On The Expansion Of Algebraic Expressions In Geometric Algebra

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Abstract. Abstract goes here...

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1. Introduction

While the expansion of algebraic expressions taken from, say, a polynomial ring, are found as a trivial matter of applying the associative and distributive properties, and combining like-terms, it is interesting to note that this is certainly not true of expressions taken from a geometric algebra. In this paper, a general stratagy, or algorithm, if you will, is given for the expansion of such expressions, and it is shown that it is perhaps just as natural to write an element of a geometric algebra as a sum of "cursors" as it is to write such an element as a sum of blades. The term "cursor" is introduced in Table 1 below, along with similar, traditional terms found in geometric algebra.

TABLE 1. Terms used in GA

Term	Definition
Blade	The outer product of zero or more linearly-independent vectors.
Versor	The geometric product of zero or more invertible vectors.
Cursor	The geometric product of zero or more vectors, not necessarily invertible.

From these it is clear that every versor is a cursor, but the converse is not generally true.

The concept of grade will be carried forward in this paper from blades to cursors. As an n-blade refers to a blade of grade n, we will let an n-cursor refer to a cursor of grade n; that is, a geometric product of precisely n vectors, none of which need be invertible. Note that blades of grade zero are indistinguishable from cursors of grade zero, each denoting the set of all scalars.

2. Symmetry Between The Outer And Geometric Products

As will be shown by the various identities established in this section, there is perhaps a lot more in common between the outer and geometric products than one might think. Certainly the outer and inner products play a complementary role in the building up or tearing down of blades, respectively, but from a purely algebraic perspective, consider the following well-known definition of the geometric product between two vectors.

$$ab = a \cdot b + a \wedge b \tag{2.1}$$

The right-hand side of equation (**??**) is a sum of blades, while the left-hand side is a sum of cursors; in this case, exactly one; namely, ab. Thus, the element ab appears naturally in blade and cursor form, but what of the element $a \wedge b$? Rearranging (**??**), we simply find that

$$a \wedge b = -a \cdot b + ab, \tag{2.2}$$

showing that it too may be written as a sum of blades or that of cursors.

2.1. From Inner Product To Sum Of Blades

Letting v denote a vector, and B_r a blade of grade r having factorization

$$B_r = \bigwedge_{i=1}^{r} b_i, \tag{2.3}$$

recall that

$$v \cdot B_r = \frac{1}{2}(vB_r - (-1)^r B_r v), \tag{2.4}$$

$$v \wedge B_r = \frac{1}{2}(vB_r + (-1)^r B_r v). \tag{2.5}$$

From these it is clear that

$$vB_r = v \cdot B_r + v \wedge B_r. \tag{2.6}$$

Now letting a, b denote vectors, it is not hard to show that

$$a \cdot (b \wedge B_r) - b \cdot (a \wedge B_r) = (a \cdot b)B_r. \tag{2.7}$$

Similarly, it can be shown (prove it) that

$$a \wedge (b \cdot B_r) - b \wedge (a \cdot B_r) = (a \cdot b)B_r. \tag{2.8}$$

Wishing now to express $v \cdot B_r$ as a sum of blades, we will find, using the princple of mathematical induction, that for all integers $r \geq 1$, we have

$$a \cdot B_r = -\sum_{i=1}^{r} (-1)^r (a \cdot b_i) \bigwedge_{j=1, j \neq i}^{r} b_j. \tag{2.9}$$

Notice that this equation holds in the case $r = 1$ if we let an empty wedge product denote the multiplicative identity. Assuming now our inductive hypothesis, we find that

2.2. From Inner Product To Sum Of Cursors

3. The Expansion Algorithm

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