(A1, B1) *(24pts)*

Exam A.
(a) False (b) False (c) True (d) True
(e) False (f) False (g) True (h) True

Exam B.
(a) False (b) False (c) True (d) True
(e) False (f) False (g) True (h) True

(A2, B2) *(14pts)*

Exam A.

(a) *(3pts)*

$$\begin{aligned} & P(B|E_1) \\ = & \frac{P(E_1|B)P(B)}{P(E_1|B)P(B)+P(E_1|B^c)P(B^c)} \\ = & \frac{(1/8)\times 0.6}{(1/8)\times 0.6+1\times 0.4} \\ = & \frac{0.075}{0.475} = 0.1579. \end{aligned}$$

(b) *(4pts)*

$$\begin{aligned} P(E_2|E_1) &= \frac{P(E_1\cap E_2)}{P(E_1)} \\ &= \frac{(1/16)\times 0.6}{(1/8)\times 0.6+1\times 0.4} = \frac{0.0375}{0.475} \\ &= 0.0789, \end{aligned}$$

where

$$\begin{aligned} P(E_1) &= P(E_1|B)P(B)+P(E_1|B^c)P(B^c) \\ &= (1/8)\times 0.6+1\times 0.4 = 0.475, \end{aligned}$$

and

$$\begin{aligned} & P(E_1\cap E_2) \\ = & P(E_1\cap E_2|B)P(B) \\ & +P(E_1\cap E_2|B^c)P(B^c) \\ = & (1/16)\times 0.6+0\times 0.4 = 0.0375. \end{aligned}$$

An alternative method to get the answer is:

$$\begin{aligned} & P(E_2|E_1) \\ = & P(B|E_1)\times P(E_2|E_1\cap B) \\ & +P(B^c|E_1)\times P(E_2|E_1\cap B^c) \\ = & P(B|E_1)\times P(E_2|B) \\ & +P(B^c|E_1)\times P(E_2|B^c) \qquad \text{(I)} \\ = & 0.1579\times(1/2)+(1-0.1579)\times 0 \\ = & 0.0789 \end{aligned}$$

Exam B.

(a) *(3pts)*

$$\begin{aligned} & P(B|E_1) \\ = & \frac{P(E_1|B)P(B)}{P(E_1|B)P(B)+P(E_1|B^c)P(B^c)} \\ = & \frac{(1/8)\times 0.3}{(1/8)\times 0.3+1\times 0.7} \\ = & \frac{0.0375}{0.7375} = 0.0508. \end{aligned}$$

(b) *(4pts)*

$$\begin{aligned} P(E_2|E_1) &= \frac{P(E_1\cap E_2)}{P(E_1)} \\ &= \frac{(1/16)\times 0.3}{(1/8)\times 0.3+1\times 0.7} = \frac{0.01875}{0.7375} \\ &= 0.0254, \end{aligned}$$

where

$$\begin{aligned} P(E_1) &= P(E_1|B)P(B)+P(E_1|B^c)P(B^c) \\ &= (1/8)\times 0.3+1\times 0.7 = 0.7375, \end{aligned}$$

and

$$\begin{aligned} & P(E_1\cap E_2) \\ = & P(E_1\cap E_2|B)P(B) \\ & +P(E_1\cap E_2|B^c)P(B^c) \\ = & (1/16)\times 0.3+0\times 0.7 = 0.01875. \end{aligned}$$

An alternative method to get the answer is:

$$\begin{aligned} & P(E_2|E_1) \\ = & P(B|E_1)\times P(E_2|E_1\cap B) \\ & +P(B^c|E_1)\times P(E_2|E_1\cap B^c) \\ = & P(B|E_1)\times P(E_2|B) \\ & +P(B^c|E_1)\times P(E_2|B^c) \qquad \text{(II)} \\ = & 0.0508\times(1/2)+(1-0.0508)\times 0 \\ = & 0.0254 \end{aligned}$$

(A2, B2) *(cont.)*

Exam A.

