Gravitational Radiation From Cosmological Turbulence

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An injection of energy into the early Universe on a given characteristic length scale will result in turbulent motions of the primordial plasma. We calculate the stochastic background of gravitational radiation arising from a period of cosmological turbulence, using a simple model of isotropic Kolmogoroff turbulence produced in a cosmological phase transition. We also derive the gravitational radiation generated by magnetic fields arising from a dynamo operating during the period of turbulence. The resulting gravitational radiation background has a maximum amplitude comparable to the radiation background from the collision of bubbles in a first-order phase transition, but at a lower frequency, while the radiation from the induced magnetic fields is always subdominant to that from the turbulence itself. We briefly discuss the detectability of such a signal.

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I. INTRODUCTION

Gravitational radiation is likely the only direct source of information about the Universe at very early times. Electromagnetic radiation has propagated freely only since the epoch of recombination at a redshift $z \simeq 1000$; any radiation produced at earlier times was quickly thermalized by Compton scattering from free electrons in the primordial plasma. Neutrinos probe to somewhat earlier epochs since they were in thermal equilibrium only until the Universe was around one second old, but detection prospects for the cosmic neutrino background are nil. In contrast, gravitational radiation was in thermal equilibrium only at temperatures approaching the Planck energy when the Universe had an age of around the Planck time. Furthermore, gravitational radiation, unlike electromagnetic radiation, propagates virtually unimpeded throughout the entire history of the Universe. These properties make gravitational radiation a powerful probe of the very early Universe, in principle. The difficulty is of course the extremely small amplitude of the propagating metric perturbations.

The most cosmologically interesting gravitational radiation sources are stochastic backgrounds produced by some event in the early evolution of the Universe. One widely discussed example is the background of tensor metric perturbations produced by quantum fluctuations during inflation [1]. However, the amplitude of temperature fluctuations in the cosmic microwave background likely limits the amplitude of an inflationary gravitational wave background to be undetectably small on scales amenable to direct detection (i.e. laboratory to solar system scales). Another possibility is a significant background from the evolution of topological defects such as cosmic strings [2]. Current measurements of the microwave background and the large-scale distribution of galaxies rule out defects as the sole structure formation mechanism, although it is conceivable that some small fraction of the microwave background fluctuations arise from defects. In this case as well, direct detection of the gravitational radiation from defects appears improbable.

The most promising source of a detectable cosmological background of stochastic gravitational waves is a phase transition in the early Universe [3,4]. A first-order phase transition proceeds via the random nucleation of bubbles of the new phase, which subsequently expand and merge, converting the old phase to the new phase. The coherent motion of the bubble walls, which contain a significant fraction of the free energy associated with the phase transition, can produce copious gravitational radiation [5–7]. The radiation spectrum generically peaks at a comoving wavelength corresponding to the Hubble length at the time of the phase transition times the bubble wall velocity in units of the speed of light. Remarkably, the horizon scale at the electroweak phase transition falls into the frequency band of the proposed Laser Interferometer Space Antenna (LISA) space-based laser interferometric gravitational radiation detector [8], and a reasonably strong electroweak phase transition (although much stronger than in the standard model) would be detectable with currently planned gravitational wave experiments [9].

Besides the bubble wall motions in a phase transition, a related source of gravitational radiation is the subsequent turbulent motion of the plasma following the phase transition. Dimensional analysis suggests that turbulence might contribute a gravitational radiation background comparable to or larger than that from bubble wall motions [9,10]. In the absence of bubble shape instabilities, the bubbles of the low-temperature phase will expand spherically until encountering other expanding bubbles. After the bubbles collide, a region of complex, turbulent plasma motions will result since large amounts of energy are being injected on a particular characteristic length scale. As the phase transition completes, the bubble wall motions sourcing the turbulence cease to be effective, and the turbulence damps away with a characteristic damping time scale depending on the plasma viscosity. If the bubbles are unstable to distortions of their shapes, then the expansion of the non-spherical bubbles can also create additional turbulence. If the turbulence is strong, with velocities some non-negligible fraction of the speed of light, significant gravitational radiation can be generated during the interval between the initial bubble collisions and the damping of the turbulence after the completion of the phase transition.

In this paper, we quantify these claims by computing the gravitational radiation resulting from an idealized turbulent source. We assume that a source of turbulence exists for some specified length of time, injecting energy on a particular length scale at a particular redshift. We model the resulting turbulence as having a Kolmogoroff energy spectrum. Details of the turbulence model and discussion of the validity of various assumptions are presented in Sec. II. We then compute the generated gravitational waves using the turbulent plasma motions as a source to the wave equation (Sec. III); the results are then converted to present-day amplitudes and energy densities as functions of frequency. Section IV derives the additional gravitational radiation generated by turbulence-induced magnetic fields, showing that the peak amplitude from this source will be far smaller than the peak amplitude from the turbulence itself, though at a higher frequency. In Sec. V, we apply the results to a generic model of first-order phase transitions, including a brief review of hydrodynamic bubble evolution. Section VI discusses the detectability of the resulting backgrounds with planned and envisioned experiments. Throughout the paper we employ natural units with $c = \hbar = k_B = 1$.

A substantial literature on cosmological turbulence appeared three decades ago, when turbulent vorticity was considered as a mechanism for initiating galaxy formation [11]. While this particular idea soon fell out of favor due to inconsistency with the microwave background isotropy [12] and nucleosynthesis [13], some formal aspects of these treatments are relevant for this work; see, e.g., [14–16], which develop phenomenological descriptions of cosmological turbulence similar in spirit to that presented in this paper. The hydrodynamic equations in an expanding Universe were derived through a transformation of the nonexpanding case in [17], a special case of a more general theorem [18].

We emphasize that our results are independent of the nature of the turbulence source. While first-order phase transitions are the only obvious source of strong turbulence in the early Universe, the calculations presented here are equally applicable to any other potential source of turbulence (see, e.g., [19]).

II. MODEL ISOTROPIC TURBULENCE

The theory of turbulence was originally formulated over sixty years ago [20,21]. But the complexity of turbulent motion makes any analysis beyond basic scaling considerations and dimensional analysis intractable. Model isotropic turbulence is experimentally tested via wind tunnel measurements on scales small compared to the size of the tunnel, and the concepts of a cascade of kinetic energy from large to small scales and the role of viscosity are well established. But classical turbulence analysis is done for non-relativistic fluid velocities and incompressible fluids. Here we need to model turbulence in a radiation-dominated plasma, potentially with moderately relativistic fluid velocities and complications like shock formation. While the theory of turbulence in highly relativistic plasmas is not well understood, we will simply extend the nonrelativistic results in the naive manner with the understanding that some corrections might apply.

