Univerzitet u Nišu
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# Matrični metodi u računarstvu
# Domaci zadatak II

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## Zadaci:

**1.** Data je $LU$ faktorizacija matrice $A$:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -6 & 4 & -1 \\ 4 & -6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

Opisati, bez izračunavanja, koje elementarne operacije nad vrstama su upotrebljene za dovođenje matrice $A$ na gornje trougaoni oblik. Voditi računa o redosledu navođenja operacija.

**Rešenje:** Kako bi prešli sa matrice $A$ na proizvod matrica $LU$, odnosno dobili gornju trougaonu matricu $U$ potrebno je:

1: Množenje prve vrste količnikom drugog i prvog elementa prve kolone $(R_1 * \frac{a_{21}}{a_{11}})$.
2: Oduzimanje tako pomnožene prve vrste od druge vrste matrice $A$ $(R_2 - R_1 * \frac{a_{21}}{a_{11}})$.
3: Množenje prve vrste matrice $A$ količnikom trećeg i prvog elementa prve kolone matrice $(R_1 * \frac{a_{31}}{a_{11}})$.
4: Oduzimanje tako pomnožene prve vrste od treće vrste matrice $A$ $(R_3 - R_1 * \frac{a_{31}}{a_{11}})$.
5: Množenje druge vrste količnikom trećeg i drugog elementa druge kolone matrice $(R_2 * \frac{a_{32}}{a_{22}})$.
6: Oduzimanje tako pomnožene druge vrste od treće vrste matrice $A$ $(R_3 - R_2 * \frac{a_{32}}{a_{22}})$.

## Napomena:

Primetiti da se oduzimanje može gledati kao sabiranje, pritom kolonu kojom oduzimamo množimo količnikom pomnoženim sa $(-1)$:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -6 & 4 & -1 \\ 4 & -6 & 7 \end{bmatrix} \begin{array}{l} {}^{*3} \quad {}^{*(-2)} \\ \longleftarrow + \\ \longleftarrow + \end{array} \simeq \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -4 & 7 \end{bmatrix} \begin{array}{l} \\ {}^{*4} \\ \longleftarrow + \end{array} \simeq \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix} = U$$

**2.** Neka su $a_i \neq 0$ međusobno različiti brojevi. Odrediti $LU$ faktorizaciju sledećih Vandermondovih matrica.

$$V_2 = \begin{bmatrix} 1 & 1 \\ a_1 & a_2 \end{bmatrix}, V_3 = \begin{bmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{bmatrix}, V_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 \end{bmatrix}$$

Opisati uočenu pravilnost.

**Rešenje:**

$$V_2 = \begin{bmatrix} 1 & 1 \\ a_1 & a_2 \end{bmatrix} \begin{array}{l} \rceil^{*(-a_1)} \\ \hookleftarrow_{+} \end{array} \implies V_2 = L_{V_2} U_{V_2} = \begin{bmatrix} 1 & 0 \\ a_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & a_2 - a_1 \end{bmatrix}$$

$$V_3 = \begin{bmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{bmatrix} \begin{array}{l} \rceil^{*(-a_1)} \quad \rceil^{*(-a_1^2)} \\ \hookleftarrow_{+} \\ \longleftarrow_{+} \end{array} \simeq \begin{bmatrix} 1 & 1 & 1 \\ 0 & a_2 - a_1 & a_3 - a_1 \\ 0 & a_2^2 - a_1^2 & a_3^2 - a_1^2 \end{bmatrix} \begin{array}{l} \rceil^{*(-\frac{a_2^2 - a_1^2}{a_2 - a_1} = -(a_1 + a_2))} \\ \hookleftarrow_{+} \end{array}$$

$$(a_3^2 - a_1^2) - ((a_3 - a_1)(a_1 + a_2)) = (a_3 - a_1)((a_3 + a_1) - (a_1 + a_2)) = (a_3 - a_1)(a_3 - a_2) = \boldsymbol{v_{33}} \tag{2.1}$$

