Graph Theory

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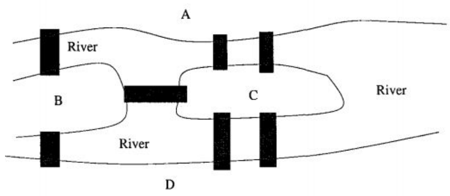
Contents

1 Introduction

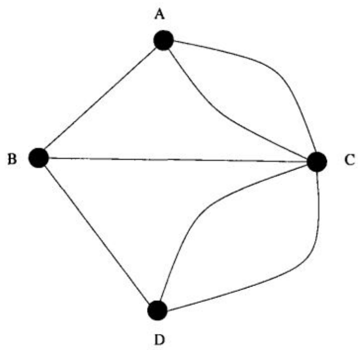
The old city of Konigsberg(currenly known as kalingrad) consisted of islands where two branches of the river Pregel joined.Seven bridges connecting dif-



ferent islands is shown in the picture. A famous mathematician Leonhard Euler became interested in solving the following question answering which led to the foundations of modern day graph theory.

> “ Is it possible to walk through the town,starting and ending at the same place ,so that we use each bridge exactly once?”

While solving the problem one thing became very clear to him ,that the only relevant pieces of information here are those of connectivity .So instead of using the whole map of Konigsberg he used the following drawing which later came to be known as a graph.[1]



2 Terminology

2.1 Graph

A graph is an ordered pair $G = (V, E)$ comprising a set V of *vertices* or *nodes* with a set of E *edges* or *lines*, which connect two vertices or technically speaking are 2-element subsets of V [3].

2.1.1 Simple Graph

If a graph has no loops and no multiple edges then it is called a simple graph.

2.1.2 Connnected Graph

If the graph has the property that for any two vertices x and y,one can find a path from x to y,then we say that the graph G is a connected

2.2 Trail

A sequence of distinct edges is called a trail if we can take a continuous walk in our graph. i.e The ending node of an edge is the starting node of the next edge in the sequence.

2.2.1 Closed trail

If in a walk the the starting node of the first edge in the sequence is equal to the ending node of the last vertex then it is a closed walk.

2.2.2 Eulerian Trail

If a trail uses all edges of a graph then it is called an Eulerian Path.

2.3 Path

If a trail does not touch any vertex twice it is a path.

3 Eulerian Trails

3.1 Closed Eulerian trail in a closed connected graph

Theorem 1. *A connected graph G has a closed Eulerian loop if and only if all vertices of G have even degree.*

Proof. First we preove the necessary condition.that is we show that if G has a closed Eulerian trail T then all its vertices have an even degree. Necessary or the only if condition Let T represent the closed eulerian trail that the graph G has.And let V be any vertex that was not where T started and assume that we visited A exactly p times.All it means is we entered vertex Vp times and also left it p times.As T is a trail it would have taken different edges to enter vertex A and also to leave it.So we used $2p$ edges in all inorder to to visit the verex p times.From the definition of Eulerian trail we know that T has all the edges of G,so a cannot have any additional edges sticking to it. The above argument shows that the degree of any vertex other than the starting vertex is even. Now let S be the starting vertex ,suppose we visited it q times apart from starting from it and ending in it then total number of edges we used to visit S is $1 + 2q + 1$ ie $2(q + 1)$ times.So its degree has to be even.And this proves our claim.

The above necessary condition was also showed by Euler whereas the proof of sufficiency which follows is something that was proved much later.

Now we consider the proof of sufficiency. Assume all vertices of G have even degree and we have to show that it has a closed Eulerian trail. Start walking from any vertex S along an edge e_1 to the other end of the edge and continue walking this way,(using new or unused vertices)until a closed cycle C_1 is formed which would happen only when we revisit a vertex.As the degree of each vertex is even,we cannot get stuck at any vertex,so each time we enter a vertex we will have to leave it except possibly the starting vertex.If $C_1 = G$ (it is a simple case where degree of each vertex is 2)then we are done.If not ,then choose a vertex V in C_1 so that it does not contain all edges adjacent to it V.

