Gravitational Stability of Vortices in Bose-Einstein Condensate Dark Matter

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Abstract. We investigate a simple model for a galactic halo under the assumption that it is dominated by a dark matter component in the form of a Bose-Einstein condensate involving an ultra-light scalar particle. In particular we discuss the possibility if the dark matter is in superfluid state then a rotating galactic halo might contain quantised vortices which would be low-energy analogues of cosmic strings. Using known solutions for the density profiles of such vortices we compute the self-gravitational interactions in such halos and place bounds on the parameters describing such models, such as the mass of the particles involved.

PACS numbers: 03.75.Nt, 11.27.+d, 95.35.+d

1. Introduction

In the standard model of galaxy formation, the visible component of a galaxy is supposed to be embedded in an invisible halo of non-baryonic matter [1, 2]. This dark component is further supposed to be *cold*, meaning that it is usually assumed to consist of very heavy particles with very low thermal velocities. However, it has been known for some time that Cold Dark Matter (CDM) models have certain problems in reproducing observable properties of galaxies, among them being the predicted presence of central density cusps and the overabundance of small scale structure [3, 4, 5]. In the light of these issues, some authors (e.g. [6]) have suggested that the Dark Matter could instead consist of ultralight particles possessing a de Broglie wavelength sufficiently large that quantum-mechanical effects might manifest themselves on astrophysically interesting scales. Such models would naturally predict smoother and less centrally concentrated galaxy haloes owing than in the CDM case.

Advocating a particular version of this idea, Silverman & Mallett [7] suggested a symmetry breaking mechanism for the production of such a particle, based upon a real-valued scalar field. Although in this case the symmetry breaking mechanism provides a nice example of particle production in a universe with a cosmological constant, symmetry breaking with a real scalar field generically produces a catastrophic domain wall problem [8], and this example would seem to be no exception [9] so this is probably not a viable scenario. However, these papers consider the possibility that the Dark Matter component resides in a Bose-Einstein Condensate (BEC). The dynamics and possible observational consequences of a Cosmological fluid with such properties has been investigated [10], using techniques developed in the field of condensed matter physics. The equation describing a BEC is known to condensed matter theorists as the Gross-Pitaevskii (GP) equation, but is probably more familiar to cosmologists as the nonlinear Schrödinger equation (NLSE).

In condensed matter theory, the term Bose-Einstein Condensate is usually applied to a dilute bosonic gas confined by an external potential, the bosons occupying the lowest available quantum state. Typically, in the limit of large particle number, the density distribution of the condensate is taken to be described by a macroscopic wave-function that is considered to be a quantum field. This field is manipulated by the Gross-Pitaevskii equation, or nonlinear Schrödinger equation, rather than working with the usual creation and annihilation operators of quantum mechanics. The density distribution of the condensate can be represented by a macroscopic wave-function of the same form as the ground state wave-function of a single particle. The momentum distribution of the condensate is obtained by taking the Fourier transform of this wave-function. In an experimental setup, the occurrence of a Bose-Einstein condensate is confirmed by a sharp peak in the momentum space distribution of the gas of particles.

More speculatively, the concept of a BEC can also be applied to such hypothetical particles as axions or ghosts. In this context, the axion field, for example, is coherent and has relatively small spatial gradients. The gradient energy can be interpreted as particle momenta, which will be the same and small for each particle, hence giving a sharp peak in the momentum space distribution as in the case of the more familiar BEC described above.

In quantum field theory, a condensate corresponds to a non-zero expectation value for some operator in the vacuum and, in the limit of large quantum number, this condensate can be considered to be a classical field. This is a good model for the condensate of Cooper pairs in a superconductor, or for helium atoms in a superfluid [11].

The usual, linear Schrödinger equation, coupled to the Poisson equation can be used to model many phenomena in Cosmology. As well as modelling a quantum mechanical system, as in [6], it has also been used as a classical wave equation to model structure formation. It has been shown that using the Condensed Matter concept of a Madelung transformation to yield the Euler and Continuity equations from the Schrödinger equation, applies as well as to a Cosmological fluid as it does to fluids in the laboratory [12, 13, 14, 15, 16, 17].

Silverman & Mallett [7] also considered the rotation of a galactic-scale dark matter halo. Using a phenomenological description taken directly from condensed matter, they concluded that a galactic halo should be threaded by a lattice of quantised vortices, as a consequence of the rotation of that galaxy. Indeed from studies of rotating BECs and quantum turbulence [18, 19], it would seem to be difficult to prevent such vortices from forming. The galaxy velocity rotation curve produced by these authors reproduces the approximate form of observed rotation curves.

A similar conclusion was reached in Yu and Morgan [20]. This paper considered stationary cylindrical solutions of a complex ϕ^4 scalar field model, coupled to gravity. These solutions are Nielson-Olesen vortices, also known as local U(1) Cosmic Strings [8]. To describe the motion of these vortices in the galaxy, Yu and Morgan's procedure was to calculate the motion of one vortex according to a gradient in the phase induced by the surrounding vortices.

There are many models using the Schrödinger-Poisson, or the relativistic Einstein-Klein-Gordon, system to describe slightly different physical processes. A non-exhaustive list includes scalar field dark matter [21, 22], boson stars [23], Oscillatons [24]; condensate stars [25], repulsive dark matter [26] and fluid dark matter [27, 28], as well as the fuzzy dark matter and classical fluid approaches that we have already mentioned, and the more established theories such as the Abelian-Higgs model in field theory, and the Landau-Ginzberg model in condensed matter. We will not attempt a thorough review of each model here, except to say that it is sometimes difficult to explicitly distinguish between them.

The effects of the interaction of gravity with a coherent state of matter, such as a BEC, have certainly been considered [29, 30], and prompted the question of whether it is actually possible for DM to be in a coherent quantum state, if the only interaction with visible matter is gravitational. Penrose has also used the Schrödinger-Poisson system during his 'Quantum State reduction' research program [31].

