(A1, B1) (*10pts*)

Exam A.

(a) (*5pts*) $\mathbf{X} \sim$ multinomial(n, m, $p_1, \ldots, p_m$) with $n = 12$, $m = 3$, and $p_1 = 1/8$, $p_2 = 2/8$, $p_3 = 5/8$.

(b) (*5pts*) $X \sim$ normal(μ, σ^2) with $\mu = 4$ and σ^2 unknown.

Exam B.

(a) (*5pts*) $\mathbf{X} \sim$ multinomial(n, m, $p_1, \ldots, p_m$) with $n = 15$, $m = 3$, and $p_1 = 3/8$, $p_2 = 4/8$, $p_3 = 1/8$.

(b) (*5pts*) $X \sim$ normal(μ, σ^2) with $\mu = 7$ and σ^2 unknown.

(A2, B2) (*20pts*)

Exam A.

(a) (*6pts*) Let X be the location of the point. When $X < L - X$ ($\Leftrightarrow X < L/2$),

$$X/(L-X) < 1/4 \Leftrightarrow X < L/5,$$

and when $X > L - X$ ($\Leftrightarrow X > L/2$),

$$(L-X)/X < 14 \Leftrightarrow X > 4L/5.$$

The question asked us to find the probability of the event $\{X < L/5\} \cup \{X > 4L/5\}$. Because $X \sim$ uniform$(0, L)$,

$$\begin{aligned} & P(\{X < L/5\} \cup \{X > 4L/5\}) \\ = \ & P(\{X < L/5\}) + P(\{X > 4L/5\}) \\ = \ & \int_0^{L/5} \frac{1}{L}\, dx + \int_{4L/5}^{L} \frac{1}{L}\, dx = 2/5. \end{aligned}$$

(b) (*7pts*) Let X and Y be the times that it takes to service the cars of A.J. and M.J. respectively. The question asked us to find the probability of the event $A = \{X > Y + t\}$. Because $X \sim \exp(1)$, $Y \sim \exp(1)$, and X, Y are independent, the joint pdf of (X, Y) is

$$f_{X,Y}(x, y) = e^{-(x+y)}, \ \text{ for } x, y > 0,$$

and zero, otherwise. The probability of interest is

$$\begin{aligned} P(X > Y + t) &= \int\int_A f_{X,Y}(x,y)\, dxdy \\ &= \int_0^\infty \int_{y+t}^\infty e^{-(x+y)}\, dxdy \\ &= \int_0^\infty \left[-e^{-(x+y)}\Big|_{x=y+t}^{\infty} \right] dy \\ &= \int_0^\infty e^{-(2y+t)}\, dy \\ &= -(1/2)e^{-(2y+t)}\Big|_0^\infty = e^{-t}/2. \end{aligned}$$

Exam B.

(a) (*5pts*) Let X be the location of the point. When $X < L - X$ ($\Leftrightarrow X < L/2$),

$$X/(L-X) < 1/5 \Leftrightarrow X < L/6,$$

and when $X > L - X$ ($\Leftrightarrow X > L/2$),

$$(L-X)/X < 1/5 \Leftrightarrow X > 5L/6.$$

The question asked us to find the probability of the event $\{X < L/6\} \cup \{X > 5L/6\}$. Because $X \sim$ uniform$(0, L)$,

$$\begin{aligned} & P(\{X < L/6\} \cup \{X > 5L/6\}) \\ = \ & P(\{X < L/6\}) + P(\{X > 5L/6\}) \\ = \ & \int_0^{L/6} \frac{1}{L}\, dx + \int_{5L/6}^{L} \frac{1}{L}\, dx = 1/3. \end{aligned}$$

(b) (*7pts*) Let X and Y be the times that it takes to service the cars of A.J. and M.J. respectively. The question asked us to find the probability of the event $A = \{X < Y + t\}$. Because $X \sim \exp(1)$, $Y \sim \exp(1)$, and X, Y are independent, the joint pdf of (X, Y) is

$$f_{X,Y}(x, y) = e^{-(x+y)}, \ \text{ for } x, y > 0,$$

and zero, otherwise. The probability of interest is

$$\begin{aligned} P(X < Y + t) &= \int\int_A f_{X,Y}(x,y)\, dxdy \\ &= \int_0^\infty \int_0^{y+t} e^{-(x+y)}\, dxdy \\ &= \int_0^\infty \left[-e^{-(x+y)}\Big|_{x=0}^{y+t} \right] dy \\ &= \int_0^\infty e^{-y} - e^{-(2y+t)}\, dy \\ &= -e^{-y} + (1/2)e^{-(2y+t)}\Big|_0^\infty = 1 - e^{-t}/2. \end{aligned}$$

(A2, B2) *(cont.)*

Exam A.

