The Tumbling Box in 3-D

Here is an example that's more or less for fun. -- the problem tumbling box. We throw a box and the question is the spinning box stable or not.

Those are the three equations came from Euler.

$$\frac{dx}{dt} = yz$$

$$\frac{dy}{dt} = -2xz$$

$$\frac{dz}{dt} = xy$$

They are not linear. And those are for the angular momentum. So there is a little physics behind the equations. And the equations will tell us short and long axes should give a stable turning. And the in between axis is unstable. We have to find this steady state, and then for each steady state we linearize. We find the derivatives at that steady state. And that gives us a constant matrix at that steady state. And then the eigenvalue is decided. So first, find the critical points. Second, find the derivatives at the critical points. Third, for that matrix of derivatives, find the eigenvalues and decide stability. That's the sequence of steps.

Before we find the critical points. notice some nice properties. If we multiply the first equation by x, the second one by y and the third one by z. And those will add to 0.

$$x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = 0$$

That's an important fact. That's telling us that the derivative of something is 0. That something will be constant.

$$\frac{1}{2}(x^2 + y^2 + z^2) = C$$

It tells that the total energy, the kinetic energy is constant. And we can also have

$$2x\frac{dx}{dy} + y\frac{dy}{dt} = 0 \qquad 2z\frac{dx}{dy} + y\frac{dy}{dt} = 0$$

So

$$x^2 + \frac{1}{2}y^2 = C \qquad z^2 + \frac{1}{2}y^2 = C$$

Another quantity that is conserved.

$$x^2 - z^2 = C$$

We can see two ellipse one sphere and one hyperbola. Now we return to the three equations and find out the critical points.

We put

$$yz = 0,\ -2xz = 0,\ xy = 0$$

They are pretty special equations. Solution can be $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ $(0, 0, 0)$. So those are our steady states. Now the step is find all the derivatives, find that Jacobian matrix of derivative.

$$\begin{bmatrix} 0 & z & y \\ -2z & 0 & -2x \\ y & x & 0 \end{bmatrix}$$

Its the eigenvalues of that matrix at previous points that decide stability. The situation for the $(1, 0, 0)$ is neutral stability and for $(0, 1, 0)$ is instable.