Problems for Lecture 12

1.

Let $D = I - S$, where S is the shift matrix with $S_{i,i-1} = 1$ for $i = 2, \ldots, n$. Then D is a bidiagonal matrix with diagonal entries 1 and subdiagonal entries -1.

$$\begin{aligned}
DD^T &= (I-S)(I-S)^T \\
&= (I-S)(I-S) && \text{(since } S \text{ is a symmetric matrix)} \\
&= I - 2S + S^2 \\
&= I - 2S + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \end{pmatrix} && \text{(since } S^2 \text{ has 1 on the } (i, i-2) \text{ entry)} \\
&= -S + 2I - S^T.
\end{aligned}$$

Thus, DD^T is equal to A except that $1 \neq 2$ in their $(1,1)$ entries.

On the other hand, $D^T D$ can be computed as:

$$\begin{aligned}
D^T D &= (I-S)^T(I-S) \\
&= (I-S^T)(I-S) \\
&= I - (S+S^T) + SS^T \\
&= I - \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} && \text{(since } S^T \text{ has 1 on the } (i, i-1) \text{ entr} \\
&= -S + 2I - S^T.
\end{aligned}$$

Thus, $D^T D$ is also equal to A except that $1 \neq 2$ in their (n,n) entries.