

# On The Use Of Blades As Representatives Of Geometry

Spencer T. Parkin

**Abstract.** Abstract goes here...

**Keywords.** Key words go here...

## 1. Introduction And Motivation

In many models of geometry based upon geometric algebra (see []), blades are used to represent geometries. Seeing a great deal of commonality between these models, a formal treatment of this idea deserves to be given in an abstract setting.<sup>1</sup> To the author's knowledge, this is the first treatment of its kind.

## 2. Foundation

To lay the foundation of our work, we introduce  $\mathbb{V}$  as denoting an  $m$ -dimensional vector space generating a geometric algebra denoted by  $\mathbb{G}$ . We leave the signature of this geometric algebra unspecified, but in cases where a proof depends upon signature, one is given as either euclidean or non-euclidean.<sup>2</sup> The scalars of  $\mathbb{V}$ , (and therefore  $\mathbb{G}$ ), are taken from the field of real numbers denoted by  $\mathbb{R}$ .<sup>3</sup> We will let  $\mathbb{R}^n$  denote  $n$ -dimensional euclidean space,<sup>4</sup> and let  $\mathbb{B}$  denote the set of all blades taken from  $\mathbb{G}$ . Lastly, we will let  $p : \mathbb{R}^n \rightarrow \mathbb{V}$  be an unspecified function we'll use in the following definition which launches us forth into a general theory of models of geometry based upon blades in

---

<sup>1</sup>For example, the use of ideals of polynomial rings as representatives of algebraic sets is studied in the setting of abstract algebra.

<sup>2</sup>The Gram-Schmidt orthogonalization process is applicable to all blades taken from and only from geometric algebras having euclidean signatures.

<sup>3</sup>To be more abstract, we could have used any field with characteristic 1, but there will be no foreseeable advantage to doing so in this paper.

<sup>4</sup>Some models of geometry find affine space to be the natural space within which to work, but this will not be the case in this paper.

a geometric algebra. In our abstract setting, the definition of this function does not matter. All that matters is that it is a function.

**Definition 2.1 (Direct And Dual Representation).** For any blade  $B \in \mathbb{B}$ , we say that  $B$  directly represents the set of all points  $x \in \mathbb{R}^n$  such that  $p(x) \wedge B = 0$ , and say that  $B$  dually represents the set of all points  $x \in \mathbb{R}^n$  such that  $p(x) \cdot B = 0$ . For convenience, we introduce the following functions using set-builder notation.

$$\begin{aligned}\hat{g}(B) &= \{x \in \mathbb{R}^n \mid p(x) \wedge B = 0\} \\ \dot{g}(B) &= \{x \in \mathbb{R}^n \mid p(x) \cdot B = 0\}\end{aligned}$$

From Definition 2.1, it's important to take away the realization that a given blade  $B \in \mathbb{B}$  represents two geometries simultaneously; namely,  $\hat{g}(B)$  and  $\dot{g}(B)$ . Which geometry we choose to think of  $B$  as being a representative of at any given time is completely arbitrary.<sup>5</sup>

It should also be clear from Definition 2.1 that the geometry represented by a blade  $B$ , (directly or dually), remains invariant under any non-zero scaling of the blade  $B$ . Something interesting happens, however, when we take the dual of  $B$ , as our first lemma shows.

**Lemma 2.2 (Dual Something).** *For any subset  $S$  of  $\mathbb{R}^n$ , if there exists  $B \in \mathbb{B}$  such that  $\hat{g}(B) = S$ , then  $\dot{g}(BI) = S$ , where  $I$  is the unit pseudo-scalar of  $\mathbb{G}$ . Similarly, if there exists  $B \in \mathbb{B}$  such that  $\dot{g}(B) = S$ , then  $\hat{g}(BI) = S$ .*

*Proof.* The first of these two statements is proven by

$$0 = p(x) \wedge B = -(p(x) \cdot BI)I \iff p(x) \cdot BI = 0,$$

while the second is proven by

$$p(x) \cdot B = 0 \iff 0 = (p(x) \cdot B)I = p(x) \wedge BI.$$

See identities (3.5) and (3.6) of Section 3. □

In words, Lemma 2.2 is telling us that for a single given geometry, the algebraic relationship between a blade directly (dually) representative of that geometry, and a blade dually (directly) representative of that geometry, is simply that, up to scale, they are duals of one another.

Of course, there will also be a geometric relationship between the geometry that is directly represented by a single given blade  $B \in \mathbb{B}$ , and the geometry that is dually represented by  $B$ , but this depends upon the definition of our function  $p$ , which we choose, in this paper, to leave open for speculation.

---

<sup>5</sup>In some literature on geometric algebra, a blade  $B$  intended to represent some piece of geometry directly or dually is referred to as a “geometry” or a “dual geometry,” respectively. This is confusing and not practiced in this paper. A blade is a blade; and when we refer to geometry, we will use proper language in identifying what represents it and how it does so. In this paper, a geometry is a subset of  $\mathbb{R}^n$  that can be represented dually or directly by some blade  $B \in \mathbb{B}$  under Definition 2.1.

### 3. Useful Identities

In this section we give a number of useful algebraic identities that would otherwise distract us from the flow of the paper if given in the main body. This section is not intended as a complete review of geometric algebra. See [ ] for such a review.

Letting  $v \in \mathbb{V}$  and  $B \in \mathbb{B}$ , recall that

$$vB = v \cdot B + v \wedge B. \quad (3.1)$$

Also recall that

$$v \wedge B = \frac{1}{2}(vB + (-1)^{\text{grade}(B)}Bv), \quad (3.2)$$

$$v \cdot B = \frac{1}{2}(vB - (-1)^{\text{grade}(B)}Bv). \quad (3.3)$$

Realizing that  $\text{grade}(I) = m$ , and that by (3.1), we have  $vI = v \cdot I$ , we can use equation (3.3) to establish the commutativity of vectors in  $\mathbb{V}$  with the unit psuedo-scalar  $I$  as

$$vI = -(-1)^m Iv. \quad (3.4)$$

Using equation (3.4) in conjunction with equation (3.3), we find that

$$(v \cdot B)I = v \wedge BI. \quad (3.5)$$

In verifying this identity, it helps to realize that for any integer  $k$ ,  $(-1)^k = (-1)^{-k}$ . Replacing  $B$  in equation (3.5) with  $BI$ , we find that

$$v \wedge B = -(v \cdot BI)I. \quad (3.6)$$

### References

Spencer T. Parkin  
102 W. 500 S.,  
Salt Lake City, UT 84101  
e-mail: [spencerparkin@outlook.com](mailto:spencerparkin@outlook.com)