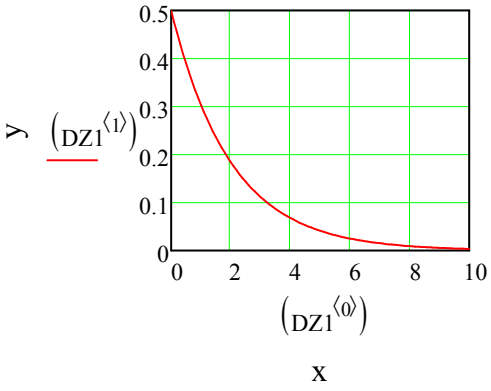
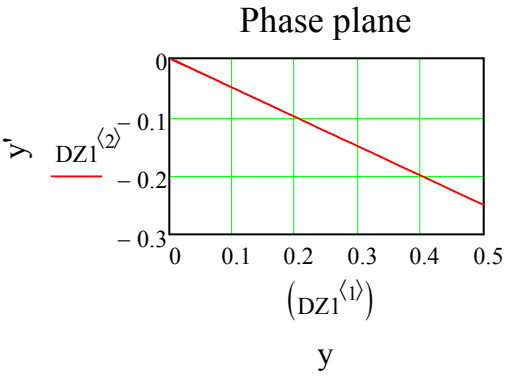
```
mkEx1(t0, tb, N) :=
    T ← 2
    a ← (T, 1)
    D(x, y) ← (−a_1 / a_0) · y_0
    IC_0 ← 0.5
    Z ← mkZ(IC, t0, tb, N, D)
    return Z
```

```
mkZ(y, t0, tk, N, D) :=
    Z0 ← rkfixed(y, t0, tk, N, D)
    n ← rows(Z0) − 1
    for i ∈ 0..n
        cy_0 ← (Z0^⟨1⟩)_i
        cy_−i ← D[(Z0^⟨0⟩)_i, cy]
    return augment(Z0, cy_)
```

$DZ1 := mkEx1(0, 10, 100)$






```
mkMatrA(va) :=
    n ← rows(va)
    for i ∈ 1..n − 1
        for j ∈ 0..i − 1
            r_{i−1, j} ← va_{i−1−j}
    1
    return r
```

```
mkVecD_Type1(b, va) :=
    n ← rows(va) − 1
    r_{n−1} ← b
    return r
```

$$mkMatrA\left[(2 \quad 4 \quad 10)^T\right] = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}$$

$$mkVecD\_Type1\left[1, (2 \quad 4 \quad 10)^T\right] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

```
mk_ΔIC_Type1(va, b) :=
    A ← mkMatrA(va)
    d ← mkVecD_Type1(b, va)
    r ← lsolve(A, d)
    return r
```

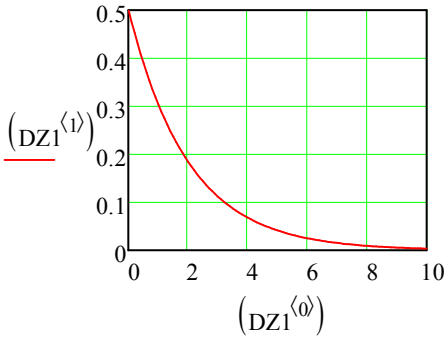
$$mk\_\Delta IC\_Type1\left[(2 \quad 4 \quad 10)^T, 1\right] = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$$

$$mk\_\Delta IC\_Type1\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}, 1\right] = (0.5)$$

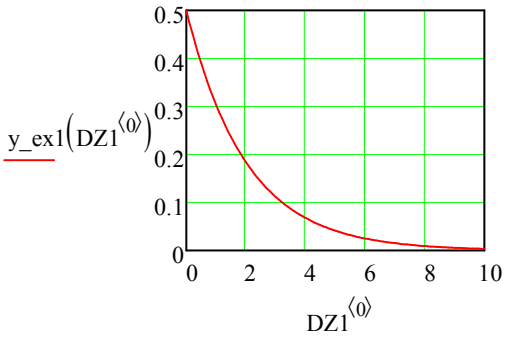
$T\_ex1 := 2$

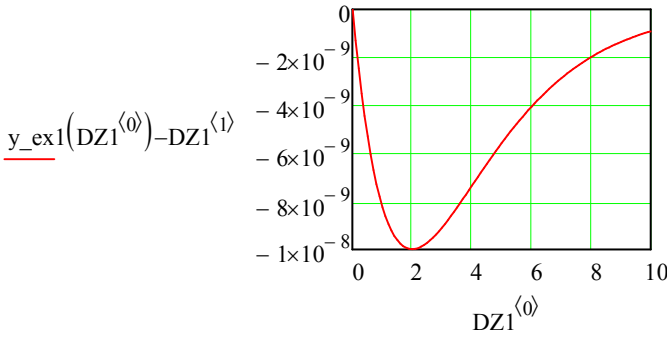
$mkEx1\_2(t0, tb, N) :=$
$zeroIC \leftarrow 0$
$va \leftarrow (T\_ex1 \quad 1)^T$
$b \leftarrow 1$
$\Delta IC \leftarrow mk\_\Delta IC\_Type1(va, b)$
$IC \leftarrow zeroIC + \Delta IC$
$Diff(x, y) \leftarrow \frac{-va_1}{va_0} y_0$
$Z \leftarrow rkfixed(IC, t0, tb, N, Diff)$
return Z

$DZ1 := mkEx1\_2(0, 10, 100)$



analitycal solution $\quad y\_ex1(t) := \frac{1}{T\_ex1} \cdot \exp\left(\frac{-t}{T\_ex1}\right)$





