1.

(a) $\because \int_{-1}^{1} f(x)dx = 1$

$\Rightarrow \int_{-1}^{1} c(1-x^2)\,dx = c\int_{-1}^{1}(1-x^2)\,dx = c\left(x\Big|_{-1}^{1} - \frac{1}{3}x^3\Big|_{-1}^{1} \right) = c\left(2-\frac{2}{3}\right) = \frac{4}{3}c = 1$

$\Rightarrow c = \frac{3}{4}$ #

(b)

$F(x) = P(X \le x) = \int_{-1}^{x} \frac{3}{4}(1-t^2)dt = \frac{3}{4}\left(t\Big|_{-1}^{x} - \frac{1}{3}t^3\Big|_{-1}^{x}\right) = \frac{3}{4}\left(x+1-\frac{1}{3}x^3-\frac{1}{3}\right)$

$= -\frac{1}{4}x^3+\frac{3}{4}x+\frac{1}{2}$ if $-1 \le x \le 1$

$F(x) = 0$ if $x < -1$

$F(x) = 1$ if $x > 1$

$$\Rightarrow F(x) = \begin{cases} 1 & x > 1 \\ -\frac{1}{4}x^3+\frac{3}{4}x+\frac{1}{2} \quad \text{if} & -1 \le x \le 1 \\ 0 & x < -1 \end{cases}$$

#

4.

(a)

$P(X > 20) = \int_{20}^{\infty} \frac{10}{x^2}\,dx = 10\left(-\frac{1}{x}\Big|_{20}^{\infty}\right) = \frac{1}{2}$ #

(b)

$$F(x) = P(X \le x) = \begin{cases} \int_{10}^{x} \frac{10}{t^2}dt = 10\left(-\frac{1}{t}\right)\Big|_{10}^{x} = 1-\frac{10}{x}, & x > 10 \\ 0, & x \le 10 \end{cases}$$

#

(c)

Assumption：每個裝置互相獨立運作

P (at least 3 will function for at least 15 hours)

$= \sum_{i=3}^{6} \binom{6}{i} P(X \ge 15)^i P(X<15)^{6-i} = \sum_{i=3}^{6} \binom{6}{i} \left(\frac{2}{3}\right)^i \left(\frac{1}{3}\right)^{6-i} \approx 0.8999$

#

6、

(a)

$$E(X) = \int_0^\infty x \frac{1}{4} x e^{-\frac{x}{2}} dx = \int_0^\infty \frac{1}{4} x^2 e^{-\frac{x}{2}} dx$$

$$= -\frac{1}{2} x^2 e^{-\frac{x}{2}} \Big|_0^\infty + \int_0^\infty \frac{1}{2} 2x e^{-\frac{x}{2}} dx$$

$$= 0 + \int_0^\infty x e^{-\frac{x}{2}} dx$$

$$= -2x e^{-\frac{x}{2}} \Big|_0^\infty + \int_0^\infty 2 e^{-\frac{x}{2}} dx$$

$$= -4 e^{-\frac{x}{2}} \Big|_0^\infty = 4 \quad \#$$

(b)

$$E(X) = \int_{-1}^{1} x c (1-x^2) dx = c \int_{-1}^{1} (x - x^3) dx = 0 \quad \#$$

(c)

$$E(X) = \int_5^\infty x \frac{5}{x^2} dx = 5 \log x \Big|_5^\infty = \infty \quad \#$$

11、

Def：X：在長度為 L 的片段上選取一點，每點被選取的機率相等

i.e. $X \sim \text{Unif}(0, L)$

$$P\left(\frac{\text{the shorter segment}}{\text{the longer segment}} < \frac{1}{4}\right) = P\left(\frac{X}{L-X} < \frac{1}{4} \text{ or } \frac{X}{L-X} > 4\right)$$

$$= P\left(X < \frac{L}{5} \text{ or } X > \frac{4}{5}L\right) = \int_0^{\frac{L}{5}} \frac{1}{L} dx + \int_{\frac{4}{5}L}^{L} \frac{1}{L} dx = \frac{1}{5} + \frac{1}{5} = \frac{2}{5} \quad \#$$

