Nailing Down The Directed Integral

Spencer T. Parkin

November 9, 2013

1 Motivation

2 Definitions

We shall let $\mathbb{R}^n$ denote n-dimensional euclidean space and let this space be represented by a vector space that is denoted by the same symbol $\mathbb{R}^n$. We will let $\mathbb{G}$ denote the geometric algebra that is generated by $\mathbb{R}^n$. To add structure to our space $\mathbb{R}^n$, we shall assume the euclidean metric which, for any pair of vectors (points) $a, b \in \mathbb{R}^n$, may be taken as $|a-b|$. It can then be shown that $\mathbb{R}^n$ is a metric space. To add further structure, we shall assume the usual topology on $\mathbb{R}^n$ for open sets.

Definition 2.1 (Tangent Vector). *Given any subset S of $\mathbb{R}^n$ and a point $x \in S$, we call a vector $t \in \mathbb{R}^n$ a tangent vector of S at x if there exists a sequence of points $\{x_i\}_{i=1}^{\infty} \subseteq S$ such that for any real number $\epsilon > 0$, there exists an integer $j > 0$ such that for all $i \geq j$, we have $|x_i - x| < \epsilon$ and*

$$\left|\frac{t}{|t|} - \frac{x_i - x}{|x_i - x|}\right| < \epsilon.$$

In light of Definition 2.1, we shall let $T(x)$ denote the set of all tangent vectors of S at the point x.

Definition 2.2 (Surface). *A subset S of $\mathbb{R}^n$ is a k-dimensional surface if for all points $x \in S$, the set $T(x) \cup \{0\}$ is a vector space of dimension k.*

With Definition 2.2 in place, it is easy to imagine examples of surfaces in $\mathbb{R}^n$, such as a hollow sphere or plane, although the typical surface may not really be anything like what we would or could imagine.

Given a surface $S \subseteq \mathbb{R}^n$, we will, for any point $x \in S$, let $G(x)$ denote the geometric algebra generated by the tangent space $T(x)$ at x.

If S is an orientable surface, then there exists a function $v : S \to \mathbb{G}$ giving, for each point $x \in S$, a consistent unit psuedo-scalar for the tangent algebra $G(x)$. The unit psuedo-scalar $v(x)$ is referred to as the tangent of S at x, while its principle dual, the normal of S at x.

Definition 2.3 (Surface Covering). *Given a surface S, a surface covering of S of radius r is a set C of open balls centered on points of S, each of radius r, with the property that for any point $x \in S$, there exists an open ball $b \in C$ such that $x \in b$.*

Letting $\text{ball}(x, r)$ denote an open ball of radius r centered at a point x, notice that if a surface S is compact, then, by the Heine-Borel property, (see []), we can always take the surface covering $\{\text{ball}(x, r) | x \in S\}$ and reduce it to a finite sub-cover. That is, find a finite subset of this cover that is also a surface cover of S.

If C is a surface covering of S, then we are going to let C' denote the set of open ball centers of all open balls in C.

Definition 2.4 (Directed Integral). *Let S be a compact surface upon which is defined a multivector field f. Then the directed integral of f over S, if it exists, is an element $L \in \mathbb{G}$, and we write*

$$L = \int_S dv f(x),$$

if for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that if C is a finite surface covering of S of radius $r < \delta$, then

$$\left| L - \sum_{x \in C'} r v(x) f(x) \right| < \epsilon.$$

We will characterize the set of all integrable functions on a compact surface S as those defined on such a surface, and for which the integral of Definition 2.4 exists over that surface.

Lemma 2.1. *The directed integral of Definition 2.4, as a function, is well defined.*

Proof. Letting f be an integrable function on S, we must show here that there are no two multivectors $L_0 \neq L_1$ of $\mathbb{G}$ that are both integrals of f over S. To that end, let $D = |L_0 - L_1|$ and choose $\epsilon = \frac{D}{2}$. (Use triangle inequality, if you can make it work.) $\square$