Chapter 15 Exercises
Gallian's Book on Abstract Algebra

Spencer T. Parkin

March 5, 2014

Exercise 1

Prove Theorem 15.1.

For the theorem, we let ϕ be a ring homomorphism from a ring R to a ring S, and we let A be a subring of R and let B be an ideal of S.

The first part of the theorem states that for any $r \in R$ and any positive integer n, that $\phi(nr) = n\phi(r)$ and $\phi(r^n) = (\phi(r))^n$.

Proof: That $\phi(nr) = n\phi(r)$ follows from Property 2 of Theorem 10.1 for group homomorphisms. Similarly,

$$\phi(r^n) = \phi(\underbrace{r \cdot \cdots \cdot r}_{n}) = \underbrace{\phi(r) \cdot \cdots \cdot \phi(r)}_{n} = (\phi(r))^n.$$

The second part of the theorem states that $\phi(A) = \{\phi(a) | a \in A\}$ is a subgring of S.

Proof: That $\phi(A)$ is an Abelian group follows from Properties 1 and 3 of Theorem 10.2 for group homomorphisms. Then, if $a, b \in \phi(A)$, then there exist $x, y \in A$ such that $\phi(x) = a$ and $\phi(y) = b$. Then, since $xy \in A$ and $\phi(xy) = \phi(x)\phi(y) = ab$, we see that $ab \in \phi(A)$. Having now shown closure of the ring multiplication of S in $\phi(A)$, we can claim that $\phi(A)$ is a subring of S.

The third part of the theorem states that if A is an ideal and ϕ is onto S, then $\phi(A)$ is an ideal.

Proof: By the second part of this theorem, $\phi(A)$ is a subring, so we need only prove now that it is an ideal of S. Let $s \in S - \phi(A)$ and $y \in \phi(A)$.

Then since ϕ is onto, there exists $r \in R$ such that $\phi(r) = s$. Let $x \in A$ such that $\phi(x) = y$. Then since $rx \in A$, (because A is an ideal of R), and $\phi(rx) = \phi(r)\phi(x) = sy$, we have $sy \in \phi(A)$. Similarly, since $xr \in A$, (again, because A is an ideal of R), and $\phi(xr) = \phi(x)\phi(r) = ys$, we have $ys \in \phi(A)$. We can now claim that $\phi(A)$ is an ideal of S.

The fourth part of the theorem states that $\phi^{-1}(B) = \{r \in R | \phi(r) \in B\}$ is an ideal of R.

Proof: By Property 7 of Theorem 10.2, $\phi^{-1}(B)$ is a subgroup of R. It must be an Abelian group since all subgroups of rings are Abelian. Now let $r \in R$ and $x \in \phi^{-1}(B)$. Then $\phi(r) \in S$ and $\phi(x) \in B$ and since B is an ideal of S, $\phi(rx) = \phi(r)\phi(x) \in B$, showing that $rx \in \phi^{-1}(B)$. Similar reasoning shows that $xr \in \phi^{-1}(B)$, so $\phi^{-1}(B)$ is an ideal of R.

The fifth part of the theorem states that if R is commutative, then $\phi(R)$ is commutative.

Proof: Letting $a, b \in R$, notice that

$$\phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a).$$

The sixth part of the theorem states that if R has a unity 1, $S \neq \{0\}$, and ϕ is onto, then $\phi(1)$ is the unity of S.

Proof: Notice that for all $r \in R$, we have $\phi(1)\phi(r) = \phi(r)$. Then since ϕ is onto, it follows that $\phi(1)s = s$ for all $s \in S$. This shows that $\phi(1)$ is either the unity of S, or that $\phi(r) = 0$ for all $r \in R$. Now if $S \neq \{0\}$ and ϕ is onto, then we can't have $\phi(r) = 0$ for all $r \in R$. So $\phi(1) = 1$.

The seventh part of the theorem states that ϕ is an isomorphism if and only if ϕ is onto and $\ker \phi = \{r \in R | \phi(r) = 0\} = \{0\}$.

Proof: This follows immediately from Property 9 of Theorem 10.2. We need only look at the statement from a purely group-theoretic stand-point and also realize that ϕ will preserve the multiplication product of the ring.

The eighth and last part of the theorem states that if ϕ is an isomorphism from R onto S, then ϕ^{-1} is an isomorphism from S onto R.

Proof: Realize that $\ker \phi^{-1}$ is the trivial subring of R. This part of the theorem then follows from the seventh part of the theorem.

Exercise 2

Prove Theorem 15.2.

Let ϕ be a homomorphism from a ring R to a ring S. Then $\ker\phi = \{r \in R|\phi(r) = 0\}$ is an ideal of R.

Proof: From group theory, we already know that $\ker\phi$ is a normal subgroup of R. Now let $r \in R$ and $x \in \ker\phi$. Then $\phi(rx) = \phi(r)\phi(x) = \phi(r)\cdot 0 = 0 \implies rx \in \ker\phi$. Similarly, we have $xr \in \ker\phi$, so $\ker\phi$ is an ideal of R.

Exercise 3

Prove Theorem 15.3.

Let ϕ be a ring homomorphism from R to S. Then the mapping from $R/\ker\phi$ to $\phi(R)$, given by $r + \ker\phi \to \phi(r)$, is an isomorphism. In symbols, $R/\ker\phi \approx \phi(R)$.

Proof: By Theorem 10.3, ϕ is a group isomorphism from $R/\ker\phi$ to $\phi(R)$. Now since $\ker\phi$ is an ideal, $R/\ker\phi$ is a factor ring by Theorem 14.2. What remains to be shown is that the mapping preserves multiplication in $\phi(R)$. To that end, see that for any pair of elements $x, y \in R$, we have

$$\Psi(x + \ker\phi)\Psi(y + \ker\phi) = \phi(x)\phi(y) = \phi(xy) = \Psi(rs + \ker\phi),$$

where $\Psi : R/\ker\phi \to \phi(R)$ is the mapping given in the theorem's statement.

Exercise 4

Prove Theorem 15.4.

Every ideal of a ring R is the kernel of a ring homomorphism of R. In particular, an ideal A is the kernel of the mapping $r \to r + A$ from R to R/A.

Proof: Define $\phi(r) = r + A$ as the natural homomorphism from R to R/A. It is not hard to see that ϕ preserves both operations of R in R/A. Clearly, $\phi(r) = A$ if and only if $r \in A$, so $\ker\phi = A$.