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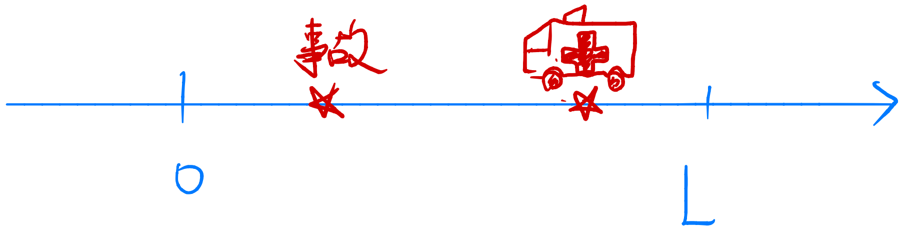


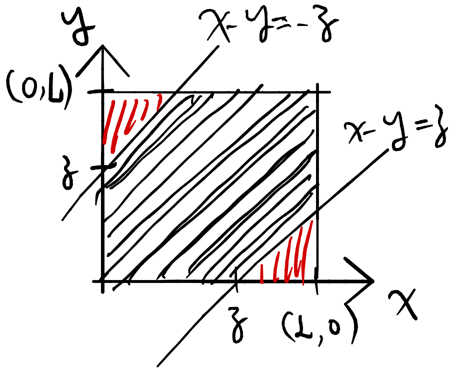
定義 X : 發生事故的位置

Y : 事故發生時救護車位置

$Z = |X-Y|$ 為事故地點與救護車距離

$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) = \frac{1}{L^2}, \quad 0<x,y<L$$

by independence



求 Z 的 CDF,

$$\begin{aligned} F_Z(z) &= P(|X-Y| \le z) \\ &= P(-z \le X-Y \le z) \\ &= 1 - \underline{2\cdot\frac{1}{L^2}\cdot\frac{(L-z)^2}{2}} \\ &= 1-\frac{(L-z)^2}{L^2}, \quad 0 \le z \le L \end{aligned}$$

紅色的面積乘上 density

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$$\perp\!\!\!\perp \begin{cases} X_1 \sim \exp(\lambda_1) \\ X_2 \sim \exp(\lambda_2) \end{cases}$$

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2)$$
$$= \lambda_1 \lambda_2 e^{-(\lambda_1 x_1 + \lambda_2 x_2)},$$
$$x_1 \ge 0,\ x_2 \ge 0$$

Z 的 CDF $F_Z(z) = P(Z \le z) = P\left(\frac{X_1}{X_2} \le z\right)$

$$= P(X_1 \le z X_2)$$
$$= \int_0^\infty \int_0^{z x_2} \lambda_1 \lambda_2 e^{-(\lambda_1 x_1 + \lambda_2 x_2)}\, dx_1\, dx_2$$
$$= \lambda_1 \lambda_2 \int_0^\infty e^{-\lambda_2 x_2} \left(\frac{1}{-\lambda_1} e^{-\lambda_1 x_1} \Big|_0^{z x_2} \right) dx_2$$
$$= \lambda_2 \int_0^\infty e^{-\lambda_2 x_2} \left(1 - e^{-\lambda_1 z x_2} \right) dx_2$$
$$= \int_0^\infty \underbrace{\lambda_2 e^{-\lambda_2 x_2}}_{\text{pdf of } Exp(\lambda_2)} - \lambda_2 e^{-(\lambda_1 z + \lambda_2) x_2}\, dx_2$$
$$= 1 - \left[\frac{\lambda_2}{-(\lambda_1 z + \lambda_2)} e^{-(\lambda_1 z + \lambda_2) x_2} \Big|_0^\infty \right]$$
$$= 1 - \frac{\lambda_2}{\lambda_1 z + \lambda_2}, \quad z \ge 0$$

$$P(X_1<X_2) = P\left(\frac{X_1}{X_2}<1\right)$$
$$= P(Z<1)$$
$$= F_Z(1)$$
$$= 1-\frac{\lambda_2}{\lambda_1+\lambda_2}$$
$$= \frac{\lambda_1}{\lambda_1+\lambda_2} \quad ※$$

