The Stability and Instability of Steady States

This is a interesting topic. It is about stability or instability of a steady state. Let's see the differential equation.

$$\boxed{\frac{dy}{dt} = f(y)}$$

It could be linear, but might be nonlinear. And when do we have the steady state?

There's steady state when the derivative is 0. So if the derivative 0 when $f(y) = 0$. Let's call those special $y's$ by a Y, i.e. $f(Y) = 0$. If we start with those point, we will stay there forever. And the question is, if we start near Y, do we approach Y as time goes on ? In that case, we would say stable -- or does the solution when we start near y go far away from Y? In that case. we would call the steady state unstable. So stable or unstable and it's vert important to know which it is. Let's see some examples.

$$\frac{dy}{dt} = ay$$

So here is first starting with a linear equation. And in this case $Y = 0$, so if we start at 0 , we will stay at 0

$$\frac{dy}{dt} = y - y^2$$

here is a second example, the logistic equation, where we have taken the coefficients to be 1. What are the steady states for the logistic equation?

$$y - y^2 = 0$$

so $Y = 0$ or 1.

$$\frac{dy}{dt} = y - y^3$$

And finally, now we let $y - y^3 = 0$, solve it we get $Y = 0, 1, -1$.

The actual problem could have sines, cosines, and exponentials, but these are clear cases, and of course, the linear case is always the good guide.

So in the linear case, when does a solution stay near 0?

Let's see the first example. It is **stable** if $a < 0$. So the thing to look at stable or unstable is the derivative.

Look at the derivative of that right hand side i.e. $\frac{df}{dy}$ at $y = Y$. And if the derivative $\frac{df}{dy} < 0$ then stable.

What about example 2 and 3? So with those two examples you will see the whole idea.

Look at the second example

$$\frac{df}{dt} = 1 - 2y$$

so $y = 0$ is now unstable and $y = 1$ will be stable. so as to the third example.

Now, we want to show why and then show an example by throwing the book and this would be an example in three dimensions that we will get to when we are doing a system of equations.

We want to look at the difference between y and the steady state

$$\frac{d}{dt}(y - Y)$$

And the question is , if that goes to 0, we have something stable. If that blows up, it's unstable