(c) (*7pts*) Let $X_{(1)}, X_{(2)}, X_{(3)}$ be the order statistics of X_1, X_2, X_3. The question asked us to find the probability of the event $A = \{X_{(3)} > X_{(2)} + X_{(1)}\}$. By the properties of mutually exclusive and symmetry, we have

$$\begin{aligned} P(X_{(3)} > X_{(2)} + X_{(1)}) &= P(X_3 > X_1 + X_2) \\ &\quad + P(X_2 > X_1 + X_3) + P(X_1 > X_2 + X_3) \\ &= 3P(X_3 > X_1 + X_2) \end{aligned}$$

Because X_1, X_2, X_3 are i.i.d. from uniform$(0, 1)$, their joint pdf is

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = 1,$$

for $0 < x_1, x_2, x_3 < 1$, and zero, otherwise. The probability of interest is

$$\begin{aligned} &3P(X_3 > X_1 + X_2) \\ &= 3\int\int\int_A f_{X_1,X_2,X_3}(x_1, x_2, x_3)\ dx_1dx_2dx_3 \\ &= 3\int_0^1\int_0^{x_3}\int_0^{x_3-x_2} 1\ dx_1dx_2dx_3 \\ &= 3\int_0^1\int_0^{x_3} x_3 - x_2\ dx_2dx_3 \\ &= 3\int_0^1 \left[x_2x_3 - (1/2)x_2^2\Big|_{x_2=0}^{x_3}\right]\ dx_3 \\ &= 3\int_0^1 (1/2)x_3^2\ dx_3 = 3\left[(1/6)x_3^3\Big|_0^1\right] = 1/2. \end{aligned}$$

Exam B.

(c) (*7pts*) Let $X_{(1)}, X_{(2)}, X_{(3)}$ be the order statistics of X_1, X_2, X_3. The question asked us to find the probability of the event $A = \{X_{(1)} < X_{(2)} + X_{(3)}\}$. By the properties of mutually exclusive and symmetry, we have

$$\begin{aligned} P(X_{(1)} < X_{(2)} + X_{(3)}) &= P(X_3 < X_1 + X_2) \\ &\quad + P(X_2 < X_1 + X_3) + P(X_1 < X_2 + X_3) \\ &= 3P(X_3 < X_1 + X_2) \end{aligned}$$

Because X_1, X_2, X_3 are i.i.d. from uniform$(-1, 0)$, their joint pdf is

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = 1,$$

for $-1 < x_1, x_2, x_3 < 0$, and zero, otherwise. The probability of interest is

$$\begin{aligned} &3P(X_3 < X_1 + X_2) \\ &= 3\int\int\int_A f_{X_1,X_2,X_3}(x_1, x_2, x_3)\ dx_1dx_2dx_3 \\ &= 3\int_{-1}^0\int_{x_3}^0\int_{x_3-x_2}^0 1\ dx_1dx_2dx_3 \\ &= 3\int_{-1}^0\int_{x_3}^0 x_2 - x_3\ dx_2dx_3 \\ &= 3\int_{-1}^0 \left[(1/2)x_2^2 - x_2x_3\Big|_{x_2=x_3}^0\right]\ dx_3 \\ &= 3\int_{-1}^0 (1/2)x_3^2\ dx_3 = 3\left[(1/6)x_3^3\Big|_{-1}^0\right] = 1/2. \end{aligned}$$

(A3, B3) *(14pts)*

Exam A.

(a) (*4pts*) Because

$$\begin{aligned} 1 &= \int\int_{\mathbb{R}^2} f_{X,Y}(x, y)\ dydx \\ &= \int_0^1\int_0^1 c\cdot x(1-x)y\ dydx \\ &= c\int_0^1 x(1-x)\left[(1/2)y^2\Big|_{y=0}^1\right]\ dx \\ &= (c/2)\int_0^1 x - x^2\ dx \\ &= (c/2)\left[(1/2)x^2 - (1/3)x^3\Big|_0^1\right] = c/12, \end{aligned}$$

we have $c = 12$.

