The Mother Minkowski Algebra of Order m

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To my dear wife Melinda.

Abstract. It is found that all polynomials of up to degree m have an encoding as m-vectors in a geometric algebra referred to as the Mother Minkowski algebra of order m. It is then shown that all conformal transformations may be applied to these m-vectors, the results of which, when converted back into polynomial form, give us the transformed surfaces in terms of the zero sets of the original and final polynomials.

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1. Motivation

Before presenting the Mother Minkowski algebra of order m, we lead up to it here with some background and motivation.[1] We begin by recalling that an algebraic set is any subset of an n-dimensional euclidean space $\mathbb{R}^n$ that is also the zero set of one or more polynomials, each in n independent variables. Given a geometric algebra $\mathbb{G}$, we can represent such sets using blades $B \in \mathbb{G}$ as the set of all points $x \in \mathbb{R}^n$ such that

$$p(x) \cdot B = 0,$$

where $p : \mathbb{R}^n \to \mathbb{V}$ maps points in $\mathbb{R}^n$ to a vector space $\mathbb{V}$ generating our geometric algebra $\mathbb{G}$. Though not necessary, $\mathbb{R}^n$ is often embedded in $\mathbb{V}$; but regardless of this, the function p is necessarily defined in such a way that the expression $p(x) \cdot B$ is a polynomial in the vector components of x when $B \in \mathbb{V}$.

[1] In all equations to follow, we let the outer product take precedence over the inner product, and the geometric product take precedence over the inner and outer products. The inner product used in this paper is the Hestenes inner product. All polynomials and geometric algebras are assumed to be defined over the field $\mathbb{R}$ of real numbers.

Letting $\mathbb{B}$ denote the set of all blades found in $\mathbb{G}$, and letting $P(\mathbb{R}^n)$ denote the power set of $\mathbb{R}^n$, we will find it useful to define the mapping $\dot{g} : \mathbb{B} \to P(\mathbb{R}^n)$ as

$$\dot{g}(B) = \{x \in \mathbb{R}^n | p(x) \cdot B = 0\}. \tag{1.1}$$

To see that $\dot{g}(B)$ is an algebraic set, we first observe that when $B \in \mathbb{V}$, $\dot{g}(B)$ is the zero set of a polynomial in the vector components of x. Secondly, we observe that if $\bigwedge_{i=1}^k b_i$ is a factorization of the k-blade B, each b_i being in $\mathbb{V}$, then

$$p(x) \cdot B = -\sum_{i=1}^k (-1)^i (p(x) \cdot b_i) B_i, \tag{1.2}$$

where each B_i is given by

$$B_i = \bigwedge_{\substack{j=1 \\ j \neq i}}^k b_j,$$

and therefore, since $\{B_i\}_{i=1}^k$ is a linearly independent set, we have

$$\dot{g}(B) = \bigcap_{i=1}^k \dot{g}(b_i).$$

This method of representing algebraic sets using blades of a geometric algebra presents some interesting properties. To begin, if $A, B \in \mathbb{B}$ are blades with $A \wedge B \neq 0$, then

$$\dot{g}(A) \cap \dot{g}(B) = \dot{g}(A \wedge B).$$

In this way, the outer product serves to take the intersection of two surfaces. But we can also look at the outer product in a different light as an operator that takes at least the union of its two given surfaces. To see this, we must consider an alternative interpretation of blades $B \in \mathbb{B}$ as being representative algebraic sets. Defining $\hat{g} : \mathbb{B} \to P(\mathbb{R}^n)$ as

$$\hat{g}(B) = \{x \in \mathbb{R}^n | p(x) \wedge B = 0\}, \tag{1.3}$$

we see that $\hat{g}(B) = \dot{g}(BI)$, where I is the unit pseudo-scalar of $\mathbb{G}$, showing that the image of $\hat{g}$, like $\dot{g}$, consists of algebraic sets. Under this new interpretation, we find that for blades $A, B \in \mathbb{B}$, we have

$$\hat{g}(A) \cup \hat{g}(B) \subseteq \hat{g}(A \wedge B).$$

Exactly what surface we get from $A \wedge B$ in terms of $\hat{g}$ can be deduced by considering the surface $(A \wedge B)I$ in terms of $\dot{g}$.

It is often useful to alternate between the interpretations provided by $\dot{g}$ and $\hat{g}$. For example, while the intersection of two surfaces having no real intersection may be perceived as an imaginary surface, it is equally useful, if not more so, to switch from the $\dot{g}$ interpretation to that of $\hat{g}$, the result of doing so being a real surface having geometric significance to the situation at hand.

