The Fundamental Theorem of Finite Abelian Groups

Spencer T. Parkin

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This fundamental theorem is traditionally proven using a greedy-algorithm approach. Here we attempt to give a proof based upon a divide-and-conquer approach. The basic idea is to show that every finite abelian group, under some condition, can be non-trivially factored into the internal direct product of two non-trivial subgroups. This is then applied recursively until the condition fails. (The condition might be based upon the group having a non-trivial subgroup.)

Lemma 0.1. *If G is a finite abelian group having a subgroup H, then for every element $g \in G$ that can be factored as $g = g'h$, where $g' \in G$ and $h \in H - \{e\}$, there exists $g'' \in G - (H - \{e\})$ and $h' \in H - \{e\}$ such that $g = g''h'$.*

Lemma 0.2. *If G is a finite abelian group having a subgroup H, then G/H is isomorphic to a subgroup K of G. Moreover, $G = KH$, the internal direct product of K and H.*

Proof. If H is a trivial subgroup, we have nothing to prove, so we may assume H is non-trivial. That said, consider the set S given by

$$S = \{g \in G | g = g'h \text{ where } g' \in G,\ h \in H - \{e\}\}.$$

We now claim that $K = (G - S) \cup \{e\}$ is a subgroup of G. If $K = \{e\}$, we're done. So we may assume $K \supset \{e\}$. Our group G being finite, we need only show closure. Consider a pair of elements $a, b \in K$. If either a or b is the identity, or if $a = b^{-1}$, then clearly $ab \in K$. We may, therefore, assume that neither of these is the case. Now suppose that $ab \in S - \{e\}$. ($S - \{e\}$ is non-empty, because H is non-trivial.) Writing $ab = g'h$, where $g' \in G$ and $h \in H$, it follows that $a = (g'b^{-1})h$, placing $a \in S$. Now since $S \cap K = \{e\}$, we must have $a = e$, which contradicts our supposition. Therefore, $ab \notin S - \{e\}$, and we have

$$ab \in G - (S - \{e\}) = (G - S) \cup \{e\} = K.$$

Having now established K as a subgroup of G, consider the homomorphism $\phi : G \to G/H$ given by $\phi(x) = xH$. We now contend that ϕ is an isomorphism from K to G/H.

We first show that $\phi(K) = G/H$. Let $x \in G$ and consider the coset xH. If $x \in K$, we're done. If $x \notin K$, then $x \in S$, and we can invoking Lemma 0.1 to say that there exists $g'' \in K$ such that $x = g''h'$ with $h' \in H$. We then have $xH = g''h'H = g''H$, and we have it.

What remains to be shown now is that ϕ, when restricted to K, is ono-to-one. Let $x, y \in K$ such that $xH = yH$. It follows that $y^{-1}x \in H$. Now since $S \supseteq H$, we have $K \cap H = \{e\}$, and therefore $y^{-1}x = e \implies x = y$. We can now conclude that ϕ is the claimed isomorphism.

The second part of our lemma now follows from the fact that

$$|HK| = \frac{|H||K|}{|H \cap K|} = |H||K|,$$

while we have just shown that $|K| = |G|/|H|$. Seeing that $|HK| = |G|$, we have $G = HK$. $\square$

We now jump straight to the fundamental theorem by way of Couchy's Theorem.

Theorem 0.1. *Every finite abelian group G is isomorphic to an external direct product of cyclic groups.*

Proof. If G is of prime order, we're done. This not being the case, let p be any prime divisor of $|G|$. By Couchy's Theorem, there exists an element $a_1 \in G$ of order p, and therefore a cyclic subgroup $\langle a_1 \rangle$ of G of order p. Applying Lemma 0.2, we may write $G = \langle a_1 \rangle K_1$, where $K_1 < G$ and has order $|G|/p$. Now write $G \approx \langle a_1 \rangle \times K_1$ and apply our procedure again with K_1 to obtain $G \approx \langle a_1 \rangle \times \langle a_2 \rangle \times K_2$. Our group G being of finite order, this process must, for some integer k, terminate with

$$G = \langle a_1 \rangle \times \cdots \times \langle a_k \rangle.$$

$\square$

There is something wrong here. The number of non-isomorphic abelian groups of order p^n is the number of partitions of n, but I'm not getting this.