Forced Harmonic Motion

This is the second lecture on second order differential equations, constant coefficients

$$m\frac{d^2y}{dt^2} + ky = \cos\omega t$$

we have a forcing frequency, ω

remember that for the null solution, there was natural frequency $\omega_n = \sqrt{\frac{k}{m}}$, it is very important are those close, are those well separated? That governs whether the bridge that you are walking over oscillates too much and eventually falls. Or in the extreme case, are they equal? If $\omega_n = \omega$, that is called **resonance**.

We would like to hope that the particular solution could be

$$y_p = Y\cos\omega t$$

this is called **forced response****. Let's plug it into the original equation and find Y.

$$-\omega^2 mY\cos\omega t + kY\cos\omega t = \cos\omega t$$

solve for Y

$$Y = \frac{1}{k - m\omega^2}$$

and the general solution

$$y(t) = y_p + y_n = \frac{\cos\omega t}{k - m\omega^2} + C_1\cos\omega_n t + C_2\sin\omega_n t$$

Remember, $\omega_n \neq \omega$, actually recalling ${\omega_n}^2 = k/m$, so

$$y_p = \cos\omega t/m(\omega_n^2 - \omega^2)$$

You will see the whole point of resonance or near resonance when the bridge is getting forced by a frequency close to its resonant frequency. And we call

$$\frac{1}{m(\omega_n^2 - \omega^2)}$$

frequency response. That is the key multiplier for when the forcing term is a pure frequency, that frequency gets exploded. And now, of course, what are C_1, C_2? We find those from the initial condition.

There is one more equation, one more forcing term I would like often and always and now to discuss. And that is a delta function, an impulse. Let's see an example.

$$my'' + ky = \delta(t)$$

y is called impulse response. In fact, it is so important, we are going to give it its own letter g, $g(t)$ is the impulse response.

I can think of it as solving it with no force $mg'' + kg = 0$ starting from $g(0) = 0, g'(0) = 1/m$.

$$g(t) = \frac{\sin\omega_n t}{m\omega_n}$$

It is a beautiful and simple answer, and every forcing function comes from this one. We will see that point.