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Abstract. Abstract...

Keywords. Key words...

1. Introduction And Motivation

It is well-known that *blades* taken from a geometric algebra are, under a
given definition (such as those given in [1]), apt representatives of geometry.
The set of all blades being closed under the meet and join operations, one
must consider what this means for the geometries represented by them. As it
turns out, these operations allow us to intersect geometries and build them
up as fitting the points of others. Furthermore, blades are subject to the
outermorphic property of the sandwitch-product between a versor and its
inverse. All sorts of wonderful transformations on the geometries represented
by blades have been discovered when taken in this product with versors (see
[]).

Thus seeing that a particular means of geometric representation lends
itself to desirable algebraic proprties, and therefore desirable geometric prop-
erties, alternative means of geometric representation are worth exploring.
Indeed, in this paper, we find that *spades* taken from a geometric algebra
are, under a similar definition, also apt representative of geometry. The set
of all spades is closed under the geometric product. Unlike the meet and
join operations, the geometric product enjoys the potential of invertability.
Furthermore, we'll find out that spades, like blades, are also subject to the
outermorphic proprety of the sandwitch-product between a versor and its
inverse.

Throughout this paper we simply let $\mathbb{V}$ denote a vector space generating
the geometric algebra $\mathbb{G}$ within which we work.

2. A Treatment Of Spades

Definition 2.1 (Spade). An element $M \in \mathbb{G}$ is a *spade* if it can be written as the geometric product of zero or more vectors.

It follows from Definition 2.1 that all versors are spades, but not all spades are versors. Furthermore, while the set of all spades of $\mathbb{G}$ enjoys closure under the geometric product, this set, unlike the set of versors of $\mathbb{G}$, does not form a group. We can also say that all null-versors are spades, but not all spades are null-versors.

Definition 2.2 (Spade Rank). The *rank* of a spade $M_r \in \mathbb{G}$, denoted $\mathrm{rank}(M_r)$, is the smallest number[1] of vectors for which M_r can be written as a geometric product of such. If a spade $M_r \in \mathbb{G}$ has factorization

$$M_r = \prod_{i=1}^{r} m_i, \tag{2.1}$$

then it is not necessarily the case that $\mathrm{rank}(M_r) = r$. However, such a factorization does exist. Clearly, it would not be unique.

Many identities involving a spade M_r hold whether or not $\mathrm{rank}(M_r) = r$. In any case, we will become interested in precisely what we can say about the vectors in $\{m_i\}_{i=1}^{r}$ when $\mathrm{rank}(M_r)$ is r.

Definition 2.3 (Dull/Sharp Spade). For any given spade $M_r \in \mathbb{G}$, we say that M_r is *sharp* if and only if $\langle M_r \rangle_s \neq 0$, where $s = \mathrm{rank}(M_r)$, and *dull* otherwise.

The following proposition is the claim that all spades are sharp.

Proposition 2.4. *For any given non-zero spade $M_r \in \mathbb{G}$ with $r > 0$,*

$$rank(M_r) = r \text{ if and only if } 0 \neq \bigwedge_{i=1}^{r} m_i.$$

While one direction of Proposition 2.4 is trivial to prove, the other, most decidedly, is not! An outline of a proof follows.

Given a spade $M_r \in \mathbb{G}$ with factorization (2.1), the set of r vectors $\{m_i\}_{i=1}^{r}$ is either a linearly independent set, or a linearly dependent set. In the former case, it is clear that $\mathrm{rank}(M_r) = r$, because $0 \neq \bigwedge_{i=1}^{r} m_i = \langle M_r \rangle_r$. In the latter case, we must show that $\mathrm{rank}(M_r) < r$. To that end, let s be the largest integer with $1 \leq s < r$ such that $\{m_i\}_{i=1}^{s}$ is a linearly independent set, and write

$$\left\langle \prod_{i=1}^{s+1} m_i \right\rangle_{s-1} = \left\langle \left(\prod_{i=1}^{s} m_i \right) \sum_{i=1}^{s} \alpha_i m_i \right\rangle_{s-1} = \sum_{i=1}^{s} \beta_i \bigwedge_{\substack{j=1 \\ j \neq i}}^{s} m_j.$$

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[1] This smallest number exists by the well-ordering principle. See [].

Just as not all scalars α_i are necessarily non-zero, we may not have all scalars β_i non-zero. Assuming for the moment that each β_i is non-zero, we may write the grade $s - 1$ part as the following $(s - 1)$-blade.

$$\sum_{i=1}^{s} \beta_i \bigwedge_{\substack{j=1 \\ j \neq i}}^{s} m_j = \bigwedge_{i=1}^{s-1} n_i,$$

where $n_1 = \beta_1 m_2 + \beta_2 m_1$, and each of the remaining n_i are given by

$$n_i = m_{i+1} + \frac{\beta_{i+1}}{\beta_i} m_i.$$

In any case, even if some β_i are zero, we can still come up with the $(s - 1)$-blade that is the grade $s - 1$ part of $\prod_{i=1}^{s+1} m_i$ in terms of vectors n_i.

