Conjugate Subgroups

Let H and K be subgroups of a group G. We say that H and K are conjugate in G if there is an element g in G such that $H = gKg^{-1} = \{gkg^{-1} : k \in K\}$.

Conjugate Subgroups are Isomorphic

Let H and K be subgroups of G and conjugate in a group G. Then H and K are isomorphic.

Proof: By hypothesis, there exists $g \in G$ such that $H = gKg^{-1}$. Define $\phi : K \to H$ by $\phi(k) = gkg^{-1}$. Letting $x, y \in K$, we see that

$$x = y \iff gxg^{-1} = gyg^{-1} \iff \phi(x) = \phi(y),$$

showing that ϕ is well defined and one-to-one. We then see that

$$\phi(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = \phi(x)\phi(y),$$

showing that ϕ is operation preserving. Now let $h \in H \subseteq gKg^{-1}$. Then there exists $k \in K$ such that $h = gkg^{-1}$, and so $\phi(k) = h$, showing that ϕ is onto. We have now shown that ϕ is an isomorphism between H and K.

Notice that H and K are not necessarily the same subset of G.

Conjugation

Let H be a subgroup of a group G. Then for any $g \in G$, the subset gHg^{-1} of G is a conjugate of H in G.

Proof: We first show that gHg^{-1} is a subgroup of G. Clearly, $e \in gHg^{-1}$. Now let $a, b \in gHg^{-1}$. Then there exists $h_a, h_b \in H$ such that $a = gh_ag^{-1}$ and $b = gh_bg^{-1}$. Then we have

$$ab^{-1} = gh_ag^{-1}(gh_bg^{-1})^{-1} = gh_ag^{-1}gh_b^{-1}g^{-1} = gh_ah_b^{-1}g^{-1} \in gHg^{-1},$$

showing that gHg^{-1} is a subgroup of G by the one-step subgroup test.

We now show that the subgroups H and gHg^{-1} of G are conjugate in G. Notice that H and gHg^{-1} have the same cardinality. If $h_1, h_2 \in H$ where $h_1 \neq h_2$, and $gh_1g^{-1} = gh_2g^{-1}$, then we have a contradiction. Choosing $g \in G$, we see that $g^{-1}(gHg^{-1})g = (g^{-1}g)H(g^{-1}g) = H$.

Notice that if H is a normal subgroup of G, then $H = gHg^{-1}$, but in general, we can only say that $H \approx gHg^{-1}$.

Now let H and K be subgroups of G and conjugate in G. Then there exists $g \in G$ such that $H = gKg^{-1}$, and we have $H \approx K \approx gKg^{-1}$. This shows that there is an automorphism at work here.

Page 413, Exercise 9

Let K be a Sylow p-subgroup of a finite group G. Prove that if $x \in N(K)$ and the order of x is a power of p, then $x \in K$.

By Sylow's Second Theorem, the cyclic subgroup $\langle x \rangle$ of G is contained in a Sylow p-subgroup of G. If we can show that $\langle x \rangle$ is a subgroup of K, then $x \in K$.

It can be shown that $N(K)$ is a subgroup of G and that K is a normal subgroup of $N(K)$. Since $x \in N(K)$, it is easy to see that $\langle x \rangle$ is a subgroup of $N(K)$. Now using the result of Exercise 51 of Page 194, we see that $K\langle x \rangle$ is a subgroup of $N(K)$. Then using the result of Exercise 7 of Page 174, we have $|K\langle x \rangle| = |K||\langle x \rangle|/|K \cap \langle x \rangle|$. At this point, realize that $|K| = |xKx^{-1}| = |xK| = |Kx| = |K\langle x \rangle|$. It follows that $|\langle x \rangle|/|K \cap \langle x \rangle| = 1$, which implies that $|\langle x \rangle| = |K \cap \langle x \rangle|$, which implies that $\langle x \rangle$ is a subgroup of K.

Sylow's Second Theorem

If H is a subgroup of a finite group G and $|H|$ is a power of a prime p, then H is contained in some Sylow p-subgroup of G.

Proof