Exam B.

(a) (*4pts*) Because

$$\begin{aligned} 1 &= \int\int_{\mathbb{R}^2} f_{X,Y}(x, y)\ dydx \\ &= \int_0^1\int_0^1 c\cdot xy(1-y)\ dydx \\ &= c\int_0^1 x\left[(1/2)y^2 - (1/3)y^3\Big|_{y=0}^1\right]\ dx \\ &= (c/6)\int_0^1 x\ dx \\ &= (c/6)\left[(1/2)x^2\Big|_0^1\right] = c/12, \end{aligned}$$

we have $c = 12$.

(A3, B3) (*cont.*)

Exam A.

(b) (*4pts*) The marginal pdf of X is

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \\ &= \int_0^1 12x(1-x)y\, dy \\ &= 12x(1-x)\left[(1/2)y^2\Big|_0^1\right] = 6x(1-x), \end{aligned}$$

for $0 < x < 1$, and zero, otherwise.

(c) (*3pts*) Let $g(x) = x(1-x)$ and $h(y) = y$. Then, the independence of X and Y can also be concluded from (i) $f_{X,Y}(x,y) \propto g(x)h(y)$ on the set $A = \{(x,y)|0 < x < 1, 0 < y < 1\}$, and (ii) the set A is a cross-product set.

(d) (*3pts*) Because X, Y are independent, we have

$$\begin{aligned} &P(X < 1/2|Y > 1/2) = P(X < 1/2) \\ &= \int_0^{1/2} f_X(x)\, dx = \int_0^{1/2} 6x(1-x)\, dx \\ &= 6\left[(1/2)x^2 - (1/3)x^3\Big|_0^{1/2}\right] = 1/2. \end{aligned}$$

Exam B.

(b) (*4pts*) The marginal pdf of Y is

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx \\ &= \int_0^1 12xy(1-y)\, dx \\ &= 12y(1-y)\left[(1/2)x^2\Big|_0^1\right] = 6y(1-y), \end{aligned}$$

for $0 < y < 1$, and zero, otherwise.

(c) (*3pts*) Let $g(x) = x$ and $h(y) = y(1-y)$. Then, the independence of X and Y can also be concluded from (i) $f_{X,Y}(x,y) \propto g(x)h(y)$ on the set $A = \{(x,y)|0 < x < 1, 0 < y < 1\}$, and (ii) the set A is a cross-product set.

(d) (*3pts*) Because X, Y are independent, we have

$$\begin{aligned} &P(Y > 1/2|X > 1/2) = P(Y > 1/2) \\ &= \int_{1/2}^{1} f_Y(y)\, dy = \int_{1/2}^{1} 6y(1-y)\, dy \\ &= 6\left[(1/2)y^2 - (1/3)y^3\Big|_{1/2}^{1}\right] = 1/2. \end{aligned}$$

(A4, B4) (*21pts*)

(a) (*3pts*) Because $X_1, \ldots, X_n$ are i.i.d. from uniform$(0,1)$ distribution, their marginal pdf is $f_X(x) = 1$, for $0 < x < 1$, and their marginal cdf is $F_X(x) = x$, for $0 < x < 1$. The joint pdf of $X_{(1)}$ and $X_{(n)}$ is

$$\begin{aligned} f_{X_{(1)},X_{(n)}}(s,t) &= \binom{n}{2} \times f_X(s) \times f_X(t) \times [F_X(t) - F_X(s)]^{n-2} \\ &= n(n-1) \times 1 \times 1 \times (t-s)^{n-2}, \end{aligned}$$

for $0 < s < t < 1$, and zero otherwise.

(b) (*8pts*) Notice that

$$F_{X_{(1)},X_{(n)}}(u,v) = P(X_{(1)} \le u, X_{(n)} \le v) = \int_{-\infty}^{u}\int_{-\infty}^{v} f_{X_{(1)},X_{(n)}}(s,t)\, dtds.$$

It is clear that $F_{X_{(1)},X_{(n)}}(u,v) = 0$ if $v < 0$ or $u < 0$, and $F_{X_{(1)},X_{(n)}}(u,v) = 1$ if $1 \le v$ and $1 \le u$. If $0 \le u < v < 1$,

$$\begin{aligned} &\int_{-\infty}^{u}\int_{-\infty}^{v} f_{X_{(1)},X_{(n)}}(s,t)\, dtds = \int_0^u \int_s^v n(n-1)(t-s)^{n-2}\, dtds \\ &= n\int_0^u \left[(t-s)^{n-1}\Big|_{t=s}^{v}\right]\, ds = n\int_0^u -(v-s)^{n-1}\, ds = (v-s)^n\Big|_{s=0}^{u} = v^n - (v-u)^n. \end{aligned}$$

