
Simplicial Neural Networks

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Abstract

We generalize graph convolutional neural networks to data that live on a class of topological spaces called simplicial complexes. These are natural multi-dimensional generalizations of graphs which encode more than just pairwise relationships, namely higher-order relationships between nodes (represented geometrically as filled triangles, tetrahedra, and so forth). We define an appropriate notion of convolutions for such data, which we leverage to construct the desired convolutional neural networks. This allows us to consider richer data than traditional methods, including n-fold collaboration networks and vector field data.

1 Introduction

- Graph Neural Networks intro
- Extension to simplicial complexes.
- Simplicial complex represent collaboration on which we have data like citations.
- Preliminary results on vector field data. Where we believe this method can be applied and work?
- Cite work on hypergraphs, message passing.
- Advantage our work: low computational cost, Laplacian an operator encodes topological information has been extensively used in other application eg page ranking. In higher dimension we believe it carries additional information on the simplicial complex.

2 Proposed Technique

FIXME

2.1 Simplicial Complexes

A *simplicial complex* is a collection of finite sets closed under taking subsets. We refer to a set in a simplicial complex as a *simplex* of *dimension* p if it has cardinality $p + 1$. Such a p -simplex has $p + 1$ *faces* of dimension $p - 1$, namely the sets omitting one element, which we will denote as $(v_0, \dots, \hat{v}_i, \dots, v_p)$ when omitting the i 'th element. While this definition is entirely combinatorial, we will soon see that there is a geometric interpretation, and it will make sense to refer to and think of 0-simplices as *vertices*, 1-simplices as *edges*, 2-simplices as *triangles*, 3-simplices as *tetrahedra*, and so forth.

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Let $C_p(K)$ be the free real vector space with basis K_p , the set of p -simplices in a simplicial complex K . The elements of $C_p(K)$ are called p -chains. These vector spaces come equipped with *boundary maps*, namely linear maps defined by

$$\partial_p : C_p \rightarrow C_{p-1}$$

$$\partial_p((v_0, \dots, v_p)) = \sum_{i=0}^p (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p)$$

[[Stefania says: Change above with definition of cochain. Cochain in a grid are pixels. Any diemnsion are the features of our simplices]]



Figure 1: Example collaboration complex

2.2 Combinatorial Laplacians

We are in this paper concerned with finite simplicial complexes, and assume that they are built in a way that encodes useful information about the data being studied. In particular, it is not necessary for it to come equipped with some embedding into Euclidean space, nor do we demand that it triangulates a Riemannian manifold. Therefore dualities like the Hodge star, which is used to construct the Hodge–de Rham Laplacian in the smooth setting [2] that motivates us, are unavailable for our method. The same is true for discrete versions of the Hodge star, such as that of Hirani [1]. To define a discrete version of the Laplacian for simplicial complexes, we simply take the linear adjoint of the boundary operator with respect to the inner product, defining $\partial_i^* : C_{i-1} \rightarrow C_i$ by

$$\langle \partial_i^* \sigma, \tau \rangle_i = \langle \sigma, \partial_i \tau \rangle_{i-1} \quad \forall \sigma \in C_{i-1}, \tau \in C_i.$$

In analogy with Hodge–de Rham theory, we then define the *degree- i simplicial Laplacian* of a simplicial complex K as the linear operator $\mathcal{L}_i : C_i(K) \rightarrow C_i(K)$ such that

$$\mathcal{L}_i = \mathcal{L}_i^{\text{up}} + \mathcal{L}_i^{\text{down}}$$

$$\mathcal{L}_i^{\text{up}} = \partial_{i+1} \circ \partial_{i+1}^* : C_i(K) \rightarrow C_i(K)$$

$$\mathcal{L}_i^{\text{down}} = \partial_i^* \circ \partial_i : C_i(K) \rightarrow C_i(K).$$

[[Stefania says: Write with co-boundary]] In case $p = 0$, \mathcal{L}_0 corresponds to the classical graph Laplacian. Observe that there are p Laplacians for a complex of dimension p . In most practical applications, the matrices for the Laplacians are very sparse and can easily be computed as a product of sparse boundary matrices and their transposes.

