“Machine Learning and Computational Statistics”

4^{th} Homework

Exercise 1:

Consider the **regression problem** y=g($\boldsymbol{x}$)+η

It is known that E[y|$\boldsymbol{x}$] is the minimum MSE estimate of y given $\boldsymbol{x}$. Consider the estimator $f(\boldsymbol{x};D)$.

(a) Under what conditions (theoretically) the quantity $\mathrm{E}_D[(f(\boldsymbol{x};D)-E[\mathrm{y}\,|\,\boldsymbol{x}])^2]$ becomes zero?

(b) Why this cannot be achieved in practice?

Exercise 2:

Consider a regression task $y = g(x) + \eta$, where y and x are modeled by the random variables y and x. The joint pdf of y and x is:

$p(x, y) = \frac{3}{2}$, for $x \in (0,1), y \in (x^2, 1)$.

Determine the optimum MSE estimate $E[y|x]$, for a given x, by performing the following steps:

(a) Verify that $p(x, y)$ is a pdf (prove that it integrates to 1).

(b) Compute the marginal pdf of x, $p_x(x)$.

(c) Compute the conditional pdf of y, given x.

(d) Compute and plot $E[y|x]$.

Hint: It is $\int_a^b x^n dx = [\frac{1}{n+1}x^{n+1}]_a^b = \frac{1}{n+1}b^{n+1} - \frac{1}{n+1}a^{n+1}$

Exercise 3 (python code + text):

Consider the **regression problem** (1-dep., 1-indep. variables)

$$y=g(x)+\eta$$

where y and x are jointly distributed according to the normal distribution $p(y,x) = N(\boldsymbol{\mu}, \Sigma)$

with $\boldsymbol{\mu} = \begin{bmatrix} \mu_y \\ \mu_x \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\quad \Sigma = \begin{bmatrix} \sigma_y^2 & \sigma_{yx} \\ \sigma_{yx} & \sigma_x^2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$

(a) Determine E[y|$\boldsymbol{x}$] and plot the corresponding curve (recall the relevant theory concerning the normal distribution case).
(b) Generate 100 data sets D_i, i=1,...100, each one consisting of N=50 randomly selected pairs (y_n,x_n), $n=1,\ldots,N$, from $p(y,x)$.
(c) Adopt a linear estimator $f(x;D)$ and determine its instances $f(x;D_1),\ldots,f(x;D_{100})$, utilizing the LS criterion.
(d) Plot in a single figure **(i)** the lines corresponding to the above 100 estimates (blue color) and **(ii)** the line corresponding to the optimal MSE estimate (green color).
(e) Repeat steps (b)-(d) where now each data set consists of N=5000 points.
(f) Discuss the results (in your discussion, take into account the decomposition of the MSE to a variance and a bias term).

Exercise 4 (python code + text)**:**

Consider the set up of exercise 2 and recall the E[y|$\boldsymbol{x}$] determined there.

(a) Generate a single data set D of 100 pairs (y_n,x_n), $n=1,\ldots,100$ from $p(y,x)$.
(b) Determine the linear estimate $f(x;D)$ that minimizes the MSE criterion, based on D.
(c) Generate randomly a set D' of additional 50 points (y'_n,x'_n), $n=1,\ldots,50$. For each x'_n determine the estimate $y_{n'} = f(x_n;D')$ (50 numbers (estimates) should be finally computed).
(d) Again, for the 50 x'_n's determine the associated estimates $\hat{y}$ = E[y|$\boldsymbol{x}$].
(e) Based on the previous derived estimates for the 50 points from both $f(x_n;D)$ and E[y|$\boldsymbol{x}$], propose and use a (practical) way for quantifying the performance of the two estimators $f(x_n;D')$ and E[y|$\boldsymbol{x}$].

Exercise 5 (python code + text)**:** Consider the setup of exercise 2. Generate a set D of $N = 100$ data pairs $\boldsymbol{z}_n=(y_n,x_n)$.

(a) For each x_n compute the optimal MSE estimate (use the results of exercise 2).
(b) Compute $\boldsymbol{\mu} = \begin{bmatrix}\mu_x \\ \mu_y\end{bmatrix} = \begin{bmatrix}\frac{1}{N}\sum_{n=1}^{N} x_n \\ \frac{1}{N}\sum_{n=1}^{N} y_n\end{bmatrix}$ and $\Sigma = \frac{1}{N}\sum_{n=1}^{N}(\boldsymbol{\mu}-\boldsymbol{z}_n)(\boldsymbol{\mu}-\boldsymbol{z}_n)^T$.
(c) Pretend that you do not know the true distribution that generates the data and you (erroneously) assume that the joint pdf of x and y is a normal one with mean and covariance matrix those computed in (b). Derive the optimum MSE estimate for this case and compute the MSE estimate for each one of the 100 x_n's.
(d) Discuss the results obtained from (a) and (c).

NOTE: Please give **brief explanations** in all **exercises**.