The Intersection Of Rays And Algebraic Surfaces

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Abstract. It is shown that for any real-valued, multi-variable polynomial defined over the real numbers that the image of a line through the domain of such a function is determined entirely by all orders of the directional derivatives of this function at any one point along the line and in a direction of the line. Though arrived at independantly in this paper, this result, in hindsight, is well known, and can be found in [3]. This paper shows a derivision using the language of geometric algebra.

The result has an application in the problem of casting rays through algebraic surfaces as it shows that such a problem, in all cases, reduces to the problem of finding the roots of a single-variable polynomial having an explicit formulation in terms of the multi-variable polynomial and ray in question.

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1. Introduction

Letting $f : \mathbb{R}^n \to \mathbb{R}$ be any polynomial equation in n variables and up to degree m, it was shown in [2] that the function $\bigwedge_{i=1}^m p_i(x)$ may be factored out of this polynomial in terms of the inner product as

$$f(x) = \bigwedge_{i=1}^{m} p_i(x) \cdot B, \tag{1.1}$$

where B is an m-vector of our geometric algebra with $\infty_i \cdot B = 0$ for all integers $i \in [1, m]$, and where the function $p_i : \mathbb{R}^n \to \mathbb{V}_i$ is given by

$$p_i(x) = o_i + x_i + \frac{1}{2} x_i^2 \infty_i, \tag{1.2}$$

having its origins in the paper [1]. Given a point $x \in \mathbb{R}^n$ and a direction vector $v \in \mathbb{R}^n$, we wish to find the set of all scalars $\lambda \in \mathbb{R}$ such that $f(x + \lambda v) = 0$.

Utilizing equation (1.1) for this purpose, we easily find that

$$f(x+\lambda v)=\bigwedge_{i=1}^{m}(p_i(x)+\lambda v_i)\cdot B, \tag{1.3}$$

because we can ignore the $\frac{1}{2}x_i^2\infty_i$ term in equation (1.2). Looking at equation (1.3), it is immediately clear that its expansion is that of a polynomial in λ of up to degree m. What we're going to show in this paper is that an explicit formula for this polynomial can be found in terms of all orders of directional derivatives of f at x and in the direction of v.

2. The Result

We begin by rewriting equation (1.3) as

$$f(x+\lambda v)=\sum_{i=0}^{m}T_i(x), \tag{2.1}$$

where $T_i(x)$ will denote the i^{th} term involving λ^i in the series expansion of (1.3). Carefully formulating this term, we get

$$T_i(x)=\lambda^i\sum_{j=1}^{\binom{m}{i}}W_{j,i}(x)\cdot B,$$

where $W_{j,i}$ is the j^{th} way to write an outer product involving i vectors taken from $\{v_k\}_{k=1}^m$ and $m-i$ vectors taken from $\{p_k(x)\}_{k=1}^m$ in an order having ascending sub-scripts. The following examples help clarify this in the case $m=3$.

$$\begin{aligned}
W_{1,0}&=p_1\wedge p_2\wedge p_3\\
W_{1,1}&=p_1\wedge p_2\wedge v_3\\
W_{2,1}&=p_1\wedge v_2\wedge p_3\\
W_{3,1}&=v_1\wedge p_2\wedge p_3\\
W_{1,2}&=p_1\wedge v_2\wedge v_3\\
W_{2,2}&=v_1\wedge p_2\wedge v_3\\
W_{3,2}&=v_1\wedge v_2\wedge v_3\\
W_{1,3}&=v_1\wedge v_2\wedge v_3
\end{aligned}$$

Having now come to terms, (no pun intended), with the general expansion of equation (1.3), we proceed now to fearlessly take the directional derivative of T_i at x and in the direction of v. Doing so, we get

$$\nabla_v T_i(x)=\lambda^i\sum_{j=1}^{\binom{m}{i}}\lim_{\delta\to 0}\frac{W_{j,i}(x+\delta v)-W_{j,i}(x)}{\delta}\cdot B,$$

knowing that each individual limit will exist. What we must realize now is that the term $W_{j,i}(x)$ will get canceled in the expansion of $W_{j,i}(x+\delta v)$, leaving only terms that are multiples of positive powers of δ. Furthermore, it is only those remaining terms that are multiples of δ itself that will survive the limit process. We are therefore left to deduce these terms in an evaluation of the limit. What we find is that all such terms are of the form $\delta W_{j,i+1}(x)$, but we need to determine just how many we have. Realizing that $\binom{m}{i}$ old terms will each contribute $m-i$ new terms of this form, of which there should be $\binom{m}{i+1}$, but that no type of term will be produced any more or less than any other, we see that

