NTHU MATH 2810 Final Examination - A January 10, 2017

Note. There are 6 problems in total. **For problems 2 to 6, to ensure consideration for partial/full scores, write down necessary intermediate steps.** Correct answers with inadequate or no intermediate steps may result in zero credit.

Some useful formula.

- The probability mass function (pmf) of a binomial(n, p) distribution is

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}, \text{ for } x = 0, 1, \ldots, n,$$

  and zero, otherwise. Its mean and variance are np and $np(1-p)$, respectively. When $n = 1$, it is called Bernoulli(p) distribution.

- The pmf of a negative binomial$(r,\ p)$ distribution is

$$p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \text{ for } x = r, r+1, r+2, \ldots,$$

  and zero, otherwise. Its mean and variance are $\frac{r}{p}$ and $\frac{r(1-p)}{p^2}$, respectively. When $r = 1$, it is called geometric(p) distribution.

- The pmf of a Poisson(λ) distribution is

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \text{ for } x = 0, 1, 2, \ldots,$$

  and zero, otherwise. Its mean and variance are λ and λ, respectively.

- The joint pmf of a multinomial$(n,\ m,\ p_1, \ldots, p_m)$, where $p_1 + \cdots + p_m = 1$, is

$$p(x_1, \ldots, x_m) = \binom{n}{x_1, \cdots, x_m} p_1^{x_1} \times \cdots \times p_m^{x_m},$$

  for $0 \leq x_i \leq n$, $i = 1, \ldots, m$; $x_1 + \cdots + x_m = n$, and zero, otherwise.

- The probability density function (pdf) of a uniform(α, β) distribution is

$$f(x) = \frac{1}{\beta - \alpha}, \text{ for } \alpha < x < \beta,$$

  and zero, otherwise. Its mean and variance are $\frac{\alpha+\beta}{2}$ and $\frac{(\beta-\alpha)^2}{12}$, respectively.

- The pdf of a gamma(α, λ) distribution is

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \text{ for } x > 0,$$

  and zero, otherwise. Its mean and variance are $\frac{\alpha}{\lambda}$ and $\frac{\alpha}{\lambda^2}$, respectively. The gamma function is defined as $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$, and $\Gamma(\alpha) = (\alpha - 1)!$ if α is a positive integer. When $\alpha = 1$, it is called exponential(λ) distribution.

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1. (*10 points*) For each of the random variable X or random vector $\mathbf{X}$ given below,

   - determine the type of the (joint) distribution (i.e., normal(μ, σ^2), exponential(λ), gamma(α, λ), beta(α, β), uniform, Weibull$(\alpha,\ \beta,\ \nu)$, Cauchy$(\mu,\ \sigma)$, Poisson(λ), hyper-geometric(n, N, R), binomial(n, p), Bernoulli(p), multinomial$(n, m, p_1, \ldots, p_m)$, negative binomial(r, p), geometric(p), …, etc.) which best models X or $\mathbf{X}$, and
   - **if possible**, identify the values of the parameters in the chosen distribution.

   (a) (*5 points*) George, Mary, and Hilary live in a certain house in which phone calls are received as a Poisson process with parameter λ per hour, where λ is unknown. Someday, in order to receive an important phone call, they came up with a schedule: George waited by the phone between 9AM-10AM, Mary waited by the pone between 10AM-12PM, and Hilary waited by the phone between 12PM-5PM. Let X_1, X_2, X_3 be the numbers of phone calls that George, Mary, and Hilary received respectively. Suppose that the total number of phone calls they received between 9AM-5PM was 12. Set $\mathbf{X} = (X_1, X_2, X_3)$.

   (b) (*5 points*) A trained flea sits on the real line at $x = 4$, and its master begins flipping a fair coin. Each time the coin shows a head, the flea hops one unit to the right, each time a tail shows it hops one unit to the left. Let X be the random variable "the flea's position after a *large number* of flips," where the number of flips is unknown. [**Hint**. Apply the normal approximation to the binomial distribution.]

2. (*20 points*) Compute the following probabilities. A correct answer without intermediate steps will receive no credit.

   (a) (*6 points*) A point is chosen at random on a line segment of length L. Compute the probability that the ratio of the shorter to the longer segment is less than $1/4$.

   (b) (*7 points*) The time that it takes to service a car is an exponential random variable with rate $\lambda = 1$. If A.J. brings his car in a time 0 and M.J. brings her car in a time $t > 0$, what is the probability that M.J.'s car is ready before A.J.'s car? (assume that service times are independent and service begins upon arrival of the car.)

