Problem 1

The outcomes of choosing two balls from the urn may be one of the following cases:

cases	white	black	orange	winings	probability
(1)	2	0	0	-2	$\binom{8}{2}/\binom{14}{2} = \frac{4}{13}$
(2)	0	2	0	4	$\binom{4}{2}/\binom{14}{2} = \frac{6}{91}$
(3)	0	0	2	0	$\binom{2}{2}/\binom{14}{2} = \frac{1}{91}$
(4)	1	1	0	1	$\binom{8}{1}\binom{4}{1}/\binom{14}{2} = \frac{32}{91}$
(5)	1	0	1	-1	$\binom{8}{1}\binom{2}{1}/\binom{14}{2} = \frac{16}{91}$
(6)	0	1	1	2	$\binom{4}{1}\binom{2}{1}/\binom{14}{2} = \frac{8}{91}$

Then, we have: $P\{X = x\} = \begin{cases} \frac{4}{13} & \text{, if } x = -2 \\ \frac{16}{91} & \text{, if } x = -1 \\ \frac{1}{91} & \text{, if } x = 0 \\ \frac{32}{91} & \text{, if } x = 1 \\ \frac{8}{91} & \text{, if } x = 2 \\ \frac{6}{91} & \text{, if } x = 4 \\ 0 & \text{, o.w.} \end{cases}$

Problem 14

Consider $P\{X = i\}$ case by case:

For $P\{X = 0\} = P\{\text{Player 1 lose to player 2}\} = \dfrac{\binom{5}{2} \times 3!}{5!} = \dfrac{1}{2} =$

$$\frac{(\text{choose 2 from 5 numbers for player 1 \& 2}) \times (\text{permutation of rest 3 numbers for player 3} \sim 5)}{\text{all permutations}}$$

(Reason: since "player 1's number < player 2's number", thus we only need to give them a pair of number, player 2 always take the larger one. The rest 3 numbers does not affect this result, thus they can be arbitrary permuted.)

For $P\{X = 1\} = P\{\text{Player 1 win against player 2, but lose to player 3}\} = \dfrac{\binom{5}{3} \times 2!}{5!} = \dfrac{1}{6} =$

$$\frac{(\text{choose 3 from 5 numbers for player 1, 2 \& 3}) \times (\text{permutation of rest 2 numbers for player 4 \& 5})}{\text{all permutations}}$$

(Reason: similarly to previous case, we have "player 2's number < player 1's number < player 3's number", thus, we give player 1 $\sim$ to 3 a set of 3 numbers, then give the rest 2 to player 4 & 5.)

For $P\{X=2\} = P\{\text{Player 1 win against player 2} \sim 4\text{, but lose to player 5}\} = \dfrac{(\begin{pmatrix} 5 \\ 4 \end{pmatrix} \times 2!)}{5!} = \dfrac{1}{12} =$

$$\frac{(\text{choose 4 from 5 numbers for player } 1 \sim 4) \times (\text{permutation among player 2 \& 3})}{\text{all permutations}}$$

(Reason: similarly to previous cases, we have "player 2 & 3 's number < player 1's number < player 4's number", thus, we give player 1 $\sim$ to 4 a set of 4 numbers, the reason of 2! here is that we may have the following 2 possibilities: "player 2's number < or > player 3's number". Next, give the rest 1 number to player 5.)

For $P\{X=3\} = P\{\text{Player 1 only lose to player 5}\} = \dfrac{3!}{5!} = \dfrac{1}{20} = \dfrac{\text{permutation among player } 2 \sim 4}{\text{all permutations}}$

(Reason: we have that "player $2 \sim 4$'s number < player 1's number < player 5's number", similarly to previous case, we will have 3! for player $2 \sim 4$.)

For $P\{X=4\} = P\{\text{Player 1 win all}\} = \dfrac{4!}{5!} = \dfrac{1}{5} = \dfrac{\text{permutation among player } 2 \sim 5}{\text{all permutations}}$

(Reason: player 1 can only take the largest number, and the rest 4 numbers can be arbitrary permuted among player $2 \sim 5$.)