If $0 \le v < 1$ and $v \le u$,

$$\begin{aligned}\int_{-\infty}^{u}\int_{-\infty}^{v} f_{X_{(1)},X_{(n)}}(s,t)\;dtds &= \int_{0}^{v}\int_{s}^{v} n(n-1)(t-s)^{n-2}\;dtds\\ = n\int_{0}^{v} &\left[(t-s)^{n-1}\big|_{t=s}^{v}\right]\;ds = n\int_{0}^{v} -(v-s)^{n-1}\;ds = (v-s)^{n}\big|_{s=0}^{v} = v^{n}.\end{aligned}$$

If $0 \le u < 1 \le v$,

$$\begin{aligned}\int_{-\infty}^{u}\int_{-\infty}^{v} f_{X_{(1)},X_{(n)}}(s,t)\;dtds &= \int_{0}^{u}\int_{s}^{1} n(n-1)(t-s)^{n-2}\;dtds\\ = n\int_{0}^{u} &\left[(t-s)^{n-1}\big|_{t=s}^{1}\right]\;ds = n\int_{0}^{u} -(1-s)^{n-1}\;ds = (1-s)^{n}\big|_{s=0}^{u} = 1-(1-u)^{n}.\end{aligned}$$

The joint cdf of $X_{(1)}$ and $X_{(n)}$ is

$$F_{X_{(1)},X_{(n)}}(u,v) = \begin{cases} 0, & \text{if } v<0 \text{ or } u<0,\\ v^{n}-(v-u)^{n}, & \text{if } 0\le u<v<1,\\ v^{n}, & \text{if } 0\le v<1 \text{ and } v\le u,\\ 1-(1-u)^{n}, & \text{if } 0\le u<1\le v,\\ 1, & \text{if } 1\le v \text{ and } 1\le u.\end{cases} \tag{I}$$

An alternative way to get the solution is given below. Because $\{X_{(1)} > u, X_{(n)} \le v\} \subset \{X_{(n)} \le v\}$, we have

$$F_{X_{(1)},X_{(n)}}(u,v) = P(X_{(1)} \le u, X_{(n)} \le v) = P(X_{(n)} \le v) - P(X_{(1)} > u, X_{(n)} \le v). \tag{II}$$

Because $X_1,\ldots,X_n$ are independent, we have

$$P(X_{(n)} \le v) = P(X_1 \le v,\cdots,X_n \le v) = \prod_{i=1}^{n} P(X_i \le v) = \begin{cases} 0, & \text{if } v<0,\\ v^{n}, & \text{if } 0\le v<1,\\ 1, & \text{if } 1\le v,\end{cases} \tag{III}$$

and

$$\begin{aligned}P(X_{(1)} > u, X_{(n)} \le v) &= P(u < X_1 \le v,\cdots,u<X_n\le v) = \prod_{i=1}^{n} P(u<X_i\le v)\\ &= \begin{cases} (v-u)^{n}, & \text{if } 0\le u<v<1,\\ (1-u)^{n}, & \text{if } 0\le u<1\le v,\\ 1, & \text{if } 1\le v \text{ and } u<0,\\ 0, & \text{otherwise.}\end{cases}\end{aligned} \tag{IV}$$

Then, we can substitute (III) and (IV) into (II) to obtain (I).

(A4, B4) *(cont.)*

Exam A.

(c) *(6pts)* The range of (R, M) is

$$\mathcal{R} = \left\{(r,m)\,\middle|\,0 < r < 1, \frac{r}{2} < m < 1 - \frac{r}{2}\right\}.$$

Because

$$X_{(1)} = \frac{2M - R}{2} \quad \text{and} \quad X_{(n)} = \frac{2M + R}{2},$$

the Jacobians is given by

$$J = \begin{vmatrix} -1/2 & 1 \\ 1/2 & 1 \end{vmatrix} = -1.$$

If $(r, m) \in \mathcal{R}$, the joint pdf of (R, M) is

$$\begin{aligned} & f_{R,M}(r,m) \\ = \; & f_{X_{(1)},X_{(n)}}\left(\frac{2m-r}{2}, \frac{2m+r}{2}\right) |J| \\ = \; & n(n-1)r^{n-2}, \end{aligned}$$

and $f_{R,M}(r, m) = 0$ if $(r, m) \notin \mathcal{R}$.

