Chapter 9 Exercises
Gallian's Book on Abstract Algebra

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Exercise 1

Let $H = \{(1), (12)\}$. Is H normal in S_3.

No, $(123)H \neq H(123)$, because $(123)(12) = (13) \neq (23) = (12)(123)$.

Exercise 2

Prove that A_n is normal in S_n.

Let $\alpha \in A_n$ and let $\beta \in S_n$. Now notice that $\beta\alpha\beta^{-1} \in A_n$, in the case that β is an even permutation, or an odd permuation. It then follows by Theorem 9.1 that A_n is a normal subgroup of S_n.

Exercise 3

Show that if G is the internal direct product of $H_1, H_2, \dots, H_n$ and $i \neq j$ with $1 \leq i \leq n$, $1 \leq j \leq n$, then $H_i \cap H_j = \{e\}$.

Without loss of generality, let $i < j$. Now notice that

$$H_i \subseteq H_1 H_2 \dots H_i \dots H_{j-2} H_{j-1}$$

and that $H_1 H_2 \dots H_i \dots H_{j-2} H_{j-1} \cap H_j = \{e\}$. It follows that $H_i \cap H_j = \{e\}$.

Finishing Theorem 9.6

We are given $\phi(h_1h_2\dots h_n) = (h_1, h_2, \dots, h_n)$. It is immediately clear that ϕ is onto $H_1 \oplus H_2 \oplus \cdots \oplus H_n$. By the uniqueness of representation of elements in $H_1H_2\dots H_n$ already proven, it follows that ϕ is one-to-one. That ϕ is operation preserving follows from the commutativity among disjoint subgroups. For all integers $i \in [1, n]$, for all $a_i, b_i \in H_i$, we have

$$\begin{aligned} &\phi(a_1a_2\dots a_nb_1b_2\dots b_n) \\ =&\phi(a_1b_1a_2b_2\dots a_nb_n) \\ =&(a_1b_1, a_2b_2, \dots, a_nb_n) \\ =&(a_1, a_2, \dots, a_n)(b_1, b_2, \dots, b_n) \\ =&\phi(a_1a_2\dots a_n)\phi(b_1b_2\dots b_n). \end{aligned}$$

Exercise 7

Prove that if H has index 2 in G, then H is normal in G.

The cosets of H in G are H and xH for any $x \in G - H$. Now consider Hx. This is H or xH. But it can't be H, because $x \notin H$. It must, therefore, be xH.

Exercise 10

Prove that a factor group of a cyclic group is cyclic.

Let H be a normal subgroup of a cyclic group $G = \langle g \rangle$. (All cyclic groups are Abelian and therefore, all subgroups of a cyclic group are normal.) We then see that

$$G/H = \{aH | a \in G\} = \{g^kH | k \in \mathbb{Z}\} = \langle gH \rangle.$$

Exercise 11

Let H be a normal subgroup of G. If H and G/H are Abelian, must G be Abelian?

No. Consider $G = D_4$ and H as the subgroup of rotations in D_4.

Exercise 12

Prove that a factor group of a cyclic group is cyclic.

Let $a, b \in G/H$ with $a = xH$ and $b = yH$ for $x, y \in G$. We then have

$$ab = xHyH = xyH = yxH = yHxH = ba.$$

Exercise 44

If $|G| = pq$, where p and q are primes that are not necessarily distinct, prove that $|Z(G)| = 1$ or pq.

If $Z(G) = \{e\}$, we're done. So suppose $Z(G) \neq \{e\}$. If G has an element of order pq, we're done, so assume no such element exists. It follows that G must have an element of order p or q. Suppose $Z(G) = \langle z \rangle$ for an element $z \in G$ of order p. Then, since q is prime, $G/Z(G)$ is cyclic, and so, by Theorem 9.3, we must have G is Abelian. But then $Z(G) = G$, which is a contradiction. A similar contradiction is reached if we suppose $Z(G) = \langle z \rangle$ for an element $z \in G$ of order q. It follows that $Z(G) = G$, and therefore, $|Z(G)| = pq$.

Exercise 46

Let G be an Abelian group and let H be the subgroup consisting of all elements of G that have finite order. Prove that every nonidentity element in G/H has infinite order.

Let $a \in G/H$ be a non-identity element. Then there exists $g \in G$ such that $a = gH$. Clearly $g \notin H$ by Property 2 of the Lemma for Theorem 7.1. It follows that $|g| = \infty$.

Exercise 50

Show that the intersection of two normal subgroups of G is a normal subgroup of G.

Let H and K be normal subgroups of G. We know that $H \cap K$ is a subgroup of G by a previous problem from a previous chapter. Let $g \in G$ and let $x \in H \cap K$. Then $gxg^{-1} \in H$ by the normality of H and $gxg^{-1} \in K$

by the normality of K. It follows that $gxg^{-1} \in H \cap K$ and so $H \cap K$ is normal in G.