Consider an event in the early Universe, presumably a first-order phase transition, which converts an energy density $\kappa\rho_{\rm vac}$ into kinetic energy of the primordial plasma in some characteristic time scale $\tau_{\rm stir}$ on some characteristic source length scale L_S. Here $\rho_{\rm vac}$ is the total free energy density liberated and κ is an efficiency factor which accounts for the fraction of the available energy which goes directly into kinetic, as opposed to thermal, energy. The length scale L_S must be connected to the Hubble length $H_*^{-1} \simeq m_{\rm Pl}/T_*^2$, which is the only cosmological length scale at early times; we write $L_S \equiv \gamma H_*^{-1}$. Here T_* is the temperature of the Universe when the event takes place and $m_{\rm Pl}$ is the Planck mass. Under suitable conditions discussed below, a turbulent cascade will develop in which energy will be transferred from larger to smaller scales as eddies of progressively smaller sizes are formed from larger ones. The cascade stops at a damping scale L_D when the fluid kinematic viscosity ν diffuses the turbulent velocities at the same rate as they are replenished from larger scales. We assume that for scales L in the range $L_D < L < L_S$ (the inertial range), the turbulence is homogeneous and isotropic. We also must know the enthalpy density $w = \rho + p$ of the (nonturbulent) plasma, which appears in the stress-energy tensor. In our simplified model, any turbulent source in the early Universe is determined completely by the physical quantities $\rho_{\rm vac}$, κ, $\tau_{\rm stir}$, L_S, T_*, w, and ν. These quantities in turn determine L_D, the damping scale, and τ, the total duration of the turbulence. Note that a given cosmological model determines w and ν from the temperature T_*. We also define the wave numbers $k_S = 2\pi/L_S$ and $k_D = 2\pi/L_D$ corresponding to the largest and smallest turbulence scales.

The turbulent energy in the cascade is characterized by the stationary Kolmogoroff spectrum

$$E(k) \equiv \frac{1}{w}\frac{d\rho_{\rm turb}}{dk} = C_k \bar{\varepsilon}^{2/3} k^{-5/3}, \tag{1}$$

where $\rho_{\rm turb}$ is the kinetic energy density of the turbulent motions. The Kolmogoroff constant C_k is of order unity and $\bar{\varepsilon}$ is the energy dissipation rate per unit enthalpy given by [22]

$$\bar{\varepsilon} = 2\nu \int_{k_S}^{k_D} dk\, k^2 E(k) \tag{2}$$

where ν is the kinematic viscosity of the plasma. This spectrum holds for a constant rate of energy flow from larger to smaller scales; the amplitude is fixed by the rate of energy dissipation. For a non-relativistic plasma, the enthalpy density w is just the mass density of the plasma, while for temperatures large compared to the masses of particles in the plasma or for any radiation-dominated plasma, w is $4/3$ times the thermal energy density of the plasma. Combining the above two equations and solving for the energy dissipation rate gives

$$\bar{\varepsilon} \simeq \frac{27}{8} k_D^4 \nu^3 \tag{3}$$

assuming $C_k = 1$ and $k_S \ll k_D$. However, $E(k)$ is not yet determined since we do not know the wave number k_D corresponding to the smallest-scale turbulent motions.

Before completing the specification of $E(k)$ in terms of the physical variables defining the phase transition, consider the time scales involved in the turbulence. Assume that the only peculiar velocities present are (relativistic) turbulent velocities with spatial distribution $\mathbf{u}(\mathbf{x})$; we employ the Fourier convention

$$\mathbf{u}(\mathbf{k}) = \frac{1}{V}\int d\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{u}(\mathbf{x}). \tag{4}$$

We retain the fiducial volume factor V to insure consistent dimensions for all quantities; all physical results will be independent of V. A statistically isotropic and homogeneous velocity field of an incompressible fluid has the two-point correlation function

$$\left\langle u_i(\mathbf{k}) u_j^*(\mathbf{k}')\right\rangle = \frac{(2\pi)^3}{V} P_{ij}(\hat{\mathbf{k}}) P(k) \delta(\mathbf{k}-\mathbf{k}'), \tag{5}$$

where

$$P_{ij}(\hat{\mathbf{k}}) \equiv \delta_{ij} - \hat{k}_i \hat{k}_j \tag{6}$$

is a projector onto the transverse plane:

$$P_{ij}P_{jk} = P_{ik}, \qquad\qquad P_{ij}\hat{k}_j = 0. \tag{7}$$

The angular brackets in Eq. (5) mean a statistical average when the velocities are considered as random variables (see Ref. [22], Volume 1, for a detailed discussion). If the fluid is compressible, a second arbitrary function appears in the correlation function, proportional to $\hat{k}_i\hat{k}_j$, describing longitudinal motions. A specific model for isotropic turbulence consists of specifying the function $P(k)$; we assume the power spectrum is a power law, $P(k) = Ak^n$, where the normalization A and the spectral index n can be deduced from the Kolmogoroff spectrum. The mean square velocity of the fluid at any point in space is given by

$$\left\langle \mathbf{u}^2(\mathbf{x})\right\rangle = \frac{V}{(2\pi)^3}\int d\mathbf{k}\, 2P(k) = \frac{V}{\pi^2}\int_{k_S}^{k_D} dk\, k^2 P(k). \tag{8}$$

But this quantity is just the kinetic energy density per unit enthalpy density of the fluid; thus we derive the connection

$$E(k) = \frac{V}{\pi^2} k^2 P(k). \tag{9}$$

For the case of a Kolmogoroff spectrum, Eq. (1) implies that

$$P(k) \simeq \frac{1}{V}\pi^2 \bar{\varepsilon}^{2/3} k^{-11/3}. \tag{10}$$

We are interested in the characteristic eddy velocity on a given scale L. From the slope of the Kolmogoroff spectrum and Eq. (8), it follows that the total turbulent velocity at a given point is dominated by the eddy velocity on the largest scale. We can thus estimate the characteristic eddy velocity on the scale L by cutting off the integral in Eq. (8) at a wave number $k_L = 2\pi/L$ corresponding to that scale:

$$\begin{aligned} u_L &\simeq \left[\int_{k_L}^{k_D} dk\, E(k)\right]^{1/2} \\ &= \left(\frac{3}{2}\right)^{1/2} (2\pi)^{-1/3} (\bar{\varepsilon} L)^{1/3}. \end{aligned} \tag{11}$$

