School of Mechanical Aerospace and Civil Engineering

TPFE MSc Advanced Turbulence Modelling

Turbulence Lengthscales and Spectra

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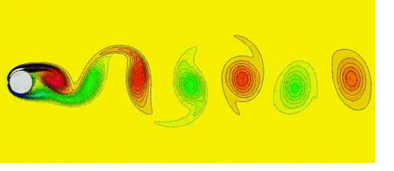
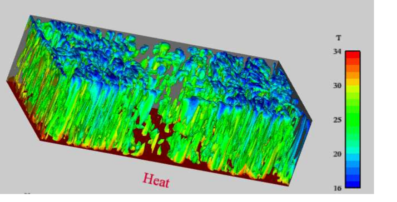
Reading:
S. Pope, *Turbulent Flows*
D. Wilcox, *Turbulence Modelling for CFD*
Closure Strategies for Turbulent and Transitional Flows, (Eds. B.E. Launder, N.D. Sandham)
Notes: Blackboard and CFD/TM web server: http://cfd.mace.manchester.ac.uk/tmcfd
- People - T. Craft - Online Teaching Material

Introduction

- RANS based schemes, studied in previous lectures, ‘average’ the effect of turbulent structures.
- They solve for the mean flow, and introduce models for the Reynolds stresses.
- None of the turbulent structures are resolved in such approaches.
- Note, however, that unsteady (URANS) can be performed, allowing large-scale unsteady structures (not strictly turbulence) to be resolved.
- Examples of such cases include vortex shedding from bluff bodies, and Raleigh-Benard type convection cells in unstably stratified flows.





- In the following lectures we consider approaches such as Large Eddy Simulation, which resolve some, or all, of the turbulent structures in a flow.
- To appreciate how these schemes work, and understand how to assess limitations, we need to to know what length and time scales are present in a turbulent flow.
- We have previously met the idea of the turbulent kinetic energy spectrum, giving information on the range of eddy sizes or lengthscales in the flow.
- Here we revisit this, to consider it and correlations between turbulent fluctuations in a more quantitative manner.

LES and DNS

- In Direct Numerical Simulation (DNS) the Navier-Stokes equations are solved with sufficient numerical resolution to represent *all* the turbulent scales accurately.
- In Large Eddy Simulations (LES) the large scales are resolved, but the effects of the smaller ones (sub-grid-scales, for example) must be modelled.
- The physical modelling input is thus less than in RANS schemes, and relatively simple, since most of the turbulence energy is contained in the larger scales.
- However, one does need to ensure the appropriate scales have actually been resolved.
- One thus needs to know what scales should be present in a flow, and be able to assess what has actually been resolved in a simulation.
- Recognising the effects of not resolving certain scales can also be important.

Spatial Correlations

- For simplicity, we begin by considering a homogeneous turbulence field.
- An example could be a large tank, initially stirred by running a grid through it. A range of turbulent eddy sizes would be generated, which would gradually decay over time.
- At any two points $\underline{x}_a$ and $\underline{x}_b$ we can examine how correlated the velocity fluctuations are by averaging their product:

$$R_{ij}(\underline{x}_a, \underline{x}_b, t) = \langle u_i(\underline{x}_a, t) u_j(\underline{x}_b, t) \rangle$$

  where $\langle \cdots \rangle$ denotes the averaging process.

- We might equally write this as a correlation between two points $\underline{x}_a$ and $\underline{x}_a + \underline{r}$, where $\underline{r} = \underline{x}_b - \underline{x}_a$:

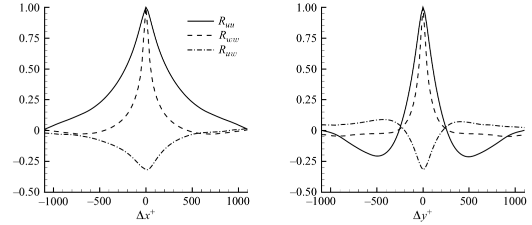
$$R_{ij}(\underline{x}_a, \underline{r}, t) = \langle u_i(\underline{x}_a, t) u_j(\underline{x}_a + \underline{r}, t) \rangle$$

- If the flow is homogeneous, it is only the separation between the two points which is important, so the two-point correlation can be written as

$$R_{ij}(\underline{r}, t) = \langle u_i(\underline{x}, t) u_j(\underline{x} + \underline{r}, t) \rangle$$

  which will be independent of the location $\underline{x}$.