[[Stefania says: Laplacian encodes important topological information (just add references)]]
[[Stefania says: Laplacian can be seen as a message passing function on the simplices]]

Our contribution is a notion of convolution for simplicial complexes using the Laplacian.

2.3 Simplicial Neural Networks

- Goal: building a convolutional NN whose input is an arbitrary p -cochain on a fixed simplicial complex K
- Definition of Fourier Transform
- Convolutional Filters: low degree polynomials in the frequency domain
- Implemented using Chebyshev polynomials
- Computational cost: good the p -Laplacian is localized and sparse.



Figure 2: Maybe figures SNN

Table 1: Number of simplices

Dimension	0	1	2	3	4	5	6	7	8	9	10
CC1	352	1474	3285	5019	5559	4547	2732	1175	343	61	5
CC2	333	4444	333	4444	333	4444	333	4444	333	4444	3333

3 Experimental results

In this section we present experimental results for the simplicial neural network. The datasets we analyze have been extracted from the Semantic Scholar datasets. The data consists of XXX papers together with their authors and number of citations. We retain paper with more than 5 citations and at most 10 authors.

An important step in preprocessing many kinds of input data in TDA is constructing a simplicial complex. Our work focus on *collaboration complexes* (or *co-authorship complexes*) [\[\[Stefania says: cite paper Alice\]\]](#), simplicial complexes where a paper with k authors is represented by a $(k - 1)$ -simplex. We constructed different co-authorship complexes by considering sub-samplings from the papers set of the Semantic Scholar dataset. The sub-samplings were obtained by performing random walks on the nodes of the graph which vertices corresponds to the papers and edges connect papers sharing at least one author. The co-authorship complexes obtained from each sub-sampling have corresponding k co-chain given by the number of shared citations.

We evaluate the performance of the SNN on the task of predicting missed input data. Specifically, given a fixed co-authorship complex missing data is introduced at random on the training co-chains at 4 levels: 10%, 20%, 30%, and 50%. We trained a SNN composed by 3-layers with 30 convolutional filters. [\[\[Stefania says: Say how laplacian is build and size input\]\]](#) The training input is given by the citations on the co-authorship complex where the random missing data is substituted by the median of the known data. We used the L_1 norm as reconstruction loss over the known elements an the Adam optimizer with learning rate of 1×10^{-3} . [\[\[Stefania says: Clarify what is test and train\]\]](#) Our evaluation focus on the accuracy of the network in predicting the missing data. A predicted citation is considered correct if the predicted value differs of at most 1 from the actual number of citations.

Figure 3 shows the accuracy results of the prediction of the missing citations on CC1 (Co-authorship Complex 1). The number of simplices per dimension in CC1 can be seen in Table ??

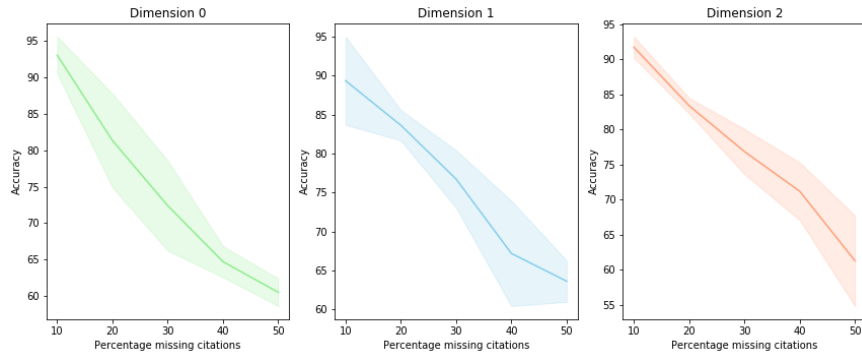


Figure 3: Accuracy of SNN in predicting missing citations

As a second assessment we evaluate how accurately a SNN pretrained on a co-authorship complex can predict citations on a different complex. The SNN with the same characteristic as above was trained on CC2 (Co-authorship Complex 2)

[[Stefania says: Write number of iterations!]]

