An efficient method to solve ODE with delta function

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This article and supporting scripts could be downloaded from:
https://github.com/stdapproach/sciArticle/tree/develop/ODE_Delta

Abstract

This article is devoted to Linear Time-Invariant (LTI) Ordinal Differential Equations (ODE) with terms consisting of Dirac delta functions. The algorithm to exchange the original non-homogenous ODE with non-null initial condition (IC) to a homogenous one with different IC is provided. The resulting ODE could be solved analytical or numerical method with. Provided 8 with known analitycal solution to check the algorythm.

Keywords:

Impulse response function, time domain, linear ODE, Delta function

Introduction

The dynamics of evolving processes is often subjected to abrupt changes such as:
• impact by hammer to a beam,
• a bat striking a ball or a bolt of lightning striking a tower.

Often these short-term perturbations are treated as having acted instantaneously or in the form of “impulses”. In this case, the output corresponding to this sudden force is referred to as the impulse response function (IFR). Mathematically, an impulse can be modeled by an initial value problem (IVP) with a special type of function known as the Dirac delta function as the external force, i.e., the non-homogeneous term. The impulse response of a system is its response to the input $\delta(t)$ when the system is initially at rest.

We’ve tried to find a method for solve such kind systems. We only founded solutions for particular First and Second order’s ODE. So we decided to (re-)invent it.

To understand this paper there is only need to have a basic knowledge of ODE, Laplace Transform and Linear Algebra.

1 Definition & Terminology

Function:

$$y(t) \text{ - function takes an argument} \in \mathbb{R} \text{ and returns a result} \in \mathbb{R}$$

Derivative:

$$y' = \frac{dy}{dt}\ , y'' = \frac{d^2y}{dt^2}\ , y^{(n)} = \frac{d^ny}{dt^n}\ , y^{(0)} = y$$

Heaviside step function

$$H(t)$$

Dirac delta function

$$\delta(t) = \frac{dH(t)}{dt}$$

Initial Value Problem (IVP), Caushy problem

LTI ODE

$$L_n(\{a\}, y) = \sum_{i=0}^{n} a_i y^{(n-i)}(t) = f(t)\ , a_i = const \in \mathbb{R}\ , i \in 0 \dots n \tag{1.1}$$

and initial conditions (IC)

$$\{y\}|_{t_0} = IC|_{t_0} = \left\{ \begin{matrix} y(t_0) \\ y'(t_0) \\ y''(t_0) \\ \vdots \\ y^{(n-1)}(t_0) \end{matrix} \right\} \tag{1.2}$$

named IVP which has an unique solution y(t) satisfied (1.1) and (1.2)
We use the following 3 equivalence short forms for IVP:

$$\begin{cases} L_n(y) = f(t) \\ \{y\}|_{t_0} \end{cases} \equiv \begin{cases} L_n(\{a\}, y) = f(t) \\ IC|_{t_0} = IC_0 \end{cases} \equiv IVP(\{a\}, f(t), t_0, IC_0)$$

Solution of IVP

$$y(t) = \begin{cases} L_n(\{a\}, y) = f(t) \\ IC|_{t_0} = IC_0 \end{cases}$$

which satisfied of (1.1) and (1.2).

2 First glimpse

Let's take a look at first order system
(order of a differential equation is the largest derivative present in the differential equation):

$$\begin{cases} x' + Ax = Bu(t) \\ x(0) = x_0 \end{cases}$$

We can write it using our notation:

$$\begin{cases} L_n(\{1, A\}, x) = Bu(t) \\ IC|_{t_0=0} = x_0 \end{cases} \tag{2.1}$$

Solution of (2.1) is a function

$$x(t) = e^{-At}x_0 + \int_0^t e^{-A(t-\tau)}Bu(\tau)d\tau \tag{2.2}$$

The expression (2.2) delivers solution for the equation (2.1), which could be rewritten in a short form

$$x(t) = \begin{cases} L_n(\{1, A\}, x) = Bu(t) \\ IC|_{t_0=0} = x_0 \end{cases}$$

The solution of homogenous system (free response) is:

$$x_{free}(t) = \begin{cases} L_n(\{1, A\}, x) = 0 \\ IC|_{t_0=0} = x_0 \end{cases} = e^{-At}x_0$$

And substitute the Dirac delta function as load, so the system (2.1) becomes as

$$\begin{cases} x' + Ax = B\delta(t), \\ x(0) = x_0 \end{cases} \tag{2.3}$$

The solution is:

$$\begin{aligned} x_\delta(t) = x_0e^{-At} + \int_0^t e^{-A(t-\tau)}B\delta(\tau)d\tau = \\ = x_0e^{-At} + Be^{-At} = e^{-At}(x_0 + B) \end{aligned} \tag{2.4}$$

Obviosely the solution of the system (2.4) is the same as solution of next one:

$$\begin{cases} x' + Ax = 0, \\ x(0) = x_0 + B \end{cases} \tag{2.5}$$

So we can write a next statement:

$$\begin{cases} L_n(\{1,A\},y) = B\delta(t) \\ IC|_{t_0=0} = x_0 \end{cases} \equiv \begin{cases} L_n(\{1,A\},y) = \mathbf{0} \\ IC|_{t_0=0} = x_0 + \mathbf{B} \end{cases} \tag{2.6}$$

For the system (2.3) the solution is the same as free response of the same system but changed IC.

About changing IC

Some books provided an analitical solution for LTI ODE with delta function as load. For example: Finan (p.57), Nagy (pp.189-190), Ogata (p.190), Zill (p.293).
Some other books noticed that the solution of IVP with delta function as load is the same as solution the similar homogenous ODE with defferernt IC.
Genta (p.180) provided the formulae for changing IC for second-order ODE in case od delta function loaded.
Rao (p.407) noticed that following systems are equal.

$$\begin{cases} y' + ay = F\delta(t), \\ y(0) = 0 \end{cases} \equiv \begin{cases} y' + ay = 0, \\ y(0) = F \end{cases}$$

Weber (p.733) noticed that following systems are equal.

$$\begin{cases} mx'' = P\delta(t), \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \equiv \begin{cases} mx'' = 0, \\ y(0) = 0 \\ y'(0) = P/m \end{cases}$$

Kelly (p.315) noticed that following systems are equal.

$$\begin{cases} mx'' + cx' + kx = \delta(t), \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \equiv \begin{cases} mx'' + cx' + kx = 0, \\ y(0) = 0 \\ y'(0) = 1/m \end{cases}$$

And Balachandran (p.286) and Genta (p.179) noticed that following systems are equal.

$$\begin{cases} mx'' + cx' + kx = f_0\delta(t), \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \equiv \begin{cases} mx'' + cx' + kx = 0, \\ y(0) = 0 \\ y'(0) = f_0/m \end{cases}$$

It looks like (but not proven yet) these two following systems are equal:

$$\begin{cases} L_n(\{a\},y) = b\delta(t) \\ IC|_{t_0} = IC_0 \end{cases} \equiv \begin{cases} L_n(\{a\},y) = 0 \\ IC|_{t_0} = IC_0 + \{0,0,\ldots,b/a_0\}^{\mathsf{T}} \end{cases} \tag{2.7}$$

3. A problem Type 0

This part ragarding the following problems.