(c) (*4pts*) Because

$$\begin{aligned} P(E_2) &= P(E_2|B)P(B) + P(E_2|B^c)P(B^c) \\ &= (1/2) \times 0.6 + 0 \times 0.4 \\ &= 0.3 \\ &\neq P(E_2|E_1), \end{aligned}$$

the events E_1 and E_2 are not independent.

(d) (*3pts*) Yes. We used $P(E_1 \cap E_2|B) = 1/16 = (1/8) \times (1/2) = P(E_1|B) \times P(E_2|B)$ and $P(E_1 \cap E_2|B^c) = 0 = 1 \times 0 = P(E_1|B^c) \times P(E_2|B^c)$ in the calculation. (We used $P(E_2|E_1 \cap B) = P(E_2|B)$ and $P(E_2|E_1 \cap B^c) = P(E_2|B^c)$ to derive (I), which also come from the conditional independence assumption.)

Exam B.

(c) (*4pts*) Because

$$\begin{aligned} P(E_2) &= P(E_2|B)P(B) + P(E_2|B^c)P(B^c) \\ &= (1/2) \times 0.3 + 0 \times 0.7 \\ &= 0.15 \\ &\neq P(E_2|E_1), \end{aligned}$$

the events E_1 and E_2 are not independent.

(d) (*3pts*) Yes. We used $P(E_1 \cap E_2|B) = 1/16 = (1/8) \times (1/2) = P(E_1|B) \times P(E_2|B)$ and $P(E_1 \cap E_2|B^c) = 0 = 1 \times 0 = P(E_1|B^c) \times P(E_2|B^c)$ in the calculation. (We used $P(E_2|E_1 \cap B) = P(E_2|B)$ and $P(E_2|E_1 \cap B^c) = P(E_2|B^c)$ to derive (II), which also come from the conditional independence assumption.)

(A3, B3) *(14pts)*

Exam A.

(a) (*4pts*) Because

$$X \sim \text{Poisson}\left(\lambda = \frac{400,000}{100,000} = 4\right),$$

we have

$$p = P(X \geq 8) = \sum_{x=8}^{\infty} \frac{e^{-4}4^x}{x!} = 1 - \sum_{x=0}^{7} \frac{e^{-4}4^x}{x!}.$$

(b) (*4pts*) Because $Y \sim \text{binomial}(12, p)$, we have

$$\begin{aligned} P(Y \geq 2) &= \sum_{y=2}^{12} \binom{12}{y} p^y (1-p)^{12-y} \\ &= 1 - \sum_{y=0}^{1} \binom{12}{y} p^y (1-p)^{12-y}. \end{aligned}$$

(c) (*4pts*) Because $Z \sim \text{geometric}(p)$, we have

$$\begin{aligned} P(Z \geq 6) &= \sum_{z=6}^{\infty} (1-p)^{z-1} p \\ &= 1 - \sum_{z=1}^{5} (1-p)^{z-1} p. \end{aligned}$$

Exam B.

(a) (*4pts*) Because

$$X \sim \text{Poisson}\left(\lambda = \frac{400,000}{50,000} = 8\right),$$

we have

$$p = P(X \geq 5) = \sum_{x=5}^{\infty} \frac{e^{-8}8^x}{x!} = 1 - \sum_{x=0}^{4} \frac{e^{-8}8^x}{x!}.$$

(b) (*4pts*) Because $Y \sim \text{geometric}(p)$, we have

$$\begin{aligned} P(Y \geq 7) &= \sum_{y=7}^{\infty} (1-p)^{y-1} p \\ &= 1 - \sum_{y=1}^{6} (1-p)^{y-1} p. \end{aligned}$$

(c) (*4pts*) Because $Z \sim \text{binomial}(12, p)$, we have

$$\begin{aligned} P(Z \geq 4) &= \sum_{z=4}^{12} \binom{12}{z} p^z (1-p)^{12-z} \\ &= 1 - \sum_{z=0}^{3} \binom{12}{z} p^z (1-p)^{12-z}. \end{aligned}$$

(A3, B3) *(cont.)*

Exam A.

(d) (*2pts*) We are making the assumptions of independence: (i) one person's act of suicide does not influence another person to commit (or not) suicide, and (ii) different Bernoulli trials (at least 8 suicides or not in every months) are independent.

