14.

5 white、7 black

(a)

黑 → 黑 → 白 → 白.

(5w7b) (5w9b) (5w11b) (7w11b)

$$\frac{7}{12} \times \frac{9}{14} \times \frac{5}{16} \times \frac{7}{18} = \frac{35}{768}$$

(b)

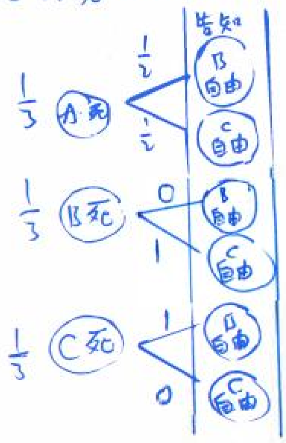
黑黑白白
黑白黑白
黑白白黑
白白黑黑
白黑白黑
白黑黑白

6種情況的機率都相同，皆為 $\frac{35}{768}$.

44.

已知有 Prisoner A、B、C 三人

Let W 為 A 被告知 B 獲得自由的事件，E 為 A 被處死的事件



$$P(E|W) = \frac{P(E\cap W)}{P(W)} = \frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{3}\cdot 0+\frac{1}{3}\cdot 1} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$$

同時，我們知道 $P(E) = \frac{1}{3} = P(E|W)$

所以事實上 E & W independent

至於為什麼 A 被告知 B 獲得自由不會令 A 被處死的機率從 $\frac{1}{3}$ 變 $\frac{1}{2}$，

其原因為限制在 A 會死或 C 會死的情況下，A 被告知 B 獲得自由的機率不相同，

若相同，$P(E|W)$ 才會為 $\frac{1}{2}$。

46.
用 M、F 分別表示一位 policyholder 為男性和女性的事件.

$P(M) = \alpha$, $P(F) = 1-\alpha$

$$P(A_1) = P(A_1|M)P(M) + P(A_1|F)P(F).$$

$$= P_m\,\alpha + P_f\,(1-\alpha)$$

假設在同一性別下，在第一年申請理賠和在第二年申請理賠是獨立事件。即 $P(A_2|A_1\cap M) = P(A_2|M)$,
$P(A_2|A_1\cap F) = P(A_2|F)$

$$P(A_2|A_1) = \frac{P(A_1\cap A_2\cap M) + P(A_1\cap A_2\cap F)}{P(A_1)}$$

$$= \frac{P(A_2|A_1\cap M)P(A_1|M)P(M) + P(A_2|A_1\cap F)P(A_1|F)P(F)}{P(A_1)}$$

$$= \frac{P(A_2|M)P(A_1|M)P(M) + P(A_2|F)P(A_1|F)P(F)}{P(A_1)}$$

$$= \frac{\alpha P_m^2 + (1-\alpha)P_f^2}{P(A_1)}$$

$P(A_2|A_1) > P(A_1)$ if and only if $\alpha P_m^2 + (1-\alpha)P_f^2 > [\alpha P_m + (1-\alpha)P_f]^2$

$$\alpha P_m^2 + (1-\alpha)P_f^2 - [\alpha P_m + (1-\alpha)P_f]^2$$

$$= \alpha P_m^2 + (1-\alpha)P_f^2 - \alpha^2 P_m^2 - (1-\alpha)^2 P_f^2 - 2\alpha(1-\alpha)P_mP_f$$

$$= (1-\alpha)\alpha P_m^2 + (1-\alpha)\alpha P_f^2 - 2\alpha(1-\alpha)P_mP_f$$

$$= \alpha(1-\alpha)(P_m - P_f)^2 > 0 \quad . \quad \because\ 0<\alpha<1 \text{ and } P_m \neq P_f$$

直觀解釋：

假設 $P_m > P_f$ ⇒ 男性比女性有更多可能發生意外而申請理賠。

若給定 A_1 的情況下，有較高的可能性為男性，因此在第二年就有更高的機會發生意外而申請理賠。

60.
(a) $\because$ Smith have brown eyes & his sister has blue eyes & his parents have brown eyes

$\therefore$ Smith's parents 的基因各有 1 brown-eyed gene & 1 blue-eyed gene

父母生下的小孩可能基因型

父 母	父 母	父 母	父 母
brown brown	brown blue	blue brown	blue blue

P(Smith possesses a blue eyed gene | Smith has brown eyes)

$$= \frac{\frac{2}{4}}{\frac{3}{4}} = \frac{2}{3} \quad \#$$

(b)

Let A be the event that the child has blue eyes.

B be the event that Smith possesses a blue-eyed gene.

First, we let

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

if Smith possesses 1 brown-eyed gene & 1 blue-eyed gene

the child's genes would be

brown blue	brown blue	blue blue	blue blue
Smith his wife	Smith his wife	Smith his wife	Smith his wife

Thus $P(A|B) = \frac{1}{2}$

if Smith don't possess any blue-eyed gene

the child's genes would be

brown blue	brown blue	brown blue	brown blue
Smith his wife	Smith his wife	Smith his wife	Smith his wife

Thus $P(A|B^c) = 0$

& we know $P(B) = \frac{2}{3}$ & $P(B^c) = \frac{1}{3}$ by (a)

Therefore $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = \frac{1}{2} \cdot \frac{2}{3} + 0 \cdot \frac{1}{3} = \frac{1}{3}$ #

(c) Let B be the event that Smith possesses a blue-eyed gene.