$$\implies L_{V_3} = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_1^2 & \frac{a_2^2 - a_1^2}{a_2 - a_1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_1^2 & a_2 + a_1 & 1 \end{bmatrix} = \left[ \begin{array}{cc|c} \multicolumn{2}{c|}{L_{V_2}} & O \\ \hline a_1^2 & a_1 + a_2 & 1 \end{array} \right]$$

$$\implies U_{V_3} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & a_2 - a_1 & a_3 - a_1 \\ 0 & 0 & \boldsymbol{v_{33}} \end{bmatrix} = \left[ \begin{array}{c|c} U_{V_2} & \begin{matrix} 1 \\ a_3 - a_1 \end{matrix} \\ \hline O & \boldsymbol{v_{33}} \end{array} \right]$$

$$V_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 \end{bmatrix} \begin{array}{l} \rceil^{*(-a_1)} \quad \rceil^{*(-a_1^2)} \quad \rceil^{*(-a_1^3)} \\ \hookleftarrow_{+} \\ \longleftarrow_{+} \\ \longleftarrow_{+} \end{array} \simeq$$

$$\simeq \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & a_2 - a_1 & a_3 - a_1 & a_4 - a_1 \\ 0 & a_2^2 - a_1^2 & a_3^2 - a_1^2 & a_4^2 - a_1^2 \\ 0 & a_2^3 - a_1^3 & a_3^3 - a_1^3 & a_4^3 - a_1^3 \end{bmatrix} \begin{array}{l} \rceil^{*(-\frac{a_2^2 - a_1^2}{a_2 - a_1} = -(a_1 + a_2))} \quad \rceil^{*(-\frac{a_2^3 - a_1^3}{a_2 - a_1} = -(a_1^2 + a_1 a_2 + a_2^2))} \\ \hookleftarrow_{+} \\ \longleftarrow_{+} \end{array}$$

$$\simeq \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & a_2 - a_1 & a_3 - a_1 & a_4 - a_1 \\ 0 & 0 & \boldsymbol{v_{33}} & \boldsymbol{v_{34}} \\ 0 & 0 & \boldsymbol{v_{43}} & \boldsymbol{v'_{44}} \end{bmatrix} \begin{array}{l} \rceil^{*(-\frac{\boldsymbol{v_{43}}}{\boldsymbol{v_{33}}})} \\ \hookleftarrow_{+} \end{array}$$

$$\begin{aligned} (a_3^3 - a_1^3) - ((a3 - a1)(a_1^2 + a_1 a_2 + a_2^2)) &= (a_3 - a_1)((a_1^2 + a_1 a_3 + a_3^2) - (a_1^2 + a_1 a_2 + a_2^2)) = \\ &= (a_3 - a_1)(a_3 - a_2)(a_1 + a_2 + a_3) = \boldsymbol{v_{43}} \end{aligned} \tag{2.2}$$

$$(a_4^2 - a_1^2) - ((a_4 - a_1)(a_1 + a_2)) = (a_4 - a_1)((a_4 + a_1) - (a_1 + a_2)) = (a_4 - a_1)(a_4 - a_2) = \boldsymbol{v_{34}} \tag{2.3}$$

$$\begin{aligned} (a_4^3 - a_1^3) - ((a4 - a1)(a_1^2 + a_1 a_2 + a_2^2)) &= (a_4 - a_1)((a_1^2 + a_1 a_4 + a_4^2) - (a_1^2 + a_1 a_2 + a_2^2)) = \\ &= (a_4 - a_1)(a_4 - a_2)(a_1 + a_2 + a_4) = \boldsymbol{v'_{44}} \end{aligned} \tag{2.4}$$