Exam B.

(d) (*2pts*) We are making the assumptions of independence: (i) one person's act of suicide does not influence another person to commit (or not) suicide, and (ii) different Bernoulli trials (at least 5 suicides or not in every months) are independent.

(A4, B4) *(18pts)*

Exam A.

(a) (*3pts*) Because

$$\begin{aligned} P(N=0) &= 1-\sum_{n=1}^{\infty} P(N=n) \\ &= 1-\sum_{n=1}^{\infty} \alpha p^n = 1-\frac{\alpha p}{1-p}, \end{aligned}$$

the probability mass function of N is

$$f_N(n) = \begin{cases} 1-\frac{\alpha p}{1-p}, & \text{if } n=0, \\ \alpha p^n, & \text{if } n=1,2,3,\ldots, \\ 0, & \text{otherwise.} \end{cases}$$

(b) (*4pts*) Denote the cumulative distribution function of N by $F_N(n)$. For $k \le n < k+1$, $k=1,2,3,\ldots$,

$$\begin{aligned} F_N(n) &= \left(1-\frac{\alpha p}{1-p}\right) + \sum_{i=1}^{k} \alpha p^i \\ &= 1-\frac{\alpha p}{1-p} + \frac{\alpha p - \alpha p^{k+1}}{1-p} = 1-\frac{\alpha p^{k+1}}{1-p}. \end{aligned}$$

So, we have

$$F_N(n) = \begin{cases} 0, & \text{if } n<0, \\ 1-\frac{\alpha p}{1-p}, & \text{if } 0 \le n < 1, \\ 1-\frac{\alpha p^{k+1}}{1-p}, & \text{if } k \le n < k+1, \\ & \quad k=1,2,3,\ldots. \end{cases}$$

(c) (*3pts*) Because the distribution of B given $N=n$ is binomial$(n,1/2)$, we have

$$P(B=b|N=n) = \binom{n}{b} (1/2)^n .$$

Exam B.

(a) (*3pts*) Because

$$\begin{aligned} P(N=0) &= 1-\sum_{n=1}^{\infty} P(N=n) \\ &= 1-\sum_{n=1}^{\infty} \alpha p^{n+1} = 1-\frac{\alpha p^2}{1-p}, \end{aligned}$$

the probability mass function of N is

$$f_N(n) = \begin{cases} 1-\frac{\alpha p^2}{1-p}, & \text{if } n=0, \\ \alpha p^{n+1}, & \text{if } n=1,2,3,\ldots, \\ 0, & \text{otherwise.} \end{cases}$$

(b) (*4pts*) Denote the cumulative distribution function of N by $F_N(n)$. For $k \le n < k+1$, $k=1,2,3,\ldots$,

$$\begin{aligned} F_N(n) &= \left(1-\frac{\alpha p^2}{1-p}\right) + \sum_{i=1}^{k} \alpha p^{i+1} \\ &= 1-\frac{\alpha p^2}{1-p} + \frac{\alpha p^2 - \alpha p^{k+2}}{1-p} = 1-\frac{\alpha p^{k+2}}{1-p}. \end{aligned}$$

So, we have

$$F_N(n) = \begin{cases} 0, & \text{if } n<0, \\ 1-\frac{\alpha p^2}{1-p}, & \text{if } 0 \le n < 1, \\ 1-\frac{\alpha p^{k+2}}{1-p}, & \text{if } k \le n < k+1, \\ & \quad k=1,2,3,\ldots. \end{cases}$$

(c) (*3pts*) Because the distribution of B given $N=n$ is binomial$(n,1/2)$, we have

$$P(B=b|N=n) = \binom{n}{b} (1/2)^n .$$

(A4, B4) *(cont.)*

Exam A.