$P($the second child has brown eyes $|$ the first child has brown eyes$)$

$$\underset{E}{\parallel} \qquad \underset{F}{\parallel}$$

$$= \frac{P(E\cap F)}{P(F)} = \frac{P(E\cap F\mid B)P(B)+P(E\cap F\mid B^c)P(B^c)}{P(F)} = \frac{P(E\mid B)P(F\mid B)P(B)+P(E\mid B^c)P(F\mid B^c)P(B^c)}{P(F)}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3}+1\cdot 1\cdot\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{6}+\frac{1}{3}}{\frac{2}{3}} = \frac{3}{4} \quad ※$$

64.

若用 a 策略，不論用誰的答案對的機率都是 p.

若用 b 策略，

$$P(\text{win}) = P(\text{win}\mid \underbrace{\text{common answer}}_{\{(0,0),(x,x)\}})\,P(\text{common answer})$$

$$+\,P(\text{win}\mid \underbrace{\text{not common answer}}_{\{(0,x),(x,0)\}})\,P(\text{not common answer})$$

$$= p\cdot\frac{1}{2} + \Big[P(\text{win}\cap\text{husband answer}\mid\text{not common answer})$$

$$+\,P(\text{win}\cap\text{wife answer}\mid\text{not common answer})\Big]\times\frac{1}{2}$$

$$= \frac{1}{2}p + \left[\frac{1}{2}p+\frac{1}{2}p\right]\times\frac{1}{2} = p$$

$\Rightarrow$ 用 a 策略或 b 策略贏的機率相同。

66.

(a)

P(a current flows between A and B)

$= P(\{1,2,5 \text{ close}\} \cup \{3,4,5 \text{ close}\})$

$= P(\{1,2,5 \text{ close}\}) + P(\{3,4,5 \text{ close}\}) - P(\{1,2,5 \text{ close}\} \cap \{3,4,5 \text{ close}\})$

$= P_1P_2P_5 + P_3P_4P_5 - P_1P_2P_3P_4P_5$ #

(b) by Hint

$P(\underset{E}{\underbrace{\text{a current flows between A and B}}})$

$= P(E \mid 3 \text{ close})P(3 \text{ close}) + P(E \mid 3 \text{ open})P(3 \text{ open})$

$P(E \mid 3 \text{ close}) = P(\{1 \text{ or } 2 \text{ close}\} \cap \{4 \text{ or } 5 \text{ close}\}) = (1-(1-P_1)(1-P_2))(1-(1-P_4)(1-P_5))$

$P(E \mid 3 \text{ open}) = P(\{1,4 \text{ close}\} \cup \{2,5 \text{ close}\}) = P_1P_4 + P_2P_5 - P_1P_4P_2P_5$

$P(3 \text{ close}) = P_3$

$P(3 \text{ open}) = 1 - P_3$

Thus P(a current flows between A and B)

$= (1-(1-P_1)(1-P_2))(1-(1-P_4)(1-P_5))P_3 + (P_1P_4 + P_2P_5 - P_1P_2P_4P_5)(1-P_3)$ #

78.

(a) 總共要比四場，所以前兩場 A、B 都要各拿下一場，否則會提早結束。

$$\left.\begin{array}{l} ① \; A \to B \to A \to A \\ ② \; A \to B \to B \to B \\ ③ \; B \to A \to A \to A \\ ④ \; B \to A \to B \to B. \end{array}\right\} \begin{array}{l} \text{A勝}: 2p^3(1-p) \\ \\ \text{B勝}: 2p(1-p)^3 \end{array}$$

$$2p^3(1-p) + 2p(1-p)^3 = 2p(1-p)[p^2 + (1-p)^2]$$

(b)

由於要贏對方兩局才算贏，可以兩局為單位來看。

	第一局	第二局	
①	A	A	→ 結束
②	A	B	重新再比
③	B	A	重新再比

$$\Rightarrow P(A\text{勝}) = P(A\text{拿下兩局}) + \underbrace{2\cdot P(A\text{、}B\text{各拿下一局})\cdot P(A\text{勝})}_{\text{若 }A\text{、}B\text{各拿下一局則重新再比}}$$

第一局A勝，第二局B勝 或第一局B勝，第二局A勝

$$= p^2 + 2p(1-p)\,P(A\text{勝})$$

$$P(A\text{勝})\left[1-2p(1-p)\right] = p^2$$

$$\Rightarrow P(A\text{勝}) = \frac{p^2}{1-2p(1-p)}$$

83.