$$\boldsymbol{v_{44}}' - \boldsymbol{v_{34}} * \frac{\boldsymbol{v_{43}}}{\boldsymbol{v_{33}}} = \boldsymbol{v_{44}}' - (a_4-a_1)(a_4-a_2) * \frac{(a_3-a_1)(a_3-a_2)(a_1+a_2+a_3)}{(a_3-a_1)(a_3-a_2)} = \tag{2.5}$$
$$= (a_4-a_1)(a_4-a_2)(a_1+a_2+a_4) - (a_4-a_1)(a_4-a_2)(a_1+a_2+a_3) =$$
$$= (a_4-a_1)(a_4-a_2)(a_1+a_2+a_4-a_1-a_2-a_3) = (a_4-a_1)(a_4-a_2)(a_4-a_3) = \boldsymbol{v_{44}}$$

$$\implies L_{V_4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_1^2 & a_1+a_2 & 1 & 0 \\ a_1^3 & a_2^2+a_2a_1+a_1^2 & \boldsymbol{v_{43}} & 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & L_{V_3} & & O \\ \hline a_1^3 & a_2^2+a_2a_1+a_1^2 & \boldsymbol{v_{43}} & 1 \end{array}\right]$$

$$\implies U_{V_4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & a_2-a_1 & a_3-a_1 & a_4-a_1 \\ 0 & 0 & \boldsymbol{v_{33}} & \boldsymbol{v_{34}} \\ 0 & 0 & 0 & \boldsymbol{v_{44}} \end{bmatrix} = \left[\begin{array}{c|c} & 1 \\ U_{V_3} & a_4-a_1 \\ & \boldsymbol{v_{34}} \\ \hline O & \boldsymbol{v_{44}} \end{array}\right]$$

**Zaključak:** Na osnovu prethodnih dekompozicija možemo zaključiti sledeće:

$$V_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_{n-1}^2 & a_n^2 \\ \vdots & \vdots & a_3^3 & \ddots & \vdots & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_{n-1}^{n-1} & a_n^{n-1} \end{bmatrix} = L_{V_n}U_{V_n} =$$

$$= \left[\begin{array}{cccc|c} & L_{V_{n-1}} & & & O \\ \hline a_1^{n-1} & \frac{a_2^{n-1}-a_1^{n-1}}{a_2-a_1} & \cdots & \boldsymbol{v_{n(n-1)}} & 1 \end{array}\right] \left[\begin{array}{c|c} & 1 \\ & a_n-a_1 \\ U_{V_{n-1}} & (a_n-a_1)(a_n-a_2) \\ & \vdots \\ & \boldsymbol{v_{(n-1)n}} \\ \hline O & \boldsymbol{v_{nn}} \end{array}\right]$$

Gde je $\boldsymbol{v_{nm}} = \begin{cases} \prod_{p=1}^{n-1}(a_m-a_p), (n \le m) \iff (\boldsymbol{v_{nm}} \in U) \\ \sum_{k=1}^{m} a_k * \prod_{p=1}^{m-1}(a_m-a_p), (n > m) \iff (\boldsymbol{v_{nm}} \in L) \end{cases}$, odnosno $\boldsymbol{v_{nm}} \in R$.

**3.** Da li su sledeća tvrđenja tačna?
- Kvadratna matrica $A$ koja ima neki element glavne dijagonale jednak nuli je singularna matrica.
- Ukoliko je neki pivot element matrice $A$ u $LU$ faktorizaciji jednak nuli, matrica $A$ je singularna.

Obrazložiti odgovore.