Hence, $P\{X=x\} = \begin{cases} \frac{1}{2} & , \text{if } x=0 \\ \frac{1}{6} & , \text{if } x=1 \\ \frac{1}{12} & , \text{if } x=2 \\ \frac{1}{20} & , \text{if } x=3 \\ \frac{1}{5} & , \text{if } x=4 \\ 0 & , o.w. \end{cases}$

Problem 17

(a)

$P\{X=1\} = F(1) - F(1^-) = \dfrac{1}{2} - \dfrac{1}{4} = \dfrac{1}{4}.$

$P\{X=2\} = F(2) - F(2^-) = \dfrac{11}{12} - (\dfrac{1}{2} + \dfrac{1}{4}) = \dfrac{1}{6}.$

$P\{X=3\} = F(3) - F(3^-) = 1 - \dfrac{11}{12} = \dfrac{1}{12}.$

(b)

$P\{\dfrac{1}{2} < X < \dfrac{3}{2}\} = P\{X < \dfrac{3}{2}\} - P\{X \le \dfrac{1}{2}\} = F(\dfrac{3}{2}) - F(\dfrac{1}{2}) = \dfrac{1}{2}.$

Problem 20

There are 5 cases in this strategy: using W to denote win(red appear); L to denote lose(red didn't appear). First game win: $\{W\}$, and 4 cases of first game lose: $\{W,W,W\}$, $\{W,W,L\}$, $\{W,L,W\}$, $\{W,L,L\}$.

The winings and probability of these cases are:

case	winings	probability
$\{W\}$	1	$\frac{18}{38} = \frac{9}{19}$
$\{L, W, W\}$	$-1+1+1=1$	$\frac{20}{38} \cdot \frac{18}{38} \cdot \frac{18}{38} = \frac{810}{19^3}$
$\{L, W, L\}$	$-1+1-1=-1$	$\frac{20}{38} \cdot \frac{18}{38} \cdot \frac{20}{38} = \frac{900}{19^3}$
$\{L, L, W\}$	$-1-1+1=-1$	$\frac{20}{38} \cdot \frac{20}{38} \cdot \frac{18}{38} = \frac{900}{19^3}$
$\{L, L, L\}$	$-1-1-1=-3$	$\frac{20}{38} \cdot \frac{20}{38} \cdot \frac{20}{38} = \frac{1000}{19^3}$

(a)

$P\{X > 0\} = \frac{9 \times 19 \times 19 + 810}{19^3} = \frac{4059}{19^3} = 0.5918.$

(b)

If we only look at the probability of wining money, part (a) told us that it is larger than half. Thus we have more chance to win the money. But, larger probability for wining money does not guarantee you make profit, which can be discuss if we calculate the expectation value.

(c)

$E[X] = 1 \times \frac{9}{19} + 1 \times \frac{810}{19^3} - 1 \times \frac{900}{19^3} - 1 \times \frac{900}{19^3} - 3 \times \frac{1000}{19^3} = \frac{3249 + 810 - 900 - 900 - 3000}{6859}$
$= -\frac{741}{6859} = -0.108.$
Note: back to problem (b), the expectation is negative, which told us that this strategy is not good.

Problem 22

(a)

The only 2 possibilities of number of games for $i = 2$ are 2 and 3, Let N be the random variable denote the number of games.

$$\begin{aligned}
P\{N=2\} &= P\{\text{A win 2 games}\} + P\{\text{B win 2 games}\} = p^2 + (1-p)^2. \\
P\{N=3\} &= 1 - p\{N=2\} = 1 - p^2 - (1-p)^2 = 2p(1-p). \\
E[N] &= 2 \cdot P\{N=2\} + 3 \cdot P\{N=3\} = 2 \cdot (p^2 + (1-p)^2) + 3 \cdot 2p(1-p) = -2p^2 + 2p + 2.
\end{aligned}$$