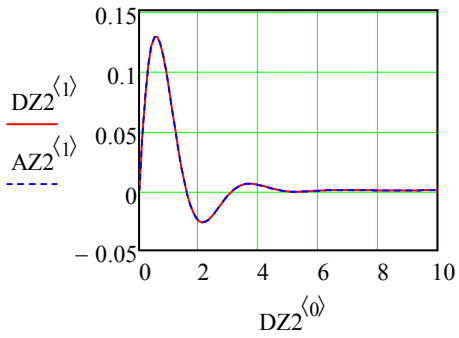
$mkEx2(t0, tb, N) :=$
$zeroIC \leftarrow (0 \quad 0)^T$
$va \leftarrow (2 \quad 4 \quad 10)^T$
$b \leftarrow 1$
$\Delta IC \leftarrow mk\_\Delta IC\_Type1(va, b)$
$IC \leftarrow zeroIC + \Delta IC$
$Diff(x, y) \leftarrow \begin{pmatrix} y_1 \\ \frac{-va_1 \cdot y_1}{va_0} + \frac{-va_2 \cdot y_0}{va_0} \end{pmatrix}$
$Z \leftarrow rkfixed(IC, t0, tb, N, Diff)$
return Z

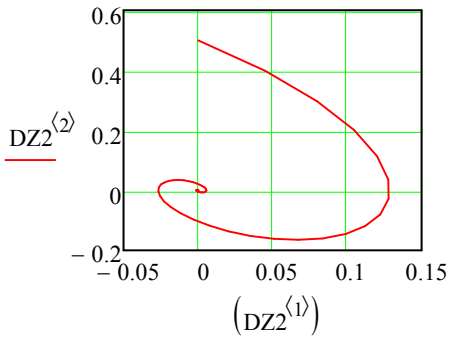
$\frac{1}{4} \cdot \exp(-t) \cdot \sin(2 \cdot t)$

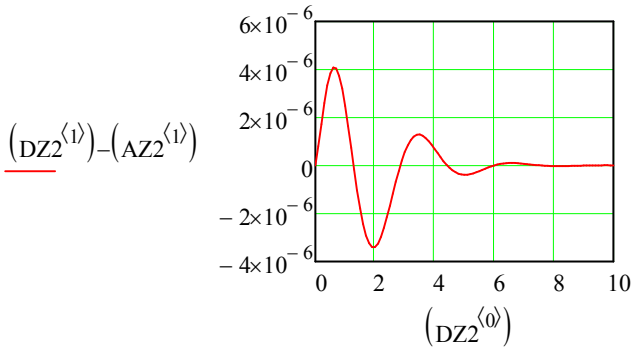
analitycal solution

$DZ2 := mkEx2(0, 10, 100)$

$AZ2 :=$
$r^{\langle 0 \rangle} \leftarrow DZ2^{\langle 0 \rangle}$
$r^{\langle 1 \rangle} \leftarrow \overrightarrow{\left(\frac{1}{4} \cdot \exp\left(-r^{\langle 0 \rangle}\right) \cdot \sin\left(2 \cdot r^{\langle 0 \rangle}\right)\right)}$
return r







$$\frac{1}{\left(2\cdot s^2 + 4\cdot s + 10\right)}\ \text{invlaplace}\ \rightarrow \frac{\sin(2\cdot t)\cdot e^{-t}}{4}$$

$$\begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 0.5 & 0 \\ -1 & 0.5 \end{pmatrix}$$

$$\text{mkIC\_Type1}(va, b, IC0) := \left| \begin{array}{l} \Delta IC \leftarrow \text{mk\_}\Delta\text{IC\_Type1}(va, b) \\ IC \leftarrow IC0 + \Delta IC \\ \text{return } IC \end{array} \right.$$

$$\frac{1}{\left(s^3 + 2\cdot s^2 + 2\cdot s\right)}\ \text{invlaplace}\ \rightarrow \frac{1}{2} - \frac{e^{-t}\cdot\sin(t)}{2} - \frac{e^{-t}\cdot\cos(t)}{2}$$

$$\text{mkSlnT1\_a}(va, b, IC0, t0, tb, N) := \left| \begin{array}{l} IC \leftarrow \text{mkIC\_Type1}(va, b, IC0) \\ D \leftarrow \text{mkDiff\_Type1}(va) \\ Z \leftarrow \text{rkfixed}(IC, t0, tb, N, D) \\ \text{return } Z \end{array} \right.$$

$$\text{mkEx3}(t0, tb, N) := \left| \begin{array}{l} \text{zeroIC} \leftarrow (0 \quad 0)^T \\ va \leftarrow (1 \quad 2 \quad 2)^T \\ b \leftarrow 1 \\ Z \leftarrow \text{mkSlnT1\_a}(va, b, \text{zeroIC}, t0, tb, N) \\ \text{return } Z \end{array} \right.$$

analitycal solution

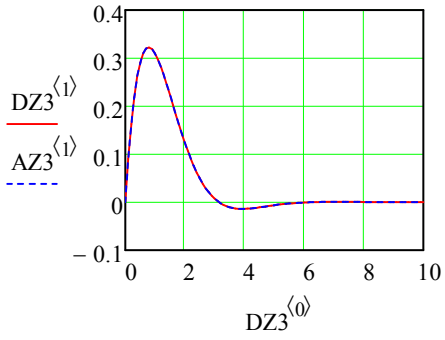
$$\frac{1}{\left(1\cdot s^2 + 2\cdot s + 2\right)}\ \text{invlaplace}\ \rightarrow e^{-t}\cdot\sin(t)$$

