The matrix Exponential

We ate still solving systems of differential equations with a matrix A in them. And now we want to create the exponential. It's just natural to produce e^{At}-- the exponential of a matrix.

So if we have one equation

$$y' = ay$$

then we know the solution is $e^{at}y(0)$. Now we have n equations with a matrix A and a vector y. And the solution should be

$$y(t) = e^{At}y(0)$$

It should be a perfect match with the above one.

$$e^{At} = I + At + \frac{1}{2!}(At)^2 + \cdots$$

The same as the exponential series -- the most important series in mathematics. And that gives an answer.

We check that by putting it into the differential equation. So we need to take the derivative.

$$\frac{de^{At}}{dt} = A + A^2t + \frac{1}{2!}A^3t^2 + \cdots = Ae^{At}$$

Just what we want. Now, is it better than what we had before, which was using eigenvalues and eigenvectors? It's better in one way. This series is totally fine whether we have n independent eigenvectors or not. We can have the repeated eigenvalues. We will see an example. For repeated eigenvalues and missing eigenvectors, e^{At} is still the correct answer. But if we want to use eigenvalues and eigenvectors to compute e^{At}, because we don't want to add up an infinite series very often, then we would want n independent eigenvectors.

Suppose we have n independent eigenvectors. And we know that means

$$A = V\Lambda V^{-1}$$

$$e^{At} = I + V\Lambda V^{-1}t + \frac{1}{2!}V\Lambda^2V^{-1}t^2 + \cdots$$

$$e^{At} = V(e^{\Lambda t})V^{-1}$$

So

$$e^{At} = c_1e^{\lambda_1 t}x_1 + \cdots + \cdots$$

Something new will be, suppose there are not a full set of n independent eigenvectors. e^{At} is still OK. But the above formula is no good.