Tumbling Box

Here are the differential equation for the angular momentum of a tumbling box.

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} yz \\ -2xz \\ xy \end{bmatrix}$$

Try throwing book, or a box, or any rectilinear object whose three dimensions are all different, into the air with a twist, to make a tumble. You could go to rotate about its longest axis or about its shortest axis. But you can't get to rotate about its middle axis. Let's examine that phenomena numerically.

Here's the anonymous function defining those system of three first ode's

```
>> F = @(t,y) [y(2)*y(3); -2*y(1)*y(3);y(1)*y(2)]
```

Now we are going to start with an initial condition that is near the first critical point.

```
>> y0 = [1 0 0]' + 0.2*randn(3,1);
   y0 = y0/norm(y0)
```

So the largest component is the first component. And the other two are small, but not too small. This is an easy problem numerically. There is no stiffness involved here.

```
>> ode23(F,[0 10], y0);
```

So we are using ode23, integrate from 0 to 10. Let's go back and take another starting condition.

```
>> y0 = [0 0 1]' + 0.2*randn(3,1);
   y0 = y0/norm(y0)
```

Here it is again. But if we take

```
>> y0 = [0 1 0]' + 0.2*randn(3,1);
   y0 = y0/norm(y0)
```

Things are completely different. This is the instability of that middle critical value.

The differential equations have these three critical points.

$$X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, Y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Any solutions to start exactly in these initial conditions stay there. But what happens if we start near these initial conditions? Well it turns out, that x and z are stable critical points. But y is an unstable critical point. If the angular momentum is near x or near z, it stays near there. But if it start near y, it move away quickly. You can think of x as the short axis, and z is the long axis. Rotation near the short axis is stable. And rotation near the long axis is stable. But rotation near the middle axis is unstable. We can see that in the following graphic.

It turns out that if a solution starts with an initial condition that has norm 1, it stays with norm 1. So the solution lives on the unit sphere.