How do we know that there is such a vertex V***?***
Assume that there is no such vertex.As C_1 contains less edges than than G, and supposedly contains all edges adjacent to all the vertices that it contains,then there must be a vertex A that is not in C_1.However, $G$is a connected graph,so there must be a path connecting A to any vertex of C_1.If we start walking from A towards any given vertex of C_1,we'll reach C_1 first in a vertex say V(of C_1),but not all the edges adjacent to it are in c1 because the one through which he just now landed in V is not.This prooves by contradiction that such a vertex always exists.

Now omit all the edges of C_1 from G.We get a graph in which again all the edges are even.(Because (even - 2) is even).Starting at V take another closed trail C_2 in the remaining graph.We can then unite C_1 and C_2 into one closed trail $C_1 \cup C_2$ in G.If the new trail $C_1 \cup C_2$ contains all the edges of G then we are done.If not, then we omit $C_1 \cup C_2$ from G and find a closed trial C_3 in the remaining graph.As G is has finite number of edges the process has to stop in a finite number of steps resulting in $C_1 \cup C_2 \cup ..... \cup C_k$ will be a closed trial containing all the edges of G. $\square$

This theorem shows that we cannot walk through all the bridges of Königsberg so that we end where we started.,and use each bridge exactly once.Because all the vertices of the graph simplified for königsberg have odd degree. [3mm]

Corollary 1. *Let G be a connected graph. Then G has an Eulerian trail starting at vertex T if and only if S and T have odd degre,and all other* *vertices of G have even degree.*

Proof. Add a new edge joining S and T, and call the new graph obtained H. Then H has a closed Eulerian trail $\iff$ G has an Eulerian trail from S to T,
so the claim follows from the Theorem 1. □

Theorem 2. *In a graph G without loops, the number of vertices of odd degree is even.*

Proof. Take such a graph with e edges. Let $d_1, d_2, ..., d_n$ be the degrees of the n vertices of G.We claim that

$$d_1 + d_2 = ... + d_n = 2e$$

The above follows because each edge contributes to two degrees one degree for each vertex it is connected to.Now the sum of degrees of odd vertices has to be even,because when we subtract sum of degrees of even degreed vertices from total number of degrees which is even the result is even.We know that the sum of odd summands is even only when there is an even number of odd summands. □

4 Isomorphism and Automorphism in Graphs

We say the graphs G and H are isomorphic if there is a bijection f from the vertex set of G onto that of H so that the number of edges between any pair of vertices X and Y of G is equal to the number of edges between vertces $f(X)$ and $f(Y)$ of H.The bijection f is called an isomorphism.

An *automorphism* of a graph G is an isomorphism between G and G itself.That is, the permutation f of the vertex set of G is an automorphism of G if for any two vertices x and y. of G,the number of edges between $f(x)$ and $f(y)$.

5 Hamiltonian Cycles

Cycle,Hamiltonian Cycle and Hamiltonian Path

A *cycle* in a graph is a closed trail that does not touch any vertex twice except the initial starting vertex.So if a cycle has k edges then it will also have k vertices.In other words, the degree of each vertex in a cycle is 2.A cycle that includes all the vertices of a graph if exists will be known as the *Hamiltonian cycle*,whereas a path that includes all vertices of a graph will be known as *Hamiltonian path.*
Whether a graph will have a hamiltonian cycle or not depends on the graph

itself.For example if each and every vertex has degree 2 and is connected to two different vertices then it certainly has a Hamiltonian cycle also if each vertex of a graph is connected to every other vertice then also it will certainly have at leat one Hamiltonian cycle,(Obviously only if number of vertices exceeds two!)

The following are theorems relating to Hamiltonian cycles some of which I directly state without proof.

Theorem 3. ***(Ore's Theorem)****Let G be a simple graph on n vertices.If n≥3,and*

$$\delta(x) + \delta(y) \geq n$$

for each pair of non adjacent vertices x and y,then G has a closed hamiltonian path.

Proof. Suppose we have been given that graph G holds the condiyion stated in the theorem that sum of degrees of every pair of non-adjacent vertices exceeds n, and now for a contradiction G does not have a Hamiltonian path.