We can also estimate an eddy turnover time scale (known as the circulation time) on a length scale L as the ratio of L to the physical velocity $v_L = u_L/(1+u_L^2)^{1/2}$. We argue below that the physical velocity will be approximately bounded by the sound speed of the fluid; for a radiation-dominated plasma, this condition is $v_L \leq 1/\sqrt{3}$. Making the simple approximation that $v_L = u_L$ until the sound speed is reached, after which time v_L is the sound speed, the circulation time is

$$\tau_L \simeq L/v_L \simeq \begin{cases} \frac{3}{2}\bar{\varepsilon}^{-1/3} L^{2/3}, & L \leq 3^{3/2}(8\bar{\varepsilon})^{-1}; \\ L\sqrt{3}, & \text{otherwise.} \end{cases} \tag{12}$$

Now the remaining undetermined quantity in the turbulence spectrum, k_D, can be fixed via energy considerations. Two different cases must be considered separately, depending on whether the duration of the turbulent source $\tau_{\rm stir}$ is long or short compared to the eddy turnover time scale τ_S on the characteristic length scale of the source L_S. First consider the simpler case where $\tau_{\rm stir} \gg \tau_S$. Fully developed turbulence is established in a time on the order of τ_S, so this case gives approximately a stationary source lasting for a time $\tau = \tau_{\rm stir}$. To keep the turbulence stationary, the energy dissipation rate must equal the mean input power of the source:

$$\bar{\varepsilon} = \frac{\kappa \rho_{\rm vac}}{w \tau_{\rm stir}}. \tag{13}$$

This expression immediately determines the amplitude of the Kolmogoroff spectrum, Eq. (1), and comparing with Eq. (3) gives

$$k_D \simeq \left(\frac{8\kappa\rho_{\rm vac}}{27\nu^3 \tau_{\rm stir} w}\right)^{1/4}. \tag{14}$$

Thus the turbulent gravitational wave source is completely determined for this case. The circulation time scale on the scale of the source is approximated by combining Eqs. (12) and (13) to give

$$\tau_S \simeq \frac{3}{2}\left(\frac{L_S^2 \tau_{\rm stir} w}{\kappa\rho_{\rm vac}}\right)^{1/3}, \tag{15}$$

so the condition for this case to be valid becomes

$$\tau_{\rm stir} \gg L_S \left(\frac{w}{\kappa\rho_{\rm vac}}\right)^{1/2}. \tag{16}$$

Finally, the Reynolds number for this turbulence is given by

$${\rm Re} = \left(\frac{k_D}{k_S}\right)^{4/3} \simeq \frac{2}{3}\left(\frac{1}{2\pi}\right)^{4/3}\left(\frac{\kappa\rho_{\rm vac} L_S^4}{\nu^3 \tau_{\rm stir} w}\right)^{1/3}. \tag{17}$$

The critical Reynolds number for the onset of stationary turbulence is around 2000. Early Universe phase transitions will generally have Reynolds numbers exceeding this value.

The alternate case, for $\tau_{\rm stir} \ll \tau_S$, is more subtle. Here, an impulsive force is imparted to the plasma, resulting in a total kinetic energy density equal to the total free energy density of the phase transition times the efficiency factor κ, coherent on the length scale L_S. The efficiency factor depends on the mechanical details of the stirring process and will be a function of mean input power $\rho_{\rm vac}/\tau_{\rm stir}$. A cascade of kinetic energy to smaller scales will occur, but stationary, isotropic turbulence will never develop because the plasma is not continually being stirred by the source.

We can estimate the time for which significant kinetic energy on a given scale lasts. On the largest scale L_S, the kinetic energy will last for a time set by the dissipation time scale, approximately equal to the eddy turnover time τ_S. As in fully developed turbulence, this kinetic energy will cascade to smaller scales. The eddies on the largest scale will act as a source for eddies on a slightly smaller scale L for a time τ_S. On the smaller scale, we assume the plasma has no kinetic energy at the moment of the impulsive force but rather acquires kinetic energy only from the cascade. The smaller-scale eddies are spun up in a time corresponding to the circulation time on the smaller scale τ_L; these eddies will last until the large-scale source becomes ineffectual and then will dissipate also on the circulation time scale τ_L. So by this argument, the eddies on a smaller scale L will exist for the same total amount of time as the eddies on the largest scale L_S, although their establishment and dissipation will be displaced to a slightly later time compared with the largest-scale eddies. The same reasoning can then be applied to eddies at successively smaller scales, with the following conclusion: on any given scale between L_S and L_D, eddies will exist for a total time τ_S. The only assumption required for this conclusion is that the time scale for establishing eddies on a given scale via the cascade from larger scales is the same as the time scale for dissipating the same eddies via the cascade to smaller scales.

The time displacements of the time intervals for the existence of eddies on different scales are essentially irrelevant for the generation of gravitational radiation, leading only to some relative phase shift between the gravitational radiation at two different frequencies. Therefore, for the purposes of modelling a gravitational wave source, we assume the plasma motion consists of kinetic energy simultaneously on all scales within the inertial range, lasting for a total time $\tau = \tau_S$, the circulation time on the scale of the turbulence source, with a kinetic energy density spectrum given by the Kolmogoroff spectrum, Eq. (1). To normalize the spectrum, we simply treat the total free energy density as being injected continually over the time τ_S rather than as an impulse. Now this case looks just like the previous one, except that $\tau_{\rm stir}$ must be replaced by τ_S in Eqs. (14) and (15):

$$k_D \simeq \left(\frac{8\kappa\rho_{\rm vac}}{27\nu^3\tau_S w}\right)^{1/4}, \qquad \tau_S = L_S\left(\frac{3}{2}\right)^{3/2}\left(\frac{w}{\kappa\rho_{\rm vac}}\right)^{1/2}. \tag{18}$$

Combining the two cases gives the simple expressions

$$k_D \simeq \left(\frac{8\kappa\rho_{\rm vac}}{27\nu^3\tau w}\right)^{1/4}, \qquad \tau_S \simeq \frac{3}{2}\left(\frac{L_S^2\tau w}{\kappa\rho_{\rm vac}}\right)^{1/3}, \tag{19}$$

valid for either case, where

$$\tau = \max(\tau_S,\, \tau_{\rm stir}). \tag{20}$$

The eventual expression for the gravitational wave amplitude is only very weakly dependent upon τ, so the distinction between the two cases is largely unimportant for our results.

