16. $X \equiv$ 雨量/年 $\quad X \sim N(40, 4^2)$

依題意，十年内未有一年年雨量超過50之機率

*假設每年年雨量兩兩獨立， assumption

P(未來十年内未有一年年雨量超過50)

$= ($P(年雨量未超過50)$)^{10}$

$$= (P(X<50))^{10} = \left(P\left(\frac{X-40}{4} < \frac{10}{4}\right)\right)^{10} = (\Phi(2.5))^{10}$$

$$= (0.9938)^{10}$$

17.

依題意 $0.25 = P\left(\frac{X-\mu}{\sigma} < \frac{180-\mu}{\sigma}\right) = \Phi\left(\frac{180-\mu}{\sigma}\right) = \Phi(-0.675)$

$0.25 = P\left(\frac{X-\mu}{\sigma} > \frac{320-\mu}{\sigma}\right) = 1-\Phi\left(\frac{320-\mu}{\sigma}\right) = 1-\Phi(0.675)$

$$\mu = \frac{320+180}{2} = 250 \quad \sigma = 103.7$$

(a) $P(X<200) = P\left(\frac{X-\mu}{\sigma} < \frac{200-250}{103.7}\right)$

$= \Phi(-0.482) = 0.315$

(b) $P(X<280) = P\left(\frac{X-\mu}{\sigma} < \frac{280-250}{103.7}\right) = 0.614$

$$\Rightarrow P(280<X<250) = 0.75 - 0.614 = 0.136$$

29. 先算到要多少次漲才能有到 0.3 的漲幅（變1.3倍）

$d^{1000} \cdot \left(\frac{u}{d}\right)^x > 1.3 = 0.99^{1000} \cdot \left(\frac{1.012}{0.99}\right)^x > 1.3$ 取log

$\Rightarrow 1000 \lg 0.99 + X(\lg 1.012 - \lg 0.99) > \lg 1.3$

$\Rightarrow$ 解 $X = 469.2$

X = S 到 US 的次數 approximate.

$X \sim Bin(1000, 0.52) \rightarrow N(520, 249.6)$

$$P(X > 469.2) = P\left(\frac{X-520}{\sqrt{249.6}} > \frac{469.2-520}{\sqrt{249.6}}\right)$$
$$\approx P(Z > -3.196)$$
$$= 0.9993 \#$$

30. region is white, $X \sim N(4, 4)$, 比例 $= 1-\alpha$

region is black, $Y \sim N(6, 9)$, 比例 $= \alpha$

$$P(\text{black}) = \frac{P(5 \mid \text{black}) \cdot \alpha}{P(5 \mid \text{black}) \cdot \alpha + P(5 \mid \text{white})(1-\alpha)} = \frac{\frac{1}{\sqrt{2\pi} \cdot 2} \exp -\frac{(5-4)^2}{2 \cdot 4} \cdot \alpha}{\frac{1}{\sqrt{2\pi}\, 2} \exp -\frac{(5-4)^2}{2 \cdot 4} \cdot \alpha + \frac{1}{\sqrt{2\pi}\, 3} \exp -\frac{(5-6)^2}{2 \cdot 9} \cdot (1-\alpha)}$$

$$= \frac{\frac{1}{2} e^{-\frac{1}{8}} \cdot \alpha}{\frac{1}{2} e^{-\frac{1}{8}} \cdot \alpha + \frac{1}{3} e^{-\frac{1}{18}} \cdot (1-\alpha)} \#$$

= 1/2, α = 0.3827

31.

(a)

$X \sim U(0, A)$

$$E(|X-a|) = \int_0^A |X-a| \frac{1}{A} dx = \int_0^a (X-a) \frac{1}{A} dx + \int_a^A (a-X) \frac{1}{A} dx$$
$$= \frac{A}{2} - \left(a - \frac{a^2}{A}\right)$$
$$\frac{d}{da} E(|X-a|) = -\left(1 - \frac{2a}{A}\right) = 0 \Rightarrow a = \frac{A}{2} \#$$

(b)

$$E(|X-a|) = \int_0^\infty |X-a| \lambda e^{-\lambda x} dx = \int_0^a (X-a) \lambda e^{-\lambda x} dx + \int_a^\infty (a-X) \lambda e^{-\lambda x} dx$$
$$= a + \frac{2}{\lambda} e^{-\lambda \cdot a} - \frac{1}{\lambda}$$
$$\frac{d}{da} E(|X-a|) = 1 + \frac{2}{\lambda} e^{-\lambda a} \cdot -\lambda \Rightarrow a = \frac{\ln 2}{\lambda} \#$$

