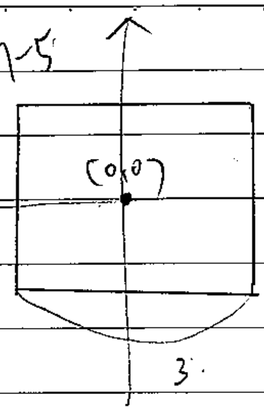
No.
Date : :

7-5



$\Rightarrow f(x,y) = \frac{1}{9}, \quad -\frac{3}{2} < x, y < \frac{3}{2}$

$E(|X|+|Y|)$

$= 4 \cdot \int_0^{\frac{3}{2}} \int_0^{\frac{3}{2}} (x+y)\frac{1}{9}\, dx\, dy$

$= \frac{4}{9} \int_0^{\frac{3}{2}} \frac{9}{8} + \frac{3}{2} y\, dy$

$= \frac{4}{9}\left(\frac{54}{16}\right) = \frac{3}{2}$

7-9

(a)
第 i 球
$i=1,2,\dots,n$

urn1
urn2
$\vdots$
urn i

$P(\text{urn } i \text{ 沒球}) = \frac{i-1}{i} \cdot \frac{i}{i+1} \cdots \frac{n-1}{n} = \frac{i-1}{n}$

let $I_i = \begin{cases} 1 & \text{if urn } i \text{ 無球} \\ 0 & \text{if urn } i \text{ 有球} \end{cases} \Rightarrow \sum_{i=1}^{n} I_i$: 有幾個沒有球

$E\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} E(I_i)$

$= \sum_{i=1}^{n} P(\text{urn } i \text{ 無球}) \times 1 + P(\text{urn } i \text{ 無球}) \times 0$

$= \sum_{i=1}^{n} \frac{i-1}{n} = \frac{1}{n}\left(\frac{n(n+1)}{2} - n\right) = \frac{n-1}{2}$

Chrys culture

(b)

∵ urn n 要有球則只有 n^{th} 球可放入

$$P(\text{每堆皆有球}) = P(n^{th}\text{ 球在 urn } n)\cdot P(n-1^{th}\text{ 球在 urn } n-1)$$

$$\cdots \times P(1^{st}\text{ 球在 urn } 1)$$

$$= \frac{1}{n}\cdot\frac{1}{n-1}\cdots\frac{1}{1} = \frac{1}{n!}$$

7-21

(a) $I_i = \begin{cases} 1, & \text{if day } i \text{ 有3个人生日} \\ 0 & \text{o.w.} \end{cases}$

$$\Rightarrow E(\text{3个人生日的天数}) = E\left(\sum_{i=1}^{365} I_i\right)$$

$$= \sum_{i=1}^{365} P(\text{day } i \text{ 有3人生日}) \times 1$$

$$= \sum_{i=1}^{365} \binom{100}{3}\left(\frac{1}{365}\right)^3\left(\frac{364}{365}\right)^{97}$$

$$= 365\cdot\binom{100}{3}\left(\frac{1}{365}\right)^3\cdot\left(\frac{364}{365}\right)^{97}$$

(b) $I_i = \begin{cases} 1, & \text{至少有1人在 day } i \text{ 生日} \\ 0 & \text{o.w.} \end{cases}$

$$E(\#\text{ of distinct birthday}) = E\left(\sum_{i=1}^{365} I_i\right)$$

$$= \sum_{i=1}^{365} E(I_i) = \sum_{i=1}^{365} P(\text{至少有1人在 day } i \text{ 生日}) \times 1$$

$$= \sum_{i=1}^{365} 1 - P(\text{無人在 day } i \text{ 生日})$$

$$= 365\left[1 - \left(\frac{364}{365}\right)^{100}\right]$$

No.
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\# 7-39 $X_1 \ldots \overset{\text{indep}}{\sim} (\mu, \sigma^2) \Rightarrow Cov(X_i, X_j) = \begin{cases} 0 & \text{if } i \neq j \\ \sigma^2 & \text{if } i = j \end{cases}$

