Chapter 6 Exercises
Gallian's Book on Abstract Algebra

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Problem 15

If G is a group, prove that $\text{Aut}(G)$ and $\text{Inn}(G)$ are groups.

It is clear that the identity function ϵ is in $\text{Aut}(G)$. Let $\alpha, \beta \in \text{Aut}(G)$. Notice that if each of α and β are bijections from G to G, then so is $\alpha\beta$. Letting $a, b \in G$, we see that

$$\alpha(\beta(ab)) = \alpha(\beta(a)\beta(b)) = \alpha(\beta(a))\alpha(\beta(b)),$$

showing that $\alpha\beta$ is operation preserving as well. Thus far we have proven closure. Now notice that if α is a bijection of G, then so is α^{-1}. To show that α^{-1} is operation preserving, notice that

$$\alpha(\alpha^{-1}(a)\alpha^{-1}(b)) = \alpha(\alpha^{-1}(a))\alpha(\alpha^{-1}(b)) = ab = \alpha(\alpha^{-1}(ab)),$$

and then take the α^{-1} of both of the furthest sides of this equation. We now that if $\alpha \in \text{Aut}(G)$, then so is α^{-1}. We can now clame that $\text{Aut}(G)$ is a group by the two-step subgroup test.

Go on...

Problem 17

Let $r \in U(n)$. Prove that the mapping $\alpha : Z_n \to Z_n$ defined by $\alpha(s) = sr$ (mod n) for all $s \in Z_n$ is an automorphism of Z_n.

It is clear that α is onto, because $\langle r \rangle = Z_n$ since $\gcd(r, n) = 1$. Then, since α maps a finite set to another of the same cardinality, it is also one-to-one. To see that α is operation preserving, we write, for any $a, b \in Z_n$,

$$\alpha(a + b) = s(a + b) = sa + sb = \alpha(a) + \alpha(b).$$

This proof was nothing more than a generalization of what was given in Example 13.

Made-Up Problems

Problem 1

For any geometric algebra $\mathbb{G}$, show that the set of all versors $V \in \mathbb{G}$ forms a group under the geometric product. Does the set of all unit-versors form a subgroup?

The geometric product is associative. Notice that 1 is the identity versor. Let $A, B \in \mathbb{G}$ be versors. There then exists a set of m invertible vectors $\{a_i\}_{i=1}^m$ and a set of n invertible vectors $\{b_i\}_{i=1}^n$ such that $A = \prod_{i=1}^m a_i$ and $B = \prod_{i=1}^n b_i$. It now follows by the definition of what a versor is, that AB is also a versor. AB is a product of one or more invertible vectors. Now notice that $A^{-1} = \prod_{i=1}^m a_{m-i+1}^{-1}$ is also, by definition, a versor. (It may also be required to say that by the zero-product property of the geometric property, inverses are unique.)

Seeing that $|A| = \prod_{i=1}^m |a_i|$, (this is not hard to prove), if $|A| = 1$, we can, without loss of generality, assume, for each a_i, that $|a_i| = 1$. (Such a set of m vectors can always be found for A.) Do the same for B. Closure then follows. Note that the identity has magnitude 1. Then, seeing that, for any a_i, we have $|a_i| = |a_i^{-1}|$, we also have inverses. So the unit-versors form a subgroup of the versor group.

Problem 2

Show that the even versors of a geometric algebra $\mathbb{G}$ form a group under the geometric product? Do the odd versors form a group?

Clearly the odd versors don't form a group, because we don't have closure. The even versors give us closure. The identity versor may be considered even. And it is clear the the inverse of an even versor is also even.

Problem 3

For a geometric algebra $\mathbb{G}$, show that the function $\phi_W(V) = WVW^{-1}$, with W a versor of $\mathbb{G}$, is an automorphism of the group of versors of $\mathbb{G}$.

Notice that we need only consider the case when W is a vector, since any composition of two or more isomorphisms is an isomorphism by Theorem 6.4.

Problem 4

For a geometric algebra $\mathbb{G}$ and a fixed positive integer k, show that the k-vectors of $\mathbb{G}$ form a group under addition. Do the k-blades form a group? (Let zero be a k-vector for any k.)

k-blades don't generally form a group, because we don't always have closure for $k \geq 2$. (In some geometric algebras, we do have closure for $k = 2$, but not all.)

k-vectors do give us closure. Zero is the zero k-vector. k-vector addition is associative. The additive inverse of a k-vector is easy to calculate and it is unique.

Problem 5

For any versor W in a geometric algebra $\mathbb{G}$, define the function $\phi_W(V) = WVW^{-1}$ on the set of k-vectors in $\mathbb{G}$, and show that it is an automorphism of the group of k-vectors.

Notice again that we need only consider the case when W is a vector, since any composition of two or more isomorphisms is an isomorphism by Theorem 6.4.