We pick any two unconnected vertices of graph G and add a new edge between them.We keep on doing this until we reach a graph G_f which has a Hamiltonian cycle.(This process has to stop after a finite number of steps because the graph is finite and ultimately we will reach a position where we get a strongly connected graph,where every possible pair of vertices are connected,and we know that this surely has a Hamiltonian path.) Let G_* be the graph obtained immediately before G_f,and suppose $\{x, y\}$ be the edge added added to G^* to make it G_f.
Let $(z_1, z_2, z_3, ..., z_n, z_1)$ be a Hamiltonian cycle in G_f.This must use the edge $\{x, y\}$ at some point.If $\{z_n, z_1\} = \{x, y\}$ then then $(z_1, z_2, .., z_n)$is a Hamiltonian Path in G^*.If that's not the case then there is some r such that $1 \leq r < n$ and $z_r = x$ and $z_{r+1} = y$;now

$$(z_{r+1}, ...z_n, z_1, ..., z_r)$$

is a Hamiltonian path in G^*.Note that in either way, all the edges used in this path appear in G_*.So it is only $\{x, y\}$ that appears in G_f but not in G_*.So we relabel the sequence of vertices in this path to be $(x_1, x_2, ..., x_n)$.

Suppose we find a vertex x_i such that x is adjacent to x_i,and y is adjacent to x_{i-1}.Then $(x, x_i, ..., x_{n-1}, y, x_{i-1}, ..., x)$ will be a Hamiltonian cycle in G^*,which contradicts our assumption.
So the only step remaining in the proof is to show that such a vertex x_i exists in the graph G.As G^* is obtained from G it still satisfies the hypothesis on degree.Now let

P={i: $2 \leq i \leq n$ and x_i is adjacent to x}
Q={i: $2 \leq i \leq$ n and x_{i-1} is adjacent to y}.

As our graphs are assumed to be simple,$\|P\|$=$\delta(x)$ and $\|Q\|$=$\delta(y)$ .x and y are non adjacent because we joined them in the final step to get G_f from G^* so the condition that $\delta(x) + \delta(y) \geq n$ holds from our assumption that the above condition is valid for all non adjacent vertices.

Now P and Q are subsets of $\{2, ..., n\}$ containing atleast n elements between them.So $\exists$ some $i \in P \cap Q$ which does the job.[4] $\square$

Theorem 4. ***(Dirac's Theorem)*** *Let $n \geq 3$,let G be a simple graoh on n vertices, and let us assume that vertices in G are of degree at least $n/2$. Then G has a Hamiltonian cycle.*

Except some special cases there is no general way,however, to test whether two graphs are isomorphic.That is,unless we verify all $n!$ bijections from G to H, where n is the number of vertices of each graph.

6 Problems

6.1 Regular Graph

A simple graph is called *regular* if all its vertices have the same degree.Let G be a connected graph with 22 edges.How many vertices can G have?

Let G have v vertices with each verice having degree d.Total number of degrees is dv which is 44 (equal to twice the number of edges) for this case.Now 44=($2^2$11) can be expressed as (2×22) or (4×11) or (1×44).As it is a simple graph it cannot have more edges than vertices so we have three cases.The first case with $v = 44, d = 1$,is clearly not possible because it would no longer be connected and would consist of disjoint connected pairs of vertices.The second case with $v = 22, d = 2$ is clearly possible as it is a simple cycle of 22 vertices.Now the last case where $v = 11, d = 4$ is also possible as maximum possible degree for it to remain a simple connected graph is 10 where each vertex is connected to every other vertex.

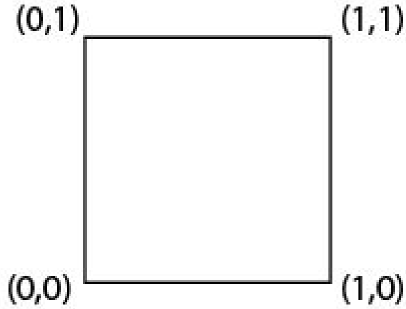
6.2 Hamiltonian cycle in a Hypercube

For graph theoretical purposes,the *the n-dimensional hypercube* Q_n is a simple graph whose vertices are the 2^n points $(x_1, x_2, ..., x_n) \in R^n$ so that for each i∈[n], either $x_i = 0$ or $x_i = 1$, and in which two vertices are adjacent if they agree in exactly *n-1* coordinates.