In this paper we seek to determine some of the properties of a quantised vortex residing in a galactic-scale Bose-Einstein Condensate dark matter. In particular, we will place bounds on the parameters that are used to describe such a vortex. For the purposes of this paper we presume that the DM does indeed consist of a BEC, formed at an earlier stage of Cosmological history and described by the coupled nonlinear Schrödinger-Poisson system, and that vortices are present in this cosmological fluid.

In Section 2 we introduce the basic formalism for describing a BEC using the Gross-Pitaevskii (nonlinear Schrödinger) equation, and vortices within it. In Section 3 we discuss coupling the NLSE to the Poisson equation. In Sections 4 and 5 we look at some of the properties of a vortex as a result of gravitational coupling. We present some results in Section 6 and a discussion in Section 7. An appendix contains some of the approximations we have used in our work, and is referenced in the main body of the paper.

2. Setup

For our discussion, we use some of the conventions and proceedures set out by Berloff & Roberts [32], and Pethick & Smith [11]. The nonlinear Schrödinger equation is written in the form

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\nabla^2\Psi + \Psi\int|\Psi(x',t)|^2V(|x-x'|)dx', \tag{1}$$

where m is the mass of a particle in the BEC, and $V(|x-x'|)$ is the interaction potential between bosons. The potential is simplified for a weakly interacting Bose system by replacing $V(|x-x'|)$ with a δ-function repulsive potential of strength V_0, giving

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\nabla^2\Psi + V_0|\Psi|^2\Psi. \tag{2}$$

Defining a state that is independent of time to be the 'laboratory frame', $\Psi = \exp(iE_v/\hbar)$, it is then possible to consider deviations from that state by considering the evolution of ψ, where $\psi = \Psi\exp(iE_vt/\hbar)$. Here, E_v is the chemical potential of a boson, in the sense that it is the increase in ground state energy when one boson is added to the system. The nonlinear Schrödinger equation used for subsequent analysis is then

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\nabla^2\psi + V_0|\psi|^2\psi - E_v\psi. \tag{3}$$

Multiplying equation (3) by ϕ^* and subtracting the complex conjugate of the resulting equation we obtain

$$\frac{\partial|\psi|^2}{\partial t} = \nabla.\left[\frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*)\right]. \tag{4}$$

We notice that this is of the form of a continuity equation.

$$\frac{\partial|\psi|^2}{\partial t} + \nabla.(|\psi|^2\mathbf{v}). \tag{5}$$

We identify $|\psi|^2$ as the number density n, and the related momentum density is given by

$$\mathbf{j} = \frac{\hbar}{2i}(\psi^*\nabla\psi - \psi\nabla\psi^*), \tag{6}$$

which is equivalent to

$$\mathbf{j} = mn\mathbf{v}. \tag{7}$$

This defines for us the mass density, as $\rho = mn = m|\psi|^2$, and the velocity

$$\mathbf{v} = \frac{\hbar}{2mi}\frac{(\psi^*\nabla\psi - \psi\nabla\psi^*)}{|\psi|^2}. \tag{8}$$

As suggested in the introduction, we can make a 'Madelung transformation'

$$\psi = \alpha\exp\left(i\phi_\omega\right), \tag{9}$$

and, from equation (8), we obtain an expression for the velocity of the condensate

$$\mathbf{v} = \frac{\hbar}{m}\nabla\phi_\omega. \tag{10}$$

Here, ϕ_ω is the velocity potential. Substituting the Madelung transformation, and taking real and imaginary parts yields the fluid equations: the continuity equation

$$\frac{\partial\left(\alpha^2\right)}{\partial t} + \frac{\hbar}{m}\nabla.(\alpha^2\nabla\phi_\omega) = 0; \tag{11}$$

and the (integrated) Euler equation:

$$\hbar\frac{\partial\phi_\omega}{\partial t} = \frac{\hbar^2}{2m}\frac{\nabla^2\alpha}{\alpha} - \frac{1}{2}m\mathbf{v}^2 - V_0\alpha^2 + E_v. \tag{12}$$

Often, the identification

$$\phi_\omega{}' = \frac{\hbar}{m}\phi_\omega \tag{13}$$

is used, to maintain contact with the more familiar form of the fluid equations:

$$\frac{\partial\left(\alpha^2\right)}{\partial t} + \nabla.(\alpha^2\nabla\phi_\omega{}') = 0, \tag{14}$$

$$\frac{\partial\phi_\omega{}'}{\partial t} = \frac{\hbar^2}{2m^2}\frac{\nabla^2\alpha}{\alpha} - \frac{(\nabla\phi_\omega{}')^2}{2} - \frac{V_0}{m}\alpha^2 + \frac{E_v}{m}. \tag{15}$$

Here the quantum nature of the fluid is evident only in the first term on the right hand side of the second equation, which is often known as the *quantum pressure* term, although dimensionally it is a chemical potential. This term is relevant only on small scales, where quantum effects become important, such as in a vortex core, or where the condensate meets a boundary. This identification rather hides the quantum nature of the fluid with respect to the fluid velocity, which will become particularly relevant when we start talking about vortices in the next section.

By assuming that the condensate reaches a stationary equilibrium state at a distance far from any disturbance, equation (3) gives us the relation

$$\psi_\infty = \left(\frac{E_v}{V_0}\right)^{\frac{1}{2}}. \tag{16}$$

When the condensate wave-function reaches a boundary, such as the wall of a container, or the core of a vortex is being considered, we can define a distance over which the wave-function changes from zero to its bulk value, or where quantum effects become important [32, 11].

$$a_0 = \frac{\hbar}{(2mE_v)^{\frac{1}{2}}} \tag{17}$$

This is known as the *coherence length*, or *healing length*, as it is the distance over which the wave-function requires ‘healing’.