**Rešenje:** -Prvu tvrdnju lako možemo opovrgnuti nalaženjem regularne matrice koja za neki od elemenata glavne dijagonale ima nulu, na primer:

$$S = \begin{bmatrix} 0 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Potražimo determinantu matrice S:

$$det(S) = \begin{vmatrix} 0 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (0*5*9)+(2*6*7)+(3*4*8)-((7*5*3)+(8*6*0)+(9*4*2)) = \tag{3.1}$$

$= 84 + 96 - (105 + 72) = 180 - 177 = 3$

Čak možemo otići i korak dalje:

$$S_1 = \begin{bmatrix} 0 & 2 & 3 \\ 4 & 0 & 6 \\ 7 & 8 & 0 \end{bmatrix} \implies det(S) = \begin{vmatrix} 0 & 2 & 3 \\ 4 & 0 & 6 \\ 7 & 8 & 0 \end{vmatrix} = (0*0*0)+(2*6*7)+(3*4*8)-((7*0*3)+(8*6*0)+(0*4*2)) = \tag{3.2}$$

$= 84 + 96 = 160$

$\implies$ Matrica koja na glavnoj dijagonali ima neke, ili čak sve elemente jednake nuli, ne mora da bude singularna.

-Za ispitivanje druge tvrdnje možemo se pozvati na prvu (3.1), s obzirom da je prvi pivot element ujedno i prvi element glavne dijagonale.
Ali hajde da vidimo kako bi izgledala LU faktorizacija matrice $S$, s obzirom da znamo vezu između $det(S)$ i $det(U)$:

$$S = LU \implies det(S) = det(LU) = det(L) * det(U) = 1 * det(U) = det(U)$$

Odavde vidimo da je $det(S) = det(U)$.

Prenego što krenemo sa računanjem matrice $U$, moramo prvo da uradimo permutaciju matrice $S$, s obzirom da nam je pivot element jednak nuli. Za to ćemo iskoristiti permutacionu matricu $P$:

$$S = \begin{bmatrix} 0 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \implies PS = \begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \implies P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*Pravilo permutacione matrice:*

$$P * P^T = P^T * P = I \implies P^T = P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P \implies det(P) = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

Više ne tražimo faktorizaciju matrice $S$, već matrice $PS$:

$$PS = \begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \;{}^{*(-7/4)}_{+}\; \simeq \begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 0 & -\frac{3}{4} & -\frac{3}{2} \end{bmatrix} \;{}^{*(3/8)}_{+}\; \simeq \begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & -\frac{3}{8} \end{bmatrix} = U$$

$$PS = \tilde{L}U \quad | \quad *(P^{-1}) \implies P^{-1}PS = P^{-1}\tilde{L}U \implies det(P^{-1}PS) = det(P^{-1}\tilde{L}U)$$

$$\implies det(S) = det(P^{-1}) * det(\tilde{L}) * det(U) \implies det(S) = (-1) * (4 * 2 * (-\frac{3}{8})) = 3$$

$\implies$ Ni matrica sa nulom za pivot element ne mora biti singularna.

**4.** Data je $LU$ faktorizacija matrice $A$ sa

$$\begin{bmatrix} a_1 \\ \hline a_2 \\ \hline a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \hline l_{21} & 1 & 0 \\ \hline l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ \hline u_2 \\ \hline u_3 \end{bmatrix}$$

Dokazati da važi:

$$L(a_1) = L(u_1)$$
$$L(a_1, a_2) = L(u_1, u_2)$$
$$L(a_1, a_2, a_3) = L(u_1, u_2, u_3)$$

Opisati i dokazati analogno tvrđenje za kolone matrica $A$ i $L$.

**Rešenje:** S obzirom da je: $A = LU$

Dobijamo sistem:

$$\iff \begin{cases} \boldsymbol{a)} \quad a_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ \hline u_2 \\ \hline u_3 \end{bmatrix} = u_1 \implies a_1 = u_1 \\ \boldsymbol{b)} \quad a_2 = \begin{bmatrix} l_{21} & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ \hline u_2 \\ \hline u_3 \end{bmatrix} = l_{21}u_1 + u_2 \implies a_2 = l_{21}u_1 + u_2 \\ \boldsymbol{c)} \quad a_3 = \begin{bmatrix} l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ \hline u_2 \\ \hline u_3 \end{bmatrix} = l_{31}u_1 + l_{32}u_2 + u_3 \implies a_3 = l_{31}u_1 + l_{32}u_2 + u_3 \end{cases} \tag{4.1}$$