(d) *(4pts)*

$$\begin{aligned} Cov(R, M) &= Cov\left(X_{(n)} - X_{(1)}, \frac{X_{(n)} + X_{(1)}}{2}\right) \\ &= \frac{1}{2}Cov(X_{(n)}, X_{(n)}) + \frac{1}{2}Cov(X_{(n)}, X_{(1)}) \\ &\quad -\frac{1}{2}Cov(X_{(1)}, X_{(n)}) - \frac{1}{2}Cov(X_{(1)}, X_{(1)}) \\ &= \frac{1}{2}Cov(X_{(n)}, X_{(n)}) - \frac{1}{2}Cov(X_{(1)}, X_{(1)}) \\ &= \left[Var(X_{(n)}) - Var(X_{(1)})\right]/2 \end{aligned}$$

Exam B.

(c) *(6pts)* The range of (R, M) is

$$\mathcal{R} = \left\{(r,m)\,\middle|\,0 < r < \frac{1}{2}, 2r < m < 2(1-r)\right\}.$$

Because

$$X_{(1)} = \frac{M - 2R}{2} \quad \text{and} \quad X_{(n)} = \frac{M + 2R}{2},$$

the Jacobians is given by

$$J = \begin{vmatrix} -1 & 1/2 \\ 1 & 1/2 \end{vmatrix} = -1.$$

If $(r, m) \in \mathcal{R}$, the joint pdf of (R, M) is

$$\begin{aligned} & f_{R,M}(r,m) \\ = \; & f_{X_{(1)},X_{(n)}}\left(\frac{m-2r}{2}, \frac{m+2r}{2}\right) |J| \\ = \; & n(n-1)(2r)^{n-2}, \end{aligned}$$

and $f_{R,M}(r, m) = 0$ if $(r, m) \notin \mathcal{R}$.

(d) *(4pts)*

$$\begin{aligned} Cov(R, M) &= Cov\left(\frac{X_{(n)} - X_{(1)}}{2}, X_{(n)} + X_{(1)}\right) \\ &= \frac{1}{2}Cov(X_{(n)}, X_{(n)}) + \frac{1}{2}Cov(X_{(n)}, X_{(1)}) \\ &\quad -\frac{1}{2}Cov(X_{(1)}, X_{(n)}) - \frac{1}{2}Cov(X_{(1)}, X_{(1)}) \\ &= \frac{1}{2}Cov(X_{(n)}, X_{(n)}) - \frac{1}{2}Cov(X_{(1)}, X_{(1)}) \\ &= \left[Var(X_{(n)}) - Var(X_{(1)})\right]/2 \end{aligned}$$

(A5, B5) *(16pts)*

Exam A.

(a) *(4pts)* By the multiplication law, the joint mixed pdf/pmf of W and N is

$$\begin{aligned} f_{W,N}(w,n) &= f_W(w) f_{N|W}(n|w) \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} w^{\alpha-1} e^{-\lambda w} \times \frac{e^{-w} w^n}{n!} \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{1}{n!} w^{(n+\alpha)-1} e^{-(\lambda+1)w} \end{aligned}$$

for $w > 0$ and $n = 0, 1, 2, \ldots,$ and $f_{W,N}(w, n) = 0$, otherwise.

Exam B.

(a) *(4pts)* By the multiplication law, the joint mixed pdf/pmf of V and M is

$$\begin{aligned} f_{V,M}(v,m) &= f_V(v) f_{M|V}(m|v) \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} v^{\alpha-1} e^{-\lambda v} \times \frac{e^{-v} v^m}{m!} \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{1}{m!} v^{(\alpha+m)-1} e^{-(\lambda+1)v} \end{aligned}$$

for $v > 0$ and $m = 0, 1, 2, \ldots,$ and $f_{W,N}(w, n) = 0$, otherwise.

(A5, B5) (*cont.*)

Exam A.

