· 3 6 8 9 11

No.
Date : :

#6-3 $Y_i = \begin{cases} 1, & \text{第}i\text{顆白球被抽中} \\ 0, & \text{o.w} \end{cases}$ $i = 1, 2, 3, 4, 5$

(a)

(y_1, y_2)	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$P_{Y_1Y_2}(y_1, y_2)$	$\frac{\binom{11}{3}}{\binom{13}{3}}$	$\frac{\binom{11}{2}}{\binom{13}{3}}$	$\frac{\binom{11}{2}}{\binom{13}{3}}$	$\frac{\binom{11}{1}}{\binom{13}{3}}$

(b)

(y_1, y_2, y_3)	$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$
$P_{Y_1Y_2Y_3}(y_1, y_2, y_3)$	$\frac{\binom{10}{3}}{\binom{13}{3}}$	$\frac{\binom{10}{2}}{\binom{13}{3}}$	$\frac{\binom{10}{2}}{\binom{13}{3}}$	$\frac{\binom{10}{2}}{\binom{13}{3}}$

(y_1, y_2, y_3)	$(1,1,0)$	$(1,0,1)$	$(0,1,1)$	$(1,1,1)$
$P_{Y_1Y_2Y_3}(y_1, y_2, y_3)$	$\frac{\binom{10}{1}}{\binom{13}{3}}$	$\frac{\binom{10}{1}}{\binom{13}{3}}$	$\frac{\binom{10}{1}}{\binom{13}{3}}$	$\frac{1}{\binom{13}{3}}$

Chuyu culture

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6-6

$$(N_1,N_2) = \left\{ \begin{array}{llll} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & \\ (3,1) & (3,2) & & \\ (4,1) & & & \end{array} \right\}$$

$P(N_1=1, N_2=1) = \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{10}$

$P(N_1=1, N_2=2) = \frac{2}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{10}$

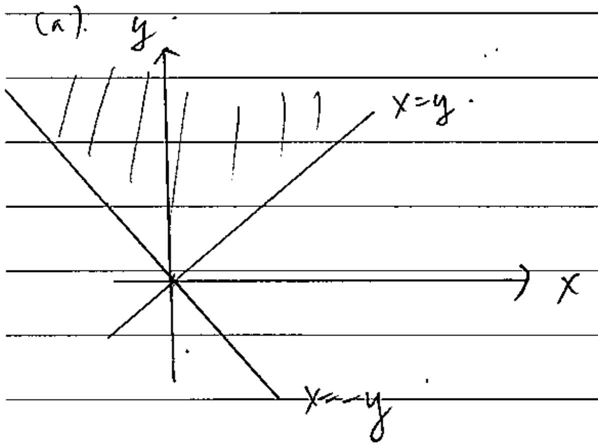
$\vdots$

$P(N_1=4, N_2=1) = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \cdot \frac{2}{2} \cdot \frac{1}{1} = \frac{1}{10}$

$\Rightarrow P(N_1=i, N_2=j) = \frac{1}{10}$, $i=1,2,3,4$.
$j=1,2,\ldots,5-i$

6-8

(a)



$\because$ 機率總和 $=1$

$\therefore \int_0^\infty \int_{-y}^{y} c(y^2-x^2)e^{-y}\,dx\,dy = \frac{4c}{3}\int_0^\infty e^{-y} y^3\,dy$

$= \frac{4c}{3}(-e^{-y})(y^3+3y^2+6y+6)\Big|_0^\infty$

$= 8c = 1 \Rightarrow c = \frac{1}{8}$ #

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#6-8

(b)

$$f_X(x) = \int_y f_{XY}(x,y)\,dy = \begin{cases} \int_{-x}^{\infty} \frac{1}{8}(y^2-x^2)e^{-y}\,dy, & x<0 \\ \int_{x}^{\infty} \frac{1}{8}(y^2-x^2)e^{-y}\,dy, & 0\le x \end{cases}$$

$$= \begin{cases} \frac{1}{4}e^{x}(1-x), & x<0 \\ \frac{1}{4}e^{-x}(1+x), & 0\le x \end{cases}$$

$$f_Y(y) = \int_x f_{XY}(x,y)\,dx$$

$$= \int_{-y}^{y} \frac{1}{8}(y^2-x^2)e^{-y}\,dx$$

$$= \frac{1}{6}y^3e^{-y}, \quad 0<y<\infty$$

(c)