What's further a benefit of using blades to represent surfaces is that of the many transformations applicable to such geometries through the use of outermorphisms; in particular, outermorphisms $f : \mathbb{B} \to \mathbb{B}$ of the form

$$f(B) = VBV^{-1},$$

where V is a versor of $\mathbb{G}$. Given such a function, we wish to compare $\dot{g}(B)$ with $\dot{g}(f(B))$. Interestingly, to understand the latter in terms of the former, we need only understand the mapping $h : \mathbb{R}^n \to \mathbb{R}^n$, if any, induced by V through p as being each point $x \in \mathbb{R}^n$ mapped to a point $y \in \mathbb{R}^n$ satisfying the condition

$$V^{-1}p(x)V = \lambda p(y), \tag{1.4}$$

λ being some non-zero scalar in $\mathbb{R}$.[2] This is, of course, only a well defined mapping, provided that for every point $x \in \mathbb{R}^n$, there exists such a point $y \in \mathbb{R}^n$, and that it is unique. Assuming that V and p collectively meet these requirements, and so do indeed induce such a mapping $h : \mathbb{R}^n \to \mathbb{R}^n$, we can show that

$$\dot{g}(f(B)) = h^{-1}(\dot{g}(B)).$$

Notice that by the symmetry of equation (1.4), any argument that can be used to show that h is a well defined mapping can also be used to show that h^{-1} exists. We now need only show that

$$\dot{g}(VBV^{-1}) = \{x \in \mathbb{R}^n | V^{-1}p(x)V \cdot B = 0\}.$$

To this end, we begin by factoring the k-blade B as $\bigwedge_{i=1}^k b_i$. Then, by substituting $V^{-1}p(x)V$ for $p(x)$ in equation (1.2), we see that

$$V^{-1}p(x)V \cdot B = 0$$

if and only if for all integers $i \in [1, k]$, we have

$$0 = V^{-1}p(x)V \cdot b_i = p(x) \cdot Vb_iV^{-1},$$

since the set of k $(k-1)$-blades $\{B_i\}_{i=1}^k$ is a linearly independent set. Then, by applying equation (1.2) again to obtain

$$p(x) \cdot VBV^{-1} = -\sum_{i=1}^k (-1)^i (p(x) \cdot Vb_iV^{-1})VB_iV^{-1},$$

we see that for all integers $i \in [1, k]$, we have $p(x) \cdot Vb_iV^{-1} = 0$ if and only if $p(x) \cdot VBV^{-1} = 0$, because the set $\{VB_iV^{-1}\}_{i=1}^k$ is also linearly independent, which linear independence follows from that of the set $\{B_i\}_{i=1}^k$. It follows that $V^{-1}p(x)V \cdot B = 0$ if and only if $p(x) \cdot VBV^{-1} = 0$, which is what we wanted to show.

[2]If no such mapping exists, then some other method, (other than the one about to be shown), must be used to determine the action of V on B that takes us from $\dot{g}(B)$ to $\dot{g}(f(B))$.

2. The Mother Minkowski Algebra of Order m

Up to this point, we have kept the definition of the function p, and the signature of the geometric algebra over which it is defined, open to speculation, because the set of all possibilities for p, in terms of the types of geometry we can consequently do, remains an open question. What might be the most interesting and significant definition of p thus far proposed is found in [3] and given by

$$p(x) = o + x + \frac{1}{2}x^2\infty. \tag{2.1}$$

Here, the vector space $\mathbb{V}$ is generated by the set of basis vectors $\{o, \infty\} \cup \{e_i\}_{i=1}^n$, where the set of n euclidean vectors $\{e_i\}_{i=1}^n$ span $\mathbb{R}^n$ as an orthonormal basis for that space, and the vectors o and ∞ are the null vectors representing the points at origin and infinity, respectively. Being null, we have $o \cdot o = \infty \cdot \infty = 0$. These basis vectors share the peculiar relationship $\infty \cdot o = o \cdot \infty = -1$. The geometric algebra generated by $\mathbb{V}$ is called a Minkowski algebra, and the resulting model of geometry imposed upon this algebra by p using functions (1.1) and (1.3) is known as the conformal model of geometric algebra. It has been shown in [3, 4, 2] that the versors of $\mathbb{G}$ generated by $\mathbb{V}$ induce the set of all conformal transformations through p. The induced mappings are well defined and invertible.

