An Introduction To Geometric Sets

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Abstract. A general theory of geometric models based upon the idea of using blades of a geometric algebra as representatives of geometry is developed. Similar to the idea of an algebraic set, the main object of study becomes the notion of a geometric set. Results about geometric sets are obtained, and their implications discussed.

Keywords. Geometric Algebra, Model Of Geometry, Geometric Set.

1. Introduction

This paper develops a general theory of geometric models based upon the idea of using blades of a geometric algebra as representatives of geometry. Specific examples include the homogeneous and conformal models. (See [].)

1.1. Motivation

Seeing a great deal of commonality between various models of geometry based upon geometric algebra, a formal treatment of the subject deserves to be given in an abstract setting in much the same way that, for example, abstract algebra provides such a setting in which algebraic sets generated by ideals of a polynomial ring can be studied. Readers familiar with algebraic geometry will no-doubt recognize at least some small overlap between that subject and the subject of this paper. For example, it will be seen that in some cases, geometric sets are algebraic sets.

1.2. Conventions

This paper uses capital letters A, B, C to denote blades of various grades, while using lower case letters a, b, c to denote vectors. Scalars are written using greek letters α, β, γ. Grades of blades A, B, C are usually denoted by lower case letters r, s, t, respectively, unlesss stated otherwise. Lower case letters i, j, k are used as indices. Capital letters R, S, T are used to denote subsets of interest of n-dimensional euclidean space $\mathbb{R}^n$. Lower case letters x, y, z are reserved for denoting points taken from $\mathbb{R}^n$. We let $P(\mathbb{R}^n)$ denote the power set of $\mathbb{R}^n$. We will use $\mathbb{G}$ to denote our geometric algebra, and $\mathbb{V}$

to denote an m-dimensional vector space generating it. $\mathbb{B}$ will denote the set of all blades taken from $\mathbb{G}$. The scalars of $\mathbb{V}$, and therefore $\mathbb{G}$, are taken from the field of real numbers $\mathbb{R}$. The capital letter I will be used to denote the unit psuedo-scalar of $\mathbb{G}$. We assume that I is invertible with respect to the geometric product.

We will let the geometric product take precedence over the inner and outer products, and the inner product take precedence over the outer product.

No specific signature of our geometric algebra $\mathbb{G}$ is assumed in this paper unless one is given in a special case.

At times a blade A may be referred to as a subspace. The notion of blades as being representatives of subspaces of $\mathbb{V}$ is a common practice and is employed throughout this paper. It may also be said that a blade A is a subspace of some other blade B; in which case, it can be understood that for all vectors $a \in \mathbb{V}$ such that $a \wedge A = 0$, we have $a \wedge B = 0$.

2. Preliminaries

The results of this paper will depend upon the following preliminary material. If desired, the reader is welcome to skip this material and refer back to it only as needed.

2.1. Identities

Given a vector $a \in \mathbb{V}$ and a blade $A \in \mathbb{B}$, central to all of geometric algebra is the identity

$$aA = a \cdot A + a \wedge A. \tag{2.1}$$

The inner and outer products of (2.1) may be written in terms of the geometric product as

$$a \wedge A = \frac{1}{2}\left(aA + (-1)^r Aa\right), \tag{2.2}$$

$$a \cdot A = \frac{1}{2}\left(aA - (-1)^r Aa\right), \tag{2.3}$$

where $r = \text{grade}(A)$. Then, realizing that $m = \text{grade}(I)$, and that by (2.1), we have $aI = a \cdot I$, we can use equation (2.3) to establish the commutativity of vectors in $\mathbb{V}$ with the unit psuedo-scalar I as

$$aI = -(-1)^m Ia. \tag{2.4}$$

Using equation (2.4) in conjunction with equation (2.3), we find that

$$(a \cdot A)I = a \wedge AI. \tag{2.5}$$

(In verifying this identity, it helps to realize that for any integer r, we have $(-1)^r = (-1)^{-r}$.) Replacing A in equation (2.5) with AI, we find that

$$(a \wedge A)I = a \cdot AI. \tag{2.6}$$

Refering back to equation (2.3), another important formulation of the inner product between a vector and a blade is given by

$$a \cdot A = -\sum_{i=1}^{r}(-1)^i(a \cdot a_i)A_i, \tag{2.7}$$

where A is factored as $\bigwedge_{i=1}^r a_i$, and we define A_i as

$$A_i = \bigwedge_{\substack{j=1 \\ j\neq i}}^{r} a_i. \tag{2.8}$$

This leads to the following recursive formulation.