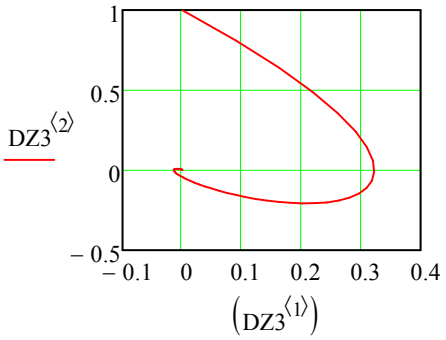
$$DZ3 := \text{mkEx3}(0, 10, 100)$$

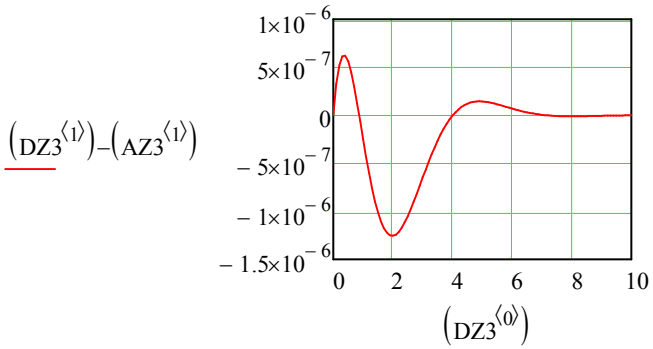
$$AZ3 := \left| \begin{array}{l} r^{\langle 0\rangle} \leftarrow DZ2^{\langle 0\rangle} \\ r^{\langle 1\rangle} \leftarrow \overrightarrow{\left(\frac{1}{1}\cdot\exp\left(-r^{\langle 0\rangle}\right)\cdot\sin\left(r^{\langle 0\rangle}\right)\right)} \\ \text{return } r \end{array} \right.$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$








```
extractIC(Z) := | n ← rows(Z)
                | lastRow ← submatrix(Z, n − 1, n − 1, 1, cols(Z) − 1)
                | lastRow

lastRow(Z) := | n ← rows(Z)
              | lastRow ← submatrix(Z, n − 1, n − 1, 0, cols(Z) − 1)
              | lastRow

mkZrowByIC(t, IC) := stack(t, IC)^T

mkSlnT1_b(va, b, c, IC0, t0, tb, N) := | if t0 = tb
                                       |     | 1
                                       |     | return mkZrowByIC(t0, IC0)
                                       | Diff ← mkDiff_Type1(va)
                                       | if c < t0 ∨ c ≥ tb
                                       |     | Z ← rkfixed(IC0, t0, tb, N, Diff)
                                       |     | return Z
                                       | if t0 = c
                                       |     | Z ← mkSlnT1_a(va, b, IC0, t0, tb, N)
                                       |     | return Z
                                       | Z1 ← rkfixed(IC0, t0, c, N, Diff)
                                       | Z2 ← | IC1_ ← extractIC(Z1)
                                       |      | ΔIC ← mk_ΔIC_Type1(va, b)
                                       |      | IC1 ← IC1_^T + ΔIC·1
                                       |      | Z2 ← mkZrowByIC(c, IC1)  if c = tb
                                       |      | Z2 ← rkfixed(IC1, c, tb, N, Diff)  otherwise
                                       | Z ← stack(Z1, Z2)
                                       | return Z
```

$$\text{Hev}(t) := \begin{cases} \text{return } 0 & \text{if } t \le 0 \\ \text{return } 1 & \text{otherwise} \end{cases}$$

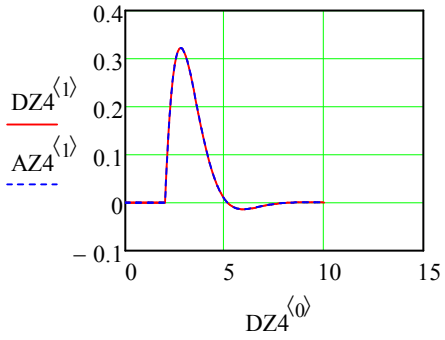
$$\text{mkEx4}(t0, tb, N, c) := \begin{cases} \text{zeroIC} \leftarrow (0 \quad 0)^T \\ \text{va} \leftarrow (1 \quad 2 \quad 2)^T \\ b \leftarrow 1 \\ Z \leftarrow \text{mkSlnT1\_b}(\text{va}, b, c, \text{zeroIC}, t0, tb, N) \\ Z \end{cases}$$

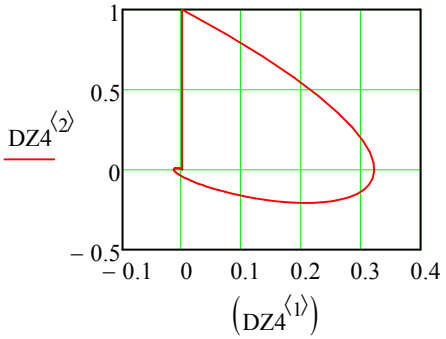
$$c\_ex4 := 2$$

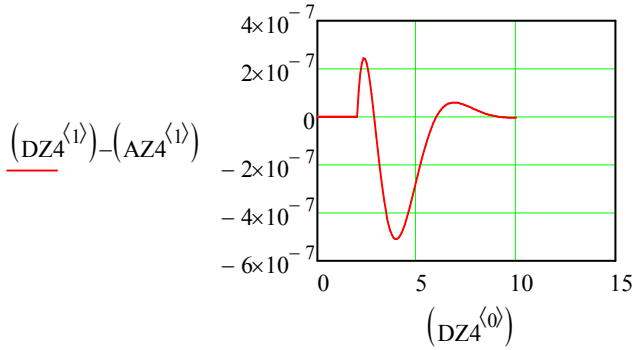
$$\text{y\_ex4}(t) := \text{Hev}(t - c\_ex4) \cdot \exp[-(t - c\_ex4)] \cdot \sin(t - c\_ex4)$$