With this $(s - 1)$-blade in hand, we now must solve, for each integer k, the system of equations given by

$$\left\langle \prod_{i=1}^{s+1} m_i \right\rangle_{s-1-2k} = \left\langle \prod_{i=1}^{s-1} \left(n_i + \sum_{j=1}^{i-1} \gamma_{i,j} n_j \right) \right\rangle_{s-1-2k}. \tag{2.2}$$

It is clear that any choice for the scalars $\gamma_{i,j}$ is acceptable in the case $k = 0$.

If a solution in the variables $\gamma_{i,j}$ can be found, then we have shown that our geometric product of r vectors can be rewritten as a geometric product of $r - 2$ vectors. We then continue this process until the set of vectors taken in the geometric product becomes a linearly independent set.

The proof outlined above is an algorithm for finding one of the smallest possible factorizations of the spade M_r; and consequently, its rank. The truthfulness of Proposition 2.4 hinges on the idea that a solution to the system of equations (2.2) can always be found. The presence of null-vectors may be trouble-some.

Lemma 2.5. *For any given spade $M_r \in \mathbb{G}$, if there exist integers $1 \leq i < j \leq r$ such that $m_i = m_j$, and m_i is invertible, then $rank(M_r) < r$.*

Proof. This is trivial in the case that $j = i + 1$. In the case that $j = i + 2$, simply notice that

$$m_i m_{i+1} m_j = m_i m_{i+1} m_i = 2(m_i \cdot m_{i+1}) m_i - m_i^2 m_{i+1}.$$

In the case that $j > i + 2$, we see that

$$m_i \left(\prod_{k=i+1}^{j-1} m_k \right) m_j = m_i^2 \prod_{k=i+1}^{j-1} m_i m_k m_i^{-1}.$$

$\square$

Lemma 2.5 is an interesting observation and implies that every spade written in the most compact form is square-free in the sense that the square of no spade will appear in the factorization. It is still, however, a far cry from helping us prove Proposition 2.4.

Lemma 2.6. *For every non-zero r-blade $B_r \in \mathbb{G}$ with $r > 1$, and having factorization*

$$B_r = \bigwedge_{i=1}^{r} b_i,$$

the set of $(r-1)$-blades $\{B_r^{(i)}\}_{i=1}^{r}$, where the notation $B_r^{(i)}$ is given by

$$B_r^{(i)} = \bigwedge_{\substack{j=1 \\ j \neq i}}^{r} b_j,$$

is a linearly independent set.

Proof. Supposing to the contrary, and without loss of generality, let

$$B_{r-1} = B_r^{(r)} = \sum_{i=1}^{r-1} \alpha_i B_r^{(i)} = \left(\sum_{i=1}^{r-1} \alpha_i B_{r-1}^{(i)} \right) \wedge b_r.$$

Now notice that

$$0 \neq B_r = B_{r-1} \wedge b_r = B_r^{(r)} \wedge b_r = \left(\sum_{i=1}^{r-1} \alpha_i B_r^{(i)} \right) \wedge b_r = 0,$$

which is clearly a contradiction. $\qquad\square$

Lemma 2.7. *Given any spade M_r, the set of all solution sets $\{\alpha_i\}_{i=1}^{r}$ of the equation*

$$0 = \sum_{i=1}^{r} \alpha_i M_r^{(i)}$$

is, for all integers $j \in [0, r]$, the intersection of all sets of solution sets of the equations

$$0 = \sum_{i=1}^{r} \alpha_i \langle M_r^{(i)} \rangle_j,$$

where the notation $M_r^{(i)}$ is given by

$$M_r^{(i)} = \prod_{\substack{j=1 \\ j \neq i}}^{r} m_j.$$

Proof. This is a simple consequence of there being no possibility of interference between elements of differing grade. $\qquad\square$

Lemma 2.8. *For any given spade $M_r \in \mathbb{G}$ with $r > 1$, if $rank(M_r) = r$, then the set of spades $\{M_r^{(i)}\}_{i=1}^{r}$ is a linearly independent set.*

Proof. This follows easily in consideration of Proposition 2.4 with Lemma 2.6 and Lemma 2.7. $\qquad\square$

References

[1] S. Parkin, *An Introduction To Geometric Sets*. Advances in Applied Clifford Algebras, Volume 25, Issue Unknown, pp. 639-655, 2015.

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