions must it contain?

Let $g \in R$ be any function that is zero at $\gamma$. (It may or may not be zero else-where.) Does there exist $h \in R$ such that $fh = g$? If we write

$$h(x) = \begin{cases} f(x)/g(x) & x \neq \gamma \\ 0 & x = \gamma \end{cases},$$

then can we argue that $h$ is a continuous function on $[0, 1]$? Unfortunately, no. There is no guarentee that $\lim_{x\to 0} h(x) = 0$.

If we could find such an $h$, then we know we have found $I = M_\gamma$, because $I$ is properly contained in $R$ and $M_\gamma$ is maximal in $R$ by Example 3.5.2.