$$DZ4 := \text{mkEx4}(0, 10, 100, c\_ex4)$$

$$AZ4 := \begin{cases} r^{\langle 0 \rangle} \leftarrow DZ4^{\langle 0 \rangle} \\ \text{for } j \in 0 .. \text{rows}\left(r^{\langle 0 \rangle}\right) - 1 \\ \quad r_{j,1} \leftarrow \text{y\_ex4}\left(r_{j,0}\right) \\ \text{return } r \end{cases}$$



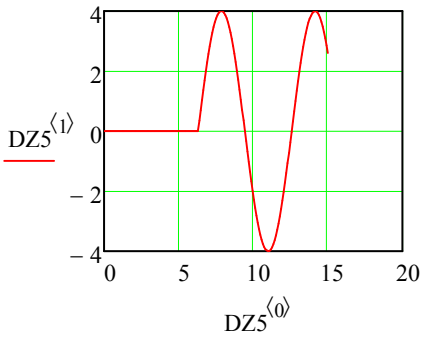


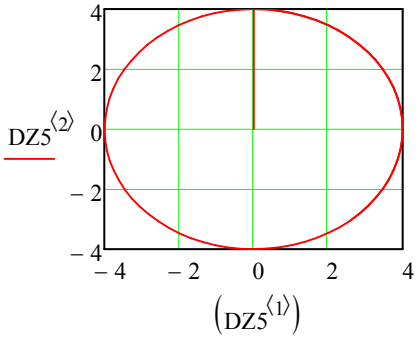


$$\text{mkEx5}(t0, tb, N, c) := \begin{cases} \text{zeroIC} \leftarrow (0 \quad 0)^T \\ \text{va} \leftarrow (1 \quad 0 \quad 1)^T \\ b \leftarrow 4 \\ Z \leftarrow \text{mkSlnT1\_b}(\text{va}, b, c, \text{zeroIC}, t0, tb, N) \\ Z \end{cases}$$

$$c\_ex5 := 2 \cdot \pi$$

$$DZ5 := \text{mkEx5}(0, 15, 100, c\_ex5)$$

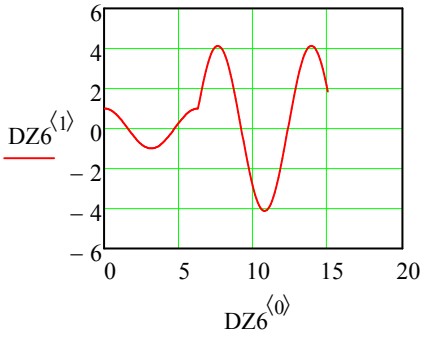


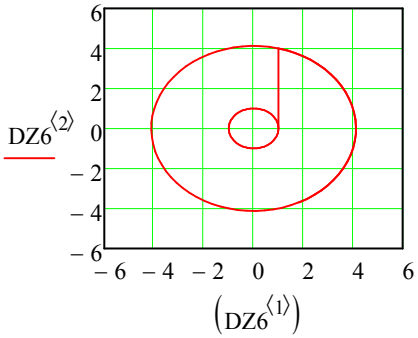


reduce_v

mkEx6(t0, tb, N, c) :=
- zeroIC ← (1 0)^T
- va ← (1 0 1)^T
- b ← 4
- Z ← mkSlnT1_b(va, b, c, zeroIC, t0, tb, N)
- Z

DZ6 := mkEx6(0, 15, 100, c_ex5)





mkReducedVcVb(vc, vb) :=
- tuple_vc_vb ← augment(vc, vb)
- sv ← csort(tuple_vc_vb, 0)
- redR ← reduce_vcb(sv)
- redVc ← redR⟨0⟩
- redVb ← redR⟨1⟩
- R_0 ← redVc, R_1 ← redVb
- R

mkReducedVcVb(0, 0) = ({1,1} / {1,1})

augment(0, 0) = (0 0)

mapZ(vZ) :=
- n ← rows(vZ)
- R ← vZ_0
- if n > 1
  - 1
  - for i ∈ 1 .. n − 1
    - R ← stack(R, vZ_i)
    - R
- R

$$\text{mkSlnT1\_c}(va, vb, vc, IC0, t0, tb, N) :=$$

$rVcVb \leftarrow \text{mkReducedVcVb}(vc, vb)$

$redVc \leftarrow rVcVb_0,\ redVb \leftarrow rVcVb_1$

$IC_0 \leftarrow IC0$

$Z_0 \leftarrow \text{mkSlnT1\_b}\left(va, redVb_0, redVc_0, IC_0, t0, redVc_0, N\right)$

$iLast \leftarrow \text{rows}(redVb) - 2$

if $iLast \geq 0$

  for $i \in 0..iLast$

    $Z_{i+1} \leftarrow$

      $IC_{-i} \leftarrow \text{extractIC}\left(Z_i\right)$

      $\Delta IC_i \leftarrow \text{mk\_}\Delta\text{IC\_Type1}\left(va, redVb_i\right)$

      $IC_{i+1} \leftarrow IC_{-i}^{T} + \Delta IC_i \cdot 0$

      $r \leftarrow \text{mkSlnT1\_b}\left(va, redVb_i, redVc_i, IC_{i+1}, redVc_i, redVc_{i+1}\right)$