Next, find the value p make $E[N]$ maximum. Let $\dfrac{dE[N]}{dp} = 0$

$$\begin{aligned} &\Rightarrow -4p + 2 = 0 \\ &\Rightarrow p = \frac{1}{2} \end{aligned}$$

(b)

There are 3 possibility of number of games for $i = 3$, which are 3,4,5. Let N_3 be the random variable denote the number of games.

$$\begin{aligned} P\{N_3 = 3\} &= P\{\text{A win 3 games}\} + P\{\text{B win 3 games}\} = p^3 + (1-p)^3. \\ P\{N_3 = 5\} &= P\{\text{A win after 5 games}\} + P\{\text{B win after 5 games}\} \\ &= \frac{4!}{2!2!} \cdot p^2(1-p)^2 \cdot p + \frac{4!}{2!2!} \cdot p^2(1-p)^2 \cdot (1-p) = 6p^2(1-p)^2. \\ P\{N_3 = 4\} &= 1 - P\{N_3 = 3\} - P\{N_3 = 5\} = -6p^4 + 12p^3 - 9p^2 + 3p. \\ E[N_3] &= 3 \cdot (p^3 + (1-p)^3) + 4 \cdot (-6p^4 + 12p^3 - 9p^2 + 3p) + 5 \cdot (6p^2(1-p)^2) \\ &= 3 \cdot (2p^4 - 4p^3 + p^2 + p + 1). \end{aligned}$$

Next, find the value p make $E[N_3]$ maximum. Let $\dfrac{dE[N_3]}{dp} = 0$

$$\begin{aligned} &\Rightarrow 8p^3 - 12p^2 + 2p + 1 = 0 \\ &\Rightarrow (2p-1)(4p^2 - 4p - 1) = 0 \\ &\Rightarrow p = \frac{1}{2} \text{ or } \frac{1 \pm \sqrt{2}}{2}. \end{aligned}$$

But the solutions $\dfrac{1+\sqrt{2}}{2} > 1$ and $\dfrac{1-\sqrt{2}}{2} < 0$ can not be a probability of Team A wining the game. Thus, the only possible p is $\dfrac{1}{2}$.

38

(a) $E(X)=1 \quad Var(X)=5.$

$E((2+X)^2) = E(4+4X+X^2) = 4+4\cdot 1+E(X^2) = 4+4+6=14$ ※

$\left(\begin{array}{l} Var(X) = E(X^2)-(E(X))^2 \Rightarrow E(X^2) = Var(X)+(E(X))^2 \\ E(X^2) = 5+1=6 \end{array} \right)$

(b) $Var(4+3X) = Var(3X) = 9\,Var(X) = 45$ ※

Theoretical Exercises

Ex: 2

$P(X<a) = F(a)$

$P(e^X<a) = P(X<\ln a) = F(\ln a), \; a>0$ ※

Ex: 4 $\quad P(X=n) = \dfrac{4}{n(n+1)(n+2)} \quad n\geq 1.$

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n(n+1)(n+2)} \overset{(1)}{=} \sum_{n=1}^{\infty} 2\left(\frac{1}{n} - \frac{1\times 2}{n+1} + \frac{1}{n+2}\right)$

$\displaystyle = \sum_{n=1}^{\infty} 2\left[\left(\frac{1}{n} - \frac{1}{n+1}\right) - \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right]$

$\displaystyle = 2\Bigg[\underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)}_{(a)} - \underbrace{\sum_{n=2}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)}_{(b)}\Bigg] = 1$ ※

$(a) = 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots$

$-)\; (b) = \qquad \frac{1}{2}-\frac{1}{3}+\cdots$

$\frac{1}{2}$

(1)