If, for the moment, we were to throw ∞ into $\mathbb{R}^n$, and let $p(\infty) = \infty$, it is not difficult to show that for any given versor $V \in \mathbb{G}$, that (2.1) satisfies the condition of equation (1.4). We first observe that the image of p over $\mathbb{R}^n$, in addition to all non-zero scalar multiples of p, gives us the set of all null-vectors in $\mathbb{V}$. That is, the set of all null vectors in $\mathbb{V}$ is given by

$$\{\lambda p(y) | \lambda \in \mathbb{R} - \{0\} \text{ and } y \in \mathbb{R}^n\}.$$

We then, without loss of generality, need only show that for a versor that is a single vector v, (which, by definition, must be invertible and therefore non-null), that the vector $v^{-1}p(x)v$ is null. Taking the inner product square of $v^{-1}p(x)v$, we get $vp^2(x)v^{-1} = 0$, (since $p(x)$ is null), showing that it is indeed null, and therefore of the form $\lambda p(y)$. To prove uniqueness, we need only show that for non-zero scalars $\lambda_1, \lambda_2 \in \mathbb{R}$ and points $x_1, x_2 \in \mathbb{R}^n$, if $\lambda_1 p(x_1) = \lambda_2 p(x_2)$, then $x_1 = x_2$. A proof of this being trivial, it is left as an exercise for the reader.

Building upon the ideas presented in [1], (specifically, those surrounding the idea of what is referred to in [1] as "the mother algebra"), we will now consider a new model of geometry based upon a geometric algebra $\mathbb{G}$ generated by a vector space $\mathbb{V}$ described in set builder notation as

$$\mathbb{V} = \left\{ \sum_{i=1}^{m} v_i \,\middle|\, v_i \in \mathbb{V}_i \right\},$$

where for each vector space $\mathbb{V}_i$, the geometric algebra $\mathbb{G}_i$ generated by $\mathbb{V}_i$ is a Minkowski algebra. For all $i \neq j$, if $a \in \mathbb{V}_i$ and $b \in \mathbb{V}_j$, we let $a \cdot b = 0$. We will refer to the geometric algebra $\mathbb{G}$ generated by this vector space $\mathbb{V}$ as the Mother Minkowski algebra of order m.

Letting $\mathbb{B}$ denote the set of all blades taken from $\mathbb{G}$, we now define the function $\dot{G} : \mathbb{B} \to P(\mathbb{R}^n)$ as

$$\dot{G}(B) = \left\{ x \in \mathbb{R}^n \,\middle|\, \bigwedge_{i=1}^{m} p_i(x) \cdot B = 0 \right\}, \tag{2.2}$$

where we define $p_i : \mathbb{R}^n \to \mathbb{V}_i$ as

$$p_i(x) = o_i + x_i + \frac{1}{2} x_i^2 \infty_i, \tag{2.3}$$

where $o_i, \infty_i \in \mathbb{V}_i$ are the familiar null vectors in the i^{th} Minkowski sub-algebra, and where x_i denotes the embedding of x in the n-dimensional euclidean sub-space of $\mathbb{V}_i$. If more precision is needed here, we can let $\mathbb{B}_i$ denote the set of all blades generated by $\mathbb{V}_i$, let $\mathbb{R}^n_i$ denote the n-dimensional euclidean sub-space of $\mathbb{V}_i$, and then work exclusively in $\mathbb{R}^n_1$ by defining an outermorphism that takes any blade in $\mathbb{B}_1$ to its corresponding blade in $\mathbb{B}_i$. The function p_i can then be defined in terms of this outermorphism. Letting $f_i : \mathbb{B}_1 \to \mathbb{B}_i$ be this outermorphism, we would, (working exclusively in $\mathbb{R}^n_1$), redefine $p_i : \mathbb{R}^n_1 \to \mathbb{V}_i$ as

$$\begin{aligned} p_i(x) &= o_i + \frac{1}{2} x^2 \infty_i + \begin{cases} f_i(x) & \text{if } i \neq 1, \\ x & \text{if } i = 1, \end{cases} \\ &= \begin{cases} f_i(p_1(x)) & \text{if } i \neq 1, \\ p_1(x) & \text{if } i = 1, \end{cases} \end{aligned}$$

with the idea that $x = x_1$ using our earlier definition (2.3).