$$E(X) = \int_{-\infty}^{0} \frac{1}{4}e^{x}(x-x^2)\,dx + \int_{0}^{\infty} \frac{1}{4}e^{-x}(x+x^2)\,dx$$

$$= \frac{1}{4}e^{x}(x-x^2-1+2x-2)\Big|_{-\infty}^{0} + \frac{1}{4}e^{-x}(-x-x^2-1-2x-2)\Big|_{0}^{\infty}$$

$$= \frac{-3}{4} + \frac{3}{4} = 0$$

Chryv culture

No.
Date a, b, c, d

#9

(a) $\int_x \int_y f(x,y)\,dy\,dx$

$= \int_0^1 \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy\,dx$

$= \int_0^1 \frac{6}{7}(2x^2 + x)\,dx = 1 \Rightarrow$ 機率總合$=1$

且 $\forall x \in (0,1)$ $y \in (0,2)$ . . $f_{XY}(x,y) \geq 0$

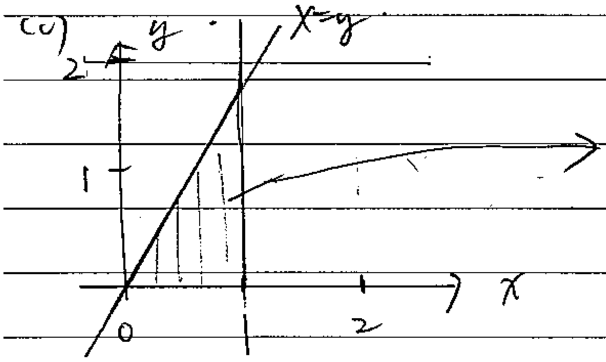
$\Rightarrow$ $f(x,y)$ is a joint pdf

(b)

$f_X(x) = \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy$

$= \frac{6}{7}(2x^2 + x), \quad 0 < x < 1$

(c)



$P(X > Y) = \int_0^1 \int_0^x \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy\,dx$

$= \int_0^1 \frac{15}{14} x^3\,dx$

$= \frac{15}{56}$

Chryculture

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#6-9

(d)

$P(Y>\frac{1}{2} \mid X<\frac{1}{2})$

$$= \frac{P(X<\frac{1}{2}, Y>\frac{1}{2})}{P(X<\frac{1}{2})} = \frac{\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{2} \frac{6}{7}(x^2+\frac{xy}{2})\,dy\,dx}{\int_0^{\frac{1}{2}} \frac{6}{7}(2x^2+x)\,dx}$$

$$= \frac{\frac{6}{7}\cdot\frac{23}{128}}{\frac{5}{28}} = \frac{69}{80} \#$$

#6-11

P(sold 2 ordinary set, 1 plasma set among 5 customers)

$$= \frac{5!}{2!\,1!\,2!}(0.45)^2(0.15)(0.4)^2 = 0.1458 \#$$

Chuyu culture

(13.) Let X defines man's arrival time $\sim Unif(15, 45)$.
Y " woman's " $\sim Unif(0, 60)$ $\Big\}$ $\perp\!\!\!\perp$.

(1). $P(|Y-X|<5) = P(-5 \le Y-X \le 5)$

$= P(-5+X \le Y \le 5+X)$

(給定X.並把X的範圍積分). $= \int_{15}^{45} P(-5+X \le Y \le 5+X \mid X=x)\,P(X=x)\,dx$

$= \int_{15}^{45}\int_{-5+x}^{5+x} \frac{1}{60}\,dy\,\frac{1}{30}\,dx$

$= \frac{1}{1800}\int_{15}^{45} 10\,dx = \frac{1}{6}$ #

(Note: 不能先給定 $Y=y$, X的值域會變成 $[-5+y, 5+y]$

但事實上若 e.g. $y=0$, 此限制值域會不在X原始值域 $[15,45]$ 中)

(2). $P(X<Y) = \int_{15}^{45} P(X<Y \mid X=x)\,P(X=x)\,dx$

$= \int_{15}^{45}\int_{x}^{60} \frac{1}{60}\,dy\,\frac{1}{30}\,dx$

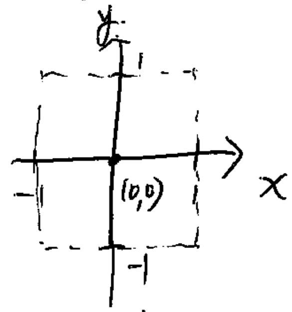
$= \frac{1}{1800}\int_{15}^{45} (60-x)\,dx$

$= \frac{1}{1800}\left(60x - \frac{1}{2}x^2\right)\Big|_{15}^{45}$

$= \frac{1}{2}$ #

15.