(b) (*6pts*) By the law of total probability, the marginal pmf of $N+\alpha$ is

$$\begin{aligned}
f_{N+\alpha}(x) &= P(N+\alpha = x) = P(N = x-\alpha) \\
&= \int_{-\infty}^{\infty} f_W(w) f_{N|W}(x-\alpha|w)\, dw \\
&= \int_0^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{1}{(x-\alpha)!} w^{x-1} e^{-(\lambda+1)w}\, dw \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{1}{(x-\alpha)!} \frac{\Gamma(x)}{(\lambda+1)^x} \\
&\quad \times \int_0^{\infty} \underbrace{\frac{(\lambda+1)^x}{\Gamma(x)} w^{x-1} e^{-(\lambda+1)w}}_{\text{pdf of gamma}(x,\lambda+1)}\, dw \\
&= \frac{\lambda^\alpha}{(\alpha-1)!} \frac{1}{(x-\alpha)!} \frac{(x-1)!}{(\lambda+1)^x} \\
&= \underbrace{\binom{x-1}{\alpha-1} \left(\frac{\lambda}{\lambda+1}\right)^{\alpha} \left(1-\frac{\lambda}{\lambda+1}\right)^{x-\alpha}}_{\text{pmf of negative binomial}\left(\alpha,\, \frac{\lambda}{\lambda+1}\right)}
\end{aligned}$$

for $x = \alpha, \alpha+1, \alpha+2, \ldots$, and zero, otherwise.

(c) (*6pts*) By the Bayes theorem, the conditional pdf of W given $N = n$ is

$$\begin{aligned}
f_{W|N}(w|n) &= \frac{f_W(w) f_{N|W}(n|w)}{\int_{-\infty}^{\infty} f_W(w) f_{N|W}(n|w)\, dw} \\
&= \frac{\frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{1}{m!} w^{(\alpha+n)-1} e^{-(\lambda+1)w}}{\binom{(\alpha+n)-1}{\alpha-1} \left(\frac{\lambda}{\lambda+1}\right)^{\alpha} \left(1-\frac{\lambda}{\lambda+1}\right)^{(\alpha+n)-\alpha}} \\
&= \frac{\frac{\lambda^\alpha}{(\alpha-1)!} \frac{1}{n!} w^{(\alpha+n)-1} e^{-(\lambda+1)w}}{\frac{(\alpha+n-1)!}{(\alpha-1)!n!} \frac{\lambda^\alpha}{(\lambda+1)^{\alpha+n}}} \\
&= \underbrace{\frac{(\lambda+1)^{\alpha+n}}{\Gamma(\alpha+n)} w^{(\alpha+n)-1} e^{-(\lambda+1)w}}_{\text{pdf of gamma}(\alpha+n,\lambda+1)}
\end{aligned}$$

for $w > 0$, and zero, otherwise.

Exam B.

(b) (*6pts*) By the law of total probability, the marginal pmf of $M+\alpha$ is

$$\begin{aligned}
f_{M+\alpha}(x) &= P(M+\alpha = x) = P(M = x-\alpha) \\
&= \int_{-\infty}^{\infty} f_V(v) f_{M|V}(x-\alpha|v)\, dv \\
&= \int_0^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{1}{(x-\alpha)!} v^{x-1} e^{-(\lambda+1)v}\, dv \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{1}{(x-\alpha)!} \frac{\Gamma(x)}{(\lambda+1)^x} \\
&\quad \times \int_0^{\infty} \underbrace{\frac{(\lambda+1)^x}{\Gamma(x)} v^{x-1} e^{-(\lambda+1)v}}_{\text{pdf of gamma}(x,\lambda+1)}\, dv \\
&= \frac{\lambda^\alpha}{(\alpha-1)!} \frac{1}{(x-\alpha)!} \frac{(x-1)!}{(\lambda+1)^x} \\
&= \underbrace{\binom{x-1}{\alpha-1} \left(\frac{\lambda}{\lambda+1}\right)^{\alpha} \left(1-\frac{\lambda}{\lambda+1}\right)^{x-\alpha}}_{\text{pmf of negative binomial}\left(\alpha,\, \frac{\lambda}{\lambda+1}\right)}
\end{aligned}$$

for $x = \alpha, \alpha+1, \alpha+2, \ldots$, and zero, otherwise.