When computing the gravitational wave signal, we will encounter unequal time velocity correlators of the form

$$\left\langle u_i(\mathbf{k},t)u_j^*(\mathbf{k}',t')\right\rangle \equiv \frac{(2\pi)^3}{V}P_{ij}(\hat{\mathbf{k}})F(k,t-t')\delta(\mathbf{k}-\mathbf{k}') \tag{21}$$

[cf. Eq. (5)]. The dependence of the function F only upon the time difference $t-t'$ follows from the assumption that the turbulence can be treated as stationary, with $F(k,0)=P(k)$. No general form is known for F. However, general physical considerations imply that F must be a decreasing function of $t-t'$, and we assume that the decay of F should have a characteristic time scale on the order of the circulation time on the scale $L=2\pi/k$. We actually will only need to guarantee that F goes to zero no faster than the light-crossing time of L, which is guaranteed by causality.

We have sidestepped the issue of relativistic versus nonrelativistic turbulence. The Kolmogoroff model of turbulence phenomenology has only been formulated and tested for turbulence with nonrelativistic velocities in plasmas with nonrelativistic equations of state. No general model exists for the opposite situation of a relativistic plasma with relativistic velocities. The plasma in our case will also be compressible, contrary to the basic assumption above. For a large enough input of energy, plasma velocities may be driven past the sound speed, leading to shock formation. We conservatively assume that the sound velocity represents an upper limit to the turbulent plasma velocity, because shocks will result in significant thermal dissipation. Note that to the extent that shock fronts retain kinetic energy, our ultimate gravitational wave background will be increased relative to the estimates made here, in the case of highly relativistic fluid velocities.

To summarize, this model for cosmological turbulence requires (i) $\kappa\rho_{\rm vac}$, the energy density converted to turbulent motion where $\rho_{\rm vac}$ is a characteristic energy density and κ is an efficiency factor; (ii) L_S, the characteristic length scale of the source producing the turbulence (the "stirring scale"); (iii) $\tau_{\rm stir}$, the duration of the source producing the turbulence; (iv) T_*, the temperature of the Universe at the onset of the turbulence, which in turn determines w, the enthalpy density, and ν, the kinematic viscosity of the plasma. The assumption of stationary homogeneous and isotropic Kolmogoroff turbulence then specifies in terms of these quantities (i) the normalization of the turbulence power spectrum, (ii) the length scale L_D at which the turbulence is dissipated by viscosity, and (iii) the circulation time for any particular turbulent length scale between L_S and L_D.

We have neglected the expansion of the Universe in this description of turbulence. If the duration of the turbulence τ is longer than the Hubble time H^{-1}, then the expansion will produce additional damping of the turbulence as the energy density is redshifted. Furthermore, if the circulation time on the stirring scale τ_S is comparable to or longer than the Hubble time, the expansion damping may inhibit the establishment of a turbulent cascade. Particular cases should be checked individually, but in general, if a phase transition is strong enough to drive turbulence producing an interestingly large gravitational radiation amplitude, it will last for a time short compared to the Hubble time and expansion damping will be negligible. This claim can be quantified using the expressions derived in Sec. V below.

III. GRAVITATIONAL RADIATION FROM TURBULENT PLASMA

A. General Considerations

The source of gravitational radiation is the transverse and traceless piece of the stress-energy tensor of a given system. For turbulent plasma, the relevant stress-energy tensor is given by

$$T_{ij}(\mathbf{x}) = w u_i(\mathbf{x}) u_j(\mathbf{x}). \tag{22}$$

The above expression drops the diagonal (trace) component of the stress-energy because it cannot source any gravitational radiation. To simplify the problem, we assume (conservatively) that the enthalpy density w remains constant throughout space, while the variation of the velocity vector describes the turbulent motions of the plasma. If this assumption does not hold, the resulting gravitational wave amplitude will increase. In Fourier space, the stress-energy is then given by the convolution

$$T_{ij}(\mathbf{k}, t) \simeq \frac{V}{(2\pi)^3} w \int d\mathbf{q}\, u_i(\mathbf{q}, t) u_j(\mathbf{k} - \mathbf{q}, t). \tag{23}$$

Gravitational radiation is produced by the transverse and traceless piece of the stress-energy tensor. Given an arbitrary stress-energy tensor in Fourier space, $T_{ij}(\mathbf{k}, t)$, the portion sourcing gravitational radiation can be obtained by applying a projection tensor (see, e.g., [23]):

$$\Pi_{ij} = \left(P_{il} P_{jm} - \frac{1}{2} P_{ij} P_{lm} \right) T_{lm}. \tag{24}$$

Once the source is specified, the gravitational wave metric perturbations h_{ij} obey the wave equation

$$\frac{d^2 h_{ij}}{d\eta^2} + \frac{2}{a}\frac{da}{d\eta}\frac{dh_{ij}}{d\eta} + \tilde{k}^2 h_{ij} = 8\pi G a^2 \Pi_{ij} \tag{25}$$

where η is conformal time, $\tilde{k}$ is the comoving wave number, and a is the scale factor of the Universe. Note that we have defined the tensor metric perturbation as $\delta g_{ij} \equiv 2h_{ij}$. For relevant phase transitions, the duration of the source will be short compared to the Hubble time, which means the expansion of the Universe can be neglected during the generation of the waves. We can thus drop the expansion drag term in Eq. (25) and change variables to physical time and physical wave number, obtaining the simple oscillator equation

$$\ddot{h}_{ij}(\mathbf{k}, t) + k^2 h_{ij}(\mathbf{k}, t) = 8\pi G \Pi_{ij}(\mathbf{k}, t), \tag{26}$$

where dots denote derivatives with respect to t. From this point on, all wave numbers will refer to physical, not comoving, quantities.

