1 Monoidal Pop Functor

Definition 1 (Monoidal Pop Functor). *The functor $Pop : \mathbf{Sphere} \to [\mathbf{Field}, \mathbf{Field}]$ is defined as:*

- *On objects:* $Pop(\Omega) = [\mathcal{F}_\Omega, \mathcal{F}_\Omega]$.
- *On morphisms:* $Pop(\sigma) = \mathcal{C}_\sigma : \mathcal{F}_{\Omega_1} \to \mathcal{F}_{\Omega_2}$.
- *Tensor product: For* $supp(\sigma_1) \cap supp(\sigma_2) = \emptyset$,

$$Pop(\sigma_1 \otimes \sigma_2) = Pop(\sigma_1) \otimes_{\mathbf{Field}} Pop(\sigma_2),$$

  where $(\mathcal{C}_{\sigma_1} \otimes_{\mathbf{Field}} \mathcal{C}_{\sigma_2})(\mathcal{F}_{\Omega_1 \sqcup \Omega_2}) = (\mathcal{C}_{\sigma_1}(\mathcal{F}_{\Omega_1}), \mathcal{C}_{\sigma_2}(\mathcal{F}_{\Omega_2}))$.
- *Unit:* $Pop(I) = id_{\mathcal{F}_\emptyset}$.

With coherence maps:

- *Associator:* $\alpha_{Pop} : Pop((\sigma_1 \otimes \sigma_2) \otimes \sigma_3) \xrightarrow{\sim} Pop(\sigma_1 \otimes (\sigma_2 \otimes \sigma_3))$.
- *Left unitor:* $\lambda_{Pop} : Pop(I \otimes \sigma) \xrightarrow{\sim} Pop(\sigma)$.
- *Right unitor:* $\rho_{Pop} : Pop(\sigma \otimes I) \xrightarrow{\sim} Pop(\sigma)$.

Theorem 1. *Pop is a monoidal functor, satisfying:*

$$Pop(\sigma_1 \otimes \sigma_2) \cong Pop(\sigma_1) \otimes_{\mathbf{Field}} Pop(\sigma_2), \quad Pop(I) \cong id_{\mathcal{F}_\emptyset}.$$

2 2-Category $\mathbf{Sphere_2}$

Definition 2 (2-Category $\mathbf{Sphere_2}$). *The 2-category* $\mathbf{Sphere_2}$ *consists of:*

- *0-cells: Regions* $\Omega \subseteq \mathbb{R}^n$.
- *1-cells: Spheres* $\sigma : \Omega_1 \to \Omega_2$, *where* $\sigma = (supp(\sigma), \mathcal{C}_\sigma)$.
- *2-cells: Natural transformations* $\tau : \sigma_1 \Rightarrow \sigma_2$, *a family of morphisms* $\tau_\mathcal{F} : \mathcal{C}_{\sigma_1}(\mathcal{F}) \to \mathcal{C}_{\sigma_2}(\mathcal{F})$, *natural in* $\mathcal{F} \in \mathbf{Field}(\Omega_1)$, *satisfying:*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\mathcal{C}_{\sigma_1}"] \arrow[d, "\mathcal{C}_{\sigma_2}"] & \mathcal{C}_{\sigma_1}(\mathcal{F}) \arrow[d, "\tau_\mathcal{F}"] \\ \mathcal{C}_{\sigma_2}(\mathcal{F}) \arrow[r, "id"] & \mathcal{C}_{\sigma_2}(\mathcal{F}) \end{tikzcd}$$

Composition:

- *Horizontal:* $(\upsilon \circ \tau)_\mathcal{F} = \upsilon_{\mathcal{C}_{\sigma_2}(\mathcal{F})} \circ \tau_\mathcal{F}$.
- *Vertical:* $(\tau' \cdot \tau)_\mathcal{F} = \tau'_\mathcal{F} \circ \tau_\mathcal{F}$.

Identities: 1-cell $id_\Omega = (\emptyset, id_{\mathcal{F}_\Omega})$, *2-cell* id_σ *with components* $id_{\mathcal{C}_\sigma(\mathcal{F})}$.

Theorem 2. $\mathbf{Sphere_2}$ *satisfies the 2-category axioms, including associativity, identity laws, and the interchange law.*

3 Topos Structure

Definition 3 (Presheaf Category)**.** *The presheaf category* $\mathbf{Sphere}^{op} = [\mathbf{Sphere}^{op}, \mathbf{Set}]$ *consists of functors* $P : \mathbf{Sphere}^{op} \to \mathbf{Set}$*, where:*

- $P(\Omega)$*: Field observations over* Ω*.*
- $P(\sigma : \Omega_1 \to \Omega_2)$*: Observation pullback* $P(\Omega_2) \to P(\Omega_1)$*.*

Theorem 3. $\mathbf{Sphere}^{op}$ *is a topos with:*

- *Subobject classifier:* $\Omega_{\mathbf{Sphere}} = \{\textit{open regions}\}$*.*
- *Exponentials:* $P^Q(\Omega) = Hom_{\mathbf{Presheaf}}(Q|_{\mathbf{Sphere}/\Omega}, P|_{\mathbf{Sphere}/\Omega})$*.*
- *Finite limits and colimits: Computed pointwise in* $\mathbf{Set}$*.*

Theorem 4 (Internal Logic)**.** $\mathbf{Sphere}^{op}$ *supports intuitionistic higher-order logic, with:*

- *Propositions: Subspheres of the truth sphere.*
- *Proofs: Sphere morphisms preserving truth.*
- *Quantification: Over regions and field states.*