Classical Runge-Kutta, ODE4

Here is the classical Runge-Kutta method. This was, by far and away, the world's most popular numerical method for over 100 years for hand computation in the first half of the 20th century, and then for computation on digital computers in the latter half of the 20th century. And it is still in use today.

You evaluate the function four times per step. First in the beginning of the interval. And then use that to step into the middle of the interval, to get s_2. Then you use s_2 to step into the middle of the interval again. And evaluate the function there again to get s_3. And then use s_3 to step clear across the interval, and get s_4. And then take a combination of those four slopes, weighting the two in the middle more heavily, to take your final step. That is the classical Runge-Kutta method.

$$
\begin{aligned}
s_1 &= f(t_n, y_n) \\
s_2 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2} s_1) \\
s_3 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2} s_2) \\
s_4 &= f(t_n + h, y_n + h s_3) \\
y_{n+1} &= y_n + \frac{h}{6}(s_1 + 2s_2 + 2s_3 + s_4)
\end{aligned}
$$

Let's see a simple model of combustion. Because the model has some important numerical properties. If you light a match, the ball of flame grows rapidly until it reaches a critical size. Then the remains at that size, because the amount of oxygen being consumed by the combustion in the interior of the ball balances the amount available through the surface.

Here is the dimensionless model.

$$
\begin{aligned}
\frac{dy}{dt} &= y^2 - y^3 \\
y(0) &= y_0 \\
0 \le t &\le \frac{2}{y_0}
\end{aligned}
$$

The match is a sphere, and y is its radius. The y^3 term is the volume of the sphere. And the y^3 accounts for the combustion in the interior. The surface of the sphere is proportional y^2. And y^2 term accounts for the oxygen that is available through the surface. The critical parameter the important parameter, is the initial radius y_0. The radius starts at y_0 and grows until the y^3 and y^2 terms balance each other, at which point the rate of growth is 0. And the radius does not grow anymore. We integrate over a long time. We integrate over a time that is inversely proportional to the initial radius. That is the model.

The critical question about the Runge-Kutta method is how reliable are the values we have around the critical point