Problems for Lecture 20

10.

The random variable A_N is the average of N independent 0-1 samples. The expected value of each sample is $\frac{1}{2}$, so by the linearity of expectation, the expected value of A_N is also $\frac{1}{2}$.

Moreover, the variance of each sample is $\frac{1}{4}$, so the variance of A_N is $\frac{1}{4N}$. Thus, the standard deviation of A_N is $\frac{1}{2\sqrt{N}}$.

The standardized variable X is defined as $X = \frac{A_N - \frac{1}{2}}{2\sqrt{N}}$. Its expected value is 0, and its variance is:

$$\begin{aligned} Var(X) &= Var\left(\frac{A_N - \frac{1}{2}}{2\sqrt{N}}\right) \\ &= \frac{1}{4N} Var(A_N) \\ &= \frac{1}{4N} \cdot \frac{1}{4N} \\ &= \frac{1}{16N^2}. \end{aligned}$$

Therefore, the standard deviation of X is $\frac{1}{4N\sqrt{N}}$.

To find the value of $A_{1000000}$, we can use the law of large numbers, which states that as the sample size increases, the sample average approaches the population mean. In our case, we have a large enough sample size (one million) for this law to apply. Therefore, the sample average $A_{1000000}$ is expected to be close to $\frac{1}{2}$.

To get a more precise estimate of $A_{1000000}$, we can use the central limit theorem, which states that the distribution of the sample average is approximately normal with mean $\frac{1}{2}$ and standard deviation $\frac{1}{2\sqrt{N}}$. Therefore, we can use a normal approximation to estimate the probability that $A_{1000000}$ is within a certain range around $\frac{1}{2}$.

Let $Z = \frac{A_{1000000} - \frac{1}{2}}{\frac{1}{2\sqrt{1000000}}}$ be the standardized variable for $A_{1000000}$. Then we want to find $P(-z < Z < z)$, where z is the number of standard deviations from the mean that defines the range of interest. For example, if we want to find the range that contains 95% of the probability mass, we would choose $z = 1.96$ (which is the 97.5th percentile of the standard normal distribution).

Using the normal approximation, we have:

$$P(-z < Z < z) \approx \Phi(z) - \Phi(-z)$$

where $\Phi(z)$ is the cumulative distribution function of the standard normal distribution evaluated at z.

12.

To find $E[x^2]$, we can use the definition of the variance:

$$Var(x) = E[(x - E[x])^2] = E[(x - m)^2] = \sigma^2$$

Expanding the square, we get:

$$E[x^2 - 2xm + m^2] = \sigma^2$$

Using linearity of expectation, we can split this into three terms:

$E[x^2] - 2mE[x] + m^2 = \sigma^2$

Substituting $E[x] = m$ and rearranging, we get:

$E[x^2] = \sigma^2 + m^2$

Therefore, the expected value of x^2 is the sum of the variance and the square of the mean.

Note that this result is true for any distribution of x with mean m and variance σ^2. It follows from the definition of the variance and the linearity of expectation, and does not depend on any specific properties of the distribution (such as whether it is symmetric or not).

3.

Using Markov Inequality, we can get

$$P[x \geq 1] = \frac{1}{2}$$

In this case, $Y = X$ is a non-negative random variable, and we want to find $P(Y \geq 2E[Y]) = P(Y \geq 1)$. Applying Markov's inequality with $a = 2E[Y] = 1$ gives

$P(Y \geq 1) \leq \frac{E[Y]}{1} = E[Y] = \frac{1}{2}$.