- At very large separation distances the correlation will be zero.



- At smaller separation distances there will be some correlation between the velocities (for example, if the points lie in the same eddy).
- As the separation distance becomes zero the correlation gives the Reynolds stress $\overline{u_i u_j}$.

Fourier Transforms

- The Fourier transform of a function $f(t)$ can be written as

$$g(\omega) = FT[f(t)] = \frac{1}{2\pi} \int f(t) \exp(-i\omega t)\, dt \qquad (1)$$

  where $i^2 = -1$.

- The inverse transformation is given by

$$f(t) = FT^{-1}[g(\omega)] = \int g(\omega) \exp(i\omega t)\, d\omega \qquad (2)$$

- By examining the transform $g(\omega)$ we can identify what frequencies are present in the original function f, and identify which are the dominant frequencies (or corresponding time periods).
- Obviously, if f is a function of space the Fourier transform gives information on the wavelengths present in it.

- One attractive feature of Fourier transforms is that transforms of function derivatives are relatively easy to obtain.
- The Fourier transform of $df(t)/dt$, for example, is given by

$$FT[df(t)/dt] = i\omega g(\omega)$$

  where $g(\omega)$ is the Fourier transform of f, as in equation (1).

- As an illustration, consider the velocity $u(x)$ in a box of side length L.
- If we assume this can be decomposed into N sinusoidal functions, then we could write

$$\begin{aligned} u(x) &= a^{(1)} \cos(2\pi x/L) + \cdots + a^{(N)} \cos(2\pi N x/L) \\ &= \sum a^{(m)} \cos(K^{(m)} x) \end{aligned}$$

- The wavelengths present are $(L, L/2, \ldots, L/N)$, and the corresponding wave numbers, K, are $(2\pi/L, \ldots, 2N\pi/L)$.

- Extending this to three dimensions, and allowing all eddy sizes, we could write

$$u_j(\underline{x}) = \int \left[\int \left[\int \hat{u}_j(K_1,K_2,K_3)\exp(iK_1x_1)dK_1 \right] \exp(iK_2x_2)dK_2 \right] \exp(iK_3x_3)dK_3$$
$$= \int\int\int \hat{u}_j(K_1,K_2,K_3)\exp(iK_1x_1+iK_2x_2+iK_3x_3)dK_1dK_2dK_3$$
$$= \int\int\int \hat{u}_j(\underline{K})\exp(i\underline{K}.\underline{x})dK_1dK_2dK_3$$

- The vectors $\underline{x}$ and $\underline{K}$ give the position in physical space and wave number vector in Fourier space.
- $\hat{u}(\underline{K})$ gives the amplitude of sinusoidal oscillations in direction $\underline{K}$, with wavelength $L = 2\pi/|K|$. It can be obtained by a Fourier transform of the velocity signal $u(\underline{x})$:

$$\hat{u}_j(\underline{K}) = \frac{1}{(2\pi)^2}\int\int\int u_j(\underline{x})\exp(-i\underline{K}.\underline{x})dx_1dx_2dx_3$$

