# An Interesting Question About The Rationals $\mathbb{Q}$

Spencer T. Parkin

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## The Question

Let $\mathbb{Q}$ denote the set of rational numbers and $\mathbb{Z}$ the integers. For any set $S$ of rationals numbers, let the span of $S$ be defined as the set

$$\text{span } S = \{z_1 q_1 + z_2 q_2 + \ldots | z_i \in \mathbb{Z}\},$$

where $S = \{q_1, q_2, \ldots\}$ is any enumeration of $S$, a finite or countably infinite set of rational numbers. If the cardinality of $S$ is greater than 1, say that such a set $S$ is linearly independent if and only if for all integers $k > 1$, $q_k \notin \text{span } \{q_1, q_2, \ldots, q_{k-1}\}$.

Now here's the question: is $\mathbb{Q}$ the span of every or any linearly independent and countably infinite set of rational numbers?

## Motivation For The Question

This question came about while trying to show that no proper subgroup of $\mathbb{Q}$ under addition is maximal. Given any proper subgroup $H$ of $\mathbb{Q}$, it is easy to find a subgroup $K$ of $\mathbb{Q}$ containing $H$ properly, but it is not at all obvious that $K$ is a proper subgroup of $\mathbb{Q}$.

Letting $H$ be a proper subgroup of $\mathbb{Q}$, if $q \in \mathbb{Q} - H$, it is easy to show that $H + Z(q)$ properly contains $H$ and is a subgroup of $\mathbb{Q}$, where $Z(q) = \{zq | z \in \mathbb{Z}\}$ and $H + Z(q) = \{h + q' | h \in H \text{ and } q' \in Z(q)\}$. Now notice that $q/2 \notin H$ and $q/2 \notin Z(q)$. But it is not at all clear whether $q/2 \in H + Z(q)$. Suppose it is. Then there exists $h \in H$ and $z \in \mathbb{Z}$ such that $q/2 = h + zq$.

Rearranging, we find that $2h = (1 - 2z)q$. Since $1 - 2z$ is never zero, this would imply that the intersection of $H$ and $Z(q)$ is a non-trivial group.

Taking a step back for a moment, realize that if $q \in H$, then $Z(q)$ is a subgroup of $H$. Now suppose that $q \notin H$, but that there exists $q' \in H \cap Z(q)$ where $q' \neq 0$. Then $Z(q')$ is a subgroup of $H \cap Z(q)$ and it follows that some multiple of $q$, (namely, $q' = zq$ for some $|z| > 1$), is in $H$.

Taking a step back again, can we always find $q \in \mathbb{Q} - H$ such that $Z(q) \cap H$ is empty? I have no idea, but if we could always find such a $q$, then a proof would go through.