Step function and Delta function

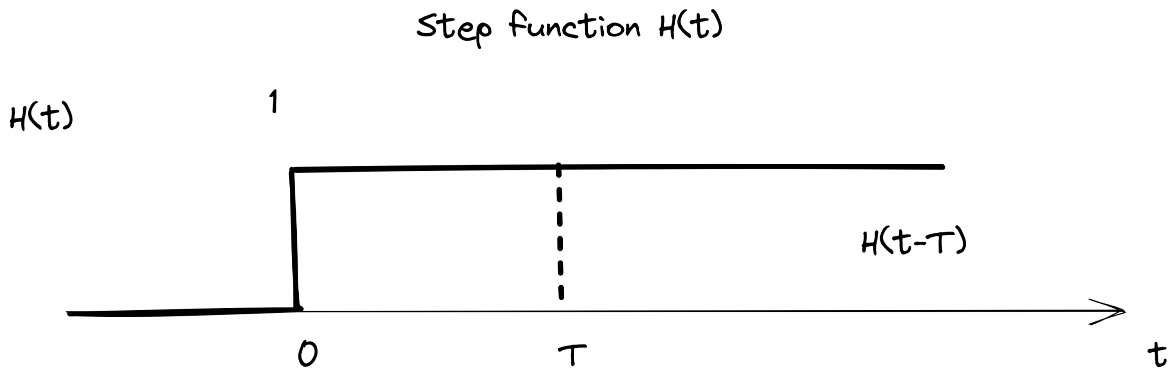
this is the notes about two neat functions -- the step function and its derivative the delta function.

They are very natural inputs to a differential equation. They happen all the time in real life. And so we need to understand how to compute these formulas.

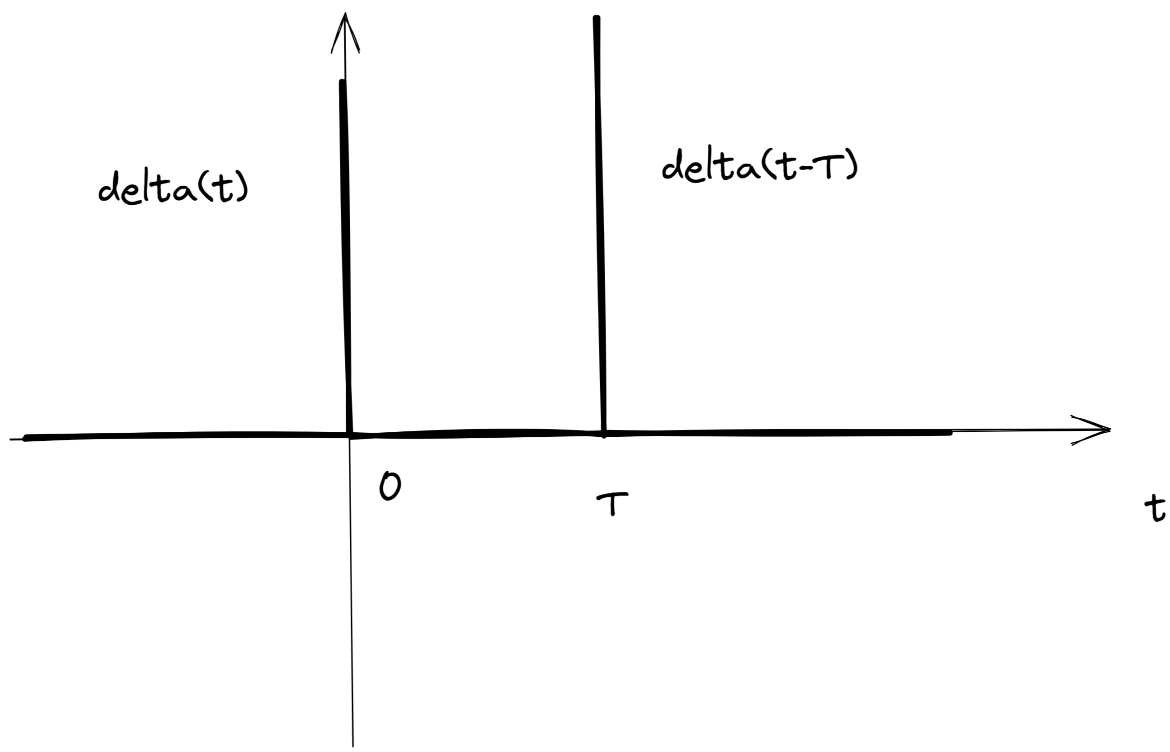
The first one is the step function. It will be called 'h' after its inventor who was an engineer name Heaviside started with an 'H'

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

just has two values and it has a jump. You could say jump function also.



the function is 0 at beginning, so the derivative is 0. The function is then constant, so the derivative is again 0. It is just at one point everything happens.



the relationship between step function and delta function is as followed

$$\frac{dH}{dt} = \delta(t)$$

the delta function runs along at 0, continues at 0, but at T and 0, the whole thing explodes. The derivative is infinite. And the point is infinity is not a sufficiently precise word to tell you exactly what is happening. We do not have really - the above graph of a delta function. it is not fully satisfactory. It is perfect for all the uninteresting boring part. But at the moment of truth, when something happens in an instant, we need to say more. And again, if it's shifted, then the infinite slope happens at t equal a T. So the infinity is just shifted over. And that would be the another delta function.

If that was the source term in the differential equation, what would that mean? i.e. if this was the $q(t)$ in the differential equation reflecting input at different times. That function would say no input except at one moment and one instant, T. At that instant of time, you put 1 in, over in an instant. And remember, that otherwise $q(t)$ has been a continuous input. Put in $\$1.00$ per year over the whole year. The delta function puts in $\$1.00$ at one moment. But of course, you see that is really what we do.

We also see

$$\int_{-\infty}^{\infty} \delta(t)\, dt = H(t)\Big|_{t=-\infty}^{t=+\infty} = 1$$

the total integral of the delta function is 1. Again, you only made the deposit at one moment, but that deposit was a full dollar. And that, adding up all deposits is just that $\$1.00$.

Now we do some other things

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt$$

That is something we will need to be able to compute. What is the right integral for that? The answer is $f(0)$.

Let look at another example.

$$\int_{-\infty}^{\infty} \delta(t-T) e^{T}\, dt = e^{T}$$

So now, let's use a delta function as the source term in our differential equation. We are seeing one last time one more -- the delta.

Solve for

$$\frac{dy}{dt} = ay + \delta(t-T)$$

start from $y(0) = 0$. The solution is

$$y(t) = \begin{cases} 0 & \text{up to t=T} \\ e^{a(t-T)} & t \geq T \end{cases}$$

it is the **impulse response** which is a very important concept in engineering. For second order differential equation, it is really a crucial function