2.1. Vortices

We have already seen that the velocity of the condensate is given by

$$\mathbf{v} = \frac{\hbar}{m}\nabla\phi_\omega. \tag{18}$$

One would expect then, that the condensate would be irrotational, as

$$\nabla\times(\nabla f) = 0 \tag{19}$$

for any scalar, f. This restricts the motion of the condensate much more than a classical fluid. The circulation around any contour then, should also be zero. By Stokes’ theorm

$$\Gamma = \oint_l \mathbf{v}.d\mathbf{l} = \int_A (\nabla\times\mathbf{v}.d\mathbf{A} = 0 \tag{20}$$

This condition, known as the Landau state, was first derived in an analysis of superfluid HeII [36], and suggests that rotation of such a condensate should be impossible. Experiments by Osbourne [37] indicated that the condensate did indeed experience rotation. Feynman [38], building on the independent work of Onsager [39], suggested that rotation and hence non-zero circulation could be explained by assuming that the condensate is threaded by a lattice of parallel vortex lines. It is possible to have circulation surrounding a region from which the condensate is excluded, and in this case, this would be the vortex core. To see this, we note that the condensate wave-function must be single valued, and so around any closed contour, the change in the phase of the wave-function $\Delta\phi$ must be a multiple of 2π.

$$\Delta\phi_\omega = \oint \nabla\phi_\omega . d\mathbf{l} = 2\pi l \tag{21}$$

where l is an integer. We immediately see that the circulation is quantised in units of h/m.

$$\Gamma = \oint \mathbf{v}.d\mathbf{l} = \frac{\hbar}{m} 2\pi l = l\frac{h}{m} \tag{22}$$

To obtain vortex solutions, we work in cylindrical coordinates (r, χ, z), and look for a static solution of the nonlinear Schrödinger equation, equation (3). To satisfy the requirement of single-valuedness, the condensate wave-function must vary as $\exp(in\chi)$, with n integer. We make the vortex ansatz

$$\psi = R(r)\exp(in\chi). \tag{23}$$

It is interesting to note the similarity between this procedure, and that used in obtaining Nielson-Olesen vortices, or Cosmic Strings, in the Abelian-Higgs model [8]. This was mentioned in Section 1, and will be useful shortly for obtaining equation (27), as shown in Appendix A.1. We can obtain an expression for the velocity of a vortex by substituting the vortex ansatz (23) into equation (8)

$$\mathbf{v}_\omega = \frac{\hbar n}{r}\frac{1}{m}\hat{\chi}, \tag{24}$$

and we note again the discrete nature of the allowed values of velocity. From now on we will consider only $n = 1$ vortices. Vortices with $n > 1$ are generally expected to be unstable, from energy considerations (see for example Chapter 9.2.2 of [11]), and will break up into several $n = 1$ vortices to make up a vortex lattice, as described above. We can note further that Cosmic Strings with winding numbers $n > 1$ are also unstable to perturbations [8]. Such defects break down to several $n = 1$ configurations in both a Condensed Matter environment, and a High Energy Field Theoretic one. Feynman initially introduced quantised vortices as a purely theoretical tool with which to explain the rotation of the condensate, but the experimental verification of the quantisation of rotational velocities (e.g. by [40]) demonstrated that these vortices were indeed real. The density profile of a vortex ($\rho(r) = m|R(r)|^2$) is defined by the vortex equation, which results from substituting the vortex ansatz into equation (3)

$$-\frac{\hbar^2}{2mE_v}\left[\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} - \frac{1}{r^2}R(r)\right] + \frac{V_0}{E_v}R(r)^3 - R(r) = 0 \tag{25}$$

From equation (16) we see that the density far from the vortex is given by

$$\rho_\infty = mR_\infty = m\frac{E_v}{V_0}. \tag{26}$$

Analytic solutions of this equation are not known so it must be solved numerically. For our anaylses we will use the approximation

$$R(r) \simeq \left(\frac{E}{V_0}\right)^{1/2} \left[1 - \exp(-r/a_0)\right], \tag{27}$$

as discussed in Appendix A.1.

3. Self-gravity of a BEC Vortex

In considering Bose-Einstein condensates on scales relevant to structure formation in the universe, we must necessarily include gravitational effects. BECs are typically sufficiently dilute that the mass densities are not very large, and so a Newtonian approximation is sufficient. Gravitational effects can be added to the BEC by including a term in the nonlinear Schrödinger equation that couples to the Poisson equation. We then have a pair of equations modelling a gravitationally coupled fluid.

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\nabla^2\psi + V_0|\psi|^2\psi - E_v\psi + m\phi_G\psi \tag{28}$$

$$\nabla^2\phi_G = 4\pi G\rho = 4\pi Gm|\psi|^2. \tag{29}$$

3.1. Vortices in Gravitationally Coupled BECs

To obtain vortex solutions, we again work in cylindrical coordinates (r, χ, z), and substitute the vortex ansatz $\psi = R(r)exp(i\chi)$ into equations (28) and (29). The system of equations describing a gravitationally coupled BEC fluid become

$$-\frac{\hbar^2}{2mE_v}\left[\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} - \frac{1}{r^2}R(r)\right] + \frac{V_0}{E_v}R(r)^3 - R(r) + m\phi_G(r) = 0 \tag{30}$$

$$\nabla^2\phi_G(r) = \frac{d^2\phi_G(r)}{dr^2} + \frac{1}{r}\frac{d\phi_{G(r)}}{dr} = 4\pi GmR(r)^2 \tag{31}$$

Ideally, we would like to find a solution describing the function $R(r)$ in this system, so we can compare the density profile of a quantum vortex, to that of one that is gravitationally coupled. However, finding a full simultaneous solution to these coupled equations is difficult. Firstly, because the nonlinear Schrödinger equation itself is not soluble analytically. Secondly, because the vortex density tends to a constant, so the Newtonian potential tends to diverge, and thirdly because these equations do not define the vortex velocity, which would be providing the centripetal force to withstand the gravitational collapse. In other words, all the variables required to provide a fully simultaneous static solution are not defined within these two equations.