The source considered here turns on at a specific time t_* and we assume no gravitational radiation exists prior to this time. The initial conditions for Eq. (26) are simply $h_{ij}(\mathbf{k}, t_*) = \dot{h}_{ij}(\mathbf{k}, t_*) = 0$. In the Euclidean space approximation we have made, the radiation generated cannot depend on the particular value of t_*, so for convenience we set $t_* = 0$ in this section. Of course, once the results are translated back into expanding spacetime, the time t_* of the phase transition fixes the energy and length scale associated with the phase transition. The Green function for the homogeneous equation is simply

$$G(t,t') = \begin{cases} 0, & 0 < t < t', \\ \frac{1}{k}\sin\left[k(t-t')\right], & 0 < t' < t, \end{cases} \tag{27}$$

with $G = \dot{G} = 0$ at $t = 0$. The general solution for the wave amplitude is then

$$h_{ij}(\mathbf{k},t) = \frac{8\pi G}{k}\int_0^\tau \Theta(t-t')\sin\left[k(t-t')\right]\Pi_{ij}(\mathbf{k},t')dt' \tag{28}$$

where Θ is a step function.

B. Time Averaging Technique

Since turbulence is a stochastic process, we cannot compute the exact gravitational waveforms. Our goal is to compute the average power spectrum or characteristic amplitude of the waves. We are concerned here only with the power spectrum, so consider the quantity

$$\begin{aligned}\left\langle h_{ij}(\mathbf{k},t)h^*_{ij}(\mathbf{k}',t)\right\rangle &= \frac{(8\pi G)^2}{V}\delta(\mathbf{k}-\mathbf{k}') \\ &\times\left\langle \frac{1}{k^2}\int_0^\tau dt_1\int_0^\tau dt_2\,\Theta(t-t_1)\Theta(t-t_2)\sin\left[k(t-t_1)\right]\sin\left[k(t-t_2)\right]\Pi_{ij}(\mathbf{k},t_1)\Pi^*_{ij}(\mathbf{k},t_2)\right\rangle.\end{aligned} \tag{29}$$

The delta-function factor is guaranteed by statistical isotropy of the gravitational waves; we have written this dependence out explicitly and then changed all factors of $\mathbf{k}'$ to $\mathbf{k}$ within the angular brackets. To make further progress, we need a practical way to deal with the averaging process. We are assuming a stationary, homogeneous and isotropic source, so we make the simple assumption that the statistical average can be estimated by either a time or space average. To evaluate Eq. (29) we use a time average, since all of the time dependence is in the Green functions and not in the source terms. Then we have

$$\begin{aligned}\left\langle h_{ij}(\mathbf{k},t)h^*_{ij}(\mathbf{k}',t)\right\rangle &= \delta(\mathbf{k}-\mathbf{k}')\frac{(8\pi G)^2}{Vk^2}\int_0^\tau dt_1\int_0^\tau dt_2\,\Pi_{ij}(\mathbf{k},t_1)\Pi^*_{ij}(\mathbf{k},t_2) \\ &\times\frac{1}{T}\int_s^{s+T} dt\,\Theta(t-t_1)\Theta(t-t_2)\sin\left[k(t-t_1)\right]\sin\left[k(t-t_2)\right],\end{aligned} \tag{30}$$

where s is some arbitrary time when the source is active, and T is an interval of time long enough for the average to be approximated by the time average. In practice, this will be some time on the order of a few circulation times on a given scale. As t_1 or t_2 approaches τ, it will not be possible to choose T large enough for a rigorously valid average, but this will not appreciably affect our estimates since we are considering only statistical averages for the source terms: the time integration is a convenient device for approximating the effect of this averaging, and the averaging itself becomes only a rough approximation for durations shorter than the circulation time on a given scale. Since we are assuming a stationary source, Eq. (30) must be independent of the chosen value of s. We choose an s which eliminates the step functions from the integral, keeping in mind the above discussion.

The integral over t is now elementary:

$$\frac{1}{T}\int_s^{s+T} dt\sin\left[k(t-t_1)\right]\sin\left[k(t-t_2)\right] = \frac{1}{2}\cos\left[k(t_2-t_1)\right] - \frac{1}{2Tk}\sin(Tk)\cos\left[k(2s+T-t_1-t_2)\right]. \tag{31}$$

We neglect the second term with respect to the first since $Tk \gg 1$: k^{-1} will be on the order of the light crossing time for a given scale, while T will be at least as long as the circulation time on the given scale k^{-1}, so the comparison will be valid on all scales except for possibly the largest, where at least the simple inequality $Tk > 1$ will hold. Since the terms are both oscillatory, the comparison really only applies to the size of the prefactors, but this is sufficient for our purpose. Now substituting Eq. (31) into Eq. (30) and making the substitution $y = t_2 - t_1$ gives

$$\left\langle h_{ij}(\mathbf{k},\tau)h^*_{ij}(\mathbf{k}',\tau)\right\rangle \simeq \delta(\mathbf{k}-\mathbf{k}')\frac{(8\pi G)^2}{2Vk^2}\int_{-t_1}^{\tau-t_1} dy\,\cos(ky)\int_0^\tau dt_1\,\Pi_{ij}(\mathbf{k},t_1)\Pi^*_{ij}(\mathbf{k},t_1+y). \tag{32}$$

We now use the t_1 integral as an estimator for the statistical average of the sources, giving

$$\left\langle h_{ij}(\mathbf{k},\tau)h^*_{ij}(\mathbf{k}',\tau)\right\rangle \simeq \frac{(8\pi G)^2\tau}{2kk'}\int_0^\tau dy\cos(ky)\left\langle \Pi_{ij}(\mathbf{k},t_1)\Pi^*_{ij}(\mathbf{k}',t_1+y)\right\rangle, \tag{33}$$

where the delta-function has been reabsorbed into the statistical average. Note that the average on the right side is independent of t_1 since the source is assumed to be stationary.

We now have an expression involving the average source correlation at different times, integrated against an oscillating function. Note that the total value is proportional to τ, the duration of the source, as it should be for an incoherent source. To make further progress, we require a more explicit form for the source average.