Prove that if $n \geq 2$,then Q_n has a Hamiltonian cycle.

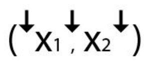
Proof. We will prove this by induction.
In Q_2 (or *square*) we have exactly one hamiltonian cycle that is

$$\{(0,0),(1,0),(1,1),(0,1),(0,0)\}.$$



Now when we go from 2-D to 3-D,we can go the following way:

Create two new sets of cycles by adding the third coordinate at any of the three positions(as shown in the figure below) in each vertice of the 2-D Hamiltonian cycle.



Suppose we add third coordinate at position 3,then the resulting sets are:

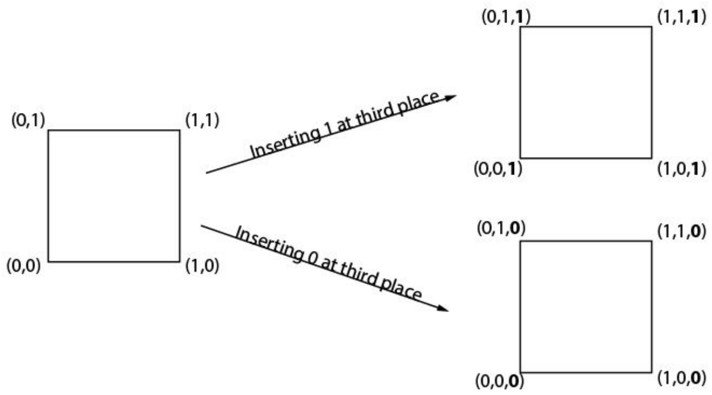
$$\{(0,0,\mathbf{0}),(1,0,\mathbf{0}),(1,1,\mathbf{0}),(0,1,\mathbf{0})\}$$

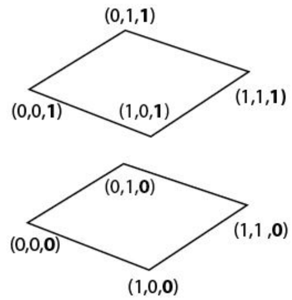
$$\{(0,0,\mathbf{1}),(1,0,\mathbf{1}),(1,1,\mathbf{1}),(0,1,\mathbf{1})\}$$

Both of the above two resulting sets are cycles in three dimensions and their union contains all the vertices of the Q_3.

Let us now define *corresponding vertices* in the two cycles created as those vertices which differ by only third coordinate or the coordinate,whose introduction created two new hamiltonian cycles.
Now,all we have, is two cycles spanning all the vertices of Q_3.So to construct a Hamiltonian we somehow have to unite these cycles into one cycle,but there is a constraint,we can only move from one vertex to its adjacent vertex.
The above problem can be overcome by following the steps given below:

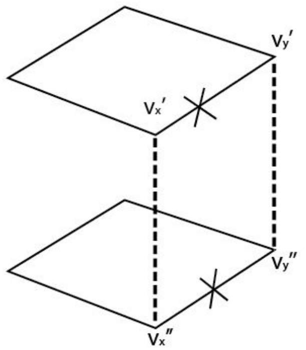
1. In any one of the two cycles created,say C_1,select a pair of adjacent vertices(*connected by an edge of the cycle*).Say we call them V_x' and V_y' and their *corresponding vertices* in the other cycle(C_2) as V_x'' and V_y''.