4. Vortex Stability in Gravitationally Coupled BECs

Rather than solving the coupled equations (28) and (29) directly, we can make some arguments regarding the stability of a gravitationally coupled BEC vortex, and consequently give some bounds on the parameters that describe it. Our analysis is based upon consideration of the circular velocity of a BEC vortex, $v_\omega(r)$, and the radial velocity induced from gravitational attraction, $v_{\rm G}(r)$. $v_\omega(r)$ is the velocity that the vortex density distribution is moving at, for a particular r, while $v_{\rm G}(r)$ would be the velocity experienced by a test particle orbiting that density distribution, at a distance r. To sustain a vortex, $v_\omega(r)$ must at least be greater than $v_{\rm G}(r)$, otherwise the quantum-mechanical forces at work in the vortex are not sufficiently strong to hold itself up against gravitational collapse. That is, the vortex is spinning too slowly to provide enough centripetal force to balance the gravitational force. For stability, we therefore have the bound,

$$v_\omega(r) \geq v_{\rm G}(r) \tag{32}$$

4.1. Gravitational Field of a Cylindrically Symmetric System

To obtain $v_{\rm G}(r)$, we turn to Gauss's Law to determine the gravitational field of a cylindrically symmetric mass distribution, and hence obtain the radial gravitational velocity of a test particle moving in the field of that system. Gauss's law is

$$\oint \mathbf{g} \cdot d\mathbf{A} = -4\pi G M_{\rm encl} \tag{33}$$

The density, $\rho(r) = m|R(r)|^2$, is already determined in terms of the cylindrical r co-ordinate, as it is a solution of the vortex equation. The mass enclosed is the density pervading a cylinder of radius r and length L.

$$M_{\rm encl} = L \int_0^r 2\pi r \rho(r) dr \tag{34}$$

The left-hand side of Gauss's law, in cylindrical co-ordinates, is

$$\int g r d\chi dz, \tag{35}$$

where the integral over the z co-ordinate is again L, the length of the vortex. Gauss's law, then, gives us

$$g r 2\pi L = -4\pi G 2\pi L \int_0^r \rho(r) r dr \tag{36}$$

giving

$$g = -\frac{4\pi G m}{r} \int_0^r |R(r)|^2 r dr. \tag{37}$$

The sign is negative as we have chosen an outward-pointing surface normal in our formulation of Gauss's Law, equation (33), which indicates that the gravitational flux will always be towards the origin. This leads to the slightly counter-intuitive conclusion that a hole (the vortex) in a constant mass density background would seem to produce a gravitational force towards it, but this is really a manifestation of the (extremely) thick shell condition. Viewed another way, this static configuration will want to act to collapse in, and close the hole. It is this force that is 'unopposed' in equations (30) and (31). This need not concern us further, as it is the magnitude that is required for our argument. The magnitude of the induced centripetal force is

$$g = \frac{{v_{\mathrm{G}}}^2}{r} \tag{38}$$

and the gravitational circular velocity profile v_{G} is given by

$$v_G(r)^2 = 4\pi G \int_0^r \rho(r) r dr = 4\pi G m \int_0^r |R(r)|^2 r dr. \tag{39}$$

5. Bounds on Parameters

We now have expressions for $v_{\mathrm{G}}(r)$ and $v_\omega(r)$, equations (24) and (39), to go in the bound given by equation (32). In Figure 1 we plot, as an example, $v_\omega(r)$ and $v_{\mathrm{G}}(r)$ and the density profile for comparison. For this example, we have used values of m $= 3.56 \times 10^{-59}$kg (2×10^{-23} eV), E$_\upsilon = 2.5 \times 10^{-49}$ J (1.56×10^{-30}eV) and $V_0 = 4.45 \times 10^{-84}$ Jm^3 (3.7×10^{-45} eV^{-2}) as explained in Appendix Appendix A.2.

The bound on stability, $v_\omega(r) \geq v_{\mathrm{G}}(r)$, will always be violated at some point, as outside the vortex core $v_\omega(r) \sim 1/r$ and $v_{\mathrm{G}}(r) \sim r$. We must specify what might be an acceptable value of r for $v_\omega(r)$ and $v_{\mathrm{G}}(r)$ to cross. For a vortex to exist, the density profile should be fully established. We take this to mean that the density has essentially reached its background level. From the scaled density profile in discussed in Appendix A.1, and plotted in Figure (A1), we see that the density reaches its background level at a value of about ten times the healing length. Using equation (27) in (39), equations (39) and (24) in (32), and substituting for E_υ from equation (17) we obtain

$$\frac{\sqrt{2\pi}}{2}\left(\frac{G\hbar^2}{V_0 {a_0}^2}\left[2r^2 + 8ra_0 e^{-\frac{r}{a_0}} + 8{a_0}^2 e^{-\frac{r}{a_0}} - 2ra_0 e^{-\frac{2r}{a_0}} - {a_0}^2 e^{-\frac{2r}{a_0}}\right]\right)^{\frac{1}{2}} \leq \frac{\hbar}{mr}. \tag{40}$$

We will fix the healing length a_0, and plot V_0 against m (fixing a_0 and m fixes E_υ, from equation (17)) to give an allowed range of parameter values. We will do this for various values of a_0, and for various values of r, which we will take to be an integer number of healing lengths, $r = na_0$, with the minimum $n = 10$ as outlined above. Equation (40) then becomes

$$V_0 \geq \frac{\pi}{2} G m^2 n^2 \left(2n^2 {a_0}^2 + 8n{a_0}^2 e^{-n} + 8{a_0}^2 e^{-n} - 2n{a_0}^2 e^{-2n} - {a_0}^2 e^{-2n}\right). \tag{41}$$