  1

$n \leftarrow \text{rows}(Z)$

$Z_n \leftarrow$

  $IC_{-n-1} \leftarrow \text{extractIC}\left(Z_{n-1}\right)$

  $\Delta IC_n \leftarrow \text{mk\_}\Delta\text{IC\_Type1}\left(va, redVb_{n-1}\right)$

  $IC_n \leftarrow IC_{-n-1}^{T}$ if $n = 1$

  $IC_n \leftarrow IC_{-n-1}^{T} + \Delta IC_n \cdot 0$ otherwise

  $r \leftarrow \text{mkSlnT1\_b}\left(va, redVb_{n-1}, redVc_{n-1}, IC_n, redVc_{n-1}, tb, N\right)$

$R \leftarrow \text{mapZ}(Z)$

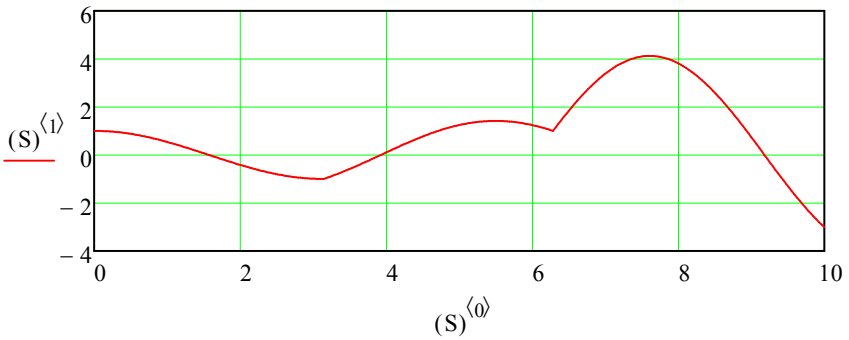
R

$$M := \text{mkSlnT1\_c}\left[\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}2\cdot\pi\\\pi\\2\pi\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, 0, 10,\right.$$

$zeroIC \leftarrow (1 \quad 0)^T \qquad va \leftarrow (1 \quad 0 \quad 1)^T$

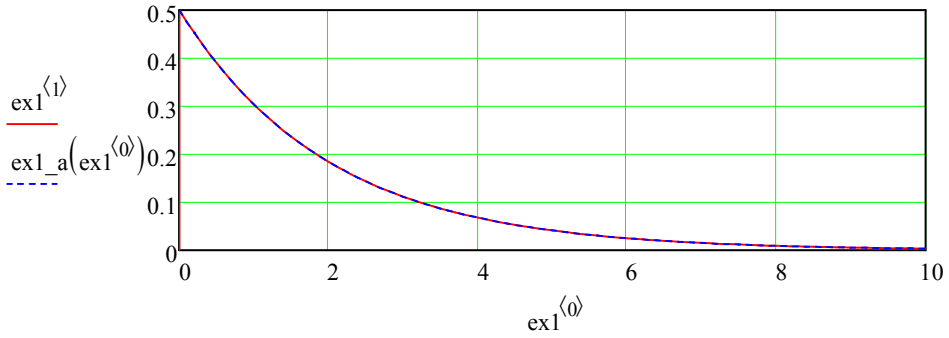
$$M1 := \text{mkSlnT1\_c}\left[\begin{pmatrix}1\\0\\1\end{pmatrix}, 0, 0, \begin{pmatrix}0\\0\end{pmatrix}, 0, 10, 100\right]$$

$S := M$

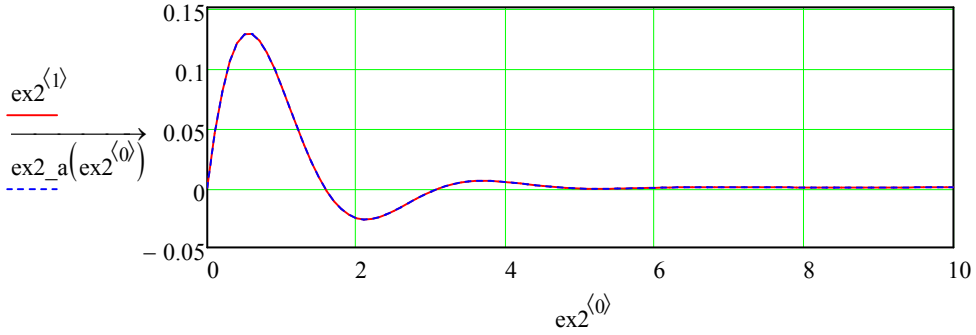


Recheck all examples with mkSlnT1_c

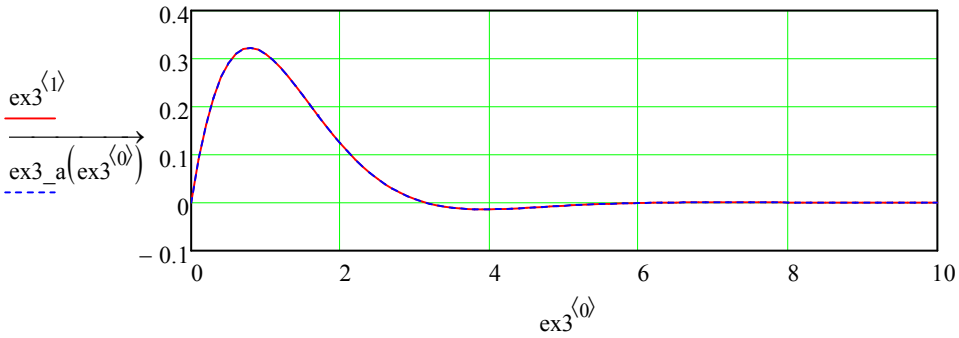
$$ex1 := mkSlnT1\_c\left[\begin{pmatrix}2\\1\end{pmatrix}, 1, 0, 0, 0, 10, 100\right] \qquad ex1\_a(t) := \frac{1}{2}\cdot e^{\frac{-t}{2}}$$