30 定義 X：Jill的分數，$X \sim N(170, 20^2)$

Y：Jack的分數，$Y \sim N(160, 15^2)$

(a) ∵ X 和 Y 獨立，$X-Y \sim N(170-160, 20^2+15^2)$

$P(\text{Jack's score is higher})$

$$= P(X-Y<0)$$
$$= P\left(\frac{X-Y-(170-160)}{\sqrt{20^2+15^2}} < \frac{0-(170-160)}{\sqrt{20^2+15^2}}\right)$$
$$= P(Z<-0.4)$$

by 查表 $= 0.34458$，其中 $Z \sim N(0,1)$

(b)

$$X+Y \sim N(170+160,\ 20^2+15^2)$$

$$P(X+Y>350)$$

$$=P\left(\frac{X+Y-(170+160)}{\sqrt{20^2+15^2}} > \frac{350-(170+160)}{\sqrt{20^2+15^2}}\right)$$

$$=P(Z>0.8)$$

by 查表 $= 0.21186$，其中 $Z \sim N(0,1)$

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(a) $P(Y=y \mid X=x) = \dfrac{1}{x}$, $\quad x=1,2,\dots,5$; $\ y=1,2,\dots,x$

$$P(x,y) = P(y\mid x)\cdot P(x) = \frac{1}{x}\cdot\frac{1}{5} = \frac{1}{5x}, \quad x=1,2,\dots,5;\ y=1,2,\dots,x$$

(b) $$P(X=i \mid Y=y) = \frac{P(X=i, Y=y)}{P(Y=y)}$$

$$= \frac{\frac{1}{5i}}{\sum_{t=y}^{5}\frac{1}{5t}}$$

$$= \frac{\frac{1}{i}}{\sum_{t=y}^{5}\frac{1}{t}}, \quad i=1,2,\dots,5;\ y=1,2,\dots,i$$

(c)

X 和 Y 獨立 $\Leftrightarrow P(X=x \mid Y=y) = P(X=x)$

$\because P(X=x \mid Y=y) = \dfrac{\frac{1}{x}}{\sum_{t=y}^{5} \frac{1}{t}} \neq P(X=x) = \frac{1}{5}$

$\Rightarrow$ X 和 Y 不獨立。

39

$$P(Y=j \mid X=i) = \frac{P(X=i, Y=j)}{P(X=i)}$$

$$P(X=i, Y=j) = \begin{cases} \frac{1}{36}, & i=j=1,2,\cdots,6 \\ \frac{1}{18}, & i>j,\ i,j=1,2,\cdots,6 \end{cases}$$

(dice1, dice2)
(1, 1)
(2, 2)
⋮
(6, 6)

(dice1, dice2)
(1, 2)
(2, 1)
event $X=3, Y=1$ $\{ (1,3), (3,1)$
⋮
event $X=6, Y=5$ $\{ (5,6), (6,5)$

$$\begin{aligned} P(X=i) &= \sum_{i \geq j} P(X=i, Y=j) \\ &= P(X=i, Y=i) + \sum_{i>j} P(X=i, Y=j) \\ &= \frac{1}{36} + (i-1)\frac{1}{18} \end{aligned}$$

$$\Rightarrow P(Y=j \mid X=i) = \begin{cases} \dfrac{\frac{1}{36}}{\frac{1}{36}+(i-1)\frac{1}{18}} = \dfrac{1}{1+(i-1)\cdot 2} = \dfrac{1}{2i-1}, & i=j=1,2,\cdots,6 \\ \dfrac{\frac{1}{18}}{\frac{1}{36}+(i-1)\frac{1}{18}} = \dfrac{2}{1+(i-1)\times 2} = \dfrac{2}{2i-1}, & i>j,\ i,j=1,2,\cdots,6 \end{cases}$$

