Linearization of Two Nonlinear Equations

Two equations, the question of stability for two equations, stability around a critical point. So the idea will be to linearize, to look very near that critical point.

Let's see an famous example -- **predator-prey**

$$\frac{dy}{dt} = y - yz$$
$$\frac{dz}{dt} = yz - z$$

Here is the linearization.

$$f(y,z) \approx f(Y,Z) + (y-Y)\frac{\partial f}{\partial y} + (z-Z)\frac{\partial f}{\partial z}$$

and notice that $f(Y,Z) = 0$. So that's why we have linear in y and linear in z. And again.

$$g(y,z) \approx (y-Y)\frac{\partial g}{\partial y} + (z-Z)\frac{\partial g}{\partial z}$$

So altogether, the linear stuff and four numbers. In the predator-prey example.

$$\begin{bmatrix} \frac{\partial f}{dy} & \frac{\partial f}{dz} \\ \frac{\partial g}{dy} & \frac{\partial g}{dz} \end{bmatrix} = \begin{bmatrix} 1-z & -y \\ z & y-1 \end{bmatrix}$$

The matrix is always named after Jacobi who studies these first. So it's called the Jacobian matrix. And we can compute two pair of critical points $(0,0)$ and $(1,1)$. For $(1,1)$, we have linearization below.

$$(y-1)' = -(z-1)$$
$$(z-1)' = (y-1)$$

If we start out with some extra foxes, then the number of rabbits will drop because foxed are eating them. The number of foxes will increase but when there are no rabbits to eat so the foxes start dropping and the number of rabbits starts increasing. We will go around and around in a circle. If you remember the pictures of the paths for 2×2 equations, there were saddle points. And $y = 0, z = 0$ was a saddle point. So it is unstable. And the another case is the neutral stability, because it doesn't blow up. So that's a case where we could see the stability, based on the linearization.