(b). by 題意，(X,Y) 的圖如下.



已知底面 也就是 R 為 $2\times2=4$.

故 $f_{XY}(x,y)=\begin{cases}\frac{1}{4}, & 當\ (x,y)\in R.\\ 0, & o.w.\end{cases}$

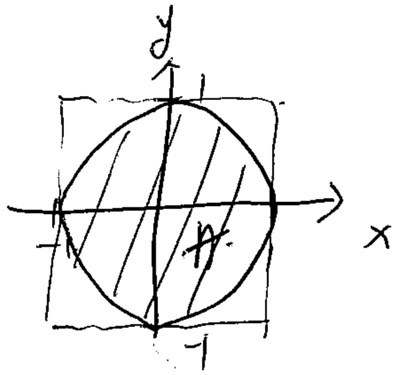
$f_X(x)=\int_{-1}^{1}\frac{1}{4}\,dy=\frac{1}{2}$, if $x\in(-1,1)$

$f_Y(y)=\int_{-1}^{1}\frac{1}{4}\,dx=\frac{1}{2}$, if $y\in(-1,1)$.

$f_X(x)f_Y(y)=\frac{1}{2}\times\frac{1}{2}=f_{XY}(x,y)$

故 $X\perp Y$ #.

(c).



A 為半徑為 1 的圓. 面積為 $1^2\pi=\pi$ 單位.

已知正方形的面積(即 R) = 4 單位.

故 $P(X^2+Y^2\le 1)=\frac{\pi}{4}$ #.

1

20.

(1). $f(x,y)=\begin{cases} xe^{-(x+y)}, & x>0,\ y>0. \\ 0, & o.w. \end{cases}$

$f_X(x)=\int_0^\infty xe^{-(x+y)}dy = xe^{-x}\int_0^\infty e^{-y}dy = xe^{-x},\ x>0.$

$f_Y(y)=\int_0^\infty xe^{-(x+y)}dx = e^{-y}\int_0^\infty xe^{-x}dx = e^{-y},\ y>0.$

($\because \Gamma(\alpha)=\int_0^\infty u^{\alpha-1}e^{-u}du$, 上式之 $\alpha=2$, 且 $\Gamma(2)=1$)

故 $f_{XY}(x,y)=xe^{-(x+y)}=f_X(x)f_Y(y)$, 則 $X \perp\!\!\!\perp Y$ #.

(2). $f_{XY}(x,y)=\begin{cases} 2, & 0<x<y,\ 0<y<1. \\ 0, & o.w. \end{cases}$

$f_X(x)=\int_x^1 2\,dy = 2(1-x).\quad 0<x<1.$

$f_Y(y)=\int_0^y 2\,dx = 2y,\quad 0<y<1.$

It's obvious that $f_{XY}(x,y)\neq f_X(x)f_Y(y)$ 故 $X \not\perp\!\!\!\perp Y$.

In fact, as we see the ranges of X and Y are dependent to each other, that is, $0<x<y$, they'll never be independent

26. (a) ∵ $A \cdot B \cdot C$ are independent.

$\therefore F_{ABC}(a \cdot b \cdot c) = P(A \le a, B \le b, C \le c) = P(A \le a) P(B \le b) P(C \le c)$

且 $P(A \le a) = \int_0^a 1\, dx = a$ 、 $P(B \le b) = b$. $P(C \le c) = c$

$\Rightarrow F_{ABC}(a \cdot b \cdot c) = abc, \ 0 < a \cdot b \cdot c < 1$ #.

(b). $B^2 - 4AC \ge 0$, implies $Ax^2 + Bx + C = 0$ 有實根.

已知 A, B, C 皆 $\in [0, 1]$.