$P(Y=j) = \frac{13-2j}{36}$, $j=1,2,\cdots,6$

(dice1, dice2) $\left.\begin{array}{l}(1,2)、(1,3)、\cdots,(1,6)\\(6,1)、(5,1)、\cdots(2,1)\end{array},(1,1)\right\}$ $y=1$, 11 種可能

$\left.\begin{array}{l}(2,3)、\cdots,(2,6)\\(6,2),\cdots,(3,2)\end{array},(2,2)\right\}$ $y=2$, 9 種可能

$\vdots$

$\left.(5,6),(6,5),(5,5)\right\}$ $y=5$, 3 種可能

$\left.(6,6)\right\}$ $y=6$, 1 種可能

$\Rightarrow P(X=3, Y=2) = \frac{1}{18}$

$P(X=3) = \frac{1}{36} + (3-1)\frac{1}{18} = \frac{5}{36}$

$P(Y=2) = \frac{13-4}{36} = \frac{9}{36}$

$\because P(X=3, Y=2) \neq P(X=3) \times P(Y=2)$

X 和 Y 不獨立

41.

(a)

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{xe^{-x(y+1)}}{\int_0^\infty xe^{-x(y+1)}dx} = (y+1)^2 xe^{-x(y+1)} & , x>0, y>0 \\ 0 & , \text{o.w.} \end{cases}$$

$$f_{Y|X} = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{xe^{-x(y+1)}}{\int_0^\infty xe^{-x(y+1)}dy} = xe^{-xy} & , x>0, y>0 \\ 0 & , \text{o.w.} \end{cases}$$

(b)

$$F_Z(z) = \begin{cases} 0 & , z \le 0 \\ P(Z \le z) = P(XY \le z) & \end{cases}$$

$$= \int_0^\infty \int_0^{\frac{z}{x}} xe^{-x(y+1)} dy\, dx = \int_0^\infty (1-e^{-z}) e^{-x} dx = 1 - e^{-z} \quad , z>0$$

$$f_Z(z) = \begin{cases} \dfrac{d}{dz} F_Z(z) = e^{-z} & , z>0 \\ 0 & , z \le 0 \end{cases}$$

Thus, $Z \sim Exp(1)$

#

44.

$X_i \overset{iid}{\sim} U(0,1)$, $i = 1, 2, 3$

$P($ max X_i is greater than the sum of other two$)$

$= P(X_1 > X_2 + X_3) + P(X_2 > X_1 + X_3) + P(X_3 > X_1 + X_2)$

$$\underset{\substack{\uparrow \\ \text{by symmetry}}}{=} 3P(X_1 > X_2 + X_3) \underset{\substack{\uparrow \\ \text{by } ①}}{=} 3 \cdot \frac{1}{6} = \frac{1}{2}$$

#

We know $P(X_1 > X_2 + X_3) = \int_0^1 \int_0^{X_1} \int_0^{(X_1 - X_3)} 1\, dx_2\, dx_3\, dx_1$

$$= \int_0^1 \frac{x_1^2}{2} dx_1 = \frac{1}{6} \quad \text{——} ①$$

48.

$X_i \overset{iid}{\sim} Exp(\lambda)$, $i=1,\dots,5$

$\therefore P(X_i \le a) = 1-e^{-\lambda a}$ & $P(X_i > a) = e^{-\lambda a}$

(a)

$$P(\min(X_1\dots,X_5)\le a) = 1-P(\min(X_1,\dots,X_5)>a)$$
$$= 1-P(X_1>a,\dots,X_5>a)$$
$$= 1-\prod_{i=1}^{5} P(X_i>a)$$
$$= 1-e^{-5\lambda a} \quad \text{if } a\ge 0$$

$$P(\min(X_1\dots X_5)\le a) = 0 \quad \text{if } a<0$$

(b)

$$P(\max(X_1\dots X_5)\le a) = P(X_1\le a,\dots X_5\le a)$$
$$= \prod_{i=1}^{5} P(X_i\le a)$$
$$= (1-e^{-\lambda a})^5, \quad \text{if } a\ge 0$$

$$P(\max(X_1\cdots X_5)\le a) = 0, \quad \text{if } a<0$$

Theoretical Exercises

22.