Interestingly, an explicit formula for this outermorphism f_i can be found and carried through all of the equations we'll present in the remainder of this paper, but there is no need to formally introduce it, because the equations still go through in its absence; nor would we want to, because it would just make the math less readable.[3] We will therefore proceed now by working in a space $\mathbb{R}^n$ that is not embedded in $\mathbb{V}_i$ for any integer $i \in [1, m]$, trusting that the reader can map $x \in \mathbb{R}^n$ to its counter-part x_i in each $\mathbb{R}^n_i$.

In this paper we are going to limit our attention to those blades $B \in \mathbb{B}$ having factorizations involving a representative from each $\mathbb{B}_i$, which is to say that for all integers $i \in [1, m]$, there exists a non-zero vector $v \in \mathbb{V}_i$ such that $v \wedge B = 0$. Doing so, we write the blade B as

$$B = \bigwedge_{i=1}^{m} B_i, \tag{2.4}$$

where each blade B_i is in $\mathbb{B}_i$, and then see that

$$\bigwedge_{i=1}^{m} p_i(x) \cdot B = (-1)^k \bigwedge_{i=1}^{m} p_i(x) \cdot B_i, \tag{2.5}$$

[3]Nevertheless, if the reader is still interested, this outermorphism is given in the appendix.

where the integer k is given by

$$k = \sum_{i=1}^{m} \sum_{j=1}^{m-i} \text{grade}(B_j), \tag{2.6}$$

letting the nested sum be zero in the case that $i = m$. Then, subscripting equation (1.1) as

$$\dot{g}_i(B_i) = \{x \in \mathbb{R}^n | p_i(x) \cdot B_i = 0\},$$

what we now find is that by equation (2.5), we have

$$\dot{G}(B) = \bigcup_{i=1}^{m} \dot{g}_i(B_i).$$

This shows that we can represent any union of up to m surfaces taken from the conformal model, (let $B_i = \infty_i$ to fill any remaining and unused blade factors), but if we extend our function $\dot{G}$ to the set of all m-vectors, we can do even better. To see why, we need only show that any monomial in up to n variables and at most degree m can be represented by the expression on the right-hand side, (and therefore the left-hand side), of equation (2.5). The n variables are taken from the components of the point $x \in \mathbb{R}^n$. Letting each B_i be a vector in $\mathbb{V}_i$, the expression becomes

$$(-1)^k \prod_{i=1}^{m} p_i(x) \cdot B_i, \tag{2.7}$$

with the integer k becoming

$$k = \frac{m(m-1)}{2}, \tag{2.8}$$

the $(m-1)^{th}$ triangle number, since for all integers $j \in [1, m]$, we have $\text{grade}(B_j) = 1$ in equation (2.6). It is clear now that for an appropriate choice of each vector B_i, we can formulate any monomial in the components of x using equation (2.7). In every such choice, notice that we may let $B_i \cdot \infty_i = 0$. Letting B be a general m-vector, (which is not necessarily an m-blade), we see now that the expression that is the left-hand side of equation (2.5) represents any polynomial of at most degree m in the vector components of x, provided we have $B_i \cdot \infty_i = 0$ for all vector factors of any blade in B. Of course, if this is not the case, what we get is a polynomial of at most degree $2m$ by the squaring that occurs in equation (2.3), but we cannot represent all polynomials of up to this degree.

As it turns out, the set of all polynomials of up to degree m is closed under the set of all planar reflections, and therefore the set of all rigid body motions. This set of polynomials, however, is not closed under the set of all spherical inversions, and therefore the set of all transversions and dilations. This is where some of the polynomials of degrees between m and $2m$ come into play.

At this point an example may be in order to make things clearer. Consider the Caylay cubic polynomial.

$$-5(x^2y + x^2z + y^2x + y^2z + z^2x + z^2y) + 2(xy + xz + yz)$$

Realizing that the symbol x in equation (2.3) is reserved in denoting a point in $\mathbb{R}^n$, another, although admittedly cumbersome way to express this polynomial, is as follows.

$$\begin{aligned} -5(x\cdot e_1)^2(x\cdot e_2) - 5(x\cdot e_1)^2(x\cdot e_3) - 5(x\cdot e_2)^2(x\cdot e_1) \\ -5(x\cdot e_2)^2(x\cdot e_3) - 5(x\cdot e_3)^2(x\cdot e_1) - 5(x\cdot e_3)^2(x\cdot e_2) \\ +2(x\cdot e_1)(x\cdot e_2) + 2(x\cdot e_1)(x\cdot e_3) + 2(x\cdot e_2)(x\cdot e_3) \end{aligned}$$