(d) (*4pts*) The question asks to find $P(B = b)$. By the law of total probability, for $b = 0$,

$$\begin{aligned}P(B=b) &= \sum_{n=0}^{\infty} P(B=b|N=n)P(N=n) \\ &= 1\times\left(1-\frac{\alpha p}{1-p}\right)+\sum_{n=1}^{\infty}\binom{n}{0}(1/2)^n\times\alpha p^n \\ &= 1-\frac{\alpha p}{1-p}+\alpha\sum_{n=1}^{\infty}(p/2)^n,\end{aligned}$$

and for $b = 1, 2, 3, \ldots,$

$$\begin{aligned}P(B=b) &= \sum_{n=b}^{\infty} P(B=b|N=n)P(N=n) \\ &= \sum_{n=b}^{\infty}\binom{n}{b}(1/2)^n\times\alpha p^n \\ &= \alpha\sum_{n=b}^{\infty}\binom{n}{b}(p/2)^n.\end{aligned}$$

(e) (*4pts*) The question asks to find $P(N = n|B = b)$. By the Bayes' rule,

$$P(N=n|B=b)=\frac{P(B=b|N=n)P(N=n)}{P(B=b)}.$$

For $b = 0$,

$$P(N=n|B=b) = \begin{cases}\frac{1-\frac{\alpha p}{1-p}}{1-\frac{\alpha p}{1-p}+\alpha\sum_{i=1}^{\infty}(p/2)^i}, & \text{if } n=0,\\ \frac{\alpha(p/2)^n}{1-\frac{\alpha p}{1-p}+\alpha\sum_{i=1}^{\infty}(p/2)^i}, & \text{if } n=1,2,3,\ldots,\end{cases}$$

and for $b = 1, 2, 3, \ldots,$ $n = b, b+1, b+2, \ldots,$

$$P(N=n|B=b)=\frac{\binom{n}{b}(p/2)^n}{\sum_{i=b}^{\infty}\binom{i}{b}(p/2)^i}$$

Exam B.

(d) (*4pts*) The question asks to find $P(B = b)$. By the law of total probability, for $b = 0$,

$$\begin{aligned}P(B=b) &= \sum_{n=0}^{\infty} P(B=b|N=n)P(N=n) \\ &= 1\times\left(1-\frac{\alpha p^2}{1-p}\right)+\sum_{n=1}^{\infty}\binom{n}{0}(1/2)^n\times\alpha p^{n+1} \\ &= 1-\frac{\alpha p^2}{1-p}+\alpha p\sum_{n=1}^{\infty}(p/2)^n,\end{aligned}$$

and for $b = 1, 2, 3, \ldots,$

$$\begin{aligned}P(B=b) &= \sum_{n=b}^{\infty} P(B=b|N=n)P(N=n) \\ &= \sum_{n=b}^{\infty}\binom{n}{b}(1/2)^n\times\alpha p^{n+1} \\ &= \alpha p\sum_{n=b}^{\infty}\binom{n}{b}(p/2)^n.\end{aligned}$$

(e) (*4pts*) The question asks to find $P(N = n|B = b)$. By the Bayes' rule,

$$P(N=n|B=b)=\frac{P(B=b|N=n)P(N=n)}{P(B=b)}.$$

For $b = 0$,

$$P(N=n|B=b) = \begin{cases}\frac{1-\frac{\alpha p^2}{1-p}}{1-\frac{\alpha p^2}{1-p}+\alpha p\sum_{i=1}^{\infty}(p/2)^i}, & \text{if } n=0,\\ \frac{\alpha p(p/2)^n}{1-\frac{\alpha p^2}{1-p}+\alpha p\sum_{i=1}^{\infty}(p/2)^i}, & \text{if } n=1,2,3,\ldots,\end{cases}$$

and for $b = 1, 2, 3, \ldots,$ $n = b, b+1, b+2, \ldots,$

$$P(N=n|B=b)=\frac{\binom{n}{b}(p/2)^n}{\sum_{i=b}^{\infty}\binom{i}{b}(p/2)^i}$$

(A5, B5) *(19pts)*

(a) (*2pts*) Unlike the X_r is to withdraw *without* replacement, in the negative binomial, the ball withdrawn is *replaced* before the next drawing. The negative binomial is constructed from a sequence of *independent* Bernoulli trials, while X_r is constructed from *dependent* Bernoulli trials.