Type 0a

$$\begin{cases} L_n(y) = b\delta(t), \\ IC_0, n \geq 1 \end{cases}$$

Type 0b

$$\begin{cases} L_n(y) = b\delta(t-c), \\ IC_0, n \geq 1 \end{cases}$$

Type 0c

$$\begin{cases} L_n(y) = \sum_{i=0}^{k} b_i \delta(t-c_i), \\ IC_0, n \geq 1 \end{cases}$$

We suppose that $t_0 = 0$, otherwise for such time-invariant systems we could just change time variable as $t^* = t - t_0$.

3.1 Problem Type 0a

Find an IC for homogenous system which delivers the equivalence for non-homogenous one.
At this case 'equivalence' means:

$$z(t) = y(t), \forall t \geq t_0$$

for the next systems: a given non homogenous system

$$\begin{cases} L_n(\{a\}, y) = b\delta(t) \\ IC\big|_{t_0=0} = IC_y \end{cases}$$

the homogenous system which IC supposed to be found

$$\begin{cases} L_n(\{a\}, z) = 0 \\ IC_z \end{cases}$$

To solve the problem Type 0a let's perform Laplace Transform (LT) for $y(t)$ with respect to non-null initial condition.

$$LT\{y(t)\} = Y(s)$$

$$Y(s) = LT\left\{\sum_{i=0}^{n}\left(a_i y^{(n-i)}(t) - b\delta(t)\right)\right\} = \sum_{i=0}^{n}\left(a_i L\{y^{(n-i)}(t)\}\right) - bL\{\delta(t)\}$$

-

$$LT\{a_0 y^{(n-0)}\} = a_0 \left[s^n Y - s^{n-1} y(0) - s^{n-2} y'(0) - s^{n-3} y''(0) - \ldots - s^1 y^{(n-2)}(0) - s^0 y^{(n-1)}(0)\right]$$

$$LT\{a_1 y^{(n-1)}\} = a_1 \left[s^{n-1} Y - s^{n-2} y(0) - s^{n-3} y'(0) - s^{n-2} y''(0) - \ldots - s^1 y^{(n-3)} - s^0 y^{(n-2)}(0)\right]$$

$$LT\{a_2 y^{(n-2)}\} = a_2 \left[s^{n-2} Y - s^{n-3} y(0) - s^{n-4} y'(0) - s^{n-5} y''(0) - \ldots - s^1 y^{(n-4)}(0) - s^0 y^{(n-3)}(0)\right]$$

$$\ldots$$

$$LT\{a_{n-1} y'\} = a_{n-1} \left[s' Y - s^0 y(0)\right]$$

$$LT\{a_n y\} = a_n (Y)$$

$$-bLT\{\delta(t)\} = -be^{(-s)0} = -b$$

Let’s using (1.2) and rewrite at this way:

$$\begin{aligned} LT\{y(t)\} = & s^n (a_0 Y) + \\ & s^{n-1} \left(a_1 Y - a_0 y_0\right) + \\ & s^{n-2} \left(a_2 Y - a_1 y_0 - a_0 y_1\right) + \\ & s^{n-3} \left(a_3 Y - a_2 y_0 - a_1 y_1 - a_0 y_2\right) + \\ & \ldots \\ & s^1 \left(a_{n-1} Y - \sum_{i=0}^{n-2} a_{n-2-i} y_i\right) + \\ & s^0 \left(a_n Y - \sum_{i=0}^{n-1} a_{n-1-i} y_i + b\right) = \\ & = \sum_{i=0}^{n} s^{n-i} a_i Y - \sum_{i=1}^{n} \left(s^{n-i} \sum_{j=0}^{i-1} a_{(i-1)-j} y_j\right) - b \end{aligned}$$

Let’s perform LT for $z(t)$ with respect to non-null initial condition.

$$LT\{z(t)\} = Z(s)$$

$$LT\{z(t)\} = \sum_{i=0}^{n} s^{n-i} a_i Z - \sum_{i=1}^{n} \left(s^{n-i} \sum_{j=0}^{i-1} a_{(i-1)-j} z_j\right)$$

$$y(t)=z(t)\implies Y(s)=Z(s)\implies$$

$$\sum_{i=0}^{n}s^{n-i}a_iY-\sum_{i=1}^{n}\left(s^{n-i}\sum_{j=0}^{i-1}a_{(i-1)-j}y_j\right)-b=\sum_{i=0}^{n}s^{n-i}a_iZ-\sum_{i=1}^{n}\left(s^{n-i}\sum_{j=0}^{i-1}a_{(i-1)-j}z_j\right)$$

Due to

$$s^{n-i}a_iY=s^{n-i}a_iZ, i=0...n\implies$$

$$\sum_{i=1}^{n}\left(s^{n-i}\sum_{j=0}^{i-1}a_{(i-1)-j}y_j\right)-b=\sum_{i=1}^{n}\left(s^{n-i}\sum_{j=0}^{i-1}a_{(i-1)-j}z_j\right)\implies$$

following equations:

$$\begin{aligned}
&i=1\rightarrow(s^{n-1}):a_0y_0=a_0z_0\\
&i=2\rightarrow(s^{n-2}):\sum_{j=0}^{1}a_{1-j}y_j=\sum_{j=0}^{1}a_{1-j}z_j\\
&\dots\\
&i=n\rightarrow(s^{n-n}):\sum_{j=0}^{n-1}a_{n-1-j}y_j+b=\sum_{j=0}^{n-1}a_{n-1-j}z_j
\end{aligned}$$