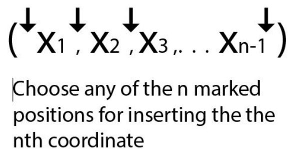


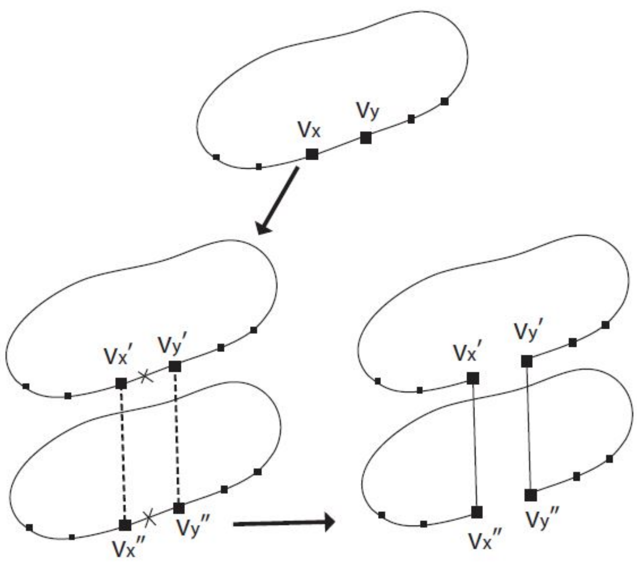
2. Construct a new cycle by:
   - Deselecting the edge between V'_x and V'_y in C_1
   - Deselecting the edge between V''_x and V''_y in C_2
   - Connecting V'_x in C_1 to V''_x in C_2
   - Connecting V'_y in C_1 to V''_y in C_2

The following figure demonstrates the Hamiltonian cycle construction for Q_3.



Now if we have a Hamiltonian cycle in Q_{n-1} a similar construction procedure can be followed to obtain a Hamiltonian cycle in Q_n ,which starts by creation of two disjoint cycles by first inserting 0 at any of the n places available and then inserting 1 at the same position and culminates with the steps enlisted above.





Thus by induction we see that at least one Hamiltonian cycle exists for every Q_n if $n \geq 2$. $\square$

Prove that if $n \geq 2$,then Q_n has atleast $\frac{n!}{2}$ Hamiltonian cycles

Proof. Here too we shall proceed by induction. Q_2 has 1(i.e.$\frac{2!}{2}$) cycle,so the base case is satisfied.

Now suppose Q_{n-1} has atleast $\frac{(n-1)!}{2}$ cycles.
Consider any one of the $n-1$ dimensional Hamiltonian cycles.As in the above problem here too we shall construct a Hamiltonian cycle inQ_n.Let C be the considered cycle.Now we proceed exactly as in the previous problem.

1. Choose one out of the n positions availible,to insert the nth coordinate.

$$(\downarrow x_1, \downarrow x_2, \downarrow x_3, \ldots x_{n-1} \downarrow)$$

2. Create a n dimensional cycle C_1 by inserting **0** at the chosen position in each of the vertices.It will have half of the vertices of Q_n
3. Create the other n dimensional cycle C_2 by inserting **1** at the chosen position in each of the vertices.This cycle will have the other half of total vertices of Q_n.
4. In say C_1,select a pair of adjacent vertices(*connected by an edge of the cycle*).Say we call them V'_x and V'_y and their *corresponding vertices* in the other cycle C_2 as V''_x and V''_y.
5. Construct a new cycle by:
   (a) Deselecting the edge between V'_x and V'_y in C_1
   (b) Deselecting the edge between V''_x and V''_y in C_2
   (c) Connecting V'_x in C_1 to V''_x in C_2
   (d) Connecting V'_y in C_1 to V''_y in C_2

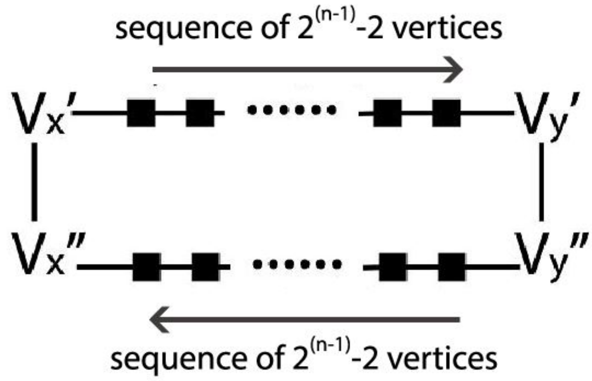
- For the above cycle,the process of selecting two adjacent vertices can be done in 2^{n-1} ways as it is a polygon with 2^{n-1} vertices which has 2^{n-1} vertices.
- It is easy to see that each selection of adjacent vertices results in a different Hamiltonian cycle.