13、

(a) X：在10:00～10:30選取一時間點，公車在這時間點會來到公車站，其中每點被選取的機率相等

i.e. $X \sim \text{Unif}(0, 30)$

$$P(X > 10) = \int_{10}^{30} \frac{1}{30} dx = \frac{2}{3} \quad \#$$

(b)

$$P(X > 25 \mid X > 15) = \frac{P(X>25, X>15)}{P(X>15)} = \frac{P(X>25)}{P(X>15)} = \frac{\int_{25}^{30} \frac{1}{30} dx}{\int_{15}^{30} \frac{1}{30} dx} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3} \quad \#$$

37

$X \sim \text{Uniform}(-1,1)$

(a)

$P\{|X| > \frac{1}{2}\} = 2P\{X > \frac{1}{2}\}$ X 的分布對稱於原點。

$= 2\int_{\frac{1}{2}}^{1} \underbrace{\frac{1}{2}}_{\text{pdf of } X} dx = \frac{1}{2}$

(b)

Let $Y = |X|$, $0 \le Y \le 1$

$P(Y \le y) = P(|X| \le y) = P(-y \le X \le y)$

$= 2P(0 \le X \le y) = 2\int_0^y \frac{1}{2} dx$

$= y, \quad 0 \le y \le 1$

40

$X \sim \text{Uniform}(0,1)$

$P(Y \le y) = P(e^X \le y) = P(X \le \ln y)$

$= \int_0^{\ln y} \underbrace{1}_{\text{pdf of } X} dx = \ln y, \quad 1 \le y \le e$

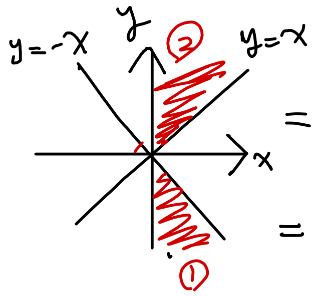
$0 \le X \le 1$

$\Rightarrow e^0 \le e^X \le e^1$

$\Rightarrow 1 \le y \le e$

(2)

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_{-\infty}^{0} y f_Y(y)\, dy + \int_{0}^{\infty} y f_Y(y)\, dy$$



$$= -\int_{-\infty}^{0} (-y) f_Y(y)\, dy + \int_{0}^{\infty} y f_Y(y)\, dy$$

$$= \underbrace{-\int_{-\infty}^{0}\int_{0}^{-y} 1\, dx\, f_Y(y)\, dy}_{(1)} + \underbrace{\int_{0}^{\infty}\int_{0}^{y} 1\, dx\, f_Y(y)\, dy}_{(2)}.$$

$$= -\int_{0}^{\infty}\int_{-\infty}^{-x} f_Y(y)\, dy\, dx + \int_{0}^{\infty}\int_{x}^{\infty} f_Y(y)\, dy\, dx$$

$$= -\int_{0}^{\infty} P(Y < -x)\, dx + \int_{0}^{\infty} P(Y > x)\, dx$$

$$= \int_{0}^{\infty} P(Y > y)\, dy - \int_{0}^{\infty} P(Y < -y)\, dy$$

(5)

$$E(X^n) = \int_{0}^{\infty} P(X^n > t)\, dt$$

let $t = x^n$

$$\frac{dt}{dx} = n x^{n-1}$$

$$= \int_{0}^{\infty} P(X^n > x^n) \cdot n x^{n-1}\, dx$$

$$= \int_{0}^{\infty} n x^{n-1} P(X > x)\, dx$$

(8) $0 \le X \le c$, $0 \le X^2 \le cX$

$$\Rightarrow 0 \le E(X^2) \le c \cdot E(X)$$

$$Var(X) = E(X^2) - [E(X)]^2 \le c E(X) - [E(X)]^2$$

$$= c \cdot (\alpha \cdot c) - (\alpha \cdot c)^2 = c^2 \alpha (1-\alpha)$$

$$\because\ X \le c \Rightarrow E(X) \le c,\ 0 \le \alpha = \frac{E(X)}{c} \le 1$$

Since $\underset{\alpha}{\arg\max}\ \alpha(1-\alpha) = \frac{1}{2}$

$$\Rightarrow \operatorname{Var}(X) \le c^2 \alpha(1-\alpha) \le c^2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{c^2}{4}.$$