$$a \cdot A = (a \cdot a_1)A_1 - a_1 \wedge (a \cdot A_1) \tag{2.9}$$

If a blade $B \in \mathbb{B}$ has grade s and factorization $\bigwedge_{i=1}^s b_i$, then we can express the product $A \cdot B$ recursively as

$$A \cdot B = \begin{cases} A_r \cdot (a_r \cdot B) & \text{if } r \leq s, \\ (A \cdot b_1) \cdot B_1 & \text{if } r \geq s. \end{cases} \tag{2.10}$$

2.2. Lemmas

The following lemmas will help us prove the results of this paper.

Lemma 2.1 (Found Factorization Of Blade). *If $A \in \mathbb{B}$ is a non-zero blade of grade $r > 0$ and $\{c_i\}_{i=1}^r$ is a set of r linearly independent vectors such that for all $c \in \{c_i\}_{i=1}^r$, we have $c \wedge A = 0$, then there exists a scalar $\beta \in \mathbb{R}$ such that $A = \beta C$, where C is an r-blade given by*

$$C = \bigwedge_{i=1}^{r} c_i. \tag{2.11}$$

Proof. Letting $\bigwedge_{i=1}^r a_i$ be a factorization of the r-blade A, it is clear that if $c_i \wedge A = 0$, then there exists a set of r scalars $\{\gamma_{i,j}\}_{j=1}^r$ such that

$$c_i = \sum_{j=1}^{r} \gamma_{i,j} a_i. \tag{2.12}$$

We then have

$$\bigwedge_{i=1}^{r} c_i = (\det M) \bigwedge_{i=1}^{r} a_i, \tag{2.13}$$

where the $r \times r$ matrix M is given by

$$M = \begin{bmatrix} \gamma_{1,1} & \gamma_{1,2} & \cdots & \gamma_{1,r} \\ \gamma_{2,1} & \gamma_{2,2} & \cdots & \gamma_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{r,1} & \gamma_{r,2} & \cdots & \gamma_{r,r} \end{bmatrix}. \tag{2.14}$$

Now since $\{c_i\}_{i=1}^r$ is a linearly independent set of vectors, $\det M \neq 0$, and we may choose $\beta = (\det M)^{-1}$ and equation (2.11) now holds. $\square$

Our next lemma is a generalization of Lemma 2.1.

Lemma 2.2 (Found Partial/Full Factorization Of Blade). *If $A \in \mathbb{B}$ is a non-zero blade of grade $r > 0$ and $\{c_i\}_{i=1}^t$ is a set of $t \leq r$ linearly independent vectors such that for all $c \in \{c_i\}_{i=1}^t$, we have $c \wedge A = 0$, then there exists a blade $B \in \mathbb{B}$ of grade $r - t$ such that $A = B \wedge C$, where C is a t-blade given by*

$$C = \bigwedge_{i=1}^{t} c_i. \tag{2.15}$$

Proof. The case of $t = r$ being handled by Lemma 2.1, and the case of $r = 1$ being trivial, we may assume that $0 < t < r$. Then, letting $\bigwedge_{i=1}^r a_i$ be a factorization of the r-blade A, it is clear that if $c_i \wedge A = 0$, then there exists a set of r scalars $\{\gamma_{i,j}\}_{j=1}^r$ such that equation (2.12) again holds. Furthermore, we can assume, without loss of generality, that for each c_i, $\gamma_{i,i} \neq 0$. We then have

$$\bigwedge_{i=1}^{t} c_i = \gamma \bigwedge_{i=1}^{t} a_i + Q, \tag{2.16}$$

where $\gamma = \prod_{i=1}^t \gamma_{i,i}$ and Q represents the remaining terms in the expansion of C. Letting

$$B = (-1)^{t(r-t)} \gamma^{-1} \bigwedge_{i=t+1}^{r} a_i, \tag{2.17}$$

it follows now that $A = B \wedge C$, since $B \wedge Q = 0$. □

Lemma 2.3. *For a non-zero r-blade A factored as $\bigwedge_{i=1}^r a_i$, the set of $(r-1)$-blades $\{A_i\}_{i=1}^r$, (see equation (2.8)), is linearly independent.*