**a)**

$a_1 = u_1 \implies$ vektori $a_1$ i $u_1$ su linearno zavisni.
$L(a_1) = \{\forall p_x \in R^x \quad | \quad p_x = \lambda a_1 \quad | \quad \lambda \in R\}$
$L(u_1) = \{\forall q_x \in R^x \quad | \quad q_x = \xi u_1 \quad | \quad \xi \in R\}$

$\implies \lambda = \xi \implies L(a_1) = L(u_1)$

**b)**

$a_2 = l_{21}u_1 + u_2 \implies$ vektori $a_2, u_1$ i $u_2$ su linearno zavisni.
$L(a_1, a_2) = \{\forall p_x \in R^x \quad | \quad p_x = \lambda_1 a_1 + \lambda_2 a_2 \quad | \quad \lambda_1, \lambda_2 \in R\}$

$L(u_1, u_2) = \{\forall q_x \in R^x \quad | \quad q_x = \xi_1 u_1 + \xi_2 u_2 \quad | \quad \xi_1, \xi_2 \in R\}$

$p_x = q_x \quad \wedge \quad a_1 = u_1 \quad \wedge \quad a_2 = l_{21} u1 + u_2$

$$\implies \lambda_1 a_1 + \lambda_2 a_2 = \xi_1 u_1 + \xi_2 u_2$$
$$\implies \lambda_1 u_1 + \lambda_2 (l_{21} u_1 + u_2) = \xi_1 u_1 + \xi_2 u_2$$
$$\implies u_1(\lambda_1 + \lambda_2 l_{21}) + u_2(\lambda_2) = u_1(\xi_1) + u_2(\xi_2)$$

$$\implies \xi_1 = \lambda_1 + l_{21}\lambda_2, \quad \xi_2 = \lambda_2 \implies L(a_1, a_2) = L(u_1, u_2)$$

**c)**

$a_3 = l_{31} u_1 + l_{32} u_2 + u_3 \implies$ vektori $a_3, u_1, u_2$ i $u_3$ su linearno zavisni.
$L(a_1, a_2, a_3) = \{\forall p_x \in R^x \quad | \quad p_x = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 \quad | \quad \lambda_1, \lambda_2, \lambda_3 \in R\}$
$L(u_1, u_2, u_3) = \{\forall q_x \in R^x \quad | \quad q_x = \xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3 \quad | \quad \xi_1, \xi_2, \xi_3 \in R\}$

$p_x = q_x \quad \wedge \quad a_1 = u_1 \quad \wedge \quad a_2 = l_{21} u_1 + u_2 \quad \wedge \quad a_3 = l_{31} u_1 + l_{32} u_2 + u_3$

$$\implies \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = \xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3$$
$$\implies \lambda_1(u_1) + \lambda_2(l_{21} u_1 + u_2) + \lambda_3(l_{31} u_1 + l_{32} u_2 + u_3) = \xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3$$
$$\implies u_1(\lambda_1 + \lambda_2 l_{21} + \lambda_3 l_{31}) + u_2(\lambda_2 + \lambda_3 l_{32}) + u_3(\lambda_3) = (\xi_1) u_1 + (\xi_2) u_2 + (\xi_3) u_3$$

$$\implies \xi_1 = \lambda_1 + \lambda_2 l_{21} + \lambda_3 l_{31}, \quad \xi_2 = \lambda_2 + \lambda_3 l_{32}, \quad \xi_3 = \lambda_3 \implies L(a_1, a_2, a_3) = L(u_1, u_2, u_3)$$

Za kolone matrica $A$ i $L$:

$$A = \begin{bmatrix} a_1 & | & a_2 & | & a_3 \end{bmatrix} = \begin{bmatrix} l_1 & | & l_2 & | & l_3 \end{bmatrix} \left[\begin{array}{c|c|c} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{array}\right]$$