Rewrite it at this way:

$$A=\begin{bmatrix}a_0&0&0&\cdots&0\\a_1&a_0&0&\cdots&0\\a_2&a_1&a_0&\cdots&0\\\vdots&\vdots&\vdots&\ddots&0\\a_{n-1}&a_{n-2}&a_{n-3}&\cdots&a_0\end{bmatrix},\{d\}=\begin{Bmatrix}0\\0\\0\\\vdots\\0\\b\end{Bmatrix}\tag{3.1}$$

$$[A]\begin{Bmatrix}y_0\\y_1\\y_2\\\vdots\\y_{n-1}\end{Bmatrix}+\{d\}=[A]\begin{Bmatrix}z_0\\z_1\\z_2\\\vdots\\z_{n-1}\end{Bmatrix}$$

$$[A]\{y\}+\{d\}=[A]\{z\},det([A])\neq0\Rightarrow$$

$$\{z\}=\{y\}+[A]^{-1}\{d\}\tag{3.2}$$

Due to $[A]$ is lower-triangle matrix and $\{d\} = \{0, 0, \ldots, b\}$ the main result is following:

$$\boxed{\begin{cases} L_n(\{a\}, y) = b\delta(t) \\ IC_0 \end{cases} \equiv \begin{cases} L_n(\{a\}, y) = \mathbf{0} \\ IC_0 + [\mathbf{A}]^{-1}\{\mathbf{d}\} \end{cases} \equiv \begin{cases} L_n(\{a\}, y) = 0 \\ IC_0 + \{0, 0, \ldots, b/a_0\}^{\mathsf{T}} \end{cases}} \tag{3.3}$$

The formula (3.3) is the mathematical notation of algorythm how to solve LTI ODE with Dirac function load. There is only need to change IC and solve the similar homogenous system using any method you like (analytical or numerical). There has been proven (2.7). Indeed, (3.3) and (2.7) are the same.
The system Type 0a could be rewritten as simple impulse differential equation (look Benchohra, Henderson and Ntouyas “Impulsive Differential Equations and Inclusions”)

3.2 Problem Type 0b

The idea how to solve the Type 0b’s problem is very simple and well explained at the part 4.4-4.6.

3.3 Problem Type 0c

The idea how to solve the Type 0c’s problem is very simple and well explained at the part 4.7.

4. Verification by examples

Let’s check the main result from previous chapter on examples Appendix B from. To prove the method we’ve created 8 Python scripts performing the calculation and generating the charts with results.

4.1 Example1 [Oliveira and Cortes, p.3]

Consider the following first order ODE (Type 0a)

$$\begin{cases} y'' + ay' = \delta(t), \\ y(0) = y'(0) = 0 \end{cases} \Rightarrow IVP(\{1\ a\ 0\}, \delta(t), t_0 = 0, y_0 = \{0, 0\}) \Rightarrow$$

according to (3.1)

$$A = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}, \{d\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \Rightarrow$$
$$A^{-1} = \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix}, A^{-1}\{d\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

According (3.2) and (3.3) these two following system are equal by solution

$$\begin{cases} y'' + ay' = \delta(t), \\ y(0) = y'(0) = 0 \end{cases} \equiv \begin{cases} y'' + ay' = 0, \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

In short form:

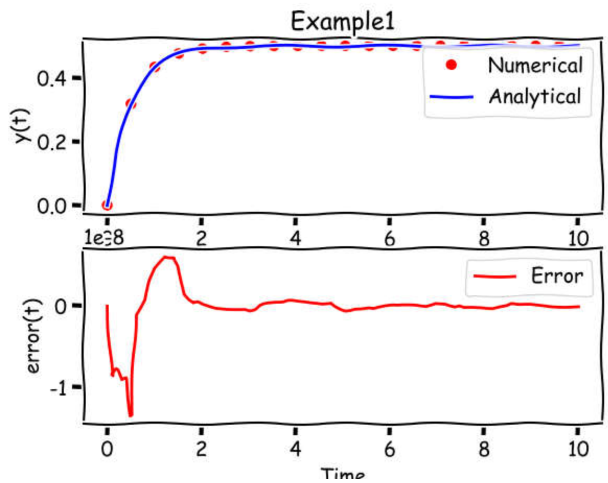
$$IVP(\{1\ a\ 0\}, \delta(t), t_0 = 0, y_0 = \{0, 0\})$$
$$\equiv IVP(\{1\ a\ 0\}, 0, t_0 = 0, y_0 = \{0, 1\})$$

Let’s check how to correspond the numerical solution for T=2 for homogenous sytem with non-null IC with analitical solution for the system

Analitycal solution taken from Appendix B:

$$y(t) = \frac{1}{a}\left(1 - e^{-at}\right)$$

Numerical solution, analytical solutions and error provided by Python’s script (example1.py):



4.2 Example2 [Finan, pp.57-58]

Considering the following second order ODE (Type 0a)

$$\begin{cases} 2y'' + 4y' + 10y = \delta(t) \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \end{cases} \Rightarrow IVP(\{2, 4, 10\}, \delta(t), t_0 = 0, y_0 = \{0, 0\}) \Rightarrow$$

$$A = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}, \{d\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \Rightarrow$$

$$A^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1 & 1/2 \end{bmatrix}, A^{-1}\{d\} = \begin{Bmatrix} 0 \\ 1/2 \end{Bmatrix} \Rightarrow$$

$$\{z\}_0 = \{y\}_0 + [A]^{-1}\{d\} = \begin{Bmatrix} 0 \\ 1/2 \end{Bmatrix}$$

These two following system are equal by solution

$$\begin{cases} 2y'' + 4y' + 10y = \delta(t) \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \end{cases} \equiv \begin{cases} 2z'' + 4z' + 10z = 0 \\ z_0 = z(0) = 0 \\ z_1 = z'(0) = 1/2 \end{cases}$$

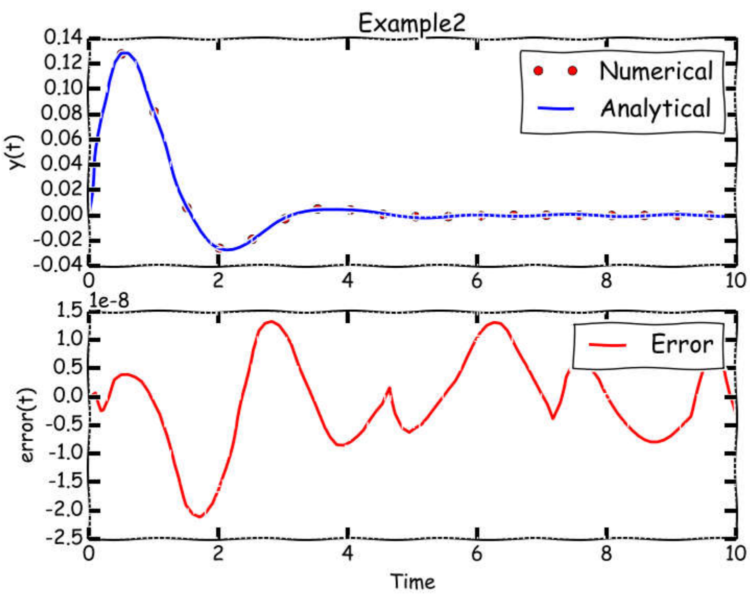
In short form:

$$IVP(\{2, 4, 10\}, \delta(t), t_0 = 0, y_0 = \{0, 0\}) \equiv IVP(\{2, 4, 10\}, 0, t_0 = 0, y_0 = \{0, 1/2\})$$