Proof. Suppose there exists a non-trivial set of r scalars $\{\alpha_i\}_{i=1}^r$ such that $0 = \sum_{i=1}^r \alpha_i A_i$. Then, without loss of generality, suppose that $\alpha_r \neq 0$, and rearrange our equation as $-\alpha_r A_r = \sum_{i=1}^{r-1} \alpha_i A_i$. Now notice that while $a_r \wedge -\alpha_r A_r \neq 0$, we have $a_r \wedge \sum_{i=1}^{r-1} \alpha_i A_i = 0$, which is a contradiction. □

Note that a perhaps more elegant proof of Lemma 2.3 could have been given under the assumption of a euclidean signature; in which case, the Gram-Schmidt orthogonalization process would have allowed us to choose, without loss of generality, an orthogonal factorization of the blade A. Doing so, A becomes the versor $\prod_{i=1}^r a_i$, and we may write

$$0 = \sum_{i=1}^{r} \alpha_i A_i \iff 0 = \left(\sum_{i=1}^{r} \alpha_i A_i \right) A = -\sum_{i=1}^{r} (-1)^{r-i} A_i^2 \alpha_i a_i. \tag{2.18}$$

The use of equation (2.18) would depend, in part, upon Lemma 2.4 to follow.

Lemma 2.4 (The Zero Product Property). *For any two non-zero blades $A, B \in \mathbb{B}$ of grades r and s, respectively, if $AB = 0$ and at least one of A and B is invertible, then at least one of A and B is zero.*

Proof. Without loss of generality, suppose B^{-1} exists. We then see that

$$A = ABB^{-1} = 0B^{-1} = 0. \tag{2.19}$$

□

Notice the requirment here of Lemma 2.4 that at least one of A and B be invertible. This requirement comes about in consideration of the square of a non-zero null-vector.

3. Results

3.1. Foundation

Our discussion begins with a non-zero, undefined function $p : \mathbb{R}^n \to \mathbb{V}$.[1] By leaving this function undefined, the results to follow generalize to the homogeneous model, conformal model, and any other model of geometry that is based upon the use of blades as representatives of geometry.

Definition 3.1 (Direct And Dual Representation). For the two functions $\hat{g} : \mathbb{B} \to P(\mathbb{R}^n)$ and $\dot{g} : \mathbb{B} \to P(\mathbb{R}^n)$, given by

$$\hat{g}(A) = \{x \in \mathbb{R}^n | p(x) \wedge A = 0\}, \tag{3.1}$$

$$\dot{g}(A) = \{x \in \mathbb{R}^n | p(x) \cdot A = 0\}, \tag{3.2}$$

we say that A directly represents the set of points $\hat{g}(A)$ and dually represents the set of points $\dot{g}(A)$.

From Definition 3.1, it is important to take away the realization that a given blade $A \in \mathbb{B}$ represents two subsets of $\mathbb{R}^n$ simultaneously; namely, $\hat{g}(A)$ and $\dot{g}(A)$. Which we choose to think of A as being a representative of at any given time is completely arbitrary.

It should also be clear from Definition 3.1 that the subset of $\mathbb{R}^n$ represented by a blade A, (directly or dually), remains invariant under any non-zero scaling of the blade A.

Finally, now enters this paper's object of study: the *geometric set.*

Definition 3.2 (Geometric Set). A subset $R \subset \mathbb{R}^n$ for which there exists a blade $A \in \mathbb{B}$ such that $\hat{g}(A) = R$ is what we'll refer to as a *geometric set.*

In the course of our study, we will find the concept of *irreducibility* important.

Definition 3.3 (Irreducible/Reducible Blade). Given an r-blade $A \in \mathbb{B}$, if there does not exist an s-blade $B \in \mathbb{B}$ with $s < r$ such that $\hat{g}(A) = \hat{g}(B)$, then A is what we'll refer to as an *irreducible* blade. A blade that is not irreducible is referred to as *reducible.*

3.2. Developments

Having given the definitions in the previous section, we may now focus on the results that follow from these definitions.

