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CH.4

~~#43~~

(1)

Majority decoding: based on the assumption that the largest number of occurrences of a symbol was the transmitted symbol.

X: # of error in a message

$X \sim b(5, 0.2)$

P(the message will be wrong when coded)

$= P(X \geq 3) = \sum_{x=3}^{5} \binom{5}{x} (0.2)^x (0.8)^{5-x}$

(2)

All of the transmission in a message are independent.
digit

#50

X: # of head result while flipping an biased coin for 10 times

$X \sim b(10, p)$

(a)

$P(\text{first three outcomes: h t t} \mid X=6) = \dfrac{P(\text{first 3 outcome: h.t.t}, X=6)}{P(X=6)}$

$= \dfrac{p(1-p)^2 \binom{7}{5} p^5 (1-p)^2}{\binom{10}{6} p^6 (1-p)^4} = \dfrac{\binom{7}{5}}{\binom{10}{6}} = \dfrac{1}{10}$

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(b)

$P(\text{first three outcomes}: t, h, t \mid X=6) = \dfrac{P(\text{first three outcomes}: t,h,t, X=6)}{P(X=6)}$

$= \dfrac{(1-p)^2 p \, C\binom{7}{5} p^5 (1-p)^2}{C\binom{10}{6} p^6 (1-p)^4} = \dfrac{1}{10}$

#53.

(a)

$P(\text{both partners were born on } 4/30) = \left(\frac{1}{365}\right)^2$

X: # of couple that both born on 4/30 within 80000 couples

$X \sim b\left(80000, \left(\frac{1}{365}\right)^2\right) \xrightarrow[\text{as } n \to \infty]{d} \text{poisson}(\lambda = 0.6)$

$\lambda = 80000 \times \left(\frac{1}{365}\right)^2 = np$

$\Rightarrow P(X \geq 1) = 1 - e^{-0.6}$ #

(b)

$P(\text{both partners celebrated their birthday on same day of the yr})$

$= \left(\frac{1}{365}\right)^2 \times 365 = \frac{1}{365}$

Y: # of couple celebrated their BD on same day of the yr within 80000 couples.

$Y \sim b\left(80000, \frac{1}{365}\right) \xrightarrow[n \to \infty]{d} \text{poisson}\left(\lambda = \frac{80000}{365}\right)$

$P(X \geq 1) = 1 - e^{-\frac{80000}{365}}$ #

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64. 令 r.v. $X \sim Poi(1)$，則 $Y = 4X \sim Poi(4)$.

(a) $P(Y \geq 8) = 1 - P(Y < 8) = 1 - \sum_{i=0}^{7} \frac{e^{-4} 4^i}{i!}$ #

(b). 令 (a) 中 $1 - \sum_{i=0}^{7} \frac{e^{-4} 4^i}{i!} = p$

令 r.v. $W \sim Bin(12, p)$，總共有12個月，每個月發生机率為 p

$P(W \geq 2) = 1 - P(W=0) - P(W=1) = 1 - \binom{12}{0} p^0 (1-p)^{12} - \binom{12}{1} p^1 (1-p)^{11}$

$= 1 - (1-p)^{12} - 12p(1-p)^{11}$ #.

(c). 題意為 令今天所在的月份為第一個月，而第一次出現有 ≥ 8 自殺的月份為第 i 月份．請問机率為？

令 r.v. $Z \sim Geo(p)$.

$P(Z = i) = (1-p)^{i-1} p$, $i \geq 1$ #.

假設：每月事件發生皆獨立．(課本 p158. 8-1 第一行).

74. (a) 令 $X \sim Bin(5, \frac{2}{3})$，總共5個人，每人願意來的机率為 $2/3$.

$\because$ interviewer 想面5個人，故全部人都要來.

$P(X=5) = \binom{5}{5} (\frac{2}{3})^5 (\frac{1}{3})^0 = \frac{32}{243}$ #.

(b) 令 $Y \sim Bin(8, \frac{2}{3})$．$\because$ interviewer 想面5個人，故 5,6,7,8人來皆可

$P(Y \geq 5) = \sum_{y=5}^{8} \binom{8}{y} (\frac{2}{3})^y (\frac{1}{3})^{8-y}$ #. (面試人數需達5人).

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74. [Cont.]

(c). interviewer 剛好與 6 個人面試，意謂前 5 個人中只面試了 4 個人，而第 6 個人必面.

於 (a) 以設 $X \sim Bin(5, \frac{2}{3})$.

則 $P(X=4) \times \frac{2}{3} = \binom{5}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) \times \frac{2}{3} = \binom{5}{4}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right) = \frac{160}{729}$ (0.2195) #

(d). 同 (c) 想法.

令 $Z \sim Bin(6, \frac{2}{3})$.

