Heat Equation

This is heat equation lecture. So this is the second of the three basic partial differential equations. We had Laplace's equation without time. Now time comes into the heat equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

And we look for solutions like

$$u(t, x) = e^{\lambda t} S(x)$$

x is the eigenvector corresponding to the eigenvalue λ. And $S(x)$ is an **eigenfunction**. And just as always, we substitute that into the differential equation to discover what determines $S(x)$.

$$\lambda e^{\lambda t} S(x) = e^{\lambda t} \frac{\partial^2 S}{\partial x^2}$$

We have an eigenvalue equation

$$\lambda S = \frac{\partial^2 S}{\partial x^2}$$

So

$$S(x) = \sin k\pi x \qquad \lambda = -k^2\pi^2$$

And the general solution is

$$u(t, x) = \sum_{k=1}^{\infty} B_k e^{-k^2\pi^2 t} S_k$$

We can find B_k with initial condition and use the technique in Fourier series

$$\sum_{k=1}^{\infty} B_k \sin k\pi x = 1$$

So the solution when $t > 0$ is

$$u(t, x) = \sum_{k=1}^{\infty} B_k e^{-k^2\pi^2 t} \sin(k\pi x)$$