Here, we're letting $\{e_l\}_{l=1}^3$ be an orthonormal basis for $\mathbb{R}^3$. Now let $\{o_i, \infty_i\} \cup \{e_{l,i}\}_{l=1}^3$ be a basis for each $\mathbb{V}_i$ with a copy of $\mathbb{R}^3$ embedded in each of these. The Caylay cubic polynomial can now be expressed by equation (2.5), if we let the trivector B be given by

$$\begin{aligned} B = {} & 5e_{1,1}\wedge e_{1,2}\wedge e_{2,3} + 5e_{1,1}\wedge e_{1,2}\wedge e_{3,3} + 5e_{2,1}\wedge e_{2,2}\wedge e_{1,3} \\ & + 5e_{2,1}\wedge e_{2,2}\wedge e_{3,3} + 5e_{3,1}\wedge e_{3,2}\wedge e_{1,3} + 5e_{3,1}\wedge e_{3,2}\wedge e_{2,3} \\ & + 2e_{1,1}\wedge e_{2,2}\wedge \infty_3 + 2e_{1,1}\wedge e_{3,2}\wedge \infty_3 + 2e_{2,1}\wedge e_{3,2}\wedge \infty_3, \end{aligned}$$

with the understanding that x_i is given by

$$x_i = (x\cdot e_1)e_{1,i} + (x\cdot e_2)e_{2,i} + (x\cdot e_3)e_{3,i}.$$

Here we're using a Mother Minkowski algebra of order 3. Notice that in such higher order algebras, and even this algebra itself, the encoding of the Caylay cubic polynomial is certainly not unique. Another way to express this fact is to say that the function $\bigwedge_{i=1}^m p_i(x)$ does not uniquely factor out of any given polynomial of at most degree m in terms of the inner product. Uniqueness of representation, up to scale, is an important feature of the conformal model, (for the purpose of composition and decomposition), that is, unfortunately, not preserved here.

Converting the trivector B above back into the Caylay cubic polynomial is simply a matter of expanding the expression $\bigwedge_{i=1}^3 p_i(x)\cdot B$. When an m-degree polynomial surface equation is expressed in vector form, its conversion to an m-vector can sometimes be given without reference to basis.

3. Conformal Transformations

While this new model certainly expands upon the set of all possible surfaces that may be represented by the conformal model, not all of the nice properties discussed in the motivating section carry over very easily, if at all. What we will show in this paper, however, is that all of the conformal transformations are preserved in the new model. It may be worth comparing this method of applying such transformations to surfaces not native to the conformal model to those found in [8, 5].

Letting $\{V_i\}_{i=1}^m$ be a set of m versors, each V_i taken from the geometric algebra generated by $\mathbb{V}_i$, and each representing the same conformal transformation, what we simply need to show is that

$$\dot{G}(VBV^{-1}) = \left\{ x \in \mathbb{R}^n \,\middle|\, \bigwedge_{i=1}^m V_i^{-1} p_i(x) V_i \cdot B = 0 \right\}, \tag{3.1}$$

where the versor V is given by

$$V = \prod_{i=1}^m V_i.$$

It is clear that the right-hand side of equation (3.1) is the surface $\dot{G}(B)$ having undergone the transformation represented by each V_i. Our result will show that this is also the surface $\dot{G}(VBV^{-1})$.

To begin, we will first prove the validity of equation (3.1) for m-blades B of $\mathbb{G}$, and then after-word consider the more general case of B as being a general m-vector. We will therefore precede by factoring B as we have in equation (2.4) with each $B_i \in \mathbb{V}_i$. We then have

$$\bigwedge_{i=1}^m V_i^{-1} p_i(x) V_i \cdot B \tag{3.2}$$

$$= (-1)^k \prod_{i=1}^m V_i^{-1} p_i(x) V_i \cdot B_i$$

$$= (-1)^k \prod_{i=1}^m p_i(x) \cdot V_i B_i V_i^{-1}$$

$$= \bigwedge_{i=1}^m p_i(x) \cdot \bigwedge_{i=1}^m V_i B_i V_i^{-1} \tag{3.3}$$

$$= \bigwedge_{i=1}^m p_i(x) \cdot \bigwedge_{i=1}^m V B_i V^{-1} \tag{3.4}$$

$$= \bigwedge_{i=1}^m p_i(x) \cdot VBV^{-1}, \tag{3.5}$$

where here, the integer k is again given by equation (2.8). The step taking us from (3.3) to (3.4) deserves some explanation. Removing the subscript i from V_i is done by inserting each of the factors $V_j V_j^{-1} = 1$ with $j \neq i$ into the appropriate position, and then commuting the V_j^{-1} to the other side of B_i into its appropriate position. Finding the net change in sign is an exercise in combinatorics, and ends up being

$$(-1)^{m(m-1)} = 1.$$

Returning now to equation (3.1) in the case that B is a general m-vector, simply notice that by the linearity of the inner, outer and geometric products, the sequence of equalities equating (3.2) with (3.5) remains valid.