Now we have at least $\frac{(n-1!)}{2}$ cycles in Q_{n-1} with each cycle giving 2^{n-1} Hamiltonian cycles of Q_n.So Q_n has atleast

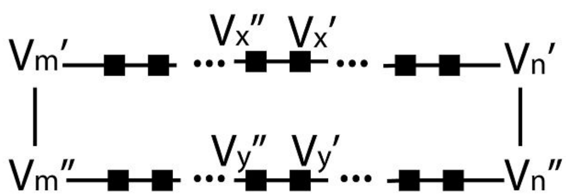
$\frac{(n-1!)}{2} \times 2^{n-1} \geq \frac{(n-1!)}{2} \times n = \frac{(n!)}{2}$ Hamiltonian cycles.

(because $2^{n-1} \geq n, \quad \forall\, n \geq 2$)

For the proof to be complete, now we have to show that the Hamiltonian cycles in Q_n resulting from one cycle in Q_{n-1} is not the same as any other Hamiltonian cycle arising from some other cycle in Q_{n-1}.In other words we have to show that the process discussed above does not have give rise to overlapping cases.
Suppose the chosen vertices are V_x' and V_y'.And the sequence of vertices excluding the inserted nth coordinate from V_x' to V_y' is S which is also the sequence of vertices of the original cycle in Q_{n-1}.



One important characteristic feature of so obtained Hamiltonian cycle is the edge connecting V_x' and V_x'' and that connecting V_y' to V_y'' because these are the only edges where the chosen coordinate of step 1 gets changed because along the sequence of vertices the inserted component (chosen in step 1 of construction procedure) doesn't change.But each cycle from which we choose vertices Vx and Vy will have a distinct sequence so the cannot result in same Hamiltonian cycle.
Now suppose we have the following case where the choosen vertices are V_m' and V_n'.Can we have $V_x' and V_x''$ and $V_y' and V_y''$ as two pairs of adjacent vertices located exactly as in the previous figure where $V_m' and V_m''$ and $V_n' and V_n''$ will be adjacent vertices in the sequence making the cycles same in the ?



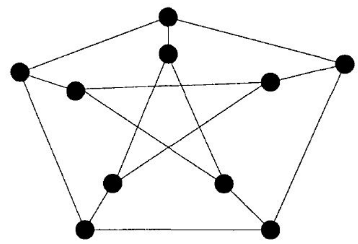
The answer is a obvious No because within the sequence the inserted nth component doesn't change as stated earlier.In other words if the edges between $V_x'andV_x''$ and $V_y'andV_y''$ are possible then the edges between $V_m'andV_m''$ and $V_n'andV_n''$ are not possible and vice versa.

Thus it can be shown by induction that if $n \geq 2$,then Q_n has atleast $\frac{n!}{2}$ Hamiltonian cycles.

$\square$

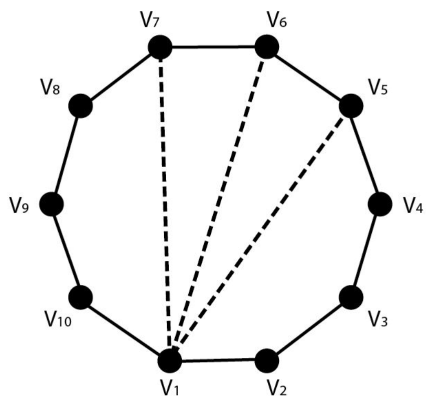
6.3 The Petersen Graph

The petersen graph is shown in the picture below.Does it have a Hamiltonian path?



First we observe that the graph has *no cycle which has less than five vertices.* We shall use the method of contradiction to show that the graph has no Hamiltonian cycle.

Now suppose that the graph has a Hamiltonian cycle.Clearly, the Hamiltonian cycle will a cycle containing all ten vertices and will look something like



where $V_1, V_2, ..., V_{10}$ are not necessarily in order.

In constructing the above cycle we have used up ten out of the total fifteen edges of the original Petersen Graph.

So we somehow should be able to connect the five leftover edges between the vertices so as to get the original graph from the cycle.Which also means that each vertex should be connected to exactly one of the five leftover edges, as we have each vertex of degree three in the Petersen graph.

Suppose we take for example the arbitrary vertex V_1,then as per the cycle given in the figure it cannot be connected to any of V_3, V_4, V_8 and V_9 as that would result in the creation of a cycle that has less than five members which according to the initial observation the original graph does not have.So the only possible vertices that V_1 can connect to is V_5, V_6 and V_7 ,also as shown in the *previous figure*.