Problem 33

Let $X \sim \exp(\frac{1}{8})$ be the random variable denote the number of years a radio functions, and pdf $f_X(x) = \frac{1}{8}e^{-\frac{x}{8}}$. Since the exponential distribution has lack of memory property, we may consider the following calculation:

$$\begin{aligned} P\{\text{radio will work after additional 8 years}\} &= P\{X > 8\} \\ &= \int_8^\infty \frac{1}{8}e^{-\frac{x}{8}}dx \\ &= e^{-\frac{8}{8}} = e^{-1}. \end{aligned}$$

Theoretical Exercise 13

(a)

$X \sim \text{Uniform}(a, b)$, the distribution function $F_X(x)$ of X is $\int_a^x \frac{1}{b-a}du$. Let m_a be the median of such X, we have $F_X(m_a) = \frac{1}{2}$. Hence, we have:

$$\begin{aligned} \frac{1}{2} &= \int_a^{m_a} \frac{1}{b-a}du = \frac{m_a - a}{b-a}. \\ \Rightarrow m_a &= \frac{a+b}{2}. \end{aligned}$$

(b)

$X \sim N(\mu, \sigma^2)$, since the normal distribution is symmetry at $x = \mu$, the median of Normal distribution is μ.

(c)

$X \sim exp(\lambda)$, the distribution function $F_X(x)$ of X is $1 - e^{-\lambda x}$. Let m_c be the median of such X, we have:

$$\begin{aligned} &\frac{1}{2} = 1 - e^{-\lambda m_c} \\ &\Rightarrow \frac{1}{2} = e^{-\lambda m_c} \\ &\Rightarrow \ln\frac{1}{2} = -\lambda m_c \\ &\Rightarrow m_c = -\frac{1}{\lambda}\ln\frac{1}{2} = \frac{\ln 2}{\lambda} \end{aligned}$$

Theoretical Exercise 19

The pdf of X: $f_X(x) = \lambda e^{-\lambda x}$.

$$\begin{aligned} E[x^n] &= \int_0^\infty x^n f_X(x)dx = \int_0^\infty \lambda x^n e^{-\lambda x}dx \\ &= \frac{\Gamma(n+1)}{\lambda^n}\int_0^\infty \frac{\lambda^{n+1}}{\Gamma(n+1)}x^n e^{-\lambda x}dx = \frac{n!}{\lambda^n} \end{aligned}$$

Theoretical Exercise 25

The cdf of X: $F_X(x) = 1 - \exp\{-\left(\frac{x-\nu}{\alpha}\right)^\beta\}$, when $x > \nu$. Then we have: (Note that we need to prove both direction.)

$$
\begin{aligned}
& X \sim weibull(\nu, \alpha, \beta) \\
\Leftrightarrow\ & P\{X < x\} = F_X(x) = 1 - \exp\{-\left(\frac{x-\nu}{\alpha}\right)^\beta\} \text{ for } x > \nu \\
\Leftrightarrow\ & P\{\left(\frac{X-\nu}{\alpha}\right)^\beta < \left(\frac{x-\nu}{\alpha}\right)^\beta\} = 1 - \exp\{-\left(\frac{x-\nu}{\alpha}\right)^\beta\} \text{ for } x > \nu, \text{ set } y = \left(\frac{x-\nu}{\alpha}\right)^\beta > 0 \\
\Leftrightarrow\ & P\{Y < y\} = 1 - \exp\{-y\} \text{ for } y > 0 \\
\Leftrightarrow\ & Y \sim exp(1)
\end{aligned}
$$

Theoretical Exercise 31

Let $f_X(x)$ denote the pdf of normal distribution of with parameter μ, σ^2, and we have:

$$
\begin{aligned}
P\{Y < y\} &= P\{e^X < y\} \\
&= \begin{cases} P\{X < \ln y\}, & \text{if } y > 0 \\ 0, & \text{if } y \le 0 \end{cases} \\
&= \begin{cases} \int_{-\infty}^{\ln y} f_X(x)dx, & \text{if } y > 0 \\ 0, & \text{if } y \le 0 \end{cases} \qquad (1)
\end{aligned}
$$

Take derivative on (1) with respect to y over $y > 0$, we will get:

$$
\frac{f_X(\ln y)}{y}, \text{ if } y > 0 = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp(-\frac{(\ln y - \mu)^2}{2\sigma^2}), \text{ if } y > 0 \qquad (2)
$$

Thus, the pdf $f_Y(y)$ of y:

$$
f_Y(y) = \begin{cases} \frac{1}{y\sqrt{2\pi\sigma^2}} \exp(-\frac{(\ln y - \mu)^2}{2\sigma^2}), & \text{if } y > 0 \\ 0, & \text{otherwise} \end{cases}
$$