(c) (*6pts*) By the Bayes theorem, the conditional pdf of V given $M = m$ is

$$\begin{aligned}
f_{V|M}(v|m) &= \frac{f_V(v) f_{M|V}(m|v)}{\int_{-\infty}^{\infty} f_V(v) f_{M|V}(m|v)\, dv} \\
&= \frac{\frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{1}{m!} v^{(\alpha+m)-1} e^{-(\lambda+1)v}}{\binom{(\alpha+m)-1}{\alpha-1} \left(\frac{\lambda}{\lambda+1}\right)^{\alpha} \left(1-\frac{\lambda}{\lambda+1}\right)^{(\alpha+m)-\alpha}} \\
&= \frac{\frac{\lambda^\alpha}{(\alpha-1)!} \frac{1}{m!} v^{(\alpha+m)-1} e^{-(\lambda+1)v}}{\frac{(\alpha+m-1)!}{(\alpha-1)!m!} \frac{\lambda^\alpha}{(\lambda+1)^{\alpha+m}}} \\
&= \underbrace{\frac{(\lambda+1)^{\alpha+m}}{\Gamma(\alpha+m)} v^{(\alpha+m)-1} e^{-(\lambda+1)v}}_{\text{pdf of gamma}(\alpha+m,\lambda+1)}
\end{aligned}$$

for $v > 0$, and zero, otherwise.

(A6, B6) *(19pts)*

Exam A.

(a) *(3pts)* Given $X = x$, the random variable I_k can only take values 0 or 1. It is clear that $I_k|X = x$ is a Bernoulli random variable. Because of the independence assumption, we have

$$
\begin{aligned}
&P(I_k = 0|X = x)\\
&= P\begin{pmatrix}\text{none of the } x \text{ people get off}\\ \text{at } k\text{th floor}\end{pmatrix}\\
&= P\begin{pmatrix}\text{1st people not get off at } k\text{th floor,}\\ \text{2nd people not get off at } k\text{th floor,}\\ \ldots,\\ x\text{th people not get off at } k\text{th floor}\end{pmatrix}\\
&= \prod_{i=1}^{x} P\begin{pmatrix}i\text{th people not get off}\\ \text{at } k\text{th floor}\end{pmatrix}\\
&= \prod_{i=1}^{x} P\begin{pmatrix}i\text{th people get off at a floor}\\ \text{other than } k\text{th floor}\end{pmatrix}\\
&= \prod_{i=1}^{x}\left(\frac{R-1}{R}\right) = \left(\frac{R-1}{R}\right)^{x}.
\end{aligned}
$$

So, $I_k|X = x \sim \text{Bernoulli}(p)$ with $p = 1 - \left(\frac{R-1}{R}\right)^x$.

(b) *(4pts)* Following the same argument as in (a), we have

$$
\begin{aligned}
&P(I_1 = 0, I_2 = 0|X = x)\\
&= P\begin{pmatrix}\text{none of the } x \text{ people get off}\\ \text{at 1st and 2nd floor}\end{pmatrix}\\
&= \prod_{i=1}^{x}\left(\frac{R-2}{R}\right) = \left(\frac{R-2}{R}\right)^{x}\\
&\neq \left(\frac{R-1}{R}\right)^{x} \times \left(\frac{R-1}{R}\right)^{x}\\
&= P(I_1 = 0|X = x) \times P(I_2 = 0|X = x)
\end{aligned}
$$

So, given $X = x$, I_1 and I_2 are not conditionally independent.

(c) *(5pts)* Because $Y = \sum_{k=1}^{R} I_k$ and $E_{I_k|X}[I_k|x] = 1 - \left(\frac{R-1}{R}\right)^x$, we have

$$
\begin{aligned}
&E_{Y|X}[Y|x] = E_{I_1,\ldots,I_R|X}\left[\left.\sum_{k=1}^{R} I_k \right| x\right]\\
&= \sum_{k=1}^{R} E_{I_k|X}[I_k|x] = \sum_{k=1}^{R}\left[1 - \left(\frac{R-1}{R}\right)^{x}\right]\\
&= R - R\left(\frac{R-1}{R}\right)^{x}.
\end{aligned}
$$

Exam B.