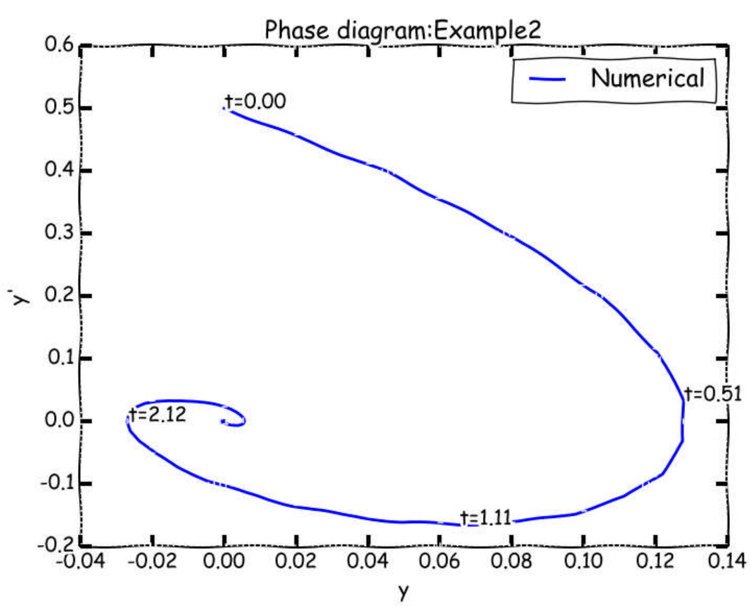
Analitycal solution:

$$y(t) = \frac{1}{4}\, e^{-t} sin(2t)$$

Numerical solution, analytical solutions and error provided by Python’s script (example2.py):



Phase diagram:



4.3 Example3 [Nagy, p.189]

Considering the following second order ODE (Type 0a)

$$\begin{cases} y'' + 2y' + 2y = \delta(t) \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \end{cases} \Rightarrow IVP(\{1, 2, 2\}, \delta(t), t_0 = 0, y_0 = \{0, 0\}) \Rightarrow$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \{d\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \Rightarrow$$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, A^{-1}\{d\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \Rightarrow$$

$$\{z\}_0 = \{y\}_0 + [A]^{-1}\{d\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

These 2 folloing system are equal by solution:

$$\begin{cases} y'' + 2y' + 2y = \delta(t) \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \end{cases} \Rightarrow \begin{cases} z'' + 2z' + 2z = \delta(t) \\ z_0 = z(0) = 0 \\ z_1 = z'(0) = 1 \end{cases}$$

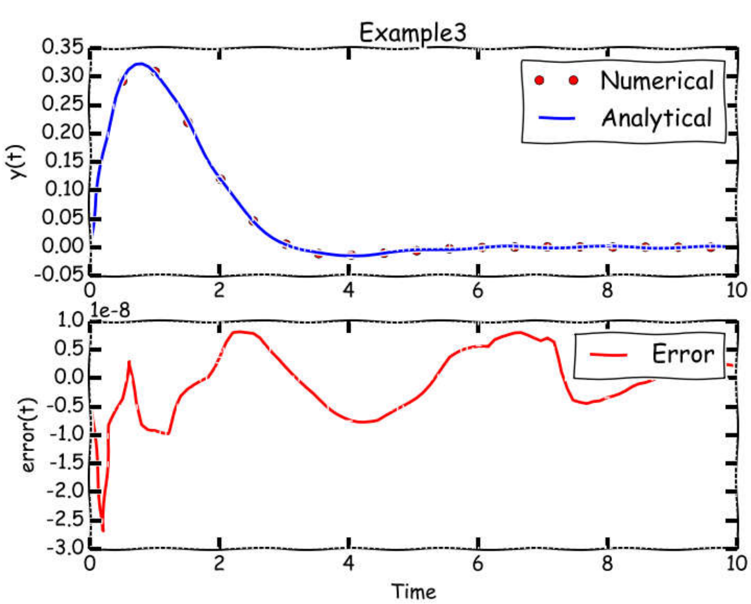
In short form:

$$IVP(\{1, 2, 2\}, \delta(t), t_0 = 0, y_0 = \{0, 0\})$$
$$\equiv IVP(\{1, 2, 2\}, 0, t_0 = 0, y_0 = \{0, 1\})$$

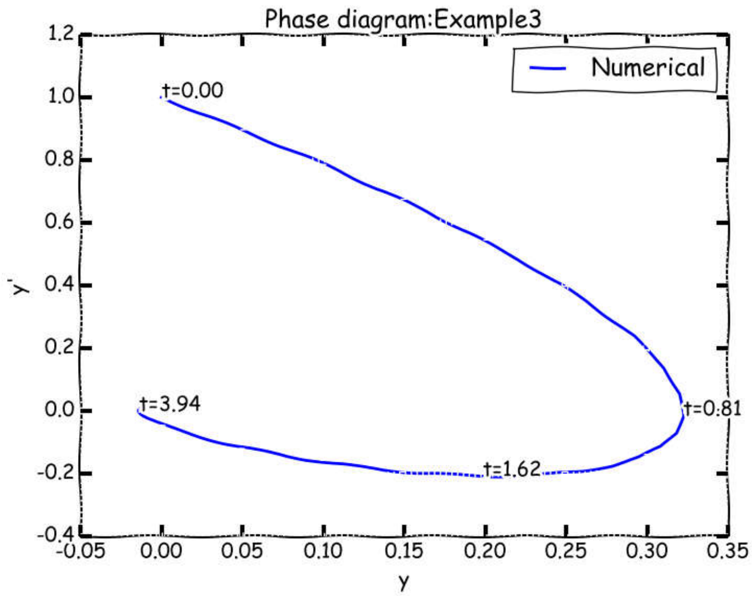
Analitycal solution:

$$y(t) = e^{-t} sin(t)$$

Numerical solution, analytical solutions and error provided by Python’s script (example3.py):