5.1. Other Bounds

We can obtain some other bounds to cut off other bits of parameter space. The asymptotic vortex density is given by

$$\rho_\infty = m\left(\frac{E_\upsilon}{V_0}\right). \tag{42}$$

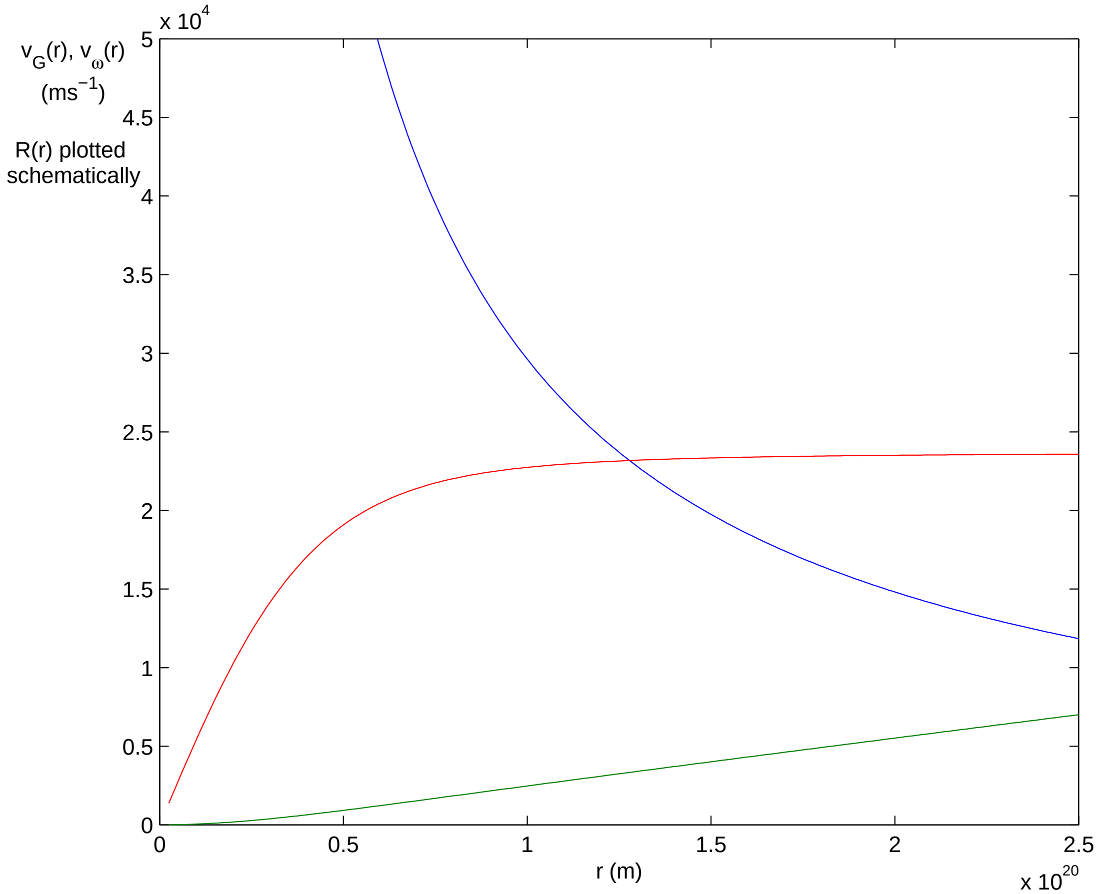

Figure 1. Velocity Profiles for v_G (green) and v_ω (blue). Density profile plotted schematically for comparison (red).

If the vortex exists as a component of a galaxy, then there is a minimum and maximum density that the vortex can have, given by the maximum and minimum known values of mass density within a galaxy:

$$\rho_{\rm min} \leq \rho_\infty \leq \rho_{\rm max}. \tag{43}$$

The value of E_v in equation (42) is fixed (as we are fixing the healing length), and so the bound on the density becomes a bound on V_0.

$$\frac{\hbar^2}{2{a_0}^2\rho_{max}} \leq V_0 \leq \frac{\hbar^2}{2{a_0}^2\rho_{min}}. \tag{44}$$

Equation (41) gives a lower bound on V_0, so to obtain an upper bound, we use the second half of the above relation.

$$V_0 \leq \frac{\hbar^2}{2{a_0}^2\rho_{min}}. \tag{45}$$

Another bound is provided because the vortex velocity should never exceed the speed of light,

$$v_\omega = \frac{\hbar}{mr} \leq c. \tag{46}$$

It can be seen from equation (24) that the vortex velocity increases with decreasing radius. This relation breaks down within the vortex core, a_0, where the vortex velocity diverges. Finding an appropriate description is a topic of some interest in Condensed Matter theory [33]. We evaluate the maximum vortex velocity at a distance of $5a_0$ from the origin. i.e. in a regime where we are sure the relation holds. This gives a bound on the mass.

$$m \geq \frac{\hbar}{5ca_0}. \tag{47}$$

5.2. Values

To see how the restriction on m and V_0 varies, we can think of a range of healing lengths that cover all possible scales in a galaxy.

$$1 \times 10^{10}\mathrm{m} \quad (3.2 \times 10^{-10}\mathrm{kpc}, \quad \sim 7 \times 10^{-2}\mathrm{AU}) \leq a_0 \tag{48}$$

$$a_0 \leq 1 \times 10^{22}\mathrm{m} \quad (324\mathrm{kpc}) \tag{49}$$

This range of scales takes us from sub solar system, to that of the largest known galaxies (e.g. IC 1101 in the Abell 2029 cluster [34]). At fixed a_0 we will also cover a large range of n; the number of healing lengths where the velocity profiles cross. For the bound given in equation (45), we take the minimum density found within a galaxy to be the cosmological density. This minimum must necessarily be close to the critical density of the universe.

$$\rho_{\min} = \rho_c = \frac{3H_0^2}{8\pi G}. \tag{50}$$

With $H_0 = 70$ km s^{-1} Mpc^{-1}, this gives a value of $\rho_{\min} = 9.2 \times 10^{-27}$ kg m^{-3}.