$\dfrac{4}{n(n+1)(n+2)} = \dfrac{a}{n}+\dfrac{b}{n+1}+\dfrac{c}{n+2}$

$\Rightarrow a(n+1)(n+2) + bn(n+2) + cn(n+1) = 4.$

$\Rightarrow a=1,\; b=-2,\; c=1$ ※

Ex:4 (b)

$$E(X)=\sum_{n=1}^{\infty}\frac{4n}{n(n+1)(n+2)}=\sum_{n=1}^{\infty}\frac{4}{(n+1)(n+2)}=\sum_{n=1}^{\infty}\left(\frac{4}{n+1}-\frac{4}{n+2}\right)$$

$$=4\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{4}{2}=2 \text{ ※}$$

(c)

$$E(X^2)=\sum_{n=1}^{\infty}\frac{4n^2}{n(n+1)(n+2)}=\sum_{n=1}^{\infty}\frac{4n}{(n+1)(n+2)}=\sum_{n=1}^{\infty}\left(-\frac{1}{n+1}+\frac{2}{n+2}\right)\cdot 4$$

$$=4\sum_{n=1}^{\infty}\left(-\frac{1}{n+1}+\frac{2}{n+2}\right)$$

$$=4\left(-\frac{1}{2}+\left(\frac{2}{3}-\frac{1}{3}\right)+\left(\frac{2}{4}-\frac{1}{4}\right)+\left(\frac{2}{5}-\frac{1}{5}\right)\cdots\right)$$

$$=-2+\sum_{i=3}^{\infty}\frac{1}{i}=\infty.$$ (微積分審斂法) $\rightarrow \int_3^{\infty}\frac{1}{x}dx$ 發散

Ex:5. $N=\{1,2,\dots\}$

show $\sum_{j=1}^{\infty}(a_1+\dots a_j)P(N=j)=\sum_{i=1}^{\infty}a_i P(N\geq i)$

$$\sum_{j=1}^{\infty}(a_1+\dots a_j)P(N=j)$$

$$=a_1P(N=1)+(a_1+a_2)P(N=2)+\dots+(a_1+\dots a_n)P(N=n)+\dots$$

$$=a_1\underbrace{\sum_{i=1}^{\infty}P(N=i)}_{P(N\geq 1)}+a_2\underbrace{\sum_{j=2}^{\infty}P(N=j)}_{P(N\geq 2)}+\dots$$

$$=\sum_{i=1}^{\infty}a_iP(N\geq i)$$

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4.5 續

$$E(N) = \sum_{j=1}^{\infty} j \cdot P(N=j)$$

$$= P(N=1) + 2 \cdot P(N=2) + \cdots$$

$$= P(N=1) + (1+1)P(N=2) + (1+1+1)P(N=3)$$

$$= \sum_{i=1}^{\infty} 1 \cdot P(N \geq i) \text{ ※ } \left(\text{by } \sum_{j=1}^{\infty} a_1 + \cdots a_j \, P(N=j) = \sum_{i=1}^{\infty} a_i P(N \geq i)\right)$$

$$E(N(N+1)) = \sum_{j=1}^{\infty} j(j+1) P(N=j)$$

$$= 2 \cdot \sum_{j=1}^{\infty} \frac{j(j+1)}{2} P(N=j)$$

$$\left(\frac{j(j+1)}{2} = 1+2+3+\cdots j\right)$$

$$= 2 \cdot \sum_{j=1}^{\infty} (1+2+3+\cdots+j) P(N=j)$$

$$\left(\text{by } \sum_{j=1}^{\infty} (a_1 + \cdots a_j) P(N=j) = \sum_{i=1}^{\infty} a_i P(N \geq i)\right)$$

$$= 2 \cdot \sum_{j=1}^{\infty} j \cdot P(N=j) \text{ ※}$$

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Ex: 7 $X \sim (\mu, \sigma^2)$ $E(X) = \mu$, $Var(X) = \sigma^2$

$$E\left(\frac{X-\mu}{\sigma}\right) = E\left(\frac{X}{\sigma}\right) - \frac{\mu}{\sigma} = \frac{1}{\sigma}E(X) - \frac{\mu}{\sigma} = 0 \text{ ※}$$

$$Var\left(\frac{X}{\sigma} - \frac{\mu}{\sigma}\right) = Var\left(\frac{X}{\sigma}\right) = \frac{1}{\sigma^2}Var(X) = \frac{\sigma^2}{\sigma^2} = 1 \text{ ※}$$