$Y_n = X_n + X_{n+1} + X_{n+2}$

if $j=1$

$Cov(Y_n, Y_{n+1}) = Cov(X_n + X_{n+1} + X_{n+2},\ X_{n+1} + X_{n+2} + X_{n+3})$

$= Cov(X_n, X_{n+1}) + Cov(X_n, X_{n+2}) + \cdots + Cov(X_{n+2}, X_{n+3})$

$= Cov(X_{n+1}, X_{n+1}) + Cov(X_{n+2}, X_{n+2}) = 2\sigma^2$

if $j=2$

$Cov(Y_n, Y_{n+2}) = Cov(X_n + X_{n+1} + X_{n+2},\ X_{n+2} + X_{n+3} + X_{n+4})$

$= Cov(X_{n+2}, X_{n+2}) = \sigma^2$

if $j > 2$

$Cov(Y_n, Y_{n+j}) = Cov(X_n + X_{n+1} + X_{n+2},\ X_{n+j} + X_{n+j+1} + X_{n+j+2})$

$= 0$

Chryv culture

40:

$$f(x,y) = \frac{1}{y} e^{-(y+\frac{x}{y})} \quad x>0,\ y>0$$

$$\begin{aligned} f(y) &= \int_0^\infty \frac{1}{y} e^{-(y+\frac{x}{y})} dx \\ &= \int_0^\infty e^{-\frac{x}{y}} dx \cdot \frac{1}{y} e^{-y} \\ &= \frac{1}{y} e^{-y} \left(\frac{e^{-\frac{x}{y}}}{-\frac{1}{y}} \Big|_{x=0}^{\infty} \right) \\ &= \frac{1}{y} e^{-y} \cdot y = e^{-y} \quad y>0 \end{aligned}$$

$$\Rightarrow f(y) = \begin{cases} e^{-y} ; & y>0 \\ 0 ; & \text{o.w.} \end{cases} \quad Y \sim \exp(1)$$

$$E(Y) = \frac{1}{\lambda} = 1 \text{ ※}$$

$$f(x|Y) = \frac{f(x,y)}{f(y)} = \frac{1}{y} e^{-\frac{x}{y}},\ x>0,\ y>0 \Rightarrow X|Y \sim \exp(\tfrac{1}{y})$$

$$E(X|Y) = \int_0^\infty \frac{x}{y} e^{-\frac{x}{y}} dx = y$$

$$E(XY) = E(E(XY|Y)) = E(Y E(X|Y)) = E(Y^2) = (E(Y))^2 + Var(Y) = 2 \text{ ※}$$

$$E(X) = EE(X|Y) = E(Y) = 1 \text{ ※}$$

$$\underbrace{E(XY) - E(X)E(Y)}_{Cov(X,Y)} = 1 \text{ ※}$$

45. (a)

$$\frac{Cov(X_1+X_2,\ X_2+X_3)}{\sqrt{Var(X_1+X_2)Var(X_2+X_3)}} = \frac{Var(X_2)}{\sqrt{[Var(X_1)+Var(X_2)]\cdot[Var(X_2)+Var(X_3)]}}$$

$(Cov(X_1 X_2)=0,\ Cov(X_2 X_3)=0)$

$$= \frac{1}{\sqrt{2\cdot 2}} = \frac{1}{2} ※$$

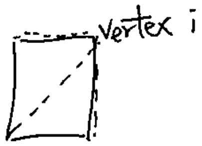
45 (b)

$$Cov(X_1+X_2,\ X_3+X_4)$$

$$= Cov(X_1, X_3) + Cov(X_1\ X_4) + Cov(X_2 X_3) + Cov(X_2, X_4) = 0$$

$$Cov(X_1+X_2, X_3+X_4) = 0 \Rightarrow Corr(X_1+X_2, X_3+X_4) = 0 ※$$

(47). (a) 試想一個正方形.



除了他自己, 他可以跟另外3点連成一線.

故 $(n-1)$ 條線 (此例為3條線).

但兩点之間有線機率為 p, 沒有機率為 $(1-p)$.

故 $D_i \sim Bin(n-1, p)$.

(b). 令 $X_{ij} = \begin{cases} 1, & i, j \text{ 之間有線} \\ 0, & i, j \text{ 之間沒線} \end{cases}$

故 $X_{ij} \sim Ber(p) \Rightarrow D_i = \sum_{i \neq k} X_{ik} \sim Bin(n-1, p)$

$D_j = \sum_{j \neq r} X_{jr} \sim Bin(n-1, p)$

$$\begin{aligned} Cov(D_i, D_j) &= \sum_{i \neq k} \sum_{j \neq r} Cov(X_{ik}, X_{jr}) \\ &= Cov(X_{ij}, X_{ji}) \\ &= Var(X_{ij}) = p(1-p). \end{aligned}$$

∵其它狀況皆互相獨立. 故 $Cov = 0$.

$Var(D_i) = Var(D_j) = (n-1)p(1-p)$.

故 $\rho(D_i, D_j) = \dfrac{p(1-p)}{(n-1)p(1-p)} = \dfrac{1}{n-1}$ #.

