Solving Linear Systems

This is the key lecture about solving a system of n linear constant coefficient equations.

$$\frac{dy}{dt} = Ay$$

y is a vector. How do we solve this system?

Eigenvectors are vectors that go in their own way. So when you have an eigenvector, it's like you have a one by one problem and the A becomes just a number λ. So for a general vector, everything is a mixed together. But for an eigenvector, everything stays one dimensional. The A changes just to λ for that special direction. And of course, as always, we need n of those eigenvectors because we want to take the starting value.

We take our starting vector, which is probably not an eigenvector. We would make it a combination of eigenvectors. And we are OK because we are assuming that we have n independent eigenvectors.

$$y(0) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

And now we follow each starting value.

$$y(t) = c_1e^{\lambda_1 t}x_1 + \cdots + c_ne^{\lambda_n t}x_n$$

You can see the stable time. point. When $\lambda_i < 0$ then the term is going to 0.

We will see an example. A Markov differential equation.

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

The column will add up to 0. So there is the eigenvalue of 1 our powers is like the eigenvalue 0 for differential equations. Because $e^{0t} = 1$. So anyway, let's find the eigenvalues of that.

The first eigenvalue is 0 and the corresponding eigenvector is $x_1 = [2, 1]^T$. And my second eigenvalue is -3 and the eigenvector is $x_2 = [1, -1]^T$. So we have done the preliminary work. Given this matrix, we have got the eigenvalues and eigenvectors. Now we take

$$y(0) = c_1x_1 + c_2x_2$$

And now the

$$y(t) = c_1e^{0t}x_1 + c_2e^{-3t}x_2$$

That's the evolution of a Markov process, a continuous Markov process. Compared to the powers of a matrix, this is a continuous evolving evolution of this vector. Now we are interested in steady state. Steady state is what happens as $t \to \infty$. Obviously, the second term goes quickly to 0. So we have c_1x_1 is the steady state. We are thinking that no matter how you start, no matter what y_0 is, as time goes on, the x_2 part is going to disappear. And if you only have the x_1 part in that ratio $2:1$. So again, if we had movement between y_1 and y_2 or we have things evolving in time, the steady state is c_1x_1. This is the differential equations happen to have a Markov matrix. And we will show why 0 is an eigenvalue of the Markov matrix.

We have now two examples of the following fact. That if all columns add to s, then $\lambda = s$ is an eigenvalue. That was the point from Markov matrices, $s = 1$. And if all rows of A add to s, then $\lambda = s$ is an eigenvalue because A^T and A share the same eigenvalues. And maybe you would like to just see why that's true.

If we want the eigenvalues of a matrix, we look at $det(A - \lambda I)$. That gives us eigenvalues of A. If we want the eigenvalues of A^T, we will look at $det(A^T - \lambda I)$. But why are they the same? Because the determinant of a matrix and the determinant of its transpose are equal.

So for special matrices, we are able to identify important fact about their eigenvalue.