$W\sim\Gamma(t,\beta)$, $X_i|W=w \sim Exp(w)$

$$f(x_1,\dots,x_n,w) = f(x_1\dots x_n|w)f(w) = w^n\left(e^{-w\sum_{i=1}^n x_i}\right)\frac{e^{-\beta w}w^{t-1}\beta^t}{\Gamma(t)}$$
$$= \frac{1}{\Gamma(t)}\beta^t w^{n+t-1}\left(e^{-w(\sum_{i=1}^n x_i+\beta)}\right), \quad x_i\ge 0,\ i=1\dots n,\ w>0$$

$$\Rightarrow f(x_1,\dots,x_n) = \int_0^\infty \frac{1}{\Gamma(t)}\beta^t w^{n+t-1}e^{-w(\sum_{i=1}^n x_i+\beta)}\,dw$$
$$= \frac{\beta^t}{\Gamma(t)}\frac{\Gamma(n+t)}{(\sum_{i=1}^n x_i+\beta)^{n+t}} \quad x_i\ge 0,\ i=1,\dots,n$$

$$\Rightarrow f(w|x_1\cdots x_n) = \frac{f(x_1,\cdots x_n,w)}{f(x_1\cdots x_n)}$$

$$= \frac{\frac{\beta^t}{\Gamma(t)} w^{n+t-1} e^{-w(\sum_{i=1}^n x_i+\beta)}}{\frac{\beta^t}{\Gamma(t)} \frac{\Gamma(n+t)}{(\sum_{i=1}^n x_i+\beta)^{n+t}}} = \frac{(\sum_{i=1}^n x_i+\beta)^{n+t}}{\Gamma(n+t)} w^{n+t-1} e^{-w(\sum_{i=1}^n x_i+\beta)}, \quad x_i \ge 0,\ w>0,\ i=1,\cdots,n$$

& $f(w|x_1\cdots x_n)=0$, o.w.

Thus $W|X_1=x_1,\cdots X_n=x_n \sim \Gamma(t+n, \sum_{i=1}^n x_i+\beta)$ #

34.

Let $Y_i = X_{(i)}$, $i=1,\cdots,n$

We know

$$f_{Y_iY_j}(y_i,y_j) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}(F(y_i))^{i-1}(F(y_j)-F(y_i))^{j-i-1}(1-F(y_j))^{n-j} f(y_i)f(y_j), \quad j>i$$

Thus $f_{Y_1,Y_n}(y_1,y_n) = \frac{n!}{(n-2)!}(y_n-y_1)^{n-2}\cdot 1\cdot 1 = n(n-1)(y_n-y_1)^{n-2}$

by $M=\frac{Y_1+Y_n}{2}$, $R=Y_n-Y_1$

$\Rightarrow Y_1=\frac{2M-R}{2}$, $Y_n=\frac{2M+R}{2}$

Thus $f_{R,M}(r,m) = f_{Y_1,Y_n}(\frac{2m-r}{2},\frac{2m+r}{2})\underbrace{|J|}_{=1} = n(n-1)r^{n-2}$, $0<r<1$, $\frac{r}{2}<m<1-\frac{r}{2}$ #

$$\left.\begin{aligned} 0<Y_1=\tfrac{2M-R}{2}<1 &\Rightarrow \tfrac{R}{2}<M<\tfrac{R+2}{2} \\ 0<Y_n=\tfrac{2M+R}{2}<1 &\Rightarrow \tfrac{-R}{2}<M<\tfrac{2-R}{2} \end{aligned}\right\} \Rightarrow \frac{R}{2}<M<1-\frac{R}{2}$$