Section 3.5 Exercises
Herstein's Topics In Algebra

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Thoughts

If R is a commutative ring and $a \in R$, then I think it's fair to define the ideal of R generated by a as

$$I = \{ra | r \in R\},$$

and write $I = \langle a \rangle$. Clearly I is non-empty. Let $x, y \in I$. Then $x = r_x a$ and $y = r_y a$, and we have $x + y = (r_x + r_y)a \in I$. Also, $-x = (-r_x)a \in I$. So I is a subgroup of R. We also have $xy = (r_x r_y a)a \in I$, so it's a subring of R. And lastly, for any $r \in R$, we have $xr = rx = (rr_x)a \in I$, so it's an ideal of R.

It should also be remarked that $\langle a \rangle$ is the smallest possible ideal of R containing a. If we knew I was an ideal of R containing a, then it must also contain all elements of the form ra. After throwing those into I, this is the soonest we form a set that is an ideal of R.

What now may be of interest is to consider any ideal I of R, choose $a \in I$, and consider the relationship

$$\langle a \rangle \subseteq I \subseteq R.$$

Notice that $\langle a \rangle$ is not only an ideal of R, but also of I. It may also be of interest to consider the case that $\langle a \rangle \neq I$. In that case, choose $b \in I - \langle a \rangle$, and see that

$$\langle a \rangle \cup \langle b \rangle \subseteq I.$$

Now let's suppose that $0 \neq x \in \langle a \rangle \cap \langle b \rangle$. Then $x = ua = vb$ for some $u, v \in R$. If we were working in a division ring, then $b = v^{-1}ua \implies b \in \langle a \rangle$, which would be a contradiction. So in a division ring, $\langle a \rangle \cap \langle b \rangle = \{0\}$.

Problem 1

Mimic proof of Lemma 3.5.1.

Problem 2

If R has a unit element, it's a division ring by Problem 1. If it doesn't, then we have to show the other conclusion. We might consider an equivalence relation and a counting principle.

Problem 3

Let J be the ring of integers, p a prime number, and $\langle p \rangle$ the ideal of J consisting of all multiples of p.

Part A

Prove that $J/\langle a \rangle$ is isomorphic to J_p, the ring of integers mod p.

Let $\phi(x) = x + \langle p \rangle$ where $\phi : J_p \to J/\langle p \rangle$. (Notice that $\phi(x + \langle p \rangle) = x$ is not well defined.) Since $\langle p \rangle$ is an ideal of J, ϕ is a homomorphism. And since $\ker \phi = \{0\}$, it's an isomorphism.

Part B

Using Theorem 3.5.1 and part A of this problem, show that J_p is a field.

By Example 3.5.1, $\langle p \rangle$ is maximal in J. Then by Theorem 3.5.1, $J/\langle p \rangle$ is a field. But since $J/\langle p \rangle \approx J_p$, J_p must be a field also.

Problem 4

Let R be the ring of all real-valued continuous functions on the closed unit interval. If M is a maximal ideal of R, prove that there exists a real number γ, $0 \le \gamma \le 1$, such that $M = M_\gamma = \{f(x) \in R | f(\gamma) = 0\}$.

Let $f \in R$ be a non-zero-valued continuous function on all of $[0, 1]$, and suppose I is an ideal of R containing it. Now letting $g \in R$ be any member of R, does there exist a function $h \in R$ such that $fh = g$? Clearly there must, since f, being non-zero on $[0,1]$, allows us to write $h = g/f$. We can now conclude that $I = R$, and that for every properly contained ideal I of R, if $f \in I$, then there exists $\gamma \in [0,1]$ such that $f(\gamma) = 0$.

Now let $f \in R$ be a function with *exactly one* zero $\gamma \in [0,1]$, and consider the ideal $\langle f \rangle$. Notice that all $g \in \langle f \rangle$ have this same zero, even if possibly others. It is not clear, however, whether $\langle f \rangle$ contains all functions of R having this zero. Persuing this, we let $g \in R$ be any such function, and ask: can we find $h \in R$ such that $fh = g$? Consider

$$h(x) = \begin{cases} g(x)/h(x) & x \neq \gamma, \\ 0 & x = \gamma. \end{cases}$$

The problem here is that h need not be continuous at γ. That is, we need not have $\lim_{x \to \gamma} h(x) = 0$. The limit may, in fact, not even exist!

Leaving this line of thinking for a moment, can there exist a proper ideal I of R with the property that there does not exist $x \in [0,1]$ such that for all $f \in I$, we have $f(x) = 0$?[1] Let's suppose for the moment that no such ideal can exist. In that case, we can claim that for every proper ideal I of R, we must have $V(I)$ non-empty, where this is defined as

$$V(I) = \{x \in [0,1] | f(x) = 0 \text{ for all } f \in I\}.$$

But then we can also establish the relationship that for any two ideals $I, J \subset R$, if $V(I) \subset V(J)$, then $I \supset J$. If $|V(I)| = 1$, then must we have $I = M_\gamma$ with $\gamma \in V(I)$? Can it be shown that if $|V(I)| > 1$, then I is not maximal? Note that if $V(I) > 1$, then I cannot contain every function of R having one of the zeros in $V(I)$, because then it must contain a function that is non-zero on all of $[0,1]$. (In such a case, $I = R$, and we have $|V(I)| = 0$, a contradiction.) For example, if $V(I) = \{\alpha, \beta\}$, we can construct $f : [0,1] \to \mathbb{R}$ that is non-zero on all of $[0,1]$ as the sum of two continuous functions h and g, each having exactly one zero: α and β, respectively. Therefore, h and g cannot co-exist in I. So if $|V(I)| > 1$, we know that I is properly contained in M_γ, where $\gamma \in V(I)$; so I is not maximal in R.

[1] I suspect that in such an ideal we would be able to construct a function that is non-zero on all of $[0,1]$, which would lead us to contradict the fact that it's a proper ideal.