(b) (*5pts*) Let A_x be the event that a red ball is obtained on the xth draw. Let Z_{x-1} be the number of red balls withdrawn on the first $x-1$ draws. Then, $Z_{x-1} \sim$ hypergeometric$(x-1, N, R)$, and

$$P(X_r = x) = P(\{Z_{x-1} = r-1\} \cap A_x) = P(Z_{x-1} = r-1)P(A_x|Z_{x-1} = r-1).$$

Because $P(A_x|Z_{x-1} = r-1) = \dfrac{R-r+1}{N-x+1}$, we have

$$p_r(x) = P(X_r = x) = \frac{\binom{R}{r-1}\binom{N-R}{x-r}}{\binom{N}{x-1}} \times \frac{R-r+1}{N-x+1}, \quad x = r, r+1, \ldots, N,$$

where $\binom{s}{t} \equiv 0$ if $s < t$.

(c) (*5pts*)

$$\begin{aligned}
E(X_r) &= \sum_{x=r}^{N} x p_r(x) = \sum_{x=r}^{N} \frac{\dfrac{R!}{(r-1)!(R-r+1)!} \times \binom{N-R}{x-r}}{\dfrac{1}{x} \times \dfrac{N!}{(x-1)!(N-x+1)!}} \times \frac{R-r+1}{N-x+1} \\
&= \frac{(N+1)r}{R+1} \sum_{x=r}^{N} \frac{\dfrac{(R+1)!}{r!(R-r+1)!} \times \binom{N-R}{x-r}}{\dfrac{(N+1)!}{x!(N-x+1)!}} \times \frac{R-r+1}{N-x+1} \\
&= \frac{(N+1)r}{R+1} \sum_{x=r}^{N} \frac{\binom{R+1}{(r+1)-1} \times \binom{(N+1)-(R+1)}{(x+1)-(r+1)}}{\binom{N+1}{(x+1)-1}} \times \frac{(R+1)-(r+1)+1}{(N+1)-(x+1)+1} \\
&= \frac{(N+1)r}{R+1} \sum_{y=r+1}^{N+1} \underbrace{\frac{\binom{R+1}{(r+1)-1} \times \binom{(N+1)-(R+1)}{y-(r+1)}}{\binom{N+1}{y-1}} \times \frac{(R+1)-(r+1)+1}{(N+1)-y+1}}_{\text{pmf of negative hypergeometric}(r+1,\ N+1,\ R+1)} \\
&\quad \textbf{(Note. } \text{let } y \text{ be } x+1\textbf{)} \\
&= \frac{(N+1)r}{R+1} \times 1 = r \times \frac{N+1}{R+1}.
\end{aligned}$$

Because $Y_r = X_r - r$,

$$E(Y_r) = E(X_r) - r = r\left(\frac{N+1}{R+1} - 1\right) = r \times \frac{N-R}{R+1}.$$

(d) (*7pts*) Because

$$\begin{aligned}
Var(X_r) &= E(X_r^2) - [E(X_r)]^2 = E(X_r^2) + E(X_r) - E(X_r) - [E(X_r)]^2 \\
&= E[X_r(X_r+1)] - E(X_r) - [E(X_r)]^2,
\end{aligned}$$

and

$$E[X_r(X_r+1)] = \sum_{x=r}^{N} x(x+1)p_r(x) = \sum_{x=r}^{N} \frac{\dfrac{R!}{(r-1)!(R-r+1)!} \times \dbinom{N-R}{x-r}}{\dfrac{1}{(x+1)x} \times \dfrac{N!}{(x-1)!(N-x+1)!}} \times \frac{R-r+1}{N-x+1}$$

$$= \frac{(N+1)(N+2)r(r+1)}{(R+1)(R+2)} \sum_{x=r}^{N} \frac{\dfrac{(R+2)!}{(r+1)!(R-r+1)!} \times \dbinom{N-R}{x-r}}{\dfrac{(N+2)!}{(x+1)!(N-x+1)!}} \times \frac{R-r+1}{N-x+1}$$

$$= \frac{(N+1)(N+2)r(r+1)}{(R+1)(R+2)} \sum_{x=r}^{N} \frac{\dbinom{R+2}{(r+2)-1} \times \dbinom{(N+2)-(R+2)}{(x+2)-(r+2)}}{\dbinom{N+2}{(x+2)-1}} \times \frac{(R+2)-(r+2)+1}{(N+2)-(x+2)+1}$$