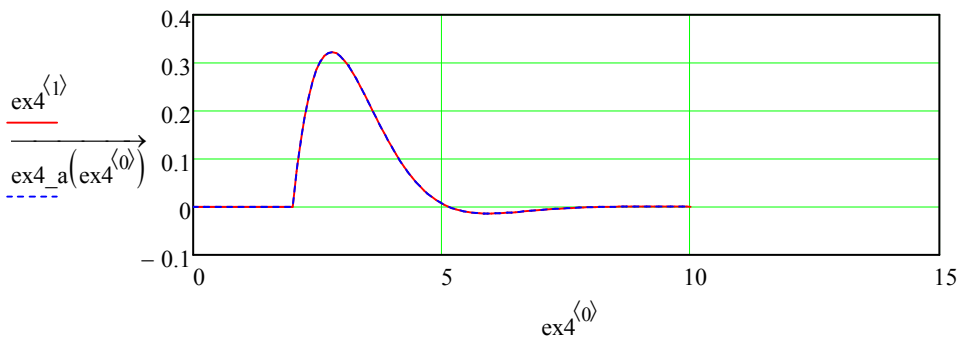
$$ex2 := mkSlnT1\_c\left[\begin{pmatrix}2\\4\\10\end{pmatrix}, 1, 0, \begin{pmatrix}0\\0\end{pmatrix}, 0, 10, 100\right] \qquad ex2\_a(t) := \frac{1}{4}\cdot e^{-t}\cdot\sin(2\cdot t)$$



$$ex3 := mkSlnT1\_c\left[\begin{pmatrix}1\\2\\2\end{pmatrix}, 1, 0, \begin{pmatrix}0\\0\end{pmatrix}, 0, 10, 100\right] \qquad ex3\_a(t) := e^{-t}\cdot\sin(t)$$

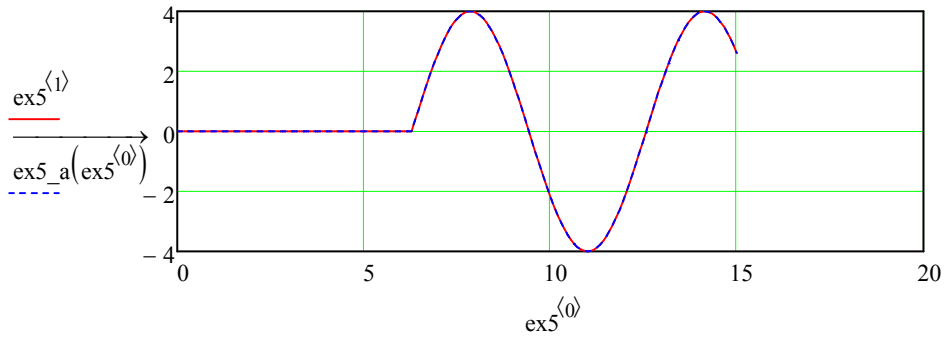


$$ex4 := mkSlnT1\_c\left[\begin{pmatrix}1\\2\\2\end{pmatrix}, 1, 2, \begin{pmatrix}0\\0\end{pmatrix}, 0, 10, 100\right] \qquad ex4\_a(t) := Hev(t-2)\cdot e^{-(t-2)}\cdot\sin(t-2)$$



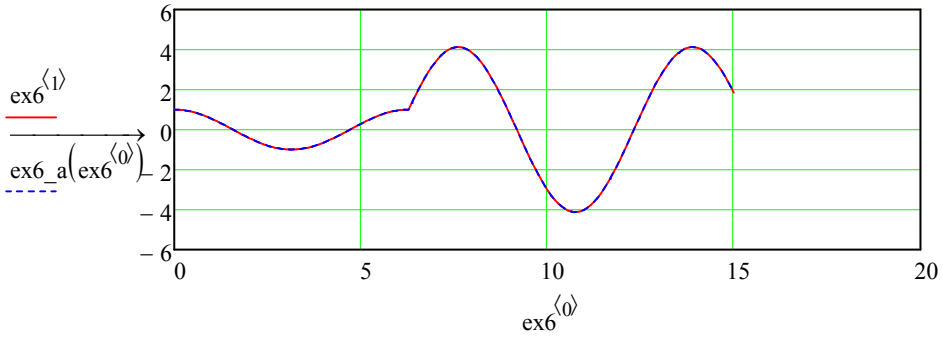
$$ex5 := mkSlnT1\_c\left[\begin{pmatrix}1\\0\\1\end{pmatrix}, 4, 2\cdot\pi, \begin{pmatrix}0\\0\end{pmatrix}, 0, 15, 100\right]$$