   (c) (*7 points*) If X_1, X_2, X_3 are independent random variables that are uniformly distribution over $(0, 1)$, compute the probability that the largest of the three is greater than the sum of the other two. [**Hint**. $\{X_{(3)} > X_{(2)} + X_{(1)}\} = \{X_3 > X_1 + X_2\} \cup \{X_2 > X_1 + X_3\} \cup \{X_1 > X_2 + X_3\}$.]

3. (*14 points*) Suppose X and Y are two jointly distributed random variables with joint probability density function (pdf):

$$f_{X,Y}(x, y) = \begin{cases} c \cdot xy(1 - x), & \text{for } 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{otherwise,} \end{cases}$$

   where c is a constant.

   (a) (*4 points*) Find the value of c.

   (b) (*4 points*) Find the marginal pdf of X.

   (c) (*3 points*) Examine whether X and Y are independent.

   (d) (*3 points*) Find $P(X < 1/2 | Y > 1/2)$. [**Hint**. Use (c) to simplify the calculation.]

4. (*21 points*) Let $X_1, \ldots, X_n$ be independent uniform$(0,1)$ random variables. Define $X_{(1)} = \min(X_1, \ldots, X_n)$ and $X_{(n)} = \max(X_1, \ldots, X_n)$. Let

$$R = X_{(n)} - X_{(1)} \quad \text{and} \quad M = \frac{X_{(n)} + X_{(1)}}{2}.$$

   (a) (*3 points*) Show that the joint pdf of $X_{(1)}$ and $X_{(n)}$ is

$$f_{X_{(1)},X_{(n)}}(s,t) = \begin{cases} n(n-1)(t-s)^{n-2}, & \text{if } 0 < s < t < 1, \\ 0, & \text{otherwise.} \end{cases}$$

   (b) (*8 points*) Find the joint cumulative distribution function (cdf) $F_{X_{(1)},X_{(n)}}(u,v)$ of $X_{(1)}$ and $X_{(n)}$.

   (c) (*6 points*) Compute the joint pdf of R and M.

   (d) (*4 points*) Show that the covariance of R and M is $\left[Var(X_{(n)}) - Var(X_{(1)})\right]\big/ 2$.

5. (*16 points*) Suppose that W, the amount of moisture in the air on a given day, is a gamma random variable with parameters (α, λ), where α is a positive integer. Suppose also that given that $W = w$, the number of accidents during that day — call it N — has a Poisson distribution with mean w.

   (a) (*4 points*) Use multiplication law to find the joint distribution of W and N.

   (b) (*6 points*) Use the law of total probability to show that $N + \alpha$ is a negative binomial random variable with parameters $\left(\alpha, \frac{\lambda}{\lambda+1}\right)$.

   (c) (*6 points*) Use Bayes theorem to show that the conditional distribution of W given that $N = n$ is the gamma distribution with parameters $(\alpha + n, \lambda + 1)$.

6. (*19 points*) Suppose that X, the number of people who enter an elevator on the ground floor, is a Poisson random variable with mean μ. Suppose that there are R floors *above* the ground floor, and each person is equally likely to get off at any one of the R floors, independently of where the others get off. Let Y be number of stops that the elevator will make before discharging all of its passengers.

   (a) (*3 points*) For $k = 1, 2, \ldots, R$, let

$$I_k = \begin{cases} 1, & \text{if the elevator stops at floor } k, \\ 0, & \text{otherwise.} \end{cases}$$

   Show that the conditional distribution of I_k given $X = x$ is Bernoulli(p) with

$$p = 1 - \left(\frac{R-1}{R}\right)^x$$

   for $k = 1, 2, \ldots, R$. [**Hint**. Calculate $P(I_k = 0 | X = x)$.]

   (b) (*4 points*) Examine whether I_1 and I_2 are conditionally independent given $X = x$. [**Hint**. Calculate $P(I_1 = 0, I_2 = 0 | X = x)$ to examine.]

   (c) (*5 points*) Identify the conditional expectation of Y given $X = x$ by expressing Y as a function of $I_1, \ldots, I_R$.

   (d) (*7 points*) Show that the expected number of stops that the elevator will make before discharging all of its passengers (i.e., $E_Y(Y)$) is $R\left(1 - e^{-\mu/R}\right)$. [**Hint**. Apply law of total expectation.]