(a) *(3pts)* Given $W = w$, the random variable I_k can only take values 0 or 1. It is clear that $I_k|W = w$ is a Bernoulli random variable. Because of the independence assumption, we have

$$
\begin{aligned}
&P(I_k = 0|W = w)\\
&= P\begin{pmatrix}\text{none of the } w \text{ people get off}\\ \text{at } k\text{th floor}\end{pmatrix}\\
&= P\begin{pmatrix}\text{1st people not get off at } k\text{th floor,}\\ \text{2nd people not get off at } k\text{th floor,}\\ \ldots,\\ w\text{th people not get off at } k\text{th floor}\end{pmatrix}\\
&= \prod_{i=1}^{w} P\begin{pmatrix}i\text{th people not get off}\\ \text{at } k\text{th floor}\end{pmatrix}\\
&= \prod_{i=1}^{w} P\begin{pmatrix}i\text{th people get off at a floor}\\ \text{other than } k\text{th floor}\end{pmatrix}\\
&= \prod_{i=1}^{w}\left(\frac{S-1}{S}\right) = \left(\frac{S-1}{S}\right)^{w}.
\end{aligned}
$$

So, $I_k|W = w \sim \text{Bernoulli}(p)$ with $p = 1 - \left(\frac{S-1}{S}\right)^w$.

(b) *(4pts)* Following the same argument as in (a), we have

$$
\begin{aligned}
&P(I_1 = 0, I_2 = 0|W = w)\\
&= P\begin{pmatrix}\text{none of the } w \text{ people get off}\\ \text{at 1st and 2nd floor}\end{pmatrix}\\
&= \prod_{i=1}^{w}\left(\frac{S-2}{S}\right) = \left(\frac{S-2}{S}\right)^{w}\\
&\neq \left(\frac{S-1}{S}\right)^{w} \times \left(\frac{S-1}{S}\right)^{w}\\
&= P(I_1 = 0|W = w) \times P(I_2 = 0|W = w)
\end{aligned}
$$

So, given $W = w$, I_1 and I_2 are not conditionally independent.

(c) *(5pts)* Because $Z = \sum_{k=1}^{S} I_k$ and $E_{I_k|W}[I_k|w] = 1 - \left(\frac{R-1}{R}\right)^w$, we have

$$
\begin{aligned}
&E_{Z|W}[Z|w] = E_{I_1,\ldots,I_S|W}\left[\left.\sum_{k=1}^{S} I_k \right| w\right]\\
&= \sum_{k=1}^{S} E_{I_k|W}[I_k|w] = \sum_{k=1}^{S}\left[1 - \left(\frac{S-1}{S}\right)^{w}\right]\\
&= S - S\left(\frac{S-1}{S}\right)^{w}.
\end{aligned}
$$

(A6, B6) (*cont.*)

Exam A.

(b) (*7pts*) By the law of total expectation and $X \sim \text{Poisson}(\mu)$, we have

$$E_Y(Y) = E_X[E_{Y|X}(Y|X)]$$
$$= E_X\left[R - R\left(\frac{R-1}{R}\right)^X\right]$$
$$= \sum_{x=0}^{\infty}\left[R - R\left(\frac{R-1}{R}\right)^x\right]\frac{\mu^x e^{-\mu}}{x!}$$
$$= R\sum_{x=0}^{\infty}\frac{\mu^x e^{-\mu}}{x!} - R\sum_{x=0}^{\infty}\frac{(\mu-\mu/R)^x e^{-\mu}}{x!}$$
$$= R - R\,e^{-\frac{\mu}{R}}\sum_{x=0}^{\infty}\underbrace{\frac{(\mu-\mu/R)^x e^{-(\mu-\mu/R)}}{x!}}_{\text{pmf of Poisson}(\mu-\mu/R)}$$
$$= R\left(1 - e^{-\mu/R}\right).$$

Exam B.

(b) (*7pts*) By the law of total expectation and $W \sim \text{Poisson}(\theta)$, we have

$$E_Z(Z) = E_W[E_{Z|W}(Z|W)]$$
$$= E_W\left[S - S\left(\frac{S-1}{S}\right)^W\right]$$
$$= \sum_{w=0}^{\infty}\left[S - S\left(\frac{S-1}{S}\right)^w\right]\frac{\theta^w e^{-\theta}}{w!}$$
$$= S\sum_{w=0}^{\infty}\frac{\theta^w e^{-\theta}}{w!} - S\sum_{w=0}^{\infty}\frac{(\theta-\theta/S)^w e^{-\theta}}{w!}$$
$$= S - S\,e^{-\frac{\theta}{S}}\sum_{w=0}^{\infty}\underbrace{\frac{(\theta-\theta/S)^w e^{-(\theta-\theta/S)}}{w!}}_{\text{pmf of Poisson}(\theta-\theta/S)}$$
$$= S\left(1 - e^{-\theta/S}\right).$$