# On The Multiplicative Inverse Of Multivectors With Respect To The Geometric Product

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We begin by extending the idea of linear independent amoung vectors to that of blades.

**Definition 0.1.** *For a given set* $\{A_i\}_{i=1}^n$ *of* $n$ *non-zero blades of various grades, we say that it is a linearly independent set if there does not exist a linear combination* $L$*, given by*

$$L = \sum_{i=1}^{k} \alpha_i A_i, \tag{1}$$

*of these blades, where* $L = 0$ *and not all scalars in* $\{\alpha_i\}_{i=1}^n$ *are zero.*

It is clear that if no two blades in $\{A_i\}_{i=1}^n$ are of the same grade, then $\{A_i\}_{i=1}^n$ must be a linearly independent set. If this is not the case, however, the set may not be linearly independent. If a set of blades is not linearly independent, we will call it linearly dependent.

**Definition 0.2.** *For any given set of* $n$ *non-zero blades* $\{A_i\}_{i=1}^n$*, we say that it is an irreducible set if there does not exist a set of* $m$ *non-zero blades* $\{B_i\}_{i=1}^m$*, with* $m < n$*, such that*

$$\sum_{i=1}^{n} A_i = \sum_{i=1}^{m} B_i. \tag{2}$$

A non-irreducible set will be referred to as reducible. In the context of Definition 0.2, we will refer to $\{B_i\}_{i=1}^m$ as a reduction of the set $\{A_i\}_{i=1}^n$.

We now find that the irreducible sets are the linearly independent sets.

**Lemma 0.1.** *A given set of $n$ non-zero blades $\{A_i\}_{i=1}^n$ is irreducible if and only if it is linearly independent.*

*Proof.* Clearly the lemma goes through in the case $n = 1$. Therefore, we will consider now only the cases $n > 1$. We prove the contrapositive in each direction.

Let $\{A_i\}_{i=1}^n$ be a linearly dependent set. Then, without loss of generality, we may write

$$A_n = \sum_{i=1}^{n-1} \alpha_i A_i. \tag{3}$$

Then, for each blade in $\{B_i\}_{i=1}^{n-1}$, letting $B_i = (\alpha_i + 1)A_i$, we have

$$\sum_{i=1}^{n} A_i = \sum_{i=1}^{n-1} B_i, \tag{4}$$

showing that $\{A_i\}_{i=1}^n$ is a reducible set.

Now assume that $\{A_i\}_{i=1}^n$ is a reducible set. Let $\{B_i\}_{i=1}^m$ be a linearly independent reduction of this set. (To find such a set, take any reduction of $\{A_i\}_{i=1}^n$ and combine any pair of linearly dependent blades until the set is linearly independent.) Considering now $\{B_i\}_{i=1}^m$ to be a basis for a linear space, we must show that $\{A_i\}_{i=1}^m$ also spans this space to establish its linear dependence. Hmmm... $\square$

For the discussion to follow, we will find it convenient to think of a multivector $M$ as a sum

$$M = \sum_{i=1}^{n} A_i, \tag{5}$$

where the set of blades $\{A_i\}_{i=1}^n$ is an irreducible set. (Clearly a multivector can always be written in terms of such a set.) Furthermore, we will overload the notation $M$ as denoting both the multivector $M$ and the linear space spanned by the set of blades $\{A_i\}_{i=1}^n$. It will therefore make sense to think of multivectors in $M$.

Consider now the multiplicative inverse of a multivector $M$ with respect to the geometric product. The geometric product not being generally commutative, we will restrict our attention in this paper to the inverse $M^{-1}$ of $M$, such that

$$1 = MM^{-1}. \tag{6}$$

Such an inverse may or may not exist. If it does exists, it is clearly unique by the zero-product property of the geometric product.

**Lemma 0.2.** *For any invertible multivector $M$, if it can be written in terms of the blades in the irreducible set $\{A_i\}_{i=1}^n$, then its inverse $M^{-1}$ is a linear combination of the blades in this set.*

*Proof.* Given such a multivector $M$, it is natural to write $M^{-1}$ as

$$M^{-1} = M_0 + M_1, \tag{7}$$

where $M_0 \in M$ and $\operatorname{span}(M_1) \cap \operatorname{span}(M) = \emptyset$. Put another way, for all multivectors $A \in M_0$, we have $A \in M$; and for all multivectors $A \in M_1$, we have $A \notin M_1$. Clearly $M^{-1}$ takes on such a form, because any multivector may be written in such a form with respect to $M$. In the case of $M^{-1}$, however, we must show that $M_1 = 0$ and that $\operatorname{span}(M_0) = \operatorname{span}(M)$.

To that end, notice that for all $B \in M_1$, it is clear that there does not exist a blade $A \in M$ such that $\langle AB \rangle_0$ is a non-zero scalar. It follows that $\langle MB \rangle_0 = 0$, and then that $\langle MM_1 \rangle_0 = 0$. It then follows that

$$1 = MM^{-1} = \langle MM^{-1} \rangle_0 = \langle MM_0 \rangle_0 + \langle MM_1 \rangle_0 \tag{8}$$

if and only if $\langle MM_0 \rangle_0 = 1$ and $\langle MM_1 \rangle_0 = 0$. Now notice that there does not exist a multivector $A \in M_0$ and a multivector $B \in M_1$ such that

$$MA + MB = 0. \tag{9}$$

It follows that for all integers $i > 0$, we must have $\langle MM_0 \rangle_i = 0$ and $\langle MM_1 \rangle_i = 0$. We can now say that

$$1 = MM^{-1} = MM_0 + MM_1 \tag{10}$$

if and only if $MM_0 = 1$ and $MM_1 = 0$, and therefore $M_1 = 0$ by the zero product property.

What we want to show now is that $\operatorname{span}(M_0) = \operatorname{span}(M)$. It is clear by definition that $\operatorname{span}(M_0) \subseteq \operatorname{span}(M)$. □

Knowing the form of which $M^{-1}$ takes on in terms of $M$ brings us a long way towards formulating $M^{-1}$ in terms of $M$. Writing $M = \sum_{i=1}^n A_i$ and then $M^{-1} = \sum_{i=1}^{n} \alpha_i A_i^{-1}$, we may now expand equation (6) to arrive at the following system of equations.

$$1 = \sum_{i=1}^{n} \alpha_i \tag{11}$$

$$0 = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left( \alpha_i A_j A_i^{-1} + \alpha_j A_i A_j^{-1} \right). \tag{12}$$

Recall that for any non-zero blade $A_i$ that $A_i^{-1}$ and $A_i$ are scalar multiples of one another. We may therefore apply Lemma 0.2 in writing $M^{-1}$ as $\sum_{i=1}^{n} \alpha_i A_i^{-1}$.