Phase diagram:



4.4 Example4 [Nagy, p.189]

Considering the following second order ODE (Type 0b)

$$\begin{cases} y'' + 2y' + 2y = \delta(t-c) \\ y(0) = y'(0) = 0 \\ c = 2 \end{cases}$$

This system could be separated on two systems and the IC for a second system based on results first one:

$$\begin{cases} y_1'' + 2y_1' + 2y_1 = 0 \\ y_1(0) = 0 \\ y_1'(0) = 0 \\ 0 \le t \le c \end{cases}$$

$$\begin{cases} y_2'' + 2y_2' + 2y_2 = \delta(t-c) \\ y_2(c) = y_1(c) \\ y_2'(c) = y_1'(c) \\ t \ge c \end{cases}$$

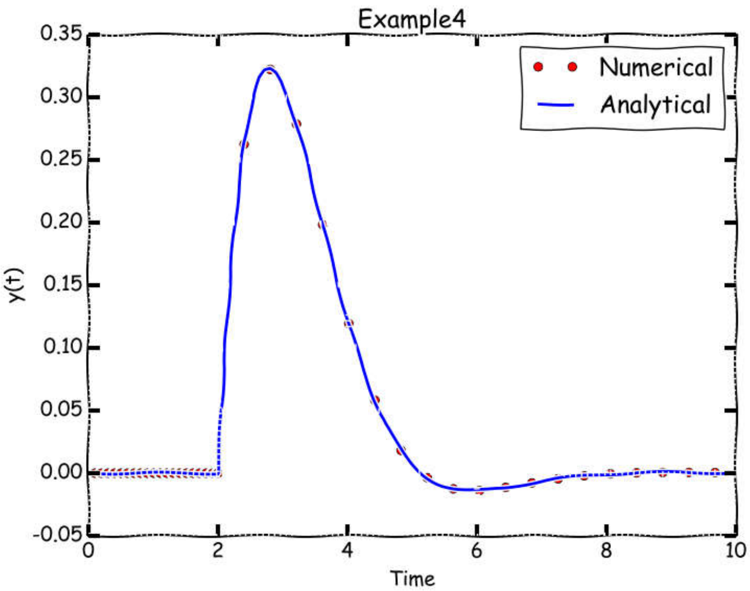
So, the solution of the original system is:

$$y(t) = \begin{cases} y_1(t), & \text{if } 0 \le t \le c \\ y_2(t), & \text{if } t \ge c \end{cases}$$

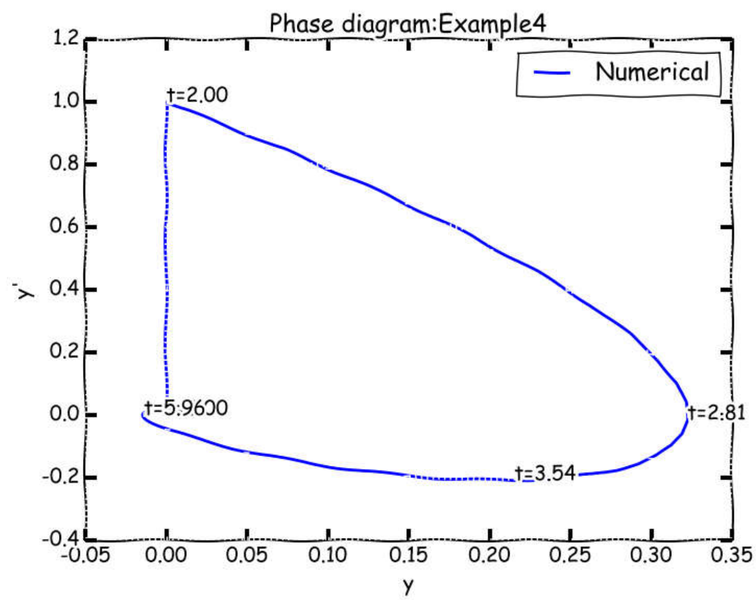
Analitycal solution:

$$y(t) = H(t-c)e^{-(t-c)}sin(t-c)$$

Numerical solution and analytical solutions provided by Python's script (example4.py):



Phase plate:



5.5 Example5 [Zill, p.293]

Considering the following second order ODE (Type 0b)

$$\begin{cases} y'' + y = 4\delta(t - 2\pi), \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \end{cases}$$

This system could be separated on two systems, and the IC for a second system based on results first one:

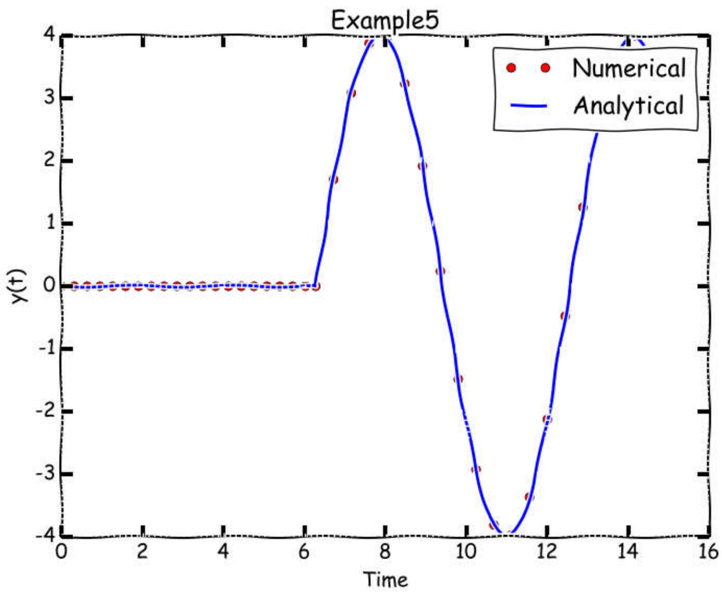
$$\begin{cases} y_1'' + y_1 = 0, \\ y_1(0) = 0 \\ y_1'(0) = 0 \\ 0 \leq t \leq 2\pi \end{cases}$$

$$
\begin{cases}
y2'' + y2 = 4\delta(t - 2\pi), \\
y_2(2\pi) = y_1(2\pi) \\
y_2'(2\pi) = y_1'(2\pi) \\
t \geq 2\pi
\end{cases}
$$