$$\frac{(m-i)\binom{m}{i}}{\binom{m}{i+1}} = i+1$$

is the number of such terms of the form $\delta W_{j,i+1}(x)$, and we may write

$$\begin{aligned}
\nabla_v T_i(x) &= \lambda^i \sum_{j=1}^{\binom{m}{i+1}} \lim_{\delta\to 0} \frac{(i+1)\delta W_{j,i+1}(x)}{\delta} \cdot B \\
&= \lambda^i (i+1) \sum_{j=1}^{\binom{m}{i+1}} W_{j,i+1}(x) \cdot B \\
&= \frac{i+1}{\lambda} T_{i+1}(x). \qquad (2.2)
\end{aligned}$$

Returning to equation (2.1), and realizing that $T_0(x) = f(x)$, we can now finally deduce the expansion of (1.3) using the recurrence relation of equation (2.2) as

$$f(x+\lambda v) = \sum_{i=0}^{m} \frac{\lambda^i}{i!} \nabla_v^i f(x), \qquad (2.3)$$

where $\nabla_v^i f(x)$ is the i^{th} order directional derivative of f at x in the direction of v with $\nabla_v^0 f(x) = f(x)$.

3. Making Use Of The Result

In its present form, equation (2.3) lacks ease of use, because knowledge of the vector v is needed to calculate all orders of the directional derivative. To solve this problem, we need to decouple the knowledge of this vector from the limit processes by generalizing the idea of the gradient to higher orders. This has already been done in [] with the use of k-forms. The manifestation of such things in the algebra we are using may be as follows.

We begin with the gradient of f, usually written ∇f. But this is ambiguous in our algebra, because we need to specify the sub-algebra over which ∇f will be taken. We will do this with an integral subscript as $\nabla_i f$, not to be confused with the directional derivative, which uses a vector subscript. The gradient can now be defined similarly to it's definition in [] as

$$\nabla_i = \sum_{j=1}^{n} e_{i,j} \nabla_{e_j}.$$

Having done this, we have, for any integer $i \in [1, m]$,

$$\nabla_v = v_i \cdot \nabla_i,$$

which is a well-known result. Generalizing this, we find that

$$\nabla_v^i = -(-1)^i \bigwedge_{j=1}^{i} v_j \cdot \bigwedge_{j=1}^{i} \nabla_j,$$

where the function operator $\bigwedge_{j=1}^{i} \nabla_j$ is defined as

$$\bigwedge_{j=1}^{i} \nabla_j = -(-1)^i \sum_{j_1=1}^{n} \cdots \sum_{j_i=1}^{n} \bigwedge_{k=1}^{i} e_{k,j_k} \prod_{k=1}^{i} \nabla_{e_{j_k}},$$

realizing that, as was assumed throughout [], the outer and inner products bind tighter than the geometric product. Here, the function operator being applied to f is an i^{th} order partial derivative.

The decoupling of equation (2.3) can now be written as

$$f(x + \lambda v) = \sum_{i=0}^{m} \frac{\lambda^i}{i!} \bigwedge_{j=1}^{i} v_j \cdot \left(\bigwedge_{j=1}^{i} \nabla_j \right) f(x), \tag{3.1}$$

where the outer product $\bigwedge_{j=1}^{i} v_j$ is one in the case $i = 0$, and where the function operator $\bigwedge_{j=1}^{i} \nabla j$ is the identity operator in the same case.

If all orders of the gradient of f are available to us, equation (3.1) becomes a convenient way to calculate the coeficients of the polynomial whose zeros give us the parameters of the intersection points of our ray with the a given algebraic surface.

Looking back again at equation (1.3), this is certainly a usable form if a symbolic calculator is available. Use of symbolic calculation was employed in []. If no such thing is available, then (3.1) may allow us to come up with the polynomial through literal evaluation, provided, again, that we have all gradients available to us. For algebraic surfaces of degrees greater than or equal to five[1], we may lose the need to come up with the polynomial expansion of $f(x + \lambda v)$ altogether in favor of root-finding methods that need know nothing more about $f(x + \lambda v)$ than that it is continuous.[2]

In any case, (3.1) appears to be, if nothing more, an interesting result about algebraic surface that may find applications in other areas.

References

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[1]There is no closed form solution to a general polynomial of degree greater than or equal to five in terms of elementary functions. See[].

[2]In such a case, the intermediate value theorem applies. Recall that the composition of two continuous functions is continuous.

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