

Chapter 24 Homework

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Problem 2

Calculate all conjugacy classes for the quaternions.

Some work shows that the conjugacy classes are

$$\{e\}, \{a, a^3\}, \{a^2\}, \{b, ba^2\}, \{ba, ba^3\}.$$

Problem 8

Find all Sylow 3-subgroups of A_4 .

The group A_4 has order $4!/2 = 24 = 2^3 \cdot 3$, so we're looking for all subgroups of order 3. Since 3 is prime, all such subgroups are cyclic. So the set of subgroups is given by $\{\langle a \rangle \mid a \in A_4 \text{ and } |a| = 3\}$. Using Table 5.1, these groups are given by

$$\begin{aligned} &\{\alpha_1, \alpha_5, \alpha_9\}, \\ &\{\alpha_1, \alpha_6, \alpha_{11}\}, \\ &\{\alpha_1, \alpha_7, \alpha_{12}\}, \\ &\{\alpha_1, \alpha_8, \alpha_{12}\}. \end{aligned}$$

This was given away in Example 2 of the chapter.

Problem 12

Show that every group of order 56 has a proper nontrivial normal subgroup.

By Sylow's First Theorem, there is at least one subgroup of order 7. By Sylow's Third Theorem, the number of such subgroups is congruent to 1 modulo 7 and divides 56. The only possible number of such subgroups is therefore 1 or 8. If there is only 1 subgroup of order 7, then it is a nontrivial normal subgroup by the corollary to Sylow's Third Theorem. If there are 8 subgroups of order 7, then because they are cyclic, they account for $8 \cdot (7 - 1) = 48$ distinct elements, leaving $56 - 48 = 8$ remaining elements, including the identity element. It follows that these 8 elements must form a subgroup of order 8, By Sylow's First Theorem. Furthermore, this is the only subgroup of order 8, because every one of the 48 elements previously mentioned have order 7, and 7 does not divide 8.

Problem 13

What is the smallest composite integer n such that there is a unique group of order n ?

For groups of order 4, 8, 9, and 12, they could all be cyclic or the external direct product of cyclic groups that results in a non-cyclic group. This is because they can all be written as the product of two non-trivial and non-coprime divisors.

For groups of order 6, 10, and 14, blah.

Now notice that all groups of order 15 are cyclic by Theorem 24.6.

Problem 17

How many Sylow 3-subgroups of S_5 are there? Exhibit five.

By Sylow's Third Theorem, there are 1, 4, 10, or 40 of them. Since we're asked to exhibit five of them, there must be 10 or 40. If there are 40 subgroups of order 3, then there are 80 elements of order 3.

Problem 24

Prove that a group of order 105 contains a subgroup of order 35.

By Sylow's First Theorem, there is a subgroup of order 5, and a subgroup of order 7. By Sylow's Third Theorem, there may be at most 21 groups of order 5, and 15 groups of order 7. Suppose there are more than one of both types of subgroups. Since these are cyclic of prime order, this would generate $21 \cdot (5 - 1) = 84$ elements of order 5, and $15 \cdot (7 - 1) = 90$ elements of order 7. But $84 + 90 > 105$, so at least one of these two types of subgroup is unique, and therefore normal by the corollary to Sylow's Third Theorem. It then follows by the result of Exercise 51 on Page 194, that if H is a subgroup of order 5, and K is a subgroup of order 7, then HK is a subgroup of G , since at least one of H and K is normal. Now notice that $|HK| = |H||K|/|H \cap K|$, but $H \cap K = \{e\}$ since H and K are cyclic groups of distinct prime orders. It follows that $|HK| = |H||K| = 5 \cdot 7 = 35$, which is what we wanted.

Problem 25

Prove that a subgroup of order 595 has a normal Sylow 17-subgroup.

Problem 28

Problem 40

Let p be prime. If the order of every element of a finite group G is a power of p , prove that $|G|$ is a power of p .

Suppose $|G|$ has a prime q other than p in its factorization. Then by Sylow's First Theorem, there must exist a cyclic subgroup of order q , and its generator is an element of order q , contradicting the fact that all elements of G have orders that are powers of p . So p is the only prime in the factorization of $|G|$, and so $|G|$ is a power of p .

Lemma 1 for Problem 41

Let H and K be subgroups of a group G and conjugate in G . Then H and K are isomorphic.

Proof: By hypothesis, there exists $g \in G$ such that $H = gKg^{-1}$. Define $\phi : K \rightarrow H$ by $\phi(k) = gkg^{-1}$. Letting $x, y \in K$, we see that

$$x = y \iff gxg^{-1} = gyg^{-1} \iff \phi(x) = \phi(y),$$

showing that ϕ is well defined and one-to-one. We then see that

$$\phi(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = \phi(x)\phi(y),$$

showing that ϕ is operation preserving. Now let $h \in H \subseteq gKg^{-1}$. Then there exists $k \in K$ such that $h = gkg^{-1}$, and so $\phi(k) = h$, showing that ϕ is onto. We have now shown that ϕ is an isomorphism between H and K .

Notice that H and K are not necessarily the same subset of G .

Lemma 2 for Problem 41

Let H be a subgroup of a group G . Then for any $g \in G$, the subset gHg^{-1} of G is a conjugate of H in G .

Proof: We first show that gHg^{-1} is a subgroup of G . Clearly, $e \in gHg^{-1}$. Now let $a, b \in gHg^{-1}$. Then there exists $h_a, h_b \in H$ such that $a = gh_ag^{-1}$ and $b = gh_bg^{-1}$. Then we have

$$ab^{-1} = gh_ag^{-1}(gh_bg^{-1})^{-1} = gh_ag^{-1}gh_b^{-1}g^{-1} = gh_ah_b^{-1}g^{-1} \in gHg^{-1},$$

showing that gHg^{-1} is a subgroup of G by the one-step subgroup test.

We now show that the subgroups H and gHg^{-1} of G are conjugate in G . Notice that H and gHg^{-1} have the same cardinality. If $h_1, h_2 \in H$ where $h_1 \neq h_2$, and $gh_1g^{-1} = gh_2g^{-1}$, then we have a contradiction. Choosing $g \in G$, we see that $g^{-1}(gHg^{-1})g = (g^{-1}g)H(g^{-1}g) = H$.

Notice that if H is a normal subgroup of G , then $H = gHg^{-1}$, but in general, we can only say that $H \approx gHg^{-1}$.

Problem 41

For each prime p , prove that all Sylow p -subgroups of a finite group are isomorphic.

It is easy to show that if two subgroups of a group are conjugates of one another, then they are isomorphic. (See Lemma 1 above.) What is hard to show is that if K is a Sylow p -subgroup of a group G , then all other Sylow p -subgroups have the form gKg^{-1} , where $g \in G$. Clearly, by Lemma 2 above, every member of the set $\{gKg^{-1} | g \in G\}$ is a subgroup of G , has the same order as K , and is isomorphic to all other members, but does this set contain all such subgroups? I do not know! It is not obvious to me!

This was proven in the second paragraph of the proof of Sylow's 3rd Theorem, though the fact was used in the proof of Sylow's 2nd Theorem. Grrr.