Dobijamo sistem analogan sistemu (4.1):

$$\iff \begin{cases} \textbf{\textit{a})} \quad a_1 = \begin{bmatrix} l_1 & | & l_2 & | & l_3 \end{bmatrix} \begin{bmatrix} u_{11} \\ 0 \\ 0 \end{bmatrix} = u_{11} l_1 \implies a_1 = u_{11} l_1 \\ \textbf{\textit{b})} \quad a_2 = \begin{bmatrix} l_1 & | & l_2 & | & l_3 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \\ 0 \end{bmatrix} = u_{12} l_1 + u_{22} l_2 \implies a_2 = u_{12} l_1 + u_{22} l_2 \\ \textbf{c)} \quad a_3 = \begin{bmatrix} l_1 & | & l_2 & | & l_3 \end{bmatrix} \begin{bmatrix} u_{13} \\ u_{23} \\ u_{33} \end{bmatrix} = u_{13} l_1 + u_{23} l_2 + u_{33} l_3 \implies a_3 = u_{13} l_1 + u_{23} l_2 + u_{33} l_3 \end{cases} \tag{4.2}$$

**a)**

$a_1 = u_{11} l_1 \implies$ vektori $a_1$ i $l_1$ su linearno zavisni.
$L(a_1) = \{\forall s_x \in R^x \quad | \quad s_x = \epsilon_1 a_1 \quad | \quad \epsilon_1 \in R\}$

$L(l_1) = \{\forall t_x \in R^x \quad | \quad t_x = \eta_1 l_1 \quad | \quad \eta_1 \in R\}$

$\Longrightarrow \; \epsilon_1 = u_{11}\eta_1 \; \Longrightarrow \; L(a_1) = L(l_1)$

**b)**

$a_2 = u_{12}l_1 + u_{22}l_2 \; \Longrightarrow$ vektori $a_2, l_1$ i $l_2$ su linearno zavisni.
$L(a_1, a_2) = \{\forall s_x \in R^x \quad | \quad s_x = \epsilon_1 a_1 + \epsilon_2 a_2 \quad | \quad \epsilon_1, \epsilon_2 \in R\}$
$L(l_1, l_2) = \{\forall t_x \in R^x \quad | \quad t_x = \eta_1 l_1 + \eta_2 l_2 \quad | \quad \eta_1, \eta_2 \in R\}$

$s_x = t_x \quad \wedge \quad a_1 = u_{11}l_1 \quad \wedge \quad a_2 = u_{12}l_1 + u_{22}l_2$

$\Longrightarrow \; \epsilon_1 a_1 + \epsilon_2 a_2 = \eta_1 u_1 + \eta_2 u_2$
$\Longrightarrow \; \epsilon_1 u_{11} l_1 + \epsilon_2(u_{12}l_1 + u_{22}l_2) = \eta_1 l_1 + \eta_2 l_2$
$\Longrightarrow \; l_1(\epsilon_1 u_{11} + \epsilon_2 u_{12}) + l_2(\epsilon_2 u_{22}) = l_1(\eta_1) + l_2(\eta_2)$

$\Longrightarrow \; \eta_1 = \epsilon_1 u_{11} + \epsilon_2 u_{12}, \quad \eta_2 = \epsilon_2 u_{22} \; \Longrightarrow \; L(a_1, a_2) = L(l_1, l_2)$

**c)**

$a_3 = u_{13}l_1 + u_{23}l_2 + u_{33}l_3 \; \Longrightarrow$ vektori $a_3, l_1, l_2$ i $l_3$ su linearno zavisni.
$L(a_1, a_2, a_3) = \{\forall s_x \in R^x \quad | \quad s_x = \epsilon_1 a_1 + \epsilon_2 a_2 + \epsilon_3 a_3 \quad | \quad \epsilon_1, \epsilon_2, \epsilon_3 \in R\}$
$L(l_1, l_2, l_3) = \{\forall t_x \in R^x \quad | \quad t_x = \eta_1 l_1 + \eta_2 l_2 + \eta_3 l_3 \quad | \quad \eta_1, \eta_2, \eta_3 \in R\}$