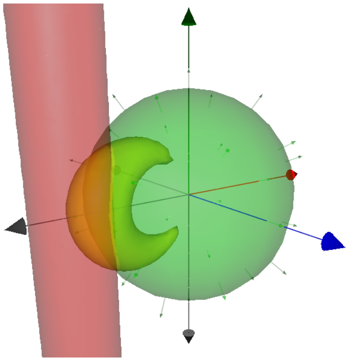

FIGURE 1. The inversion of a cylinder in a sphere. A mesh generation algorithm was used to skin the cylindrical and inverted surfaces. The inverted surface mesh suffers where the curvature of the object becomes extreme.

This result suggests the following process of transforming a polynomial f into another polynomial f' by a conformal transformation represented by a versor V.

$$\begin{tikzcd} B \arrow[r] & VBV^{-1} \arrow[d] \\ f \arrow[u] \arrow[r, dashed] & f' \end{tikzcd}$$

Here, the polynomial f is converted into an m-vector B, the versor V is applied to this m-vector as the m-vector VBV^{-1}, which is in turn converted back into the polynomial f'. The conversion process is straightforward and can be easily handled by a computer algebra system, as well as that of the computation of VBV^{-1}.

Using the vector-based equations for quadric surfaces found in [6], a piece of software[4] was written that implements the method given in this paper of applying conformal transformations to polynomials. This software was used to generate Figure 1. The input polynomial, whose zero set is the cylindrical surface, was given by

$$x^2 + z^2 + 14x + 45,$$

and the output polynomial, whose zero set is the inversion of the cylinder in the sphere, came out to be

$$\begin{gathered} 28.8x^2 + 11.2x^3 + x^4 + 11.2xy^2 + 2x^2y^2 + \\ 11.2xz^2 + 2x^2z^2 + y^4 + 2y^2z^2 + 28.8z^2 + z^4. \end{gathered}$$

[4]This software can be found at `https://github.com/spencerparkin/GAVisTool`.

The sphere was centered at origin and had a radius of 6. The input polynomial was not actually fed to the software, but its conversion to a bivector was, because it has an easy formulation when expressed in a vector-based form. These formulations are comparable to those of the conformal model for planes and spheres.

4. Appendix

As promised, the following function $f : \mathbb{G}_i \to \mathbb{G}_j$ is an outermorphism that can be used to transform any element in $\mathbb{G}_i$ to its associated element in $\mathbb{G}_j$, provided $i \neq j$. It is given by

$$f(E) = S_1 E S_1^{-1},$$

where S_k is the element given by

$$S_k = (1 - (-1)^k e_{-,i} e_{-,j})(1 + (-1)^k e_{+,i} e_{+,j}) \prod_{l=1}^{n} (1 + (-1)^k e_{l,i} e_{l,j}),$$

where $\{e_{l,i}\}_{l=1}^n$ and $\{e_{l,j}\}_{l=1}^n$ are the orthonormal bases for $\mathbb{R}_i^n$ and R_j^n, respectively, and where $e_{-,i}$ and $e_{+,i}$ are each given by

$$e_{-,i} = \frac{1}{2}(\infty_i + o_i),$$
$$e_{+,i} = \frac{1}{2}(\infty_i - o_i).$$

Take note that

$$S_1^{-1} = \frac{S_0}{2^{n+2}}.$$

Although S_k is not a versor, it can be shown that f is an outermorphism.

For a vector $v_i \in \mathbb{V}_i$, to say that it's associated vector $v_j \in \mathbb{V}_j$ is $f(v_i)$ is to say that for any integer $l \in [1, n]$, we have

$$e_{l,i} \cdot v_i = e_{l,j} \cdot v_j,$$

as well as

$$o_i \cdot v_i = o_j \cdot v_j,$$
$$\infty_i \cdot v_i = \infty_j \cdot v_j.$$

Notice that while $v_j = f(v_i)$, we have $v_i = -f(v_j)$.

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