Turbulence Energy Spectrum

- From its definition, the turbulent kinetic energy can be written as

$$k = 0.5\overline{u_iu_i} = \langle u_i(\underline{x},t)u_i(\underline{x},t)\rangle$$

using the averaging notation employed above.
- It can be shown that for the two-point correlation

$$R_{ij}(\underline{r}) = \langle u_i(\underline{x})u_j(\underline{x}+\underline{r})\rangle$$

the corresponding Fourier transform is given by

$$\hat{R}_{ij}(\underline{K}) = \langle \hat{u}_i(\underline{K})\hat{u}_j(-\underline{K})\rangle$$

- From the above definition of k, we can thus write

$$k = \int\int\int E_{3D}(\underline{K})dK_1dK_2dK_3$$

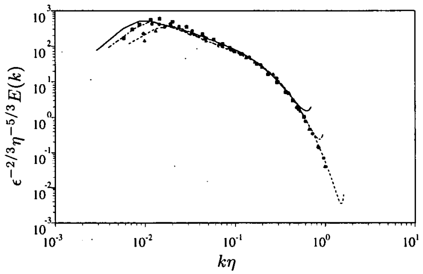
where the 3D energy spectrum is given by

$$E_{3D}(\underline{K}) = 0.5\hat{R}_{ii}(\underline{K}) = 0.5\langle \hat{u}_i(\underline{K})\hat{u}_i(-\underline{K})\rangle$$

- The energy spectrum, E_{3D}, represents the contribution to k of a sinusoidal form in direction $\underline{K}$ with wavelength $L = 2\pi/|\underline{K}|$.
- We can define the spectral density of turbulent kinetic energy, $E(|K|)$ as the total contribution from $E_{3D}(\underline{K})$ over a sphere of radius $|\underline{K}|$ (ie. all contributions with wavelength $L = 2\pi/|\underline{K}|$), so

$$k = \int_0^\infty E(K)dK$$

- The energy spectrum, $E(K)$, typically shows a maximum for some wave number K_M, corresponding to the larger energy-containing eddies.
- $E(K)$ decays at higher wave numbers (smaller eddies).



Dissipation Rate Spectrum

- Recall the dissipation rate is defined as

$$\varepsilon = \nu\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}$$

- In the results quoted above on Fourier transforms, we noted that

$$FT[df(t)/dt] = i\omega g(\omega)$$

where $g(\omega)$ was the transform of $f(t)$.
- Applying this in 3-D to obtain the transform of ε leads to

$$\varepsilon = \nu\int\int\int |\underline{K}|^2\hat{u}_i(\underline{K})\hat{u}_i(-\underline{K})dK_1dK_2dK_3$$

or

$$\varepsilon = \int\int\int D_{3D}(\underline{K})dK_1dK_2dK_3$$

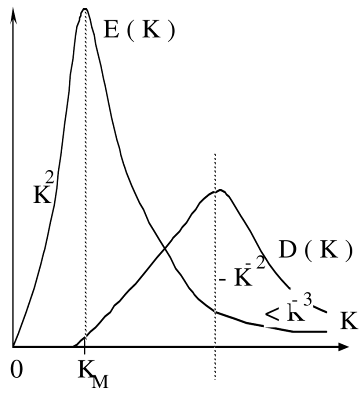
where $D_{3D}(\underline{K}) = 2\nu|\underline{K}|^2E_{3D}(\underline{K})$.

- We can again express this in terms of a spectral density of dissipation, to give

$$\varepsilon = \int_0^\infty D(K)dK = \int_0^\infty 2\nu K^2 E(K)dK$$

- Comparing to the behaviour of the energy spectrum density, $D(K)$ only decreases with wave number once $E(K)$ is decaying faster than K^{-2}, and will thus have its peak at a higher wave number than $E(K)$.

- This confirms previous statements that dissipation is mainly associated with the smaller eddies, whilst the turbulence energy is mostly associated with the larger eddies.