(5). $f_Y(y) = \int_0^y \frac{e^{-y}}{y} dx = e^{-y}$

$f_{X|Y}(x|y) = \dfrac{\frac{e^{-y}}{y}}{e^{-y}} = \dfrac{1}{y}, \quad 0 < y < \infty.$

故 $E[X^3 | Y=y] = \int_0^y x^3 \frac{1}{y} dx = \frac{1}{4} x^4 \frac{1}{y} \Big|_0^y = \frac{1}{4} y^3$ #. $0 < y < \infty$.

55.

Let N be the number of ducks & $N \sim \text{Poisson}(6)$

Given $N=n$, let $I_i = \begin{cases} 1 & \text{if duck } i \text{ is hit} \\ 0 & \text{, o.w.} \end{cases}$ $\forall i = 1, \dots, n$

Given $N=n$, each hunter will independent hit duck i with prob. $\frac{0.6}{n}$

$E(\text{Number of hit} \mid N=n) = E(\sum_{i=1}^{n} I_i) = \sum_{i=1}^{n} E(I_i) = n \cdot P(\text{duck } i \text{ is hit})$

$$= n \cdot \left(1-(1-\frac{0.6}{n})^{10}\right)$$

$E(\text{Number of hit}) = E(E(\text{Number of hit} \mid N)) = E\left(N(1-(1-\frac{0.6}{N})^{10})\right)$

$$= \sum_{n=0}^{\infty} n\left(1-(1-\frac{0.6}{n})^{10}\right)\frac{e^{-6}6^n}{n!}$$

#

65.

Let X be the number of winter storms next year and let $Y=1$ if next year is a good year and $Y=0$ o.w.

Then $E(X \mid Y=1) = 3$

$E(X \mid Y=0) = 5$

So, $E(X) = E(E(X \mid Y)) = E(X \mid Y=1) \cdot P(Y=1) + E(X \mid Y=0) P(Y=0)$

$$= 3 \times 0.4 + 5 \times 0.6 = 4.2$$

#

Similarly, $Var(X \mid Y=1) = 3$ & $Var(X \mid Y=0) = 5$

We compute: $Var(E(X \mid Y)) = (3-4.2)^2 \times 0.4 + (5-4.2)^2 \times 0.6 = 0.96$

$E(Var(X \mid Y)) = 3 \times 0.4 + 5 \times 0.6 = 4.2$

Thus, $Var(X) = Var(E(X \mid Y)) + E(Var(X \mid Y)) = 0.96 + 4.2 = 5.16$

#

(theoretical)

5

$X = 1, 2, \cdots, n$

Let $X_i = \begin{cases} 1, & \text{if } A_i \text{ occurs} \\ 0, & \text{if } A_i \text{ not occur} \end{cases}$

$\Rightarrow X = \sum_{i=1}^{n} X_i$

$E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} P(A_i)$

and $E(X) = \sum_{k=1}^{n} P(X \geq k) = \sum_{k=1}^{n} P(C_k)$

$\Rightarrow \sum_{k=1}^{n} P(C_k) = \sum_{k=1}^{n} P(A_k)$

17

$$\begin{aligned} Var(\lambda X_1 + (1-\lambda) X_2) &= \lambda^2 Var(X_1) + (1-\lambda)^2 Var(X_2) \\ &= \lambda^2 \sigma_1^2 + (1-\lambda)^2 \sigma_2^2 \end{aligned}$$

Let $\frac{d}{d\lambda}(\lambda^2 \sigma_1^2 + (1-\lambda)^2 \sigma_2^2) = 2\lambda\sigma_1^2 - 2(1-\lambda)\sigma_2^2$
$= 0$

$- (2-2\lambda)\sigma_2^2$
$-2+$
$= -2\sigma_2^2 + 2\lambda\sigma_2^2$

$\cancel{2}\lambda\sigma_1^2 = \cancel{2}\sigma_2^2 - \cancel{2}\lambda\sigma_2^2$

$\lambda = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$

$\frac{d^2}{d\lambda^2}(\lambda^2\sigma_1^2 + (1-\lambda)^2\sigma_2^2)\Big|_{\lambda = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}} = 2\sigma_1^2 + 2\sigma_2^2 > 0$

$\Rightarrow \lambda = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$ 使得 $\lambda X_1 + (1-\lambda) X_2$ 有最小的 variance

這個估計量本身已經具不偏性 (bias = 0)，若可以讓它的 variance 愈小則其會愈相對有效。