$$= \frac{(N+1)(N+2)r(r+1)}{(R+1)(R+2)} \sum_{y=r+2}^{N+2} \underbrace{\frac{\dbinom{R+2}{(r+2)-1} \times \dbinom{(N+2)-(R+2)}{y-(r+2)}}{\dbinom{N+2}{y-1}} \times \frac{(R+2)-(r+2)+1}{(N+2)-y+1}}_{\text{pmf of negative hypergeometric}(r+2,\, N+2,\, R+2)}$$

(**Note**. let y be $x+2$)

$$= \frac{(N+1)(N+2)r(r+1)}{(R+1)(R+2)},$$

we have

$$Var(X_r) = \frac{(N+1)(N+2)r(r+1)}{(R+1)(R+2)} - \frac{r(N+1)}{R+1} - \frac{r^2(N+1)^2}{(R+1)^2} = r \times \frac{(N-R)(N+1)(R-r+1)}{(R+1)^2(R+2)}.$$

Because adding a constant to a random variable does not change its variance and $Y_r = X_r - r$, we have $Var(Y_r) = Var(X_r)$.

(A6, B6) *(11pts)*

Exam A.

(a) *(5pts)* We have

$$E = ((T_1 \cap T_2) \cup (T_3 \cap T_4)) \cap T_5.$$

Because $T_1, \ldots, T_5$ are independent events,

$$\begin{aligned} P(E) =& P((T_1 \cap T_2) \cup (T_3 \cap T_4))P(T_5) \\ =& \{1 - [1 - P(T_1 \cap T_2)][1 - P(T_3 \cap T_4)]\} \\ & \times P(T_5) \\ =& \{1 - [1 - P(T_1)P(T_2)][1 - P(T_3)P(T_4)]\} \\ & \times P(T_5) \\ =& [1 - (1-p_1p_2)(1-p_3p_4)]p_5 \\ =& (p_1p_2 + p_3p_4 - p_1p_2p_3p_4)p_5. \end{aligned}$$

Exam B.

(a) *(5pts)* We have

$$E = ((T_1 \cap T_3) \cup (T_2 \cap T_4)) \cap T_5.$$

Because $T_1, \ldots, T_5$ are independent events,

$$\begin{aligned} P(E) =& P((T_1 \cap T_3) \cup (T_2 \cap T_4))P(T_5) \\ =& \{1 - [1 - P(T_1 \cap T_3)][1 - P(T_2 \cap T_4)]\} \\ & \times P(T_5) \\ =& \{1 - [1 - P(T_1)P(T_3)][1 - P(T_2)P(T_4)]\} \\ & \times P(T_5) \\ =& [1 - (1-p_1p_3)(1-p_2p_4)]p_5 \\ =& (p_1p_3 + p_2p_4 - p_1p_2p_3p_4)p_5. \end{aligned}$$

(A6, B6) *(cont.)*

Exam A.

(b) (*6pts*) When T_3 occurs (relay 3 closes), the system works (current can flow between A and B) if the event

$$G_1 = (T_1 \cup T_2) \cap (T_4 \cup T_5)$$

also occurs, and because $T_1, \ldots, T_5$ are independent events,

$$\begin{aligned} P(G_1) =& P(T_1 \cup T_2)P(T_4 \cup T_5) \\ =& \{1 - [1 - P(T_1)][1 - P(T_2)]\} \\ & \times \{1 - [1 - P(T_4)][1 - P(T_5)]\} \\ =& [1 - (1 - p_1)(1 - p_2)] \\ & \times [1 - (1 - p_4)(1 - p_5)] \\ =& (p_1 + p_2 - p_1p_2)(p_4 + p_5 - p_4p_5). \end{aligned}$$