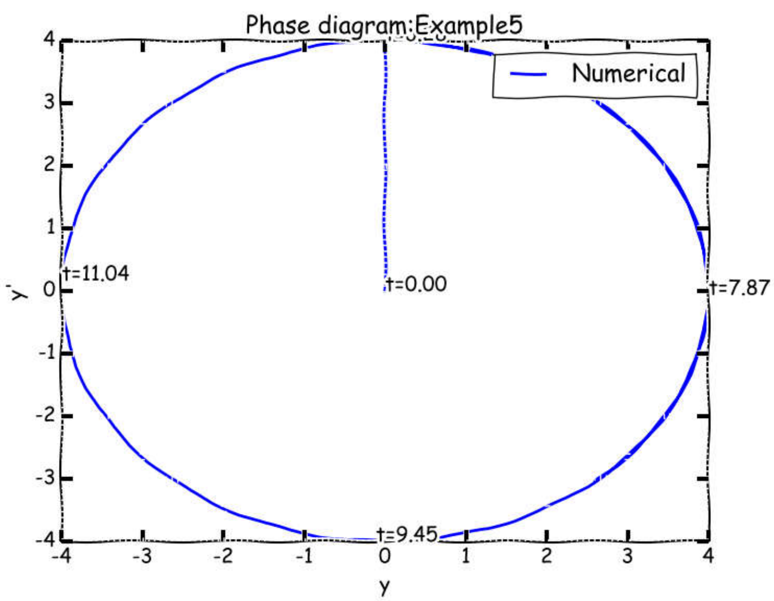
Analitycal solution:

$$y(t) = H(t - 2\pi)4sin(t)$$

Numerical solution and analytical solutions provided by Python's script (example5.py):



Phase plate:



4.6 Example6 [Zill, p.293]

Considering the following second order ODE (Type 0b)

$$
\begin{cases} y'' + y = 4\delta(t - 2\pi), \\ y_0 = y(0) = 1 \\ y_1 = y'(0) = 0 \end{cases}
$$

This system could be separated on two systems, moreover the IC for a second system based on results first one:

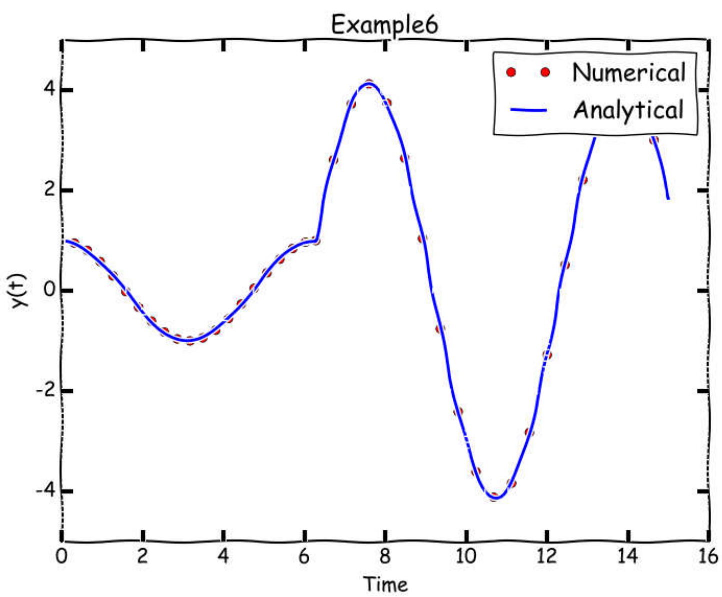
$$
\begin{cases} y_1'' + y_1 = 0, \\ y_1(0) = 1 \\ y_1'(0) = 0 \\ 0 \le t \le 2\pi \end{cases}
$$

$$
\begin{cases} y_2'' + y_2 = 4\delta(t - 2\pi), \\ y_2(2\pi) = y_1(2\pi) \\ y_2'(2\pi) = y_1'(2\pi) \\ t \ge 2\pi \end{cases}
$$

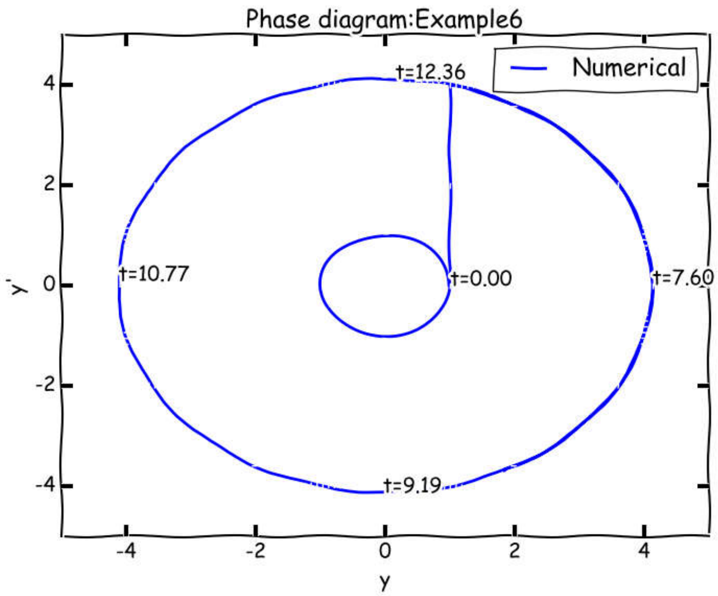
Analitycal solution:

$$
y(t) = cos(t) + 4H(t, 2\pi)sin(t)
$$

Numerical solution and analytical solutions provided by Python’s script (example6.py):



Phase plate:



4.7 Example7 [Nagy, p.190]

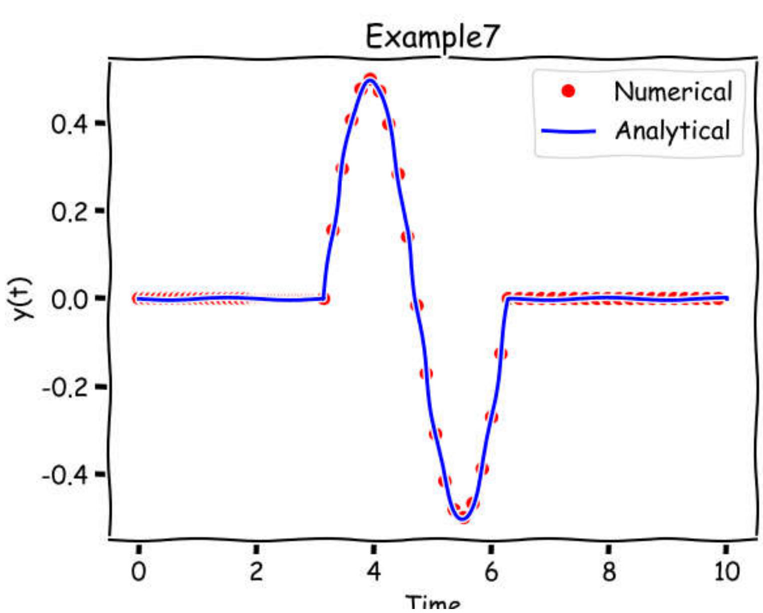
Considering the following second order ODE (Type 0c)

$$\begin{cases} y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \\ y(0) = y'(0) = 0 \end{cases}$$