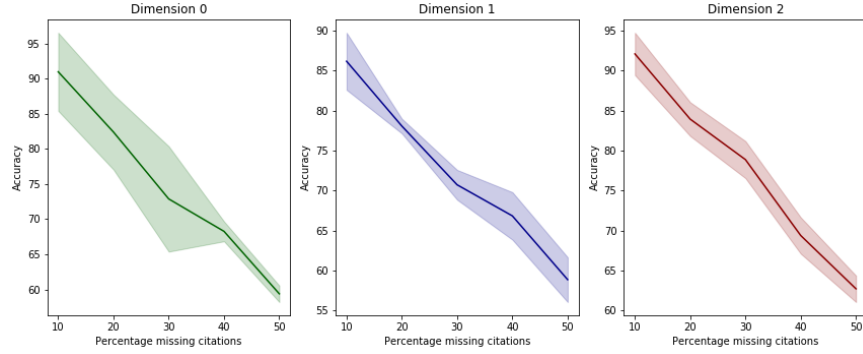


Figure 4: Accuracy in predicting missing citations with a pretrained SNN [[Stefania says: Transfer learning?]]

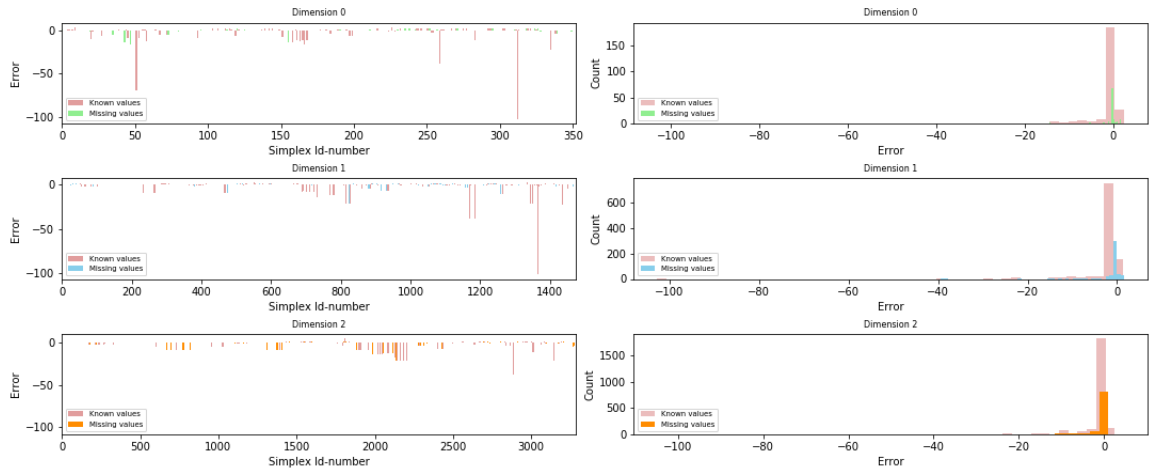


Figure 5: Distribution of the prediction's error

[[Stefania says: PLOT only error distribution in log-scale]]

4 Conclusion and Future Work

FIXME

- Define pooling
- Better baseline for the above work
- Preliminary results on vector field data. Where we believe this method can be applied?
- transfer learning accuracy is it linked to similar structure of complexes

References

- [1] Anil N. Hirani. “Discrete Exterior Calculus”. PhD thesis. California Insitute of Technology, 2003.
- [2] Ib Madsen and Jørgen Tornehave. *From calculus to cohomology: de Rham cohomology and characteristic classes*. Cambridge University Press, 1997.