6. Results

In Figure (2), we show a region of the $V_0 - m$ parameter space for the healing length $a_0 = 1 \times 10^{16}$ m (~ 1 pc). The lower bound on V_0 is given when v_ω and v_G cross at a value of ten times the healing length, $n = 10$. A vortex could be considered more stable if v_ω and v_G cross at a greater number of n, moving us up into the allowed triangular region. However, this can soon reach the minimum density bound on V_0. A value of $n = 10^6$ is also plotted, and it is clear that this is outside the bounded region. The lines bounding the region of allowed parameter values are given by equations (41), (47) and (45).

Figure (3) shows allowed regions for various healing lengths, all at a value of $n = 10$. We see that as we move to smaller values of a_0, the allowed bounds on m and V_0 both move up, as expected from equations (45) and (47). More physically, as the mass of the

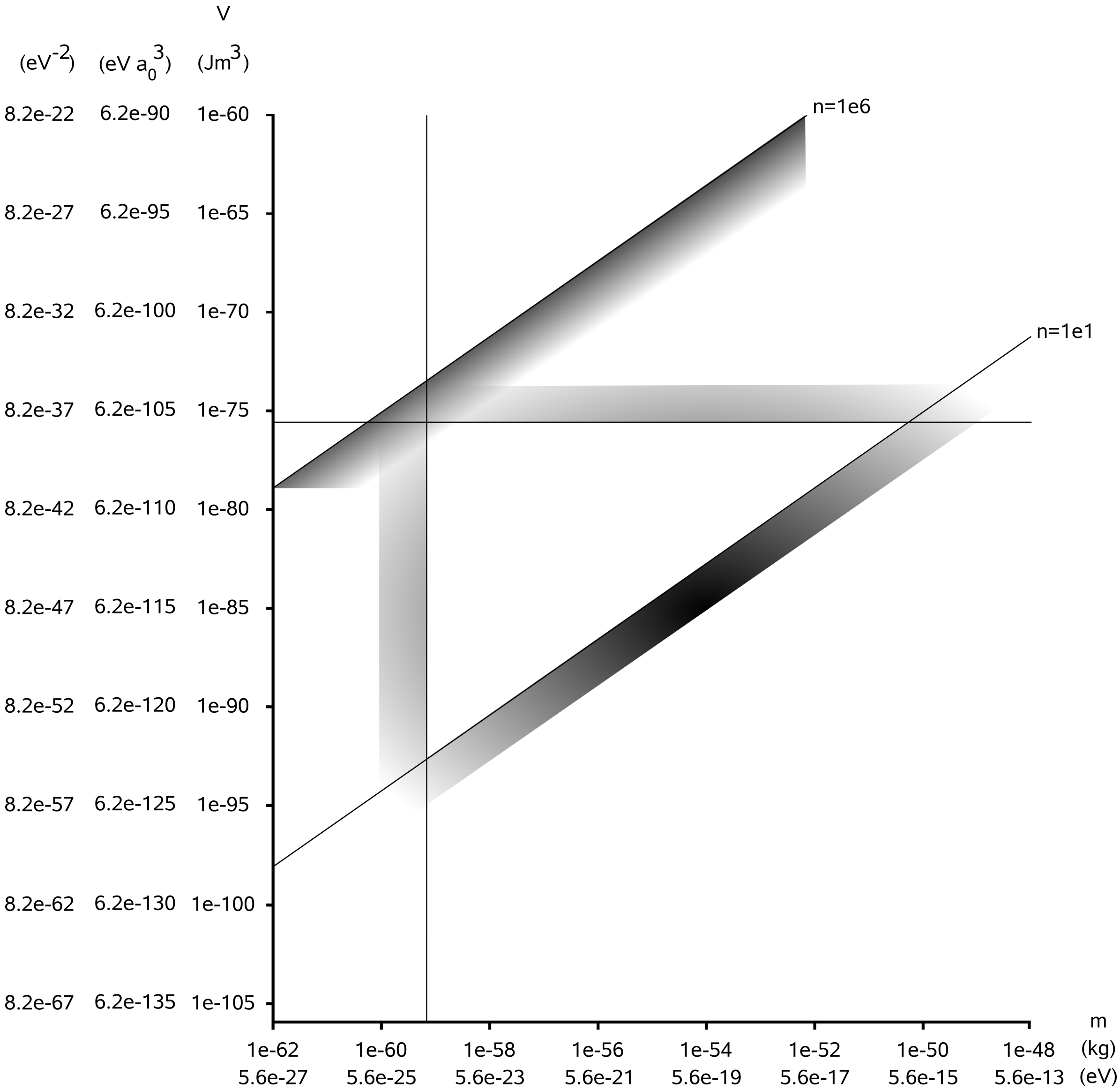

Figure 2. Allowed region in $V_0 - m$ parameter space, for a healing length of $a_0 = 1 \times 10^{16}$ m (~ 1 parsec)

particle is increased, the repulsive potential V_0 must increase to balance the stronger gravitational force.