Problem similar to Exercise 50

Gallian himself helped me with this in relation to another problem. We prove the following statement. If K and N are subgroups of a group G with N normal in G, then $K \cap N$ is normal in G. (K is not necessarily normal in G.)

Let x belong to $K \cap N$ and let k belong to K. Then, by closure, $kxk^{-1} \in K$, and by the normality of N, we have $kxk^{-1} \in N$. It follows now that $N \cap K$ is normal in K.

Exercise 51

Let N be a normal subgroup of G and let H be any subgroup of G. Prove that NH is a subgroup of G. Give an example to show that NH need not be a subgroup of G if neither N nor H is normal.

Notice that $e \in NH$. Let $a, b \in NH$. Then there exist elements $n_a, n_b \in N$ and $h_a, h_b \in H$ such that $a = n_a h_a$ and $b = n_b h_b$. Then, by the normality of N, there exists an element $n_b' \in N$ such that

$$ab^{-1} = n_a h_a (n_b h_b)^{-1} = n_a h_a h_b^{-1} n_b = n_a n_b' h_a h_b^{-1} \in NH.$$

I'm terrible at finding examples.

Exercise 53

Let N be a normal subgroup of a group G. If N is cyclic, prove that every subgroup of N is also normal in G.

Let H be a subgroup of $N = \langle n \rangle$. Then, for some integer i, we have $H = \langle n^i \rangle$. Then, for all $g \in G$, and any integer j, we have

$$g(n^i)^j g^{-1} = g(n^j)^i g^{-1} = (gn^j g^{-1})^i = (n^k)^i = (n^i)^k \in N,$$

where here, $gn^j g^{-1} = n^k$ for some integer k by virtue of N being a normal subgroup of G. It follows now by Theorem 9.1, the normal subgroup test, that H is normal in G.

Exercise 54

Without looking at inner automorphisms of D_n, determine the number of such automorphisms.

By Theorem 9.4, we know that $\text{Inn}(D_n) \approx D_n/Z(D_n)$. By Example 11 of Chapter 3, we know that $|Z(D_n)|$ is 2 if n is even, and 1 if n is odd. Then, knowing that $|D_n| = 2n$, we have

$$|\text{Inn}(D_n)| = \frac{|D_n|}{|Z(D_n)|} = \begin{cases} n & \text{if } n \text{ even,} \\ 2n & \text{if } n \text{ odd.} \end{cases}$$

Exercise 55

Let H be a normal subgroup of a finite group G and let $x \in G$. If $\gcd(|x|, |G/H|) = 1$, show that $x \in H$.

Consider the cyclic subgroup of G/H generated by xH. It is clear that $|\langle xH \rangle| = |x|$, but we must have $|\langle xH \rangle|$ dividing $|G/H|$. This means that $|x|$ must divide $|G|/|H|$. Therefore, $|x| = 1$, because $\gcd(|x|, |G|/|H|) = 1$, and we see that $xH = H \implies x \in H$.

Exercise 61

Suppose that H is a normal subgroup of a finite group G. If G/H has an element of order n, show that H has an element of order n. Show, by example, that the assumption that G is finite is necessary.

The case $n = 1$ is trivial, so let $n > 1$. Let $a \in G$ such that $|aH| = n$. Clearly $a \neq e$. It follows that the mapping $\phi : H \to H$, given by $\phi(h) = a^n h$ is a non-trivial permutation of the elements of H and so ϕ is a member of the group of permutations of H. We then see that $a^{|\phi|n} = e$. But it is easy to see that for all integers $i \in [1, |\phi|n - 1]$, we have $a^i \neq e$. So $|a^{|\phi|}| = n$.

Exercise 62

Do it...

Exercise 65

If $|G| = 30$ and $|Z(G)| = 5$, what is the structure of $G/Z(G)$?.

Note that $|G/Z(G)| = 30/5 = 6 = 2 \cdot 3$. It follows from Theorem 7.2 that $G/Z(G)$ is isomorphic to Z_6 or D_3. Erf...which one? Think about it.

Made-up Problems

Problem 1

Let G be a group and let N be a normal subgroup of G. Then, if H_1 and H_2 are subgroups of G containing N, and H_2/N is a subgroup of H_1/N, is H_2 a subgroup of H_1?

It is clear that H_2 is a subset of H_1. But this is all we need, right? We already know that H_2 is a group, and that H_1 is a group.

Problem 2

Let G be a group and let N be a normal subgroup of G. If N is a maximal subgroup of G, is G/N a cyclic group of prime order?

If true, we would need to show that G/N has no non-trivial and proper subgroups. Suppose G/N does have a non-trivial and proper subgroup. Call it K. Then K can be written as H/N where H is a proper subgroup of G, right? (I think so by Problem 1 above.) Then, since N is maximal in G, we must have $H = N$. It follows that $K = H/N = N/N = \{N\}$ is a trivial factor subgroup of G/N, which is a contradiction.

Problem 3

How about the converse of the statement given in Problem 2? Is it true?