[1] By definition, a function is well-defined even if it is left unspecified.

3.2.1. Representation. Our initial developments reveal results about the representations of geometric sets.

Lemma 3.4 (Dual Relationship Between Representations). *For any geometric set $R \subset \mathbb{R}^n$, if $A \in \mathbb{B}$ is a blade such that $\hat{g}(A) = R$, then $\dot{g}(AI) = R$, and similarly, if $A \in \mathbb{B}$ is a blade such that $\dot{g}(A) = R$, then $\hat{g}(AI) = R$.*

Proof. By the identity of equation (2.6), and Lemma 2.4, the first of these two latter statements is proven by

$$0 = p(x) \wedge A = -(p(x) \cdot AI)I \iff p(x) \cdot AI = 0, \tag{3.3}$$

while the second, by the identity of equation (2.5), and again Lemma 2.4, is proven by

$$p(x) \cdot A = 0 \iff 0 = (p(x) \cdot A)I = p(x) \wedge AI. \tag{3.4}$$

□

In other words, Lemma 3.4 is telling us that for a single given geometric set, the algebraic relationship between a blade directly (dually) representative of that set, and a blade dually (directly) representative of that set, is simply that, up to scale, they are duals of one another.

Of course, there will also be a geometric relationship between the geometric set that is directly represented by a single given blade $A \in \mathbb{B}$, and the geometric set that is dually represented by A, but this would depend upon how we choose to define the function $p : \mathbb{R}^n \to \mathbb{V}$.

Lemma 3.5. *For any geometric set $R \subset \mathbb{R}^n$, there exists a blade $A \in \mathbb{B}$ such that $\dot{g}(A) = R$.*

Proof. Letting $B \in \mathbb{B}$ be a blade such that $\hat{g}(B) = R$, simply let $A = BI$, and our lemma goes through by Lemma 3.4. □

Our next lemma shows that there is no overlap between the geometric sets dually and directly represented by a blade.

Lemma 3.6. *For all invertible blades $A \in \mathbb{B}$, we have*

$$\hat{g}(A) \cap \dot{g}(A) = \emptyset. \tag{3.5}$$

Proof. Supposing $x \in \hat{g}(A) \cap \dot{g}(A)$, we see that

$$0 = p(x) \cdot A + p(x) \wedge A = p(x)A, \tag{3.6}$$

but $p(x)$ is non-zero and A is invertible and therefore non-zero. We therefore reach a contradiction by Lemma 2.4. □

Lemma 3.7 (The Point-Fitting Lemma). *If $A \in \mathbb{B}$ is an irreducible r-blade with $\hat{g}(A) \neq \emptyset$, then there exists a set of r points $\{x_i\}_{i=1}^r \subset \mathbb{R}^n$ and a scalar $\beta \in \mathbb{R}$ such that*

$$A = \beta \bigwedge_{i=1}^{r} p(x_i). \tag{3.7}$$

Proof. Let t be the largest integer for which there exists a set of t points $\{x_i\}_{i=1}^t \subset \hat{g}(A)$ such that $\bigwedge_{i=1}^t p(x_i) \neq 0$. Clearly $t \geq 1$, because $\hat{g}(A)$ is non-empty; and clearly $t \leq r$ because of the requirement that $\{p(x_i)\}_{i=1}^t$ be a linearly independent set with each $p(x_i) \wedge A = 0$. Now if $t = r$, we're done by Lemma 2.1. Therefore, supposing $t < r$, there must exist, by Lemma 2.2, a factorization of A of the form $A = B \wedge C$, where B is a blade of grade $r - t$, and C is a t-blade given by

$$C = \bigwedge_{i=1}^{t} p(x_i), \tag{3.8}$$

Now realize that $\hat{g}(A) \subseteq \hat{g}(C)$ or else t is not the largest of its kind[2], and that $\hat{g}(C) \subseteq \hat{g}(A)$, because C is a subspace of A. It now follows that $\hat{g}(A) = \hat{g}(C)$, which contradicts the irreducibility of the blade A. □

Notice that in the proof of Lemma 3.7 that $\hat{g}(B) = \emptyset$. It is important to realize, however, that although there is no point $x \in \hat{g}(A)$ such that $x \in \hat{g}(B)$, this does not imply that $x \in \hat{g}(C)$. To see why, consider equation (3.10) below.[3]

Interestingly, Lemma 3.7 may be analagous to the fact in algebraic geometry that every algebraic set is generated by a finite set of polynomials. Here, we say, by Lemma 3.7, that every geometric set is generated by a finite subset of its points.