$$ex5\_a(t) := Hev(t - 2\cdot\pi)\cdot 4\cdot\sin(t)$$



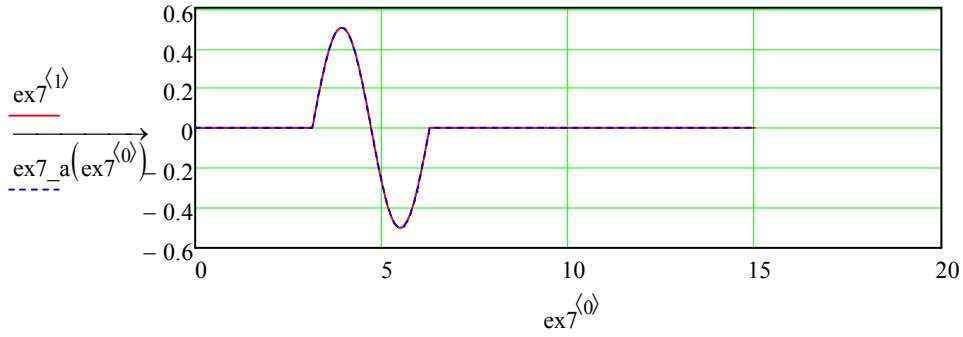
$$ex6 := mkSlnT1\_c\left[\begin{pmatrix}1\\0\\1\end{pmatrix}, 4, 2\cdot\pi, \begin{pmatrix}1\\0\end{pmatrix}, 0, 15, 100\right]$$

$$ex6\_a(t) := \cos(t) + Hev(t - 2\cdot\pi)\cdot 4\cdot\sin(t)$$

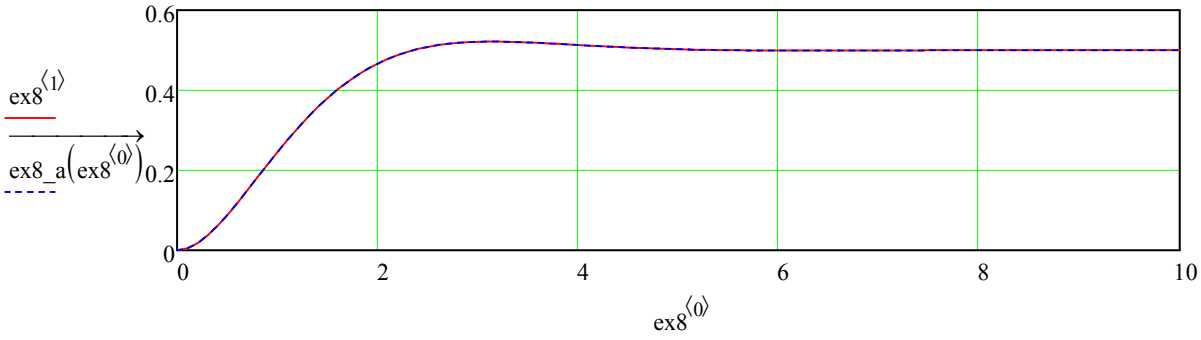


$$ex7 := mkSlnT1\_c\left[\begin{pmatrix}1\\0\\4\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}\pi\\2\cdot\pi\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, 0, 15, 100\right]$$

$$ex7\_a(t) := \frac{1}{2}\cdot(Hev(t - 1\cdot\pi) - Hev(t - 2\cdot\tau$$



$$ex8 := mkSlnT1\_c\left[\begin{pmatrix}1\\2\\2\\0\end{pmatrix}, 1, 0, \begin{pmatrix}0\\0\\0\end{pmatrix}, 0, 10, 100\right] \qquad ex8\_a(t) := \frac{1}{2} - \frac{1}{2}\cdot e^{-t}\cdot(\sin(t) + \cos(t))$$



$$mkMatrA\left(\begin{pmatrix}1\\2\\2\\0\end{pmatrix}\right) = \begin{pmatrix}1 & 0 & 0\\2 & 1 & 0\\2 & 2 & 1\end{pmatrix} \qquad mkVecD\_Type1\left[1, \begin{pmatrix}1\\2\\2\\0\end{pmatrix}\right] = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$mk\_\Delta IC\_Type1\left[\begin{pmatrix}1\\2\\2\\0\end{pmatrix}, 1\right] = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$mkMatrA\left(\begin{pmatrix}1\\2\\2\\0\end{pmatrix}\right)^{-1} = \begin{pmatrix}1 & 0 & 0\\-2 & 1 & 0\\2 & -2 & 1\end{pmatrix}$$

$$mkMatrA\left(\begin{pmatrix}1\\2\\2\\0\end{pmatrix}\right)^{-1}\cdot mkVecD\_Type1\left[1, \begin{pmatrix}1\\2\\2\\0\end{pmatrix}\right] = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\text{mkDiff\_Type1}(va) := \left| \begin{array}{l} \text{Diff1}(x,y) \leftarrow \dfrac{-va_1}{va_0}\, y_0 \\ n \leftarrow \text{rows}(va) \\ \text{return Diff1 if } n = 2 \\ \text{Diff2}(x,y) \leftarrow \begin{bmatrix} y_1 \\ \left| \begin{array}{l} \text{iMax} \leftarrow n - 2 \\ -\displaystyle\sum_{i=0}^{\text{iMax}} \dfrac{y_i \cdot va_{(\text{iMax}+1)-i}}{va_0} \end{array} \right. \end{bmatrix} \\ \text{return Diff2 if } n = 3 \\ \text{Diff3}(x,y) \leftarrow \begin{bmatrix} y_1 \\ y_2 \\ \left| \begin{array}{l} \text{iMax} \leftarrow n - 2 \\ -\displaystyle\sum_{i=0}^{\text{iMax}} \dfrac{y_i \cdot va_{(\text{iMax}+1)-i}}{va_0} \end{array} \right. \end{bmatrix} \\ \text{Diff3} \end{array} \right.$$

$$\text{Diff3}(x, y) \leftarrow \left| \begin{array}{l} \text{for } i \in 0 .. n - 2 \\ \quad r_i \leftarrow y_{i+1} \\ iMax \leftarrow n - 2 \\ r_{n-1} \leftarrow -\sum_{i=0}^{iMax} \frac{y_i \cdot va_{(iMax+1)-i}}{va_0} \\ r \end{array} \right.$$