7. Discussion

In this paper we have used techniques from condensed matter theory in a cosmological setting to place bounds on parameters describing a Dark Matter candidate, on the assumption that the Dark Matter halo consists of a Bose-Einstein Condensate, in which quantised vortices reside. In the case of a laboratory BEC, self-gravitational forces are not important and even in that case analytical progress is limited. Using a simple

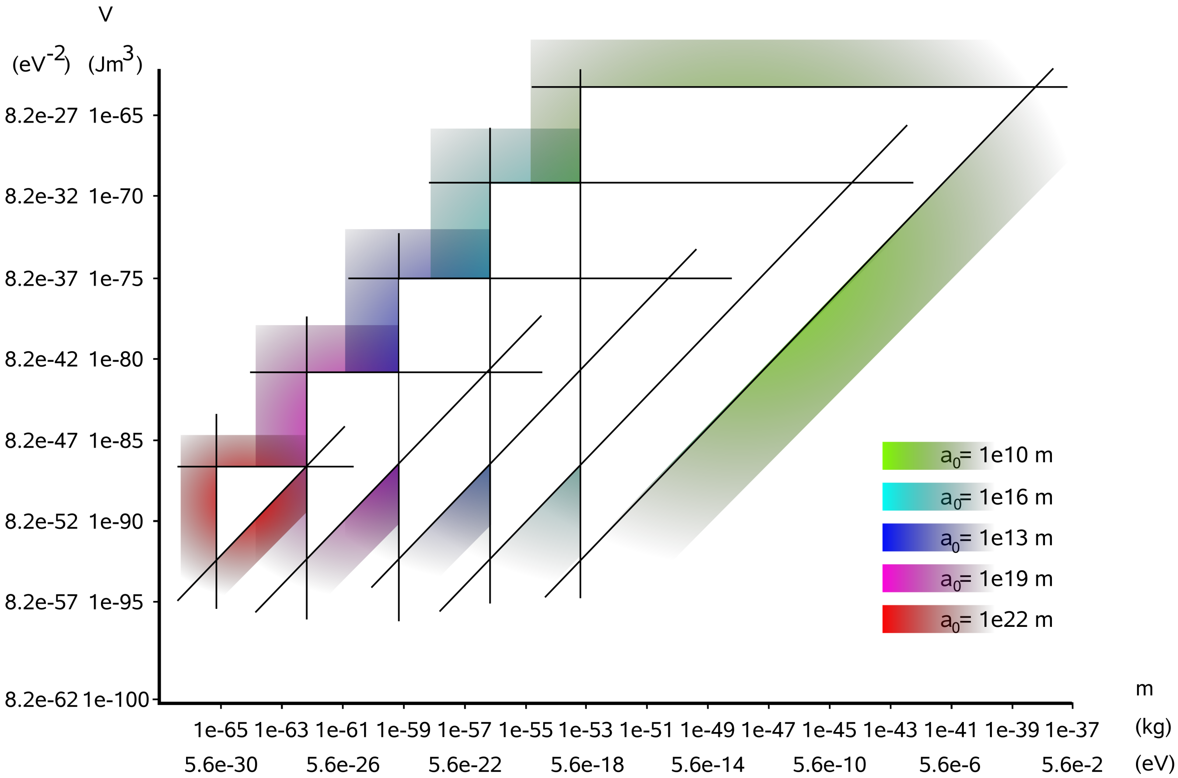

Figure 3. Allowed regions in $V_0 - m$ parameter space, with $n = 10$. Healing lengths as labelled.

physical argument, however, we have shown how rough limits on the consistency of such a model can be imposed. Considering a Dark Matter particle of a particular mass, and a vortex of a certain radius, places constraints on the values that the chemical potential, and interaction potential can take. There remain sizeable regions of parameter space in which the model appears to be viable.

In future work, it would be interesting to investigate further whether a Dark Matter candidate could reside in a coherent quantum state, if the only interaction was gravitational. A less ambitious undertaking would be to see if the Madelung transformation provides a solution to the problem of defining all the relevant variables, as suggested in Section 3.1. This would give a set of fluid equations that includes the velocity giving rise to the stabilising centripetal force. One problem to be anticipated in such a solution, would be that the velocity in the vortex core would still be ill-defined, as alluded to in Section 5. The system would therefore have to be solved by a more complete numerical method than we have been able to implement so far.

Acknowledgements

Mark Brook acknowledges support from the Science & Technology Facilities Council, and useful comments from Sean Carroll and David Tong.

Appendix A. Approximations

Appendix A.1. Approximations to the Density Profile

The numerical solution to the NLSE can be cumbersome to work with, so we provide some discussion of some approximations that can be used. It is possible to scale the the variables r and $R(r)$ in equation (25) to obtain a scale-free equation. Scaling r by the healing length, $r' = r/a_0$, and $R(r)$ by the steady state value, $R'(r') = R(r)/R_\infty$ we obtain

$$\frac{d^2R'(r')}{dr'^2} + \frac{1}{r'}\frac{dR'(r')}{dr'} - \frac{1}{r'^2}R'(r') - R'(r')^3 + R'(r') = 0. \tag{A.1}$$

Our first idea for an approximation comes from the field of cosmic strings. The method of approximation is detailed in [8]. Looking at the profile of the Higgs field in a Nielson-Olesen vortex we see that it can be written, in a similarly scaled way, as

$$\frac{d^2f'(r')}{dr'^2} + \frac{1}{r'}\frac{df'(r')}{dr'} - \frac{1}{r'^2}f(r')(\alpha(r')-1)^2 - \frac{\lambda}{2}f'(r')(f'(r')^2-1) = 0 \tag{A.2}$$

Here α is a gauge term arising from the coupling to Electromagnetism, and λ is determined by the potential term of the theory. It is possible to linearise equation (A.2) to obtain a modified Bessel function as the first order approximation to f'(r') - the zeroth order being 1. This happens in the string case, because the gauge contributions serve to cancel one of the terms, leaving the modified Bessel's equation. The linearised version of equation (A.1) does not quite reduce to a modified Bessel's equation, but taking our lead from the cosmic string example, we write

$$R'(r') \sim 1 - \exp(-r'). \tag{A.3}$$

Another approximation, which might seem to be more accurate, was developed by Berloff [35] in a condensed matter context. The Padé approximation has the same asymptotics at $r = 0$ and $r = \infty$ as the function one is trying to approximate. The Padé approximation in this case gives

$$R'(r') \sim \sqrt{\frac{r'^2(0.3437 + 0.0286r'^2)}{1 + 0.3333r'^2 + 0.0286r'^4}}. \tag{A.4}$$