It was noted earlier that if $A, B \in \mathbb{B}$ are blades such that for a scalar $\beta \in \mathbb{R}$, we have $A = \beta B$, then $\hat{g}(A) = \hat{g}(B)$. The converse of this statement, however, is not generally true, but leads us to an important and fundamental theorem.

Theorem 3.8 (The Fundamental Theorem Of Geometric Set Representation). *For every geometric set $R \subset \mathbb{R}^n$, there exists, up to scale, a unique, irreducible blade $A \in \mathbb{B}$ such that $\hat{g}(A) = R$. Moreover, this blade A is a subspace of every blade B directly representative of R.*

Proof. By the proof of Lemma 3.7, it is not hard to see that, given any blade $B \in \mathbb{B}$ directly representative of the geometric set R, a subspace A of B can be found that is also representative of R while having the property of being irreducible.

Suppose now that A and A' are two independently found blades directly representative of R and each irreducible. It can be easily understood that $\text{grade}(A) = \text{grade}(A')$ by Defintion 3.3. By Lemma 3.7, the r-blade A has a factorization of the form $\alpha \bigwedge_{i=1}^r p(x_i)$. But now for each x_i, we have $p(x_i) \wedge$

[2]Suppose there exists $x \in \hat{g}(A)$ with $x \notin \hat{g}(C)$. Then $p(x) \wedge C \neq 0$ and we have found $t+1$ points, (namely, those in $\{x\} \cup \{x_i\}_{i=1}^t$), for which the set of vectors $\{p(x)\} \cup \{p(x_i)\}_{i=1}^t$ is a linearly independent set.

[3]Put another way, realize that while $(e_1 + e_2) \wedge e_1 \wedge e_2 = 0$, the vector $e_1 + e_2$ is not in the subspace spanned by e_1, nor that of e_2.

$A' = 0$; so by Lemma 2.1, there exists a scalar $\alpha' \in \mathbb{R}$ such that $A' = \alpha' \bigwedge_{i=1}^{r} p(x_i)$. It is clear now that

$$A = \frac{\alpha}{\alpha'} A'. \tag{3.9}$$

□

The importance of Theorem 3.8 can be realized in the utility of the conformal model. (See [] for information about the conformal model.) By Theorem 3.8, we can be justified in algebraically relating two independently made formulations of a given geometric set. For example, we may equate the intersection of two spheres as some scalar multiple of the canonical intersection of a sphere centered on a plane. The former formulation is what we may wish to calculate, while the latter formulation lends itself to analysis through decomposition. The applicability of Theorem 3.8 comes in realizing that each formulation is irreducible. Reducible blades directly representative of the same geometric set may sometimes be equated as scalar multiples of one another, but this is not always the case.

The counter-part of Theorem 3.8 in algebraic geometry is Hilbert's Nullstellensatz, (see []), which implies that for every algebraic set, there exists a unique radical ideal representative of that set.

3.2.2. Intersections/Unions. Speaking of algebraic sets, knowing that the intersection or union of any two such sets is also algebraic, one must wonder if there is an analogous result with regards to geometric sets. It is easy to find a counter-example in the conformal model showing that the union of two geometric sets need not be geometric. Proving or disproving that the intersection of any two geometric sets is geometric, however, is not obvious. In any case, the following lemma gives us some insight into the union and intersection of geometric sets.

Lemma 3.9. *For blades $B, C \in \mathbb{B}$, if a non-zero blade $A \in \mathbb{B}$ has a factorization $A = B \wedge C$, then*

$$\hat{g}(A) \supseteq \hat{g}(B) \cup \hat{g}(C), \tag{3.10}$$
$$\dot{g}(A) = \dot{g}(B) \cap \dot{g}(C). \tag{3.11}$$

Proof. The first of these two equations (equation (3.10)) being an obvious statement, we need only show here the validity of equation (3.11).

