UNIVERSITY DEPARTMENT OF MATHEMATICS

Tilka Manjhi Bhagalpur University, Bhagalpur

Assignment – III

Due Date: 10–09-19 PAPER – III Session: 2018–20

1. Problems on Inner Product Space
   (a) Show that the set of all square matrices of order n over $\mathbb{R}$ forms an inner product space with the innner product is given by
   $$\langle A, B \rangle = \text{trace } B^T A$$
   (b) Show that the set of all continuous function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, forms an inner product space with inner product given by
   $$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx$$

2. On the vector space over $\mathbb{R}$, consider three vectors
   $$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$
   Suppose that $\mathbf{v}_4$ is another vector which is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_3$, and satisfying
   $$\mathbf{v}_2 \cdot \mathbf{v}_4 = -3.$$
   i. $\mathbf{v}_1 \cdot \mathbf{v}_2$
   ii. $\mathbf{v}_3 \cdot \mathbf{v}_4$
   iii. $(2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3) \cdot \mathbf{v}_4$
   iv. $\|\mathbf{v}_1\|$, $\|\mathbf{v}_2\|$, $\|\mathbf{v}_3\|$
   v. What is the distance between $\mathbf{v}_1$ and $\mathbf{v}_2$?

3. A set of nonzero orthogonal vectors forms a linearly independent set.

4. Show that the set of all bilinear forms over a vector space V over a field F, also forms a vector space over the same field F.

5. Problems on Gram Schmidt Orthogonalizaton
   (a) Let W be the subspace of $\mathbb{R}^4$ spanned by
   $$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$
   Construct an orthonormal basis of W.
   (b) Let $\mathcal{P}_2$ be the set of all polynomials of degree less than 3 over real numbers with the inner product given by
   $$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)\, dx.$$
   Starting with a basis $(1, x, x^2)$, find an orthonormal basis.

6. Problems on Matrix of Bilinear Form

(a) Let $\mathcal{P}_2$ be the set of all polynomials of degree less than 3 over real numbers with the bilinear transformation given by

$$H(f,g) = \int_{-1}^{1} f(x)g(x)\,dx.$$

Find the matrix of bilinear forms with respect to the following bases.

i. $\beta_1 = (1, x, x^2)$

ii. $\beta_2 = (1, 1+x, 1+x^2)$

7. Problems on quadratic forms

(a) Write the following quadratic form in matrix form. Find it's rank and signature.

i. $K(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$

ii. $K(x_1, x_2) = 5x_1^2 - 10x_1x_2 + x_2^2$

iii. $K(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 - 6x_1x_3 + 8x_2x_3$

(b) Reduce the quadratic form into the canonical form and decide the definiteness of a quadratic form

i. $-x^2 - y^2 - 2z^2 + 2xy$

ii. $x^2 - 2xy + xz + 2yz + 2z^2 + 3zx$

iii. $-4x^2 - y^2 + 4xz - 2z^2 + 2yz$

iv. $-x^2 - y^2 + 2xz + 4yz + 2z^2$

v. $-x^2 + 2xy - 2y^2 + 2xz - 5z^2 + 2yz$

vi. $y^2 + xy + 2xz$