(a) $P(\text{red at any throw}) = P(\text{red} \mid \text{head})P(\text{head}) + P(\text{red} \mid \text{tail})P(\text{tail})$

$$= \frac{4}{6}\cdot\frac{1}{2} + \frac{2}{6}\cdot\frac{1}{2} = \frac{2}{6}+\frac{1}{6} = \frac{1}{2} \quad \#$$

(b) let r 表示這次骰子結果為 red

$$P(r \mid rr) = \frac{P(rrr)}{P(rr)} = \frac{\frac{1}{2}\cdot\left(\frac{4}{6}\right)^3 + \frac{1}{2}\left(\frac{2}{6}\right)^3}{\frac{1}{2}\cdot\left(\frac{4}{6}\right)^2 + \frac{1}{2}\left(\frac{2}{6}\right)^2} = \frac{\frac{12}{6}}{\frac{20}{6}} = \frac{3}{5} \quad \#$$

(c) Let A be die A that is being used

$$P(A \mid rr) = \frac{P(A \cap rr)}{P(rr)} = \frac{P(rr \mid A)P(A)}{P(rr)} = \frac{\left(\frac{4}{6}\right)^2\cdot\frac{1}{2}}{\frac{1}{2}\left(\frac{4}{6}\right)^2+\frac{1}{2}\left(\frac{2}{6}\right)^2} = \frac{16}{20} = \frac{4}{5} \quad \#$$

86. 根據 Hint, Let $N(B)$ denote the number of elements in B.

$$P(A\subset B) = \sum_{i=0}^{n} P(A\subset B \mid N(B)=i)\, P(N(B)=i)$$

$$P(N(B)=i) = \frac{\binom{n}{i}\ \} \rightarrow \text{選出B的elements}}{2^n\ \} \rightarrow \text{B的所有可能性.}}$$

$P(A\subset B \mid N(B)=i)$: 當 $i=0$, $P(A\subset B \mid N(B)=0) = \frac{1}{2^n}$ (A是ϕ)

$i=1$, $P(A\subset B \mid N(B)=1) = \frac{2}{2^n}$ (A是ϕ或A只有一個 element)

$i=2$, $P(A\subset B \mid N(B)=2) = \frac{4}{2^n}$ (A是ϕ或A只有一個element或A有兩個elements)

以此類推, $P(A\subset B \mid N(B)=i) = \frac{2^i}{2^n}$

$$P(A\subset B) = \sum_{i=0}^{n} \frac{2^i}{2^n}\cdot\frac{\binom{n}{i}}{2^n} = \frac{1}{4^n}\underbrace{\sum_{i=0}^{n}\binom{n}{i}2^i\cdot 1^{n-i}}_{\text{by 二項式定理}} = \frac{1}{4^n}\cdot 3^n = \left(\frac{3}{4}\right)^n$$

$\because$ $B+B^c=S$, B^c 也是 S 的一任意子集

$P(AB=\phi) = P(A\subset B^c) = \left(\frac{3}{4}\right)^n$ by (a).

90.

(a)

P(judge 3 vote guilty | judge 1 and 2 vote guilty)

被告是犯人的機率

$$= \frac{P(\text{judge } 1,2,3 \text{ vote guilty})}{P(\text{judge } 1,2 \text{ vote guilty})} = \frac{0.7\cdot(0.7)^3+(0.3)\cdot(0.2)^3}{(0.7)\cdot(0.7)^2+(0.3)\cdot(0.2)^2} = \frac{0.2425}{0.355} = \frac{97}{142} \quad ※$$

(b)

P (judge 3 vote guilty | one of judge 1, 2 vote guilty)

judge 1 or judge 2 vote guilty

$$= \frac{2\cdot(0.7)\cdot(0.7)^2\cdot(0.3)+2\cdot(0.3)\cdot(0.2)^2\cdot(0.8)}{2\cdot(0.7)\cdot(0.7)(0.3)+2\cdot(0.3)(0.2)(0.8)} = \frac{0.225}{0.39} = \frac{15}{26} \quad ※$$

(c)

P(judge 3 vote guilty | judge 1, 2 both cast not guilty votes)

$$= \frac{(0.7)\cdot(0.7)\cdot(0.3)^2+(0.3)\cdot(0.2)\cdot(0.8)^2}{(0.7)\cdot(0.3)^2+(0.3)\cdot(0.8)^2} = \frac{0.0825}{0.255} = \frac{11}{34} \quad ※$$

(d)

(i) 由 (a)(b)(c) 可知，judge 1, 2 vote guilty 與否，會影響 judge 3 vote guilty 的機率，judge 1, 2 均 vote guilty 的條件下，judge 3 vote guilty 的機率最大；反之，judge 1, 2 均不 vote guilty 的條件下，judge 3 vote guilty 的機率最小。
Similarly, judge 1、2 vote guilty 的機率也會受其他 judge 的影響，所以 E_1、E_2、E_3 彼此間不獨立。

(ii) 但若限制在確定已知被告是犯人的情形下，依據題目，judge 1、2、3 vote guilty 的機率均為 0.7 且各自獨立，所以 E_1、E_2、E_3 是條件獨立。

※對冤罪有興趣的同學，可以閱讀森炎的「冤罪論：關於冤罪的一百種可能」。