Problems for Lecture 18

4.

We know that the eigenvectors of a symmetric matrix form an orthonormal basis. Let's use this fact to find the eigenvectors that form the basis for Y.

Since $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, we know that λ_i is the ith largest eigenvalue of S. Let's assume that q_i is the eigenvector corresponding to λ_i. Then, we can construct a subspace Y using the first i eigenvectors:

$$Y = \text{span} q_1, q_2, \cdots, q_i$$

To show that $q_1, q_2, \cdots, q_i$ form a basis for Y, we need to show that they are linearly independent and span Y.

Linear independence:

Suppose there exist scalars $c_1, c_2, \cdots, c_i$ such that

$$c_1 q_1 + c_2 q_2 + \cdots + c_i q_i = 0$$

Then, we can left-multiply both sides by q_j^T to get

$$c_1 q_j^T q_1 + c_2 q_j^T q_2 + \cdots + c_i q_j^T q_i = 0$$

Since $q_1, q_2, \cdots, q_i$ form an orthonormal basis, $q_j^T q_k = \delta_{jk}$ (Kronecker delta), so the above equation reduces to

$$c_j = 0$$

This shows that $q_1, q_2, \cdots, q_i$ are linearly independent.

Spanning:

Let $x \in Y$. Then, we can write x as a linear combination of $q_1, q_2, \cdots, q_i$:

$$x = a_1 q_1 + a_2 q_2 + \cdots + a_i q_i$$

where $a_1, a_2, \cdots, a_i$ are scalars. Then,

$$x^T S x = (a_1 q_1 + a_2 q_2 + \cdots + a_i q_i)^T S (a_1 q_1 + a_2 q_2 + \cdots + a_i q_i)$$

Since $q_1, q_2, \cdots, q_i$ are eigenvectors of S, we have

$$S q_k = \lambda_k q_k$$

for $k = 1, 2, \cdots, i$. Using this fact, we can simplify the above equation as follows:

$$x^T S x = \lambda_1 a_1^2 + \lambda_2 a_2^2 + \cdots + \lambda_i a_i^2$$

Since $\lambda_1 > \lambda_2 > \cdots > \lambda_i$, the minimum value of $x^T S x$ for $x \in Y$ occurs when $a_i = 1$ and $a_k = 0$ for $k \neq i$. In other words, the minimum value of $x^T S x$ for $x \in Y$ is λ_i, and this minimum is achieved when $x = q_i$.

Therefore, we have shown that $q_1, q_2, \cdots, q_i$ form a basis for Y, and that the minimum value of $x^T S x / x^T x$ for x in Y

10.

use elimination.