C. Evaluation of the Source Average

We have expressions for averages of the fluid velocities in the turbulent source; we need to connect these with the particular average required in Eq. (33). Writing out the projectors in Eq. (24) gives

$$\begin{aligned}\left\langle \Pi_{ij}(\mathbf{k},t)\Pi^*_{ij}(\mathbf{k}',t+y)\right\rangle &= \left[P_{ia}(\hat{\mathbf{k}})P_{jb}(\hat{\mathbf{k}})-\frac{1}{2}P_{ij}(\hat{\mathbf{k}})P_{ab}(\hat{\mathbf{k}})\right]\left[P_{ic}(\hat{\mathbf{k}}')P_{jd}(\hat{\mathbf{k}}')-\frac{1}{2}P_{ij}(\hat{\mathbf{k}}')P_{cd}(\hat{\mathbf{k}}')\right]\\ &\quad\times\left\langle T_{ab}(\mathbf{k},t)T^*_{cd}(\mathbf{k}',t+y)\right\rangle.\end{aligned} \tag{34}$$

We need to evaluate the expectation value of the stress tensor product. Equation (23) shows that this product will involve the expectation value of four velocity vectors evaluated at two different times. No general solution is known for such expectation values for turbulent flow. The simplest (and most conservative) assumption is that the correlation function factors into products of pairs of velocities, as for a Gaussian field. Then Wick's theorem applies and we have

$$\begin{aligned}\left\langle T_{ab}(\mathbf{k},t)T^*_{cd}(\mathbf{k}',t+y)\right\rangle &= \frac{V^2}{(2\pi)^6}w^2\int d\mathbf{q}\,d\mathbf{s}\Big[\left\langle u_a(\mathbf{q},t)u^*_b(\mathbf{q}-\mathbf{k},t)\right\rangle\left\langle u_c(-\mathbf{s},t+y)u^*_d(\mathbf{k}'-\mathbf{s},t+y)\right\rangle\\ &+\left\langle u_a(\mathbf{q},t)u^*_c(\mathbf{s},t+y)\right\rangle\left\langle u_b(\mathbf{k}-\mathbf{q},t)u^*_d(\mathbf{k}'-\mathbf{s},t+y)\right\rangle+\left\langle u_a(\mathbf{q},t)u^*_d(\mathbf{k}'-\mathbf{s},t+y)\right\rangle\left\langle u_b(\mathbf{k}-\mathbf{q},t)u^*_c(\mathbf{s},t+y)\right\rangle\Big].\end{aligned} \tag{35}$$

This expression can be simplified using the correlation functions in Eqs. (5) and (21), giving

$$\left\langle T_{ab}(\mathbf{k},t)T^*_{cd}(\mathbf{k}',t+y)\right\rangle = w^2\delta(\mathbf{k}-\mathbf{k}')\int d\mathbf{q}\left[P_{ac}(\hat{\mathbf{q}})P_{bd}(\widehat{\mathbf{k}-\mathbf{q}})+P_{ad}(\hat{\mathbf{q}})P_{bc}(\widehat{\mathbf{k}-\mathbf{q}})\right]F(q,y)F(|\mathbf{k}-\mathbf{q}|,y). \tag{36}$$

The first of the three terms in Eq. (35) does not contribute, since it is nonzero only for the constant offset mode with $\mathbf{k}=\mathbf{k}'=0$. After substituting the explicit form for the projectors, Eq. (6), setting $\mathbf{k}=\mathbf{k}'$ from the delta function, and simplifying the contractions, we obtain

$$\left\langle \Pi_{ij}(\mathbf{k},t)\Pi^*_{ij}(\mathbf{k}',t+y)\right\rangle = w^2\delta(\mathbf{k}-\mathbf{k}')\int d\mathbf{q}\,F(q,y)F(|\mathbf{k}-\mathbf{q}|,y)(1+\gamma^2)(1+\beta^2), \tag{37}$$

where we have defined the auxiliary quantities $\gamma=\hat{\mathbf{k}}\cdot\hat{\mathbf{q}}$ and $\beta=\hat{\mathbf{k}}\cdot\widehat{\mathbf{k}-\mathbf{q}}$.

Substituting this simple form for the unequal time source correlation into Eq. (33) gives

$$\left\langle h_{ij}(\mathbf{k},\tau)h^*_{ij}(\mathbf{k}',\tau)\right\rangle = \frac{(8\pi G)^2\tau w^2}{2k^2}\delta(\mathbf{k}-\mathbf{k}')\int d\mathbf{q}\,(1+\gamma^2)(1+\beta^2)\int_0^\tau dy\cos(ky)F(q,y)F(|\mathbf{k}-\mathbf{q}|,y). \tag{38}$$

Now $F(k,0)=P(k)$ so we make the further assumption that F can be separated as

$$F(k,y)=P(k)D(yk^{2/3}); \tag{39}$$

that is, we have assumed a universal form for the time decay for all k values, with the time argument of F scaling with the circulation time on the length scale $2\pi/k$, and D is some monotonically decreasing function of its argument. This is likely a reasonable assumption for fully developed turbulence. On the other hand, we are only concerned with the time dependence to the extent that it is integrated against the oscillatory function $\cos(kt)$ in Eq. (38). Since F or D is everywhere positive, the integral itself is oscillatory. If our crude turbulent model were exact, the induced power spectrum of gravitational waves would exhibit oscillations. But this is an artifact of the assumption that the turbulence begins and ends at precisely defined times. For the present task of estimating characteristic amplitudes for a realistic turbulence source, we instead approximate the time integral by its root-mean-square value. The $\cos(ky)$ term will always oscillate on a time scale shorter than the characteristic time for $D(yk^{2/3})$, as seen from a simple comparison of the circulation time to the light crossing time for a given scale L. Thus regardless of the particular time dependence of D, we approximate

$$\int_0^\tau dy \cos(ky) F(q,y) F(|\mathbf{k}-\mathbf{q}|,y) \simeq \int_0^\tau dy \cos(ky) P(q) P(|\mathbf{k}-\mathbf{q}|) \simeq \frac{\sqrt{2}}{2k} P(q) P(|\mathbf{k}-\mathbf{q}|). \tag{40}$$

This approximation replaces the time-dependent function D by the constant $D(0)$. Actually D will decrease with time. This will *increase* the mean value of the integral unless the characteristic time scale for the decrease of D is less than k^{-1}, which we have argued will never be obtained, so the approximation in Eq. (40) is actually a conservative one.

Substituting this result into Eq. (38) and replacing γ^2 and β^2 by their average values of 1/2 over the integral gives the simple approximate form

$$\left\langle h_{ij}(\mathbf{k},\tau) h^*_{ij}(\mathbf{k}',\tau)\right\rangle \simeq \frac{9\sqrt{2}(8\pi G)^2 \tau w^2}{16k^3} \delta(\mathbf{k}-\mathbf{k}') \int d\mathbf{q}\, P(q) P(|\mathbf{k}-\mathbf{q}|). \tag{41}$$

We now have an expression which can be evaluated for the particular turbulent power spectrum to give the final expression for the power in gravitational radiation in terms of the turbulence parameters.