When T_3^c occurs (relay 3 opens), the system works if the event

$$G_2 = (T_1 \cap T_4) \cup (T_2 \cap T_5)$$

also occurs, and because $T_1, \ldots, T_5$ are independent,

$$\begin{aligned} P(G_2) =& 1 - [1 - P(T_1 \cap T_4)][1 - P(T_2 \cap T_5)] \\ =& 1 - [1 - P(T_1)P(T_4)][1 - P(T_2)P(T_5)] \\ =& 1 - (1 - p_1p_4)(1 - p_2p_5) \\ =& p_1p_4 + p_2p_5 - p_1p_2p_4p_5. \end{aligned}$$

So, we have

$$\begin{aligned} E &= (G_1 \cap T_3) \cup (G_2 \cap T_3^c) \\ &= \{[(T_1 \cup T_2) \cap (T_4 \cup T_5)] \cap T_3\} \\ & \quad \cup \{[(T_1 \cap T_4) \cup (T_2 \cap T_5)] \cap T_3^c\}, \end{aligned}$$

and note that $G_1 \cap T_3$ and $G_2 \cap T_3^c$ are disjoint because T_3 and T_3^c form a partition of the sample space. Because $T_1, \ldots, T_5$ are independent, we have

$$\begin{aligned} P(E) =& P(G_1 \cap T_3) + P(G_2 \cap T_3^c) \\ =& P(G_1|T_3)P(T_3) + P(G_2|T_3^c)P(T_3^c) \\ =& P(G_1)P(T_3) + P(G_2)P(T_3^c) \\ =& (p_1 + p_2 - p_1p_2)(p_4 + p_5 - p_4p_5)p_3 \\ & + (p_1p_4 + p_2p_5 - p_1p_2p_4p_5)(1 - p_3). \end{aligned}$$

Exam B.

(b) (*6pts*) When T_3 occurs (relay 3 closes), the system works (current can flow between A and B) if the event

$$G_1 = (T_1 \cup T_4) \cap (T_2 \cup T_5)$$

also occurs, and because $T_1, \ldots, T_5$ are independent events,

$$\begin{aligned} P(G_1) =& P(T_1 \cup T_4)P(T_2 \cup T_5) \\ =& \{1 - [1 - P(T_1)][1 - P(T_4)]\} \\ & \times \{1 - [1 - P(T_2)][1 - P(T_5)]\} \\ =& [1 - (1 - p_1)(1 - p_4)] \\ & \times [1 - (1 - p_2)(1 - p_5)] \\ =& (p_1 + p_4 - p_1p_4)(p_2 + p_5 - p_2p_5). \end{aligned}$$

When T_3^c occurs (relay 3 opens), the system works if the event

$$G_2 = (T_1 \cap T_2) \cup (T_4 \cap T_5)$$

also occurs, and because $T_1, \ldots, T_5$ are independent,

$$\begin{aligned} P(G_2) =& 1 - [1 - P(T_1 \cap T_2)][1 - P(T_4 \cap T_5)] \\ =& 1 - [1 - P(T_1)P(T_2)][1 - P(T_4)P(T_5)] \\ =& 1 - (1 - p_1p_2)(1 - p_4p_5) \\ =& p_1p_2 + p_4p_5 - p_1p_2p_4p_5. \end{aligned}$$

So, we have

$$\begin{aligned} E &= (G_1 \cap T_3) \cup (G_2 \cap T_3^c) \\ &= \{[(T_1 \cup T_4) \cap (T_2 \cup T_5)] \cap T_3\} \\ & \quad \cup \{[(T_1 \cap T_2) \cup (T_4 \cap T_5)] \cap T_3^c\}, \end{aligned}$$

and note that $G_1 \cap T_3$ and $G_2 \cap T_3^c$ are disjoint because T_3 and T_3^c form a partition of the sample space. Because $T_1, \ldots, T_5$ are independent, we have

$$\begin{aligned} P(E) =& P(G_1 \cap T_3) + P(G_2 \cap T_3^c) \\ =& P(G_1|T_3)P(T_3) + P(G_2|T_3^c)P(T_3^c) \\ =& P(G_1)P(T_3) + P(G_2)P(T_3^c) \\ =& (p_1 + p_4 - p_1p_4)(p_2 + p_5 - p_2p_5)p_3 \\ & + (p_1p_2 + p_4p_5 - p_1p_2p_4p_5)(1 - p_3). \end{aligned}$$