The Logistic Equation

Now, finally, a nonlinear equation. Growth, but it is -- the growth is cut off by competition. Let see the equation. In a way, it is the simplest nonlinear equation we can think of.

$$\frac{dy}{dt} = ay - by^2$$

It is called the logistic equation. It's got to be a famous example. And it has a neat trick that allows you to solve it easily. Let's see the trick.

The trick is to let z -- bring a new z as $1/y$. Then, if we write the equation for z. It would turn out to be linear. So you see, We are always hoping, and here succeeding, to get back to a simple linear equation. And this device happens to work for this problem. We need, in the end, a more systematic way that would work for other problems, like the quadratic term could be some constant term there from harvesting, from the changing the equation.

$$\frac{dz}{dt} = -\frac{1}{y^2}\frac{dy}{dt}$$

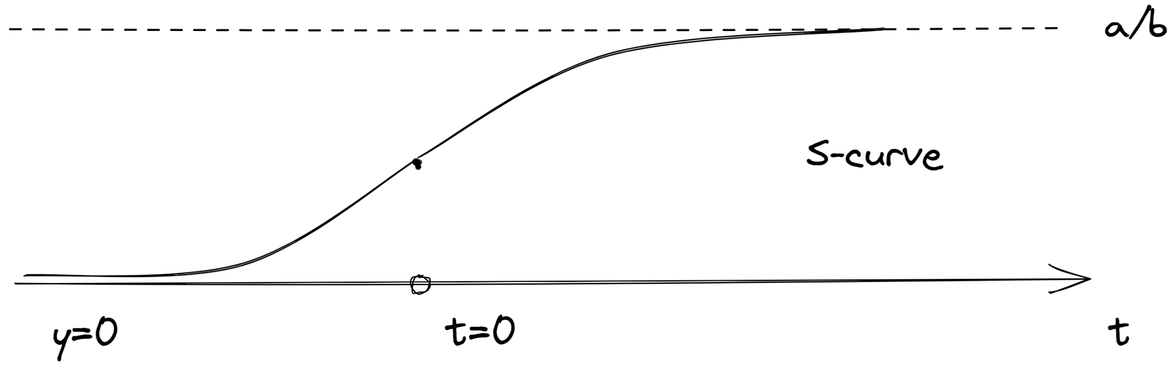
and now $dy/dt = ay - by^2$ so we get

$$\frac{dz}{dt} = -\frac{1}{y^2}(ay - by^2) = -az + b$$

And we know the solution to that equation. Then, we just take y to be $1/z$.

$$y(t) = \frac{a}{de^{-at} + b}$$

d depends on the initial condition.



Notice that the value of $y(t) \to \frac{a}{b}$ when $t \to +\infty$. And there has a turning point to the function. It is often called a **logistic curve** or $''S''$ curve, In some way --or a Sigmoidal curve.

Those two possibilities y equals 0 and a/b are the key things to see for this equation. Those are **steady states**.

A steady states is a value of y where the derivative is 0. Nothing happen, It just sits there. We use Y for the steady states. They are the numbers where this is 0.

$$aY - bY^2 = 0$$

the solution of this equation are $y = 0$ $y = \frac{a}{b}$ exactly two point on the picture.

If we start at 0, we stay at 0. No population, nothing can happen. But if we move a little away, if we get two people, we grow exponentially for a while but then the by^2 term takes it and slows it down to a/b. If we had started at a/b, we would have stayed at a/b.

So that S curve picture is really a nice graph of the solution.

actually, $y = 0$ steady state is called unstable because if we start near $y = 0$, we take off. And $y = a/b$ is called stable because if we are near a/b, we get closer and closer to a/b . Let's see an example.

$$\frac{dy}{dt} = f(y)$$

That's will be our next lecture to understand those equations.