D. The Power Spectrum

For a power law power spectrum, the remaining integral in Eq. (41) is elementary. Using the general form $P(k) = Ak^n$,

$$\begin{aligned} \int d\mathbf{q}\, P(q) P(|\mathbf{k}-\mathbf{q}|) &= 2\pi A^2 \int_{k_S}^{k_D} dq\, q^{n+2} \int_{-1}^{1} d\gamma\, (k^2+q^2-2kq\gamma)^{n/2} \\ &= 4\pi A^2 \left[\frac{k^{2n+3} n}{(n+3)(2n+3)} + \frac{k_D^{2n+3}}{2n+3} - \frac{k^n k_S^{n+3}}{n+3} \right]. \end{aligned} \tag{42}$$

For the specific case of Kolmogoroff turbulence, the power law is $n = -11/3$; then the last term in Eq. (42) is dominant. Keeping only this term and inserting Eq. (10) for the power spectrum gives

$$\left\langle h_{ij}(\mathbf{k},\tau) h^*_{ij}(\mathbf{k}',\tau)\right\rangle \simeq \frac{216\pi^7 \sqrt{2} G^2}{V^2} \tau w^2 \bar{\varepsilon}^{4/3} k^{-20/3} k_S^{-2/3} \delta(\mathbf{k}-\mathbf{k}'). \tag{43}$$

To make contact with measurable quantities, we evaluate the real-space correlation function

$$\begin{aligned} \langle h_{ij}(\mathbf{x},\tau) h_{ij}(\mathbf{x},\tau)\rangle &= \frac{V^2}{(2\pi)^6} \int d\mathbf{k}\, d\mathbf{k}'\, e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} \left\langle h_{ij}(\mathbf{k},\tau) h^*_{ij}(\mathbf{k}',\tau)\right\rangle \\ &\simeq \frac{27\sqrt{2}\pi^2}{2} G^2 \tau w^2 \bar{\varepsilon}^{4/3} k_S^{-2/3} \int_{k_S}^{k_D} dk\, k^{-14/3}. \end{aligned} \tag{44}$$

Now we need to convert this expression to one involving the gravitational wave frequency f. The frequency is determined by the scale of time variation corresponding to the spatial Fourier mode k, the circulation time τ_L given in Eq. (12). Writing $f = \tau_L^{-1}$ and changing variables in the k integral gives

$$\langle h_{ij}(\mathbf{x},\tau) h_{ij}(\mathbf{x},\tau)\rangle \simeq \frac{2^{2/3}}{3^{5/2}\pi^{7/3}} \bar{\varepsilon}^{7/2} G^2 \tau w^2 f_S^{-1} \int_{f_S}^{f_D} df\, f^{-13/2}; \tag{45}$$

the numerical prefactor is about 0.007. We define the characteristic gravitational wave amplitude $h_c(f)$ per unit logarithmic frequency interval (following Maggiore [24]) via

$$\langle h_{ij}(\mathbf{x},\tau) h_{ij}(\mathbf{x},\tau)\rangle \equiv \frac{1}{2} \int_0^\infty \frac{df}{f} h_c^2(f). \tag{46}$$

Note that Eq. (46) is smaller than the corresponding expression in Ref. [24] by a factor of 4 since the tensor metric perturbation in Ref. [24] is defined as $\delta g_{ij} \equiv h_{ij}$ whereas ours is $\delta g_{ij} \equiv 2h_{ij}$ [see comments after Eq. (25)]. Comparing with Eq. (45) gives

$$h_c(f) = 0.12 G\bar{\varepsilon}^{7/4} w \tau^{1/2} f_S^{-1/2} f^{-11/4} \tag{47}$$

for frequencies between f_S and f_D, with the frequency at the stirring scale

$$f_S \simeq \frac{2}{3}\bar{\varepsilon}^{1/3} L_S^{-2/3}. \tag{48}$$

E. Relic Gravitational Radiation

The above expressions apply to the waves generated at the time of the phase transition. We then stretch the waves with the expansion of the Universe: the frequency and amplitude are both inversely proportional to the scale factor. The latter follows from the fact that the total energy density in gravitational radiation scales like a^{-4} with the expansion, and the energy density is proportional to $\langle \dot{h}\dot{h} \rangle$. For turbulence generated at a time when the temperature of the Universe was T_*, the ratio of the scale factor then to the scale factor now is

$$\frac{a_*}{a_0} = 8.0 \times 10^{-16} \left(\frac{100}{g_*}\right)^{1/3} \left(\frac{100\,\mathrm{GeV}}{T_*}\right) \tag{49}$$

where g_* is the number of relativistic degrees of freedom at the temperature T_*. The Hubble parameter at this time is

$$H_*^2 = \frac{8\pi G}{3}\rho_{\mathrm{rad}} = \frac{8\pi^3 g_* T_*^4}{90 m_{\mathrm{Pl}}^2} \tag{50}$$

with m_{Pl} the Planck mass. This gives the relation

$$\tilde{f} = 1.65 \times 10^{-5}\,\mathrm{Hz} \left(\frac{f_*}{H_*}\right) \left(\frac{T_*}{100\,\mathrm{GeV}}\right) \left(\frac{g_*}{100}\right)^{1/6} \tag{51}$$

where f_* is a radiation frequency at the cosmic temperature T_* and $\tilde{f}$ is the corresponding frequency of the radiation today. Scaling Eqs. (47) and (48) by the expansion of the Universe and substituting $w = 4\rho_{\mathrm{rad}}/3$ and $\bar{\varepsilon} = \kappa\rho_{\mathrm{vac}}/(w\tau)$ gives

$$h_c(\tilde{f}) = 5.6 \times 10^{-17} \left(\frac{\kappa\rho_{\mathrm{vac}}}{w}\right)^{2/3} \left(\frac{\tau}{H_*^{-1}}\right)^{-1/6} \left(\frac{L_S}{H_*^{-1}}\right)^{13/6} \left(\frac{100\,\mathrm{GeV}}{T_*}\right) \left(\frac{100}{g_*}\right)^{1/3} \left(\frac{\tilde{f}}{\tilde{f}_S}\right)^{-11/4}, \tag{52}$$

for the characteristic amplitude, which holds for $\tilde{f} > \tilde{f}_S$, and

$$\tilde{f}_S = 1.1 \times 10^{-5}\,\mathrm{Hz} \left(\frac{\kappa\rho_{\mathrm{vac}}}{w}\right)^{1/3} \left(\frac{\tau}{H_*^{-1}}\right)^{-1/3} \left(\frac{L_S}{H_*^{-1}}\right)^{-2/3} \left(\frac{T_*}{100\,\mathrm{GeV}}\right) \left(\frac{g_*}{100}\right)^{1/6}. \tag{53}$$