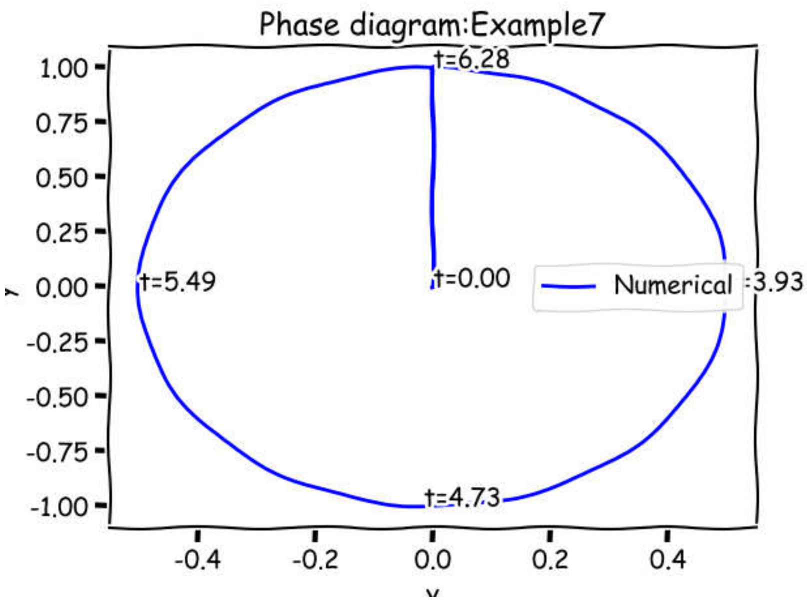
To solve this type of system we recommend to use the same approach as used for Type 0b (see 4.4). I.e. separate the orginal system on time line to some number similar system. And the IC for the next system should be taken from previous one.
Analitycal solution:

$$y(t) = \frac{1}{2}\,[H(t-\pi) - H(t-2\pi)]\, sin(2t)$$

Numerical solution and analytical solutions provided by Python's script (example7.py):



Phase plate:



Note: this example shows that impulse load could be using to generate of vibration and to damper it.

4.8 Example8

Considering the following third order ODE (Type 0a)

$$\begin{cases} y''' + 2y'' + 2y' = \delta(t) \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \\ y_2 = y''(0) = 0 \end{cases} \Rightarrow IVP(\{1,2,2,0\}, \delta(t), t_0 = 0, y_0 = \{0,0,0\})$$

$$\Rightarrow$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}, \{d\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \Rightarrow$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}, A^{-1}\{d\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \Rightarrow$$

$$\{z\}_0 = \{y\}_0 + [A]^{-1}\{d\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

These 2 folloing system are equal by solution

$$\begin{cases} y''' + 2y'' + 2y' = \delta(t) \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \\ y_2 = y''(0) = 0 \end{cases} \equiv \begin{cases} y''' + 2y'' + 2y' = 0 \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \\ y_2 = y''(0) = 1 \end{cases}$$

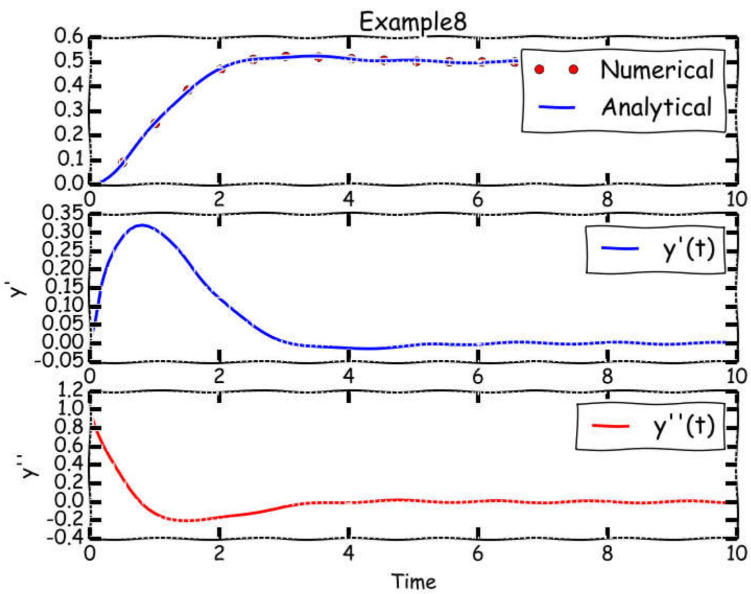
In short form:

$$IVP(\{1, 2, 2, 0\}, \delta(t), t_0 = 0, y_0 = \{0, 0, 0\})$$
$$\equiv IVP(\{1, 2, 2, 0\}, 0, t_0 = 0, y_0 = \{0, 0, 1\})$$

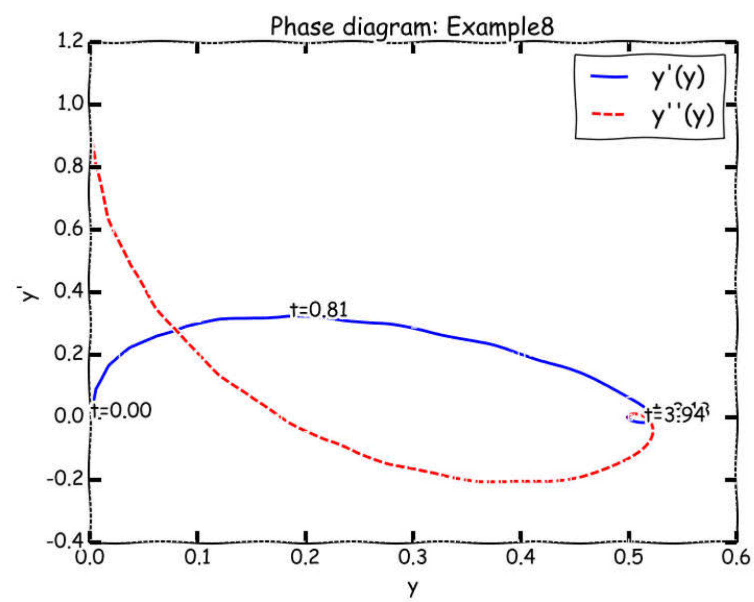
Analitycal solution:

$$y(t) = \frac{1}{2} - \frac{1}{2}\, e^{-t}\, (sin(t) + cos(t))$$

Numerical solution and analytical solutions provided by Python's script (example8.py):