$s_x = t_x \quad \wedge \quad a_1 = u_{11}l_1 \quad \wedge \quad a_2 = u_{12}l_1 + u_{22}l_2 \quad \wedge \quad a_3 = u_{31}l_1 + u_{23}l_2 + u_{33}l_3$

$\Longrightarrow \; \epsilon_1 a_1 + \epsilon_2 a_2 + \epsilon_3 a_3 \; = \eta_1 l_1 + \eta_2 l_2 + \eta_3 l_3$
$\Longrightarrow \; \epsilon_1(u_{11}l_1) + \epsilon_2(u_{12}l_1 + u_{22}l_2) + \epsilon_3(u_{13}l_1 + u_{23}l_2 + u_{33}l_3) = \eta_1 l_1 + \eta_2 l_1 + \eta_3 l_1$
$\Longrightarrow \; l_1(\epsilon_1 u_{11} + \epsilon_2 u_{12} + \epsilon_3 u_{13}) + l_2(\epsilon_2 u_{22} + \epsilon_3 u_{23}) + l_3(\epsilon_3 u_{33}) = (\eta_1)l_1 + (\eta_2)l_2 + (\eta_3)l_3$

$\Longrightarrow \; \eta_1 = \epsilon_1 u_{11} + \epsilon_2 u_{12} + \epsilon_3 u_{13}, \quad \eta_2 = \epsilon_2 u_{22} + \epsilon_3 u_{23}, \quad \eta_3 = \epsilon_3 u_{33}$
$\Longrightarrow \; L(a_1, a_2, a_3) = L(l_1, l_2, l_3)$

**5.** Neka je $A$ regularna matrica i $u, v$ vektori takvi da važi $1 + v^T A^{-1} u \neq 0$. Dokazati formulu Šermana i Morisona:

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1} u}. \tag{5}$$

**Rešenje:** Inverzna matrica je definisana kao:

$MM^{-1} = M^{-1}M = I$

**1°**

$I = (A + uv^T)(A + uv^T)^{-1} = (A + uv^T)(A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1} u}) =$

$= AA^{-1}+uv^TA^{-1}-\frac{AA^{-1}uv^TA^{-1}+uv^TA^{-1}uv^TA^{-1}}{1+v^TA^{-1}u} = I+uv^TA^{-1}-\frac{Iuv^TA^{-1}+u(v^TA^{-1}u)v^TA^{-1}}{1+v^TA^{-1}u}$

S obzirom da je $v^TA^{-1}u \in R \neq -1$

$\implies I + uv^TA^{-1} - uv^TA^{-1}\frac{1+v^TA^{-1}u}{1+v^TA^{-1}u} = I + uv^TA^{-1} - uv^TA^{-1} * 1 = I$

Ostaje još da dokažemo i: $(A+uv^T)^{-1}(A+uv^T) = I$ kako bi polazna jednačina (5) važila.

**2°**

$I = (A+uv^T)^{-1}(A+uv^T) = (A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u})(A+uv^T) =$

$= A^{-1}A+A^{-1}uv^T-\frac{A^{-1}uv^TA^{-1}A+A^{-1}uv^TA^{-1}uv^T}{1+v^TA^{-1}u} = I+A^{-1}uv^T-\frac{A^{-1}uv^TI+A^{-1}u(v^TA^{-1}u)v^T}{1+v^TA^{-1}u} =$

$= I + A^{-1}uv^T - A^{-1}uv^T\frac{1+v^TA^{-1}u}{1+v^TA^{-1}u} = I + A^{-1}uv^T - A^{-1}uv^T * 1 = I$

Na osnovu (1°) i (2°) $\implies (A+uv^T)^{-1} = (A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u})$.