If $\bigwedge_{i=1}^{s} b_i$ is a factorization of the s-blade B, and $\bigwedge_{i=1}^{t} c_i$ is a factorization of the t-blade C, we have, by the identity of equation (2.7) and Lemma 2.3,

$$\dot{g}(A) = \bigcap_{i=1}^{s} \dot{g}(b_i) \cap \bigcap_{i=1}^{t} \dot{g}(c_i) = \dot{g}(B) \cap \dot{g}(C). \tag{3.12}$$

□

In light of Lemma 3.9, we can begin to understand the difficulties found in determining whether a given intersection of geometric sets is geometric.

Letting $S, T \subset \mathbb{R}^n$ be a pair of geometric sets, let $B, C \in \mathbb{B}$ be a pair of blades dually representative of them, respectively. Then, if $A = B \wedge C \neq 0$, we have found, by Lemma 3.9, a blade A dually representative of the intersection $\dot{g}(B) \cap \dot{g}(C)$, showing that it is a geometric set. If, on the other hand, $B \wedge C = 0$, we can come to no such conclusion.

3.2.3. Transformations. In this section we will take an interest in the set of all versors of $\mathbb{G}$ that preserve the form of our function $p : \mathbb{R}^n \to \mathbb{V}$.

Definition 3.10. A versor $V \in \mathbb{G}$ with the property that for all $x \in \mathbb{R}^n$, there exists a unique $y \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ such that

$$V^{-1} p(x) V = \gamma p(y), \tag{3.13}$$

is what we will refer to as a *preservative versor.*

The significance of such versors is that the question of how they transform a geometric set from $\hat{g}(A)$ to $\hat{g}(VAV^{-1})$ is reduced to the question of how the versor maps points from $\mathbb{R}^n$ to $\mathbb{R}^n$ through our function $p : \mathbb{R}^n \to \mathbb{V}$ as mapping a point $x \in \mathbb{R}^n$ to the point $y \in \mathbb{R}^n$ satisfying equation (3.13).

The following lemma shows us that in considering the transformation of a geometric set by a preservative versor, we need only look at the irreducible blades directly representative of that geometric set.

Lemma 3.11. *Let $V \in \mathbb{G}$ be a preservative versor. Then for every blade $A \in \mathbb{B}$, if $C \in \mathbb{B}$ is an irreducible blade such that $\hat{g}(A) = \hat{g}(C)$, then*

$$\hat{g}(VAV^{-1}) = \hat{g}(VCV^{-1}). \tag{3.14}$$

Proof. Letting $A = B \wedge C$ by Lemma 2.2, consider the equation

$$VAV^{-1} = VBV^{-1} \wedge VCV^{-1}. \tag{3.15}$$

What we must show now is that VCV^{-1} is, in terms of dimension, the largest irreducible subspace of VAV^{-1}. Supposing for the moment that it isn't, this would allow us, by the preservative property of V, to easily find an irreducible subspace of A that is larger than that of C, which contradicts the fact that C is the largest irreducible subspace of A. □

Lemma 3.12. *Let $V \in \mathbb{G}$ be a preservative versor and $A \in \mathbb{B}$. Then if $\hat{g}(A) = \emptyset$, we have*

$$\hat{g}(VAV^{-1}) = \emptyset. \tag{3.16}$$

Proof. Showing the contrapositive of our lemma, let $x \in \hat{g}(VAV^{-1})$. Then, by Lemma 2.2, there exists a blade $B \in \mathbb{B}$ such that $VAV^{-1} = B \wedge p(x)$. We then see that for some $y \in \mathbb{R}^n$ and a scalar $\gamma \in \mathbb{R}$, we have

$$A = V^{-1}(B \wedge p(x))V = V^{-1}BV \wedge V^{-1}p(x)V = V^{-1}BV \wedge \gamma p(y). \tag{3.17}$$

We now see that $y \in \hat{g}(A)$, showing that $\hat{g}(A)$ is non-empty. □

3.3. Example

3.4. Reduction

4. Closing

References

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