則 $P(Z=4) \times \frac{2}{3} = \binom{6}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2 \times \frac{2}{3} = \binom{6}{4}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^2 = \frac{160}{729}$ # (0.2195)

78. 令 $X \sim Geo(p)$. 要做到 n 次 selection 亦即前 $(n-1)$ 次皆沒抽到 2W 2B.

其中 $p = \dfrac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = \frac{36}{70} = \frac{18}{35} \doteqdot 0.5143$

則 $P(X=n) = \left(\frac{17}{35}\right)^{n-1} \frac{18}{35} = 18(17)^{n-1}/35^n$ #

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Theoretical Exercises

#10

$X \sim b(n, p)$

$$E\left(\frac{1}{X+1}\right) = \sum_{x=0}^{n} \left(\frac{1}{x+1}\right) \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= \frac{1}{p(n+1)} \sum_{x=0}^{n} \frac{(n+1)!}{(x+1)!(n-x)!} p^{x+1} (1-p)^{n-x}$$

$$\overset{z=x+1}{=} \frac{1}{p(n+1)} \sum_{z=1}^{n+1} \frac{(n+1)!}{z!(n+1-z)!} p^z (1-p)^{n+1-z}$$

$$= \frac{1}{p(n+1)} \left((1-p+p)^{n+1} - \frac{(n+1)!}{0!(n+1-0)!} p^0 (1-p)^{n+1-0} \right)$$

$$= \frac{1}{p(n+1)} \left(1 - (1-p)^{n+1} \right) \text{ \#}$$

#16.

$X \sim \text{Poisson}(\lambda)$

$$\frac{P(X=i)}{P(X=i-1)} = \frac{\frac{\lambda^i e^{-\lambda}}{i!}}{\frac{\lambda^{i-1} e^{-\lambda}}{(i-1)!}} = \frac{\lambda}{i}$$

$\frac{\lambda}{i} \geq 1$ if $\lambda \geq i$, indicates that $P(X=i)$ increase monotonically, as i increases, $\longrightarrow$ ①

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$$\frac{P(i-1)}{P(i)} = \frac{\frac{e^{-\lambda}\lambda^{i-1}}{(i-1)!}}{\frac{e^{-\lambda}(\lambda)^{i}}{i!}} = \frac{i}{\lambda}$$

$\Rightarrow \frac{i}{\lambda} \leq 1$ iff $i \leq \lambda$ indicates that $P(X=i)$ decreases monotonically, as i decreases. $\rightarrow$ (2)

by (1), (2),

$P(X=i)$ reaches its maximum when i is the largest integer not exceeding λ.

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(theoretical exercise)

19. $X \sim Poi(\lambda)$, $x=0,1,2,\ldots$

$$E(X^n)=\sum_{x=0}^{\infty}x^n\frac{e^{-\lambda}\lambda^x}{x!}=\sum_{x=0}^{\infty}x^{n-1}\frac{e^{-\lambda}\lambda^x}{(x-1)!}=\lambda\sum_{x=0}^{\infty}x^{n-1}\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}$$

$$=\lambda\sum_{x=1}^{\infty}x^{n-1}\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}=\lambda\sum_{y=0}^{\infty}(y+1)^{n-1}\frac{e^{-\lambda}\lambda^y}{y!}$$

(∵當 $x=0$ 時，此項為0)．(令 $y=x-1$)

$$=\lambda\sum_{x=0}^{\infty}(x+1)^{n-1}\frac{e^{-\lambda}\lambda^x}{x!}=\lambda E[(X+1)^{n-1}] \#$$

(再將 y 轉回 x)．

故 $E[X^3]=\lambda E[(X+1)^2]=\lambda E(X^2)+2\lambda E(X)+\lambda$

$$=\lambda(Var(X)+[E(X)]^2)+2\lambda^2+\lambda$$

$$=\lambda^3+3\lambda^2+\lambda \#$$ 已知 $Var(X)=E(X)=\lambda$．

20. 題意為每輪丟 n 個銅板，若 n 個銅板中有至少一個正面則停止，若無正面則再丟一輪，直到出現至少一個正面．

故 $P(X=1)=\binom{n}{1}p(1-p)^{n-1}+(1-p)^n\binom{n}{1}p(1-p)^{n-1}+(1-p)^{2n}\binom{n}{1}p(1-p)^{n-1}+\ldots$

$$=\binom{n}{1}p(1-p)^{n-1}[1+(1-p)^n+(1-p)^{2n}+\ldots].$$

$$=\binom{n}{1}p(1-p)^{n-1}\frac{1}{1-(1-p)^n}=np(1-p)^{n-1}\frac{1}{1-(1-p)^n}$$

$(\because p=\frac{\lambda}{n})$ $$=\lambda\left(1-\frac{\lambda}{n}\right)^{n-1}\frac{1}{1-(1-\frac{\lambda}{n})^n}\xrightarrow[n\to\infty]{}\frac{\lambda e^{-\lambda}}{1-e^{-\lambda}} \#$$ (b).