Section 2.3 Exercises
Hertein's Topics In Algebra

Spencer T. Parkin

February 27, 2016

Problem 8

If G is a finite group, show that threre exists a postive integer N such that $a^N = e$ for all $a \in G$.

It is not hard to show that for every $a \in G$, there exists an integer $k(a)$ such that $a^{k(a)} = e$. Further, $a^{nk(a)} = e$ for every integer n. Now let

$$N = \prod_{a \in G} k(a).$$

Problem 11

If G is a group of even order, prove it has an element $a \neq e$ satisfying $a^2 = e$.

Clearly G has an odd number of non-identity elements. Pluck such an element from G. If $a^2 = e$, we're done. If not, pluck its inverse out of G as well. This leaves us yet a smaller pool of odd elements to choose from. Continue this process until we either find an element being its own inverse, or we're left with just one non-identity element. Clearly this last remaining non-identity element must be its own inverse.

Exercise 26

Part (a)

Let G be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo p, p a prime number, such that $ad - bc \neq 0$. G forms a group relative to matrix multiplication. What is $o(G)$?

We find $|G|$ by counting matrices of the said dimension and then subtracting from that the number of such being singular. Clearly there are p^4 matrices of the desired dimension. How many of them are singular? The singularity of a 2×2 matrix occurs whenever a row (column) is a scalar multiple of the other row (column). Consider the following expression.

$$(1)p^2 + (p-1)p + (p-1)p + (p-1)^2 p.$$

This expression, having 4 terms, represents 4 cases. In the first case, one row is zero, leaving p^2 choices for the other row. In the second case, a row is axis-aligned in $p - 1$ ways with p ways the other row is zero or parallel to it. The third cases is counted like the second, but using the other axis. In the fourth case, there are $(p-1)^2$ ways a row is non-zero and non-axis-aligned with p ways the other row is zero or parallel to it.

Putting it all together, we get

$$|G| = p^4 - p^3 - p^2 + p.$$

Part (b)

Let H be the subgroup of the G of part (a) defined by

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \,\middle|\, ad - bc = 1 \right\}.$$

What is $o(H)$?

Figure it out...