Phase diagram:



Apendix A

General formulas

Unit impulse function (Dirac's function)

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a)$$

$$\int_{-\infty}^{\infty} \delta(x-a) dx = \int_{-\infty}^{\infty} \delta(x) dx = 1$$

$$\int_{-\infty}^{\infty} f(x) \delta^{(n)}(x-a) dx = (-1)^n f^{(n)}(a)$$

$$\int_{-\infty}^{\infty} f(x) \delta'(x-a) dx = -f'(a)$$

$$\delta(-x) = \delta(x)$$

$$\delta(ax) = \frac{1}{|a|} \delta(x), a \neq 0$$

$$x\delta(x) = 0$$

Laplas transform

$$LT\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t) dt = F(s)$$

$$LT\{\delta(t-t_0)\} = e^{-st_0}$$

$$LT\{\delta(t)\} = 1$$

$$LT\{f'\} = sL\{f\} - f(0)$$

$$LT\{f''\} = s^2 L\{f\} - sf(0) - f'(0)$$

$$\begin{aligned} LT\{f^{(n)}\} &= s^n LT\{f\} - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s^0 y^{(n-1)}(0) \\ &= s^n LT\{f\} - \sum_{i=0}^{n-1} s^i f^{(n-1-i)}(0) \end{aligned}$$

Apendix B

Examples

Example1 [Oliveira and Cortes, p.3]

$$\begin{cases} y'' + ay' = \delta(t), \\ y(0) = y'(0) = 0 \end{cases}$$

Solution:

$$y(t) = \frac{1}{a}\left(1 - e^{-at}\right)$$

Example2 [Finan, pp.57-58]

$$\begin{cases} 2y'' + 4y' + 10y = \delta(t), \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \end{cases}$$

Solution (! there was a typo at original book, here’s the proper formula which we’ve checked by MathCad14 and WolframAlfa!):

$$y(t) = \frac{1}{4}\, e^{-t} sin(2t)$$

Example3 [Nagy, p.189]

$$\begin{cases} y'' + 2y' + 2y = \delta(t), \\ y(0) = y'(0) = 0 \end{cases}$$

Solution:

$$y(t) = e^{-t} sin(t)$$

Example4 [Nagy, p.189]

$$\begin{cases} y'' + 2y' + 2y = \delta(t - c), \\ y(0) = y'(0) = 0 \end{cases}$$

Solution:

$$y(t) = H(t - c)e^{-(t-c)} sin(t - c)$$

Example5 [Zill, p.293]

$$\begin{cases} y'' + y = 4\delta(t - 2\pi), \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \end{cases}$$

Solution:

$$y(t) = \begin{cases} 0, & 0 \le t < 2\pi \\ 4sin(t), & t \ge 2\pi \end{cases}$$

Example6 [Zill, p.293]

$$\begin{cases} y'' + y = 4\delta(t - 2\pi), \\ y_0 = y(0) = 1 \\ y_1 = y'(0) = 0 \end{cases}$$

Solution:

$$y(t) = cos(t) + 4H(t, 2\pi)sin(t)$$

Example7 [Nagy p.190]

$$\begin{cases} y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \\ y(0) = y'(0) = 0 \end{cases}$$

Solution:

$$y(t) = \frac{1}{2}\,[H(t - \pi) - H(t - 2\pi)]\,sin(2t)$$

Example8

$$\begin{cases} y''' + 2y'' + 2y = \delta(t), \\ y(0) = y'(0) = y''(0) = 0 \end{cases}$$

Solution, getting from WolframAlfa

$$y(t) = \frac{1}{2} - \frac{1}{2}\,e^{-t}\,(sin(t) + cos(t))$$

REFERENCES

[1] Balachandran B., Magrab Edward B. VIBRATIONS, SECOND EDITION
©2009 Cengage Learning ISBN-13: 978-0-534-55206-0 ISBN-10: 0-534-55206-4

[2] M. Benchohra, J. Henderson, and S. Ntouyas
Impulsive Differential Equations and Inclusions
© 2006 Hindawi Publishing Corporation ISBN 977-5945-50-X

[3] Bottega William J. Engineering Vibrations
© 2006 by Taylor & Francis Group, LLC ISBN-13: 978-1-4200-2009-0

[4] De Silva, Clarence W. Vibration : fundamentals and practice
© 2000 by CRC Press LLC ISBN 0-8493-1808-4

[5] Finan Marcel B. Laplace Transforms: Theory, Problems, and Solutions
Arkansas Tech University

[6] Genta Giancarlo Vibration Dynamics and Control
© Springer Science+Business Media, LLC 2009 ISBN: 978-0-387-79579-9

[7] KELLY S. GRAHAM Mechanical Vibrations. THEORY AND APPLICATIONS
© 2012Cengage Learning ISBN-13: 978-1-4390-6212-8 ISBN-10: 1-4390-6212-9

[8] NAGY GABRIEL ORDINARY DIFFERENTIAL EQUATIONS
Mathematics Department, Michigan State University, East Lansing, MI, 48824.

[9] Ogata Katsuhiko Modern Control Engineering (5th edition)
Copyright © 2010, 2002, 1997, 1990, 1970 Pearson Education, Inc.
ISBN 10: 0-13-615673-8 ISBN 13: 978-0-13-615673-4

[10] Mauricio de Oliveira and Jorge Cortes
Computing the Impulse Response, January 24, 2011
http://control.ucsd.edu/mauricio/courses/mae143a/lectures/computingImpulseResponse.pdf

[11] Rao Singiresu S. Mechanical Vibrations (5th Edition)
ISBN-13: 978-0132128193 ISBN-10: 0132128195

[12] Weber, Arfken Essential Mathematical Methods for Physicists
Academic Press, 2003, ISBN: 0-12-059877-9

[13] ZILL DENNIS G. A FIRST COURSE IN DIFFERENTIAL EQUATIONS with Modeling Applications (9th edition)
Loyola Marymount University © 2009, 2005 Brooks/Cole, Cengage Learning
ISBN-13: 978-0-495-10824-5 ISBN-10: 0-495-10824-3