Equations (52) and (53) are our fundamental results. Converting to the characteristic energy density in gravitational radiation via the relation

$$h_c(\tilde{f}) = 1.3 \times 10^{-18} \left(\frac{\mathrm{Hz}}{\tilde{f}}\right) \sqrt{\Omega_{\mathrm{GW}}(\tilde{f}) h^2}, \tag{54}$$

where h is the current Hubble parameter in units of 100 km/s Mpc^{-1} and $\Omega_{\mathrm{GW}}(\tilde{f})$ is the energy density in gravitational waves per logarithmic frequency interval in units of the current critical density, gives

$$\Omega_{\mathrm{GW}}(\tilde{f}) h^2 = 2.2 \times 10^{-7} \left(\frac{\kappa\rho_{\mathrm{vac}}}{w}\right)^{2} \left(\frac{\tau}{H_*^{-1}}\right)^{-1} \left(\frac{L_S}{H_*^{-1}}\right)^{3} \left(\frac{g_*}{100}\right)^{-1/3} \left(\frac{\tilde{f}}{\tilde{f}_S}\right)^{-7/2}. \tag{55}$$

IV. GRAVITATIONAL RADIATION FROM INDUCED MAGNETIC FIELDS

In addition to the turbulent motions, gravitational radiation also may be generated by magnetic fields arising from a turbulent dynamo mechanism: generically, the turbulence will exponentially amplify any seed magnetic fields until the field strength saturates at equipartition with the turbulent kinetic energy. The characteristic e-folding time scale on a give length scale L will be simply the circulation time τ_L. The mechanism of seed field generation is not clear, but seed fields might naturally arise during a phase transition due to bubble wall instabilities combined with surface charge densities on the bubble walls and magnetohydrodynamic amplification [25]. Once a magnetic field is generated, the high conductivity of the primordial plasma will keep the field frozen in.

It is reasonable to suspect that such a field may give a significant background of gravitational radiation: since the magnetic field has a nonzero stress, it will provide a coherent source term in Eq. (25). Such a magnetic field will act as a gravitational radiation source from the time of the phase transition until the field is damped (or until matter-radiation equality, if the field lasts that long), rather than just during the brief period of turbulence. The following calculation, however, shows that induced magnetic fields produce a maximum characteristic amplitude of gravitational radiation which is always much smaller than the maximum amplitude from the turbulence which generated them. The magnetic field gravitational radiation peaks at a much higher frequency, though, and can have a larger amplitude than the turbulence-induced gravitational radiation at that frequency. As in the previous section, quantities below are physical, except for comoving quantities denoted with a tilde.

A. General Magnetic Field Considerations

First, we assume the turbulence-induced magnetic fields are generated almost instantaneously during the time of the phase transition. To a good approximation, the turbulence-induced magnetic fields are generated within a tiny fraction of the Hubble time H_*^{-1}, thus we can normalize the magnetic field power spectrum at the time of the phase transition. Second, we assume the turbulence-induced magnetic fields are saturated at an equipartition value up to a physical scale L_B at the time of the phase transition. We will leave the ratio between the magnetic field physical saturation scale to the turbulence stirring scale, i.e. L_B/L_S, as a free parameter in our final expressions. This will make the comparison with the previous turbulence results easier. Generally, we expect L_B/L_S to be on the order of 0.003: the turbulence circulation time scales like $L^{2/3}$, so $(L_B/L_S)^{2/3}$ gives the ratio of e-foldings of the magnetic field on the scales L_S and L_B. Conservatively estimating that the turbulence lasts for a single circulation time on the largest scale, a range of L_B/L_S between 0.003 and 0.0017 gives a range of magnetic field amplification factors between 10^{20} and 10^{30}. The exponential amplification makes this estimate robust: making the seed fields smaller by a factor of 10^{10} only reduces L_B by a modest fraction. Third, we assume the turbulence-induced magnetic fields are just frozen into the plasma and retain the form of the spectrum until they are damped away by neutrino viscosity. Damping of magneto-hydrodynamic (MHD) modes by neutrino viscosity is most efficient before and around nucleosynthesis ($T \sim 0.1\,\mathrm{MeV}$). At the time of neutrino decoupling ($T \sim 1\,\mathrm{MeV}$), the neutrino physical mean free path ($l_{\nu\,\mathrm{dec}} \approx 10^{11}\,\mathrm{cm}$) and the Hubble length ($H^{-1}_{\nu\,\mathrm{dec}} \approx 5 \times 10^{10}\,\mathrm{cm}$) are comparable, hence all the subhorizon magnetic perturbations generated during the electroweak phase transition will be damped away by the time of nucleosynthesis (see, e.g., [18,26]). We do not consider any kind of inverse-cascade mechanism that will transfer small-scale magnetic fields to larger scales. Invoking an inverse cascade will spread the magnetic energy to scales larger than L_B and reduce the overall gravity wave amplitude. This will also push the gravitational radiation frequencies to smaller values than those obtained below.

A statistically homogeneous and isotropic stochastic magnetic field has a two-point correlation function given by Eq. (5) with a power spectrum we denote $P_B(k)$. We assume that the turbulence-induced magnetic field exists on scales between the saturation scale L_B and the turbulence damping scale L_D. The mean-square value of the magnetic field is [see, e.g., Eq. (2.7) of Ref. [27]]

$$B^2 = \frac{V}{\pi} \int_{k_B}^{k_D} dk\, k^2 P_B(k). \tag{56}$$

We now normalize $P_B(k)$ using the fact that the turbulence-induced magnetic field energy density is half of the turbulent kinetic energy density

$$\frac{1}{2} w u_B^2 = \frac{B^2}{8\pi}. \tag{57}$$

Using Eqs. (8) and (56), we obtain

$$P_B(k) = 4\pi w P(k) = