Linearization at Critical Points

We are concentrating now on the key question of stability. Do the solutions approach 0 in the case of linear equations? Do they approach some constant some steady state in the case of non-linear equations? So today is the beginning of non-linear. We will start with one equation.

$$\frac{dy}{dt} = f(y)$$

And first question, what is a steady state or critical point? Easy question.

We are looking at special points $f(Y) = 0$. And we call those critical points or steady states. We can plug in $y(t) = Y$. It satisfies the equation. So if we start at a critical point, we stay there. That's not out central question. Our key question is, if we start at another points, do we approach a critical point, or do we go away from it. So the way to answer that equation is to look at the equation when you are very near the critical point so that we can linearize the equation.

So what does linearize mean? Every function is linear if you look at it through a microscope.

$$f(y) = f(Y) + (y - Y)\frac{df}{dy}(Y)$$

And $f(Y) = 0$. So that we have just a linear approximation with the slope and a simple function. We will use this approximation and put it into the equation and then we have a linear equation, which we can easily solve.

$$\frac{dy}{dt} \approx (y - Y)\frac{df}{dy}$$

and notice that

$$\frac{dy}{dt} = \frac{d(y - Y)}{dy}$$

so

$$y - Y \approx Ce^{at}$$

$a < 0$ is stable. Let's see an example -- the logistic equation

$$\frac{dy}{dt} = 3y - y^2$$

so two critical point $y = 0, y = 3$. And each critical point has its own linearization, its slope at that critical point.

$$\frac{df}{dy} = 3 - 2y = 3 \; or \; -3$$

3 for unstable and -3 for stable.