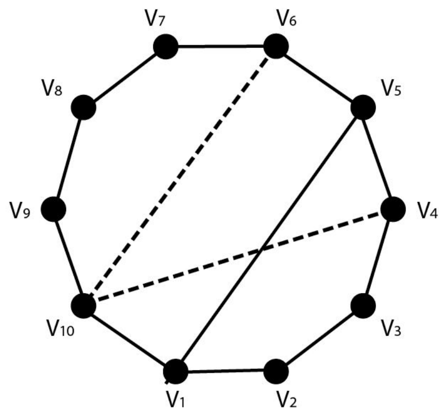
In other words, a vertex can only be connected to a vertex such that there are atleast three vertices between them.So through one of the five leftover edges, a vertex should be able to connect either to

1. A vertex such that there are three vertices in between them.

2. The diametrically opposite vertex

Case 1:When we attempt to connect a vertex to another such that there are three vertices between them in the shorter route.

We shall proceed through an example.i.e. V_1 connecting to V_5. Now that

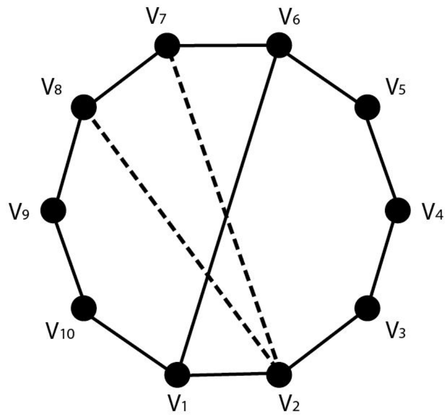


V_1 is connected to V_5 we see that its adjacent vertex V_{10} can connect to either V_6 or V_4,because V_5 is already taken . But we see that both of these possibilites lead to creation of four cycles i.e. (V_1 ,V_5 V_6, V_{10}) and (V_1 ,V_5 V_4, V_{10}) when V_{10} connects to V_6 and V_4 respectively.

So a vertex cannot connect to any other vertex such that there are three vertices in between them as it would lead to creation of four cycles which dont *exist in the original Petersen graph which we are trying to reconstruct by doing so.*

Case 2:When a vertex connects to its diametrically opposite vertex.

When for example it is attempted to connect V_1 to V_7:



Consider any of $V_1's$ adjacent vertices say V_2.

Now V_2 can either connect to V_7 or V_8.

Clearly it cannot connect to V_7 as it leads to formation of the four cycle (V_1,V_6,V_7,V_2).

Also V_2 cannot connect to V_8 as it would fall in *Case-1(with V_2 as the considered vertex)* leading to formation of four cycle which do not exist in the Petersen graph which we are trying to obtain by doing so.

Both of the above cases which attempt to connect a vertex to another,using any the five leftover vertices,in the Hamiltonian graph,result in the creation of four cycles which do not exist in the original Petersen graph.

So by any means, the Petersen graph cannot be obtained from the Hamiltonian cycle,thus its easy to conclude that the Petersen graph has no Hamiltonian cycle.

Is it possible to omit edges from the *Petersen graph* of the previous question so that the remaining graph has a closed Eulerian trail in the remaining 10 vertices ?

The initial Petersen graph has all vertices of degree 3. Suppose it is possible to omit edges such that the resulting graph has a closed Eulerian trail. In our discussion regarding the closed Eulerian trail we saw that in order for a graph to have a closed it has to have all vertices of even degree. Now the degree of each vertex has to be 2(as it has to be both even and ≤ 3)after omission.But that would mean our closed Eulerian trail is nothing but a Hamiltonian cycle,which is not possible as we saw in the previous problem.

Thus it is not possible to obtain a closed Eulerian cycle of ten vertices by omitting edges from the Petersen graph.

References

[1] **Miklos Bona**, *A Walk Through Combinatorics*, 2011

[2] **Adobe Illustrator 2016**,*For all illustrations except for Konigsberg Bridge and Petersen Graph.*

[3] https://en.wikipedia.org/wiki/Graph_Theory

[4] http://www.ma.rhul.ac.uk/ uvah099/Maths/Combinatorics07/Old/Ore.pdf.