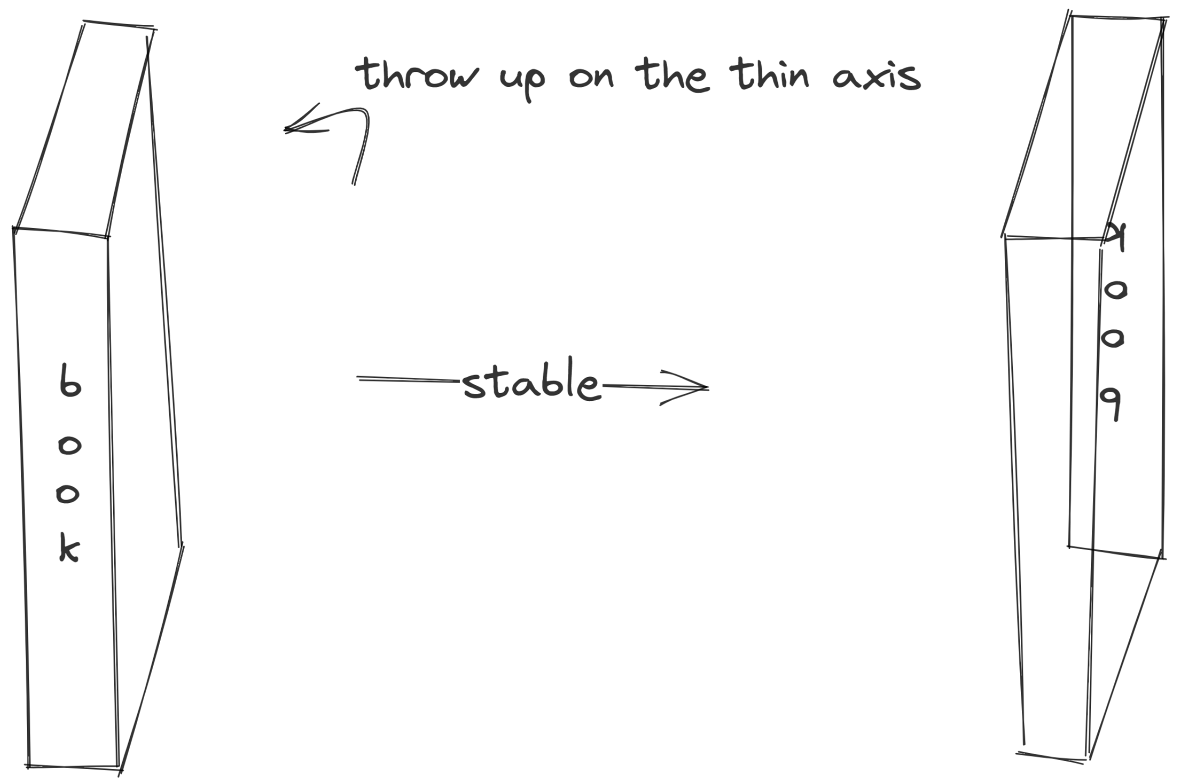
$$\frac{d}{dt}(y - Y) = f(y) - f(Y)$$

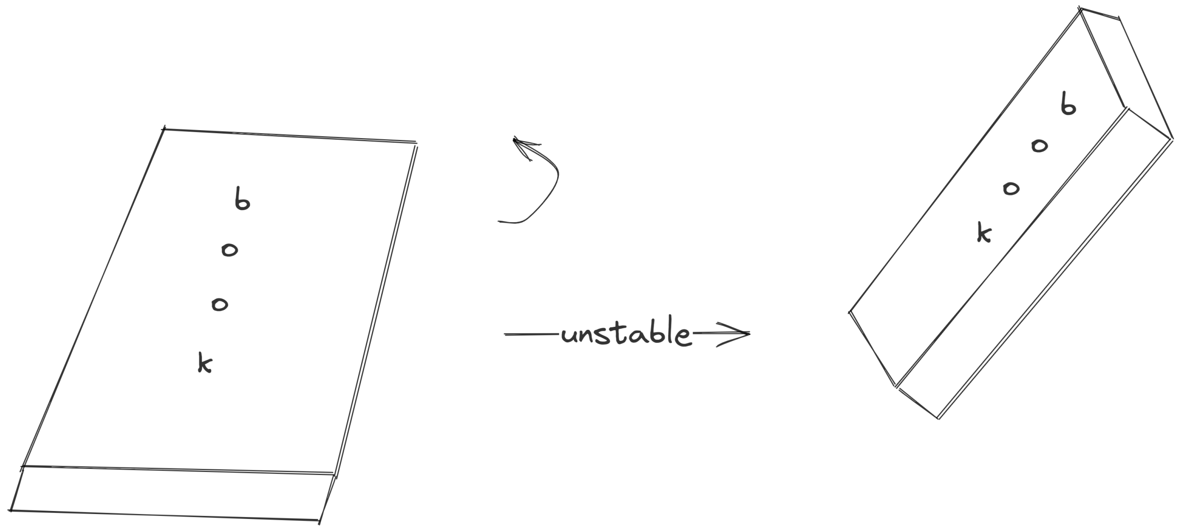
We are going to use calculus. The difference between the function at a point and the function at a nearby point it approximately -- and the mean value theorem tells me that it really is -- is approximately the $\frac{df}{dy}(y - Y)$ i.e.

$$f(y) - f(Y) \approx \frac{df}{dy}(y - Y)$$

so in other words, what we have is approximately the linear equation. so the question is do we get closer or do we not get close. And the answer is that when that dy/dt is negative. So that is the reasoning behind the beautiful, simple, easy to apply test, which is, if the derivative is negative, then stable.

Now let's see the example of a tumbling box. So the point is when we throw the book up, does it wobble all over the place -- unstable, or does it turn nicely on its axis?





We can also use stability line to tell what direction the solution moves