$$\underset{\sim}{Y} \sim N(\underline{X}\,\underline{\beta}, \underline{\Sigma}) \qquad \underline{Y} = \underset{n \times 1}{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}} \qquad \underline{\beta} = \underset{k+1 \times 1}{\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{k+1} \end{pmatrix}}$$

$$\underset{n \times k+1}{\underline{X}} = \begin{pmatrix} 1 & x_{11} & x_{21} & \cdots & x_{k1} \\ 1 & x_{12} & x_{22} & \cdots & x_{k2} \\ 1 & x_{13} & x_{23} & \cdots & x_{k3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{1n} & x_{2n} & \cdots & x_{kn} \end{pmatrix} \qquad \underset{n \times n}{\underline{\Sigma}} = \begin{pmatrix} \sigma^2 & & & 0 \\ & \sigma^2 & & \\ & & \ddots & \\ 0 & & & \sigma^2 \end{pmatrix}$$

$$= \sigma^2 \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \; 1 \end{pmatrix}$$

$$\ell(y,x,\beta,\Sigma) = -\frac{n}{2}\log 2\pi - \frac{1}{2}\log\left(\det(\Sigma)\right) - \frac{1}{2}(y-X\beta)'\,\Sigma^{-1}(y-X\beta)$$

$$\propto -\frac{1}{2}\log\left(\det\left(\sigma^2 I\right)\right) - \frac{1}{2}(y-X\beta)'\,{}^{1}\!/\!\sigma^{2}\, I^{-1}(y-X\beta)$$

$$\propto -\frac{1}{2}\log\left(\sigma^2\right)^n - \frac{1}{2\sigma^2}(\underline{y}-\underline{X}\underline{\beta})'(\underline{y}-\underline{X}\underline{\beta})$$

$\boxed{\text{MLE} - \beta}$

$$\frac{\partial\, \ell(\cdot)}{\partial \underline{\beta}} = -\frac{1}{2\sigma^2}\frac{\partial}{\partial \underline{\beta}}\left[(y-X\beta)'(y-X\beta)\right]$$

$$= \frac{1}{2\sigma^2}\frac{\partial}{\partial\beta}\left[y'y - \underbrace{y'X\beta}_{1\times 1} - \underbrace{\beta X' y}_{1\times 1} + \beta'X'X\beta\right]$$

$$= \frac{1}{2\sigma^2}\left[-\cancel{2}\beta'X'y + \cancel{2}X'X\beta\right] = 0$$

$$-X'y + X'X\beta = 0$$

$$X'X\beta = X'y$$

$$(X'X)^{-1}(X'X)\beta = (X'X)^{-1}(X'y)$$

$$\hat{\beta}_{MLE} = (X'X)^{-1}X'y$$

$$\hat{y} = X\hat{\beta}_{MLE} = X(X'X)^{-1}X'y$$

$$\ell(\cdot) \propto -\frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(\underline{Y}-\underline{X}\underline{\beta})'(\underline{Y}-\underline{X}\underline{\beta})$$

$$\frac{\partial \ell(\cdot)}{\partial\sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2(\sigma^2)^2}(Y-X\beta)'(Y-X\beta)$$

$$= \frac{1}{2\sigma^2}\left(-n + \frac{1}{\sigma^2}(\underline{Y}-\underline{X}\underline{\beta})'(\underline{Y}-\underline{X}\underline{\beta})\right) = 0$$

$\boxed{\text{Assume } \sigma^2 > 0}$

$$-n + \frac{1}{\sigma^2}(\underline{Y}-\underline{X}\underline{\beta})'(\underline{Y}-\underline{X}\underline{\beta}) = 0$$

$$\sigma^2 = \frac{1}{n}(Y-X\beta)'(Y-X\beta) \quad \text{plug in } \hat{\beta} \Rightarrow X\hat{\beta} = \hat{Y}$$

$$= \frac{1}{n}(Y-\hat{Y})'(Y-\hat{Y}) = \frac{1}{n}\sum_{i=1}^{n}(Y_i-\hat{Y}_i)^2$$