Kolmogorov Model

- Based on empirical observations and dimensional analysis, the Kolmogorov model allows us to derive some quantitative relations for the spectral densities of k and ε.
- The model is based on the following assumptions:
  - Turbulent kinetic energy is carried by the larger eddies, which are unaffected by molecular viscosity, ν.
  - The viscosity, ν, affects only the smaller eddies, responsible for dissipation.
  - The energy dissipated by the small eddies comes from the larger ones, which in turn obtain energy from the mean field.
- We denote the large, energy-containing, eddies' characteristic lengthscale as L_t, and that of the small dissipating eddies by η.

- The above assumptions lead to

$$L_t = f(k, \varepsilon) \qquad \text{and} \qquad \eta = f(\nu, \varepsilon)$$

- The first implies that the dissipation rate is determined by the flow of energy down the cascade from the large scales (hence the level of ε depends on what happens in the large scales).
- The second shows that the lengthscale of the small eddies, η, must then adapt to be consistent with the dissipation rate.
- Dimensional analysis leads to

$$L_t \propto \frac{k^{3/2}}{\varepsilon} \qquad \text{and} \qquad \eta \propto \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

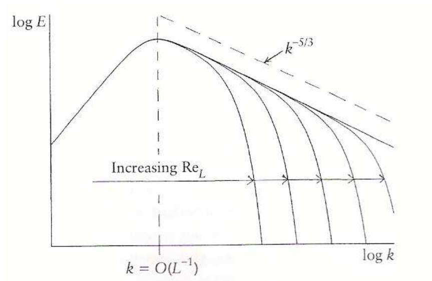
- η is called the Kolmogorov scale. Eddies smaller than this will be instantly dissipated by viscosity.
- L_t, the scale of the large eddies, is called the integral scale.

- From the lengthscale expressions above we obtain

$$\frac{L_t}{\eta} = \frac{k^{3/2}}{\varepsilon^{3/4}\,\nu^{3/4}} = R_t^{3/4}$$

where R_t is the turbulent Reynolds number met earlier.

- As noted before, as R_t increases, the range of scales present in the flow thus increases.



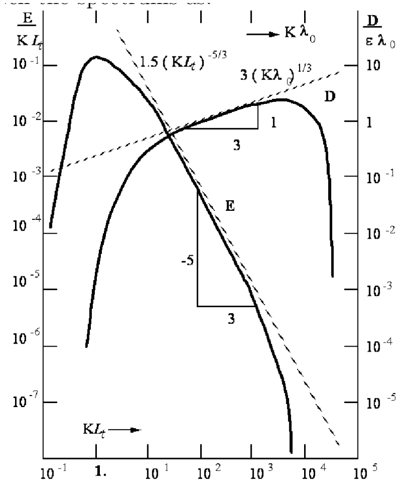
- Note that the turbulent Reynolds number is typically much smaller than the mean or bulk Reynolds number.
- For example, in a pipe flow we might have $k^{1/2} \approx 0.03U_m$, and $L_t \approx 0.1D$, giving $R_t \approx 0.003 U_m D/\nu$.

- At high enough Reynolds numbers there will be a range of wave numbers called the inertial range where eddies simply pass on energy to the smaller structures at the same rate they receive it from the larger ones.
- In this range the spectral density cannot depend on k or ν, so we must have $E = f(\varepsilon, K)$, and the only dimensionally correct combination is

$$E(K) = c_k \varepsilon^{2/3} K^{-5/3}$$

- This is often referred to as Kolmogorov's model, with a constant $c_k = 1.5$, and is generally well-verified by measurements.
- The corresponding dissipation spectral density is given by

$$D(k) = 2\nu c_k \varepsilon^{2/3} K^{1/3}$$



- The picture given by the model for high Reynolds number flows, as the Reynolds number is increased (eg. by decreasing viscosity) is as follows:
- The mean flow and large scale structures remain unchanged.
- The value of ε also remains unchanged, representing the rate at which energy is extracted from the mean field and transferred across the inertial range of wave numbers.
- The bandwidth of the inertial range increases with Re, as smaller and smaller eddies are created until the amount of energy cascading from the larger scales is dissipated to heat.