This solution is plotted in Figure A1 along with the numeric solution given by equation (A.1), and the previous approximation, equation (A.3). The Padé approximation is indeed much more accurate in the small and large r regions. However, the Padé approximation has the tendency to overestimate the density in the central region, producing a density function whose derivative is negative in this region. As discussed in the main body of this paper, the gravitational potential is proportional to the density, and so the gravitational force will be proportional to the derivative of the density function. If we chose to use the Padé approximation for our density profile, we could be potentially misled by its behaviour in the central region.

We will use the approximation

$$R(r) = \left(\frac{E_v}{V_0}\right)^{\frac{1}{2}} \left[1 - \exp[-r/a_0]\right). \tag{A.5}$$

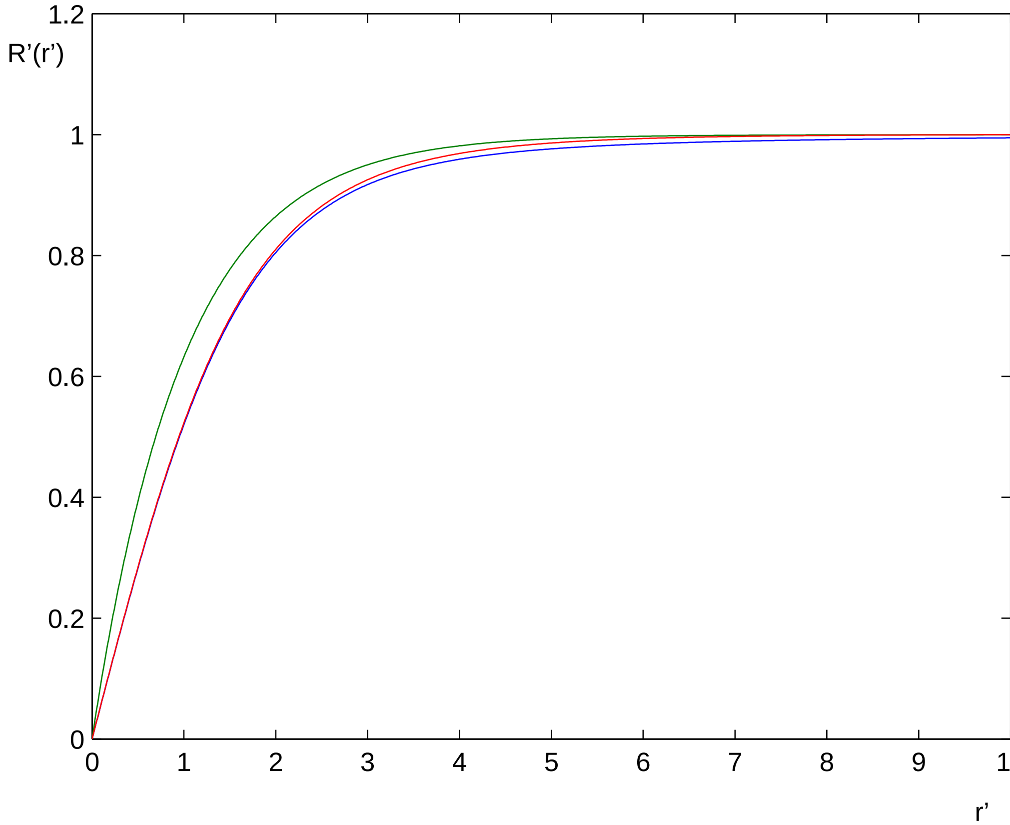

Figure A1. Numeric solution to Equation (A.1) (blue), the Padè approximation equation (A.4) (red), and the scaled approximation used in this analysis, equation (A.3) (green).

Appendix A.2. Approximations for Parameters Defining the BEC

To enable us to obtain actual values for the velocity and density profiles that we are considering, we must provide values for the parameters m, V_0, and E_v. The properties of Dark Matter particles are, by their very nature, unknown, so we must make some approximations. We use the analysis in [7] to provide us with some data values. The mass of the Bose Einstein Condensate Dark Matter particle in that paper is 3.56×10^{-59} kg (2×10^{-23} eV). Their analysis is based on the mass and angular rotation of the Andromeda galaxy. The mean density is given as 2×10^{-24}kg m^{-3}, and they estimate that the vortex line density in the galaxy would be about 1 vortex per 208 kpc^2. This gives a vortex radius of $r_\omega \sim 2.5 \times 10^{20}$ m. We again turn to vortex lattices in condensed matter systems to provide us with some further estimates of vortex properties in a BEC.

Taking the distance between two vortices to be twice the vortex radius, we note from experimental observations of vortex lattices in a BEC that the vortex density reaches the normal density at about half the vortex radius; see, for example, Figure 9.3

in [11], taken from [41]. From Figure (A1), we also see that the vortex density reaches the normal condensate density at around five healing lengths. This gives us an estimate of $r_\omega/2 = 5a_0$. We then use $r_\omega \sim 2.5 \times 10^{20}$ m, $a_0 = \hbar/(2mE_v)^{\frac{1}{2}}$, and $\rho_\infty = mE_v/V_0$ to give estimates for E_v and V_0. With these approximations we find values of $E_v = 2.5 \times 10^{-49}$ J (1.56×10^{-30}eV) and $V_0 = 4.45 \times 10^{-84}$ J m^3 (3.7×10^{-45} eV^{-2}).

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