DZ4 =

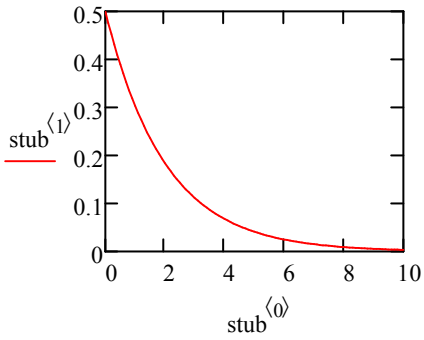
	0	1	2
0	0	0	0
1	0.02	0	0
2	0.04	0	0
3	0.06	0	0
4	0.08	0	0
5	0.1	0	0
6	0.12	0	0
7	0.14	0	0
8	0.16	0	0
9	0.18	0	0
10	0.2	0	0
11	0.22	0	0
12	0.24	0	0
13	0.26	0	0
14	0.28	0	0
15	0.3	0	...

```
rcb(vcb) := | n ← rows(vcb)
            | j ← 0
            | r_{j,0} ← vcb_{j,0}
            | r_{j,1} ← vcb_{j,1}
            | if n = 1
            |     | 1
            |     | return r
            | for i ∈ 1..n − 1
            |     | r_{j,1} ← r_{j,1} + vcb_{i,1}  if r_{j,0} = vcb_{i,0}
            |     | otherwise
            |     |     | j ← j + 1
            |     |     | r_{j,0} ← vcb_{j,0}
            |     |     | r_{j,1} ← vcb_{j,1}
            | 1
            | return r
```

$$stub\_ := mkSlnT1\_a\left[\begin{pmatrix}2\\1\end{pmatrix}, 1, 0, 0, 10, 100\right]$$

$$stub := mkSlnT1\_b\left[\begin{pmatrix}2\\1\end{pmatrix}, 1, 0, 0, 0, 10, 100\right]$$

,N)

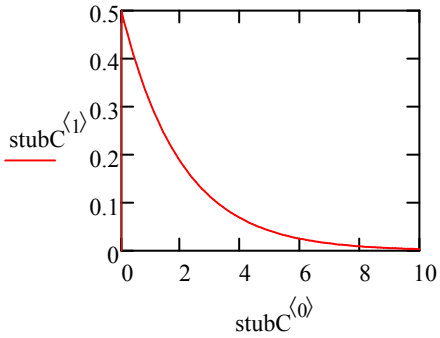


stub =

	0	1
0	0	0.5
1	0.1	0.476
2	0.2	0.452
3	0.3	0.43
4	0.4	0.409
5	0.5	0.389
6	0.6	0.37
7	0.7	0.352
8	0.8	0.335
9	0.9	0.319
10	1	0.303
11	1.1	0.288
12	1.2	0.274
13	1.3	0.261

$$stubC := mkSlnT1\_c\left[\begin{pmatrix}2\\1\end{pmatrix}, 1, 0, 0, 0, 10, 100\right]$$

100]



stubC =

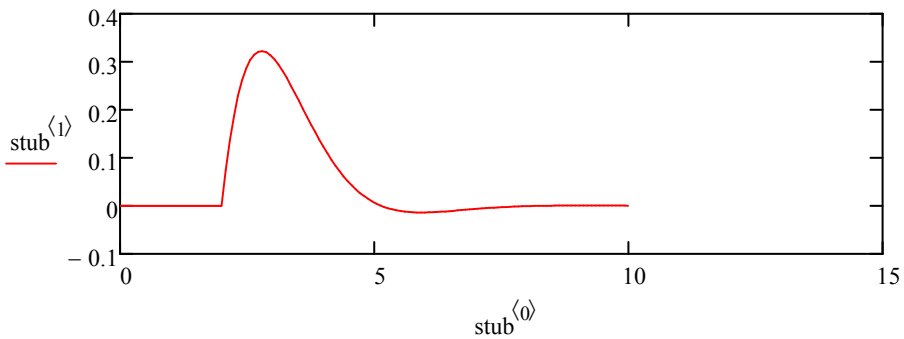
	0	1
0	0	0
1	0	0.5
2	0.1	0.476
3	0.2	0.452
4	0.3	0.43
5	0.4	0.409
6	0.5	0.389
7	0.6	0.37
8	0.7	0.352
9	0.8	0.335
10	0.9	0.319
11	1	0.303
12	1.1	0.288
13	1.2	0.274
14	1.3	0.261
15	1.4	...

M =

	0	1	2
0	0	1	0
1	0.031	1	-0.031
2	0.063	0.998	-0.063
3	0.094	0.996	-0.094
4	0.126	0.992	-0.125
5	0.157	0.988	-0.156
6	0.188	0.982	-0.187
7	0.22	0.976	-0.218
8	0.251	0.969	-0.249
9	0.283	0.96	-0.279
10	0.314	0.951	-0.309
11	0.346	0.941	-0.339
12	0.377	0.93	-0.368
13	0.408	0.918	-0.397
14	0.44	0.905	-0.426
15	0.471	0.891	

15	0.471	0.691	...

$$\text{stub} := \text{mkSlnT1\_b}\left[\begin{pmatrix}1\\2\\2\end{pmatrix}, 1, 2, \begin{pmatrix}0\\0\end{pmatrix}, 0, 10, 100\right]$$



τ))·sin(2·t)