故 $0 \le 4AC \le 1$，$A \cdot C$ 不獨立 須求 AC 的 joint pdf.

$$P(AC \le x) = \underset{(當\ \frac{x}{a} > 1)}{\int_0^x \int_0^1 dc\, da} + \underset{(當\ \frac{x}{a} < 1)}{\int_x^1 \int_0^{x/a} dc\, da}.$$

$$= \int_0^x 1\, da + \int_x^1 \frac{x}{a}\, da$$

$$= x + x \ln a \Big|_x^1 = x - x\ln x$$

$$\Rightarrow f_{AC}(x) = 1 - (\ln x + 1) = -\ln x, \ 0 < x < 1.$$

$$P(B^2 - 4AC \ge 0) = P\left(AC \le \frac{B^2}{4}\right)$$

$$= \int_0^1 P\left(AC \le \frac{B^2}{4} \,\middle|\, B = b\right) P(B = b)\, db.$$

$$= \int_0^1 \int_0^{\frac{b^2}{4}} -\ln x\, dx\, db = \int_0^1 \left(\frac{b^2}{4} - \frac{b^2}{4}\ln\left(\frac{b^2}{4}\right)\right) db$$

$$= \frac{\ln 2}{6} + \frac{5}{36} \ \#$$

by 分部積分.

$u = \ln x \Rightarrow du = \frac{1}{x}dx$

$dv = x^2 dx \Rightarrow v = \frac{x^3}{3}$

(已知 $\int x^2 \ln x\, dx = \frac{x^3 \ln x}{3} - \frac{x^3}{9}$) 將 $\frac{b^2}{4}\ln\left(\frac{b^2}{4}\right) = \left(\frac{b}{2}\right)^2 \left(\ln \frac{b}{2}\right) \times 2$

Theoretical Exercise.

(1). (a)(b) $I = P(X_1 < X_2 < X_3 < X_4 < X_5)$

$$= \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{x_2}^{\infty}\int_{x_3}^{\infty}\int_{x_4}^{\infty} f(x_1)f(x_2)f(x_3)f(x_4)f(x_5)\,dx_5\,dx_4\cdots dx_1$$

$$= \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{x_2}^{\infty}\int_{x_3}^{\infty} f(x_1)f(x_2)f(x_3)f(x_4)(1-F(x_4))\,dx_4\cdots dx_1.$$

(令 $1-F(x_4) = u_4 \Rightarrow -f(x_4)dx_4 = du_4$)

$$= \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{x_2}^{\infty}\int_{0}^{1-F(x_3)} f(x_1)f(x_2)f(x_3)\,u_4\left(\frac{f(x_4)}{-f(x_4)}\right)du_4\,dx_3\,dx_2\,dx_1$$

$$= -\int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{x_2}^{\infty} f(x_1)f(x_2)f(x_3)\frac{1}{2}(1-F(x_3))^2\,dx_3\,dx_2\,dx_1$$

(令 $1-F(x_3) = u_3 \Rightarrow -f(x_3)dx_3 = du_3$)

$$= \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\int_{0}^{1-F(x_2)} f(x_1)f(x_2)\frac{1}{2}u_3^2\,du_3\,dx_2\,dx_1$$

$$= \int_{-\infty}^{\infty}\int_{x_1}^{\infty} f(x_1)f(x_2)\frac{1}{6}(1-F(x_2))^2\,dx_2\,dx_1$$

⋮ 同样操作.

$$= \int_{-\infty}^{\infty} f(x_1)\frac{1}{4!}(1-F(x_1))^4\,dx_1.$$

(令 $1-F(x_1) = u_1 \Rightarrow -f(x_1)dx_1 = du_1$)

$$= \int_0^1 \frac{1}{4!}u_1^4\,du_1$$

$$= \frac{1}{5!}u_1^5\Big|_0^1$$

$$= \frac{1}{5!}$$ # . (not depend on F).

(c). 試想一個簡單例子.

$X_1, X_2 \cdots X_5$ 為5個不同數字. 且這5個數字絕對不會相等.

∵ 例 令 $X_2 = x_2$. $p(X_1 = x_2) = 0$. 連續型 r.v. 單點無機率.

故 $p(X_i = x_j) = 0$, $i = 1 \cdots 5$, $j \ne i$ 成立.

∴ 對這5個相異數字來說. 排列方式有 $5! = 120$ 種.

(∵ $\underset{5}{\square}\ \underset{4}{\square}\ \underset{3}{\square}\ \underset{2}{\square}\ \underset{1}{\square} = 5!$).

而 $X_1 < X_2 < X_3 < X_4 < X_5$ 只是其中一種.

故 $I = p(X_1 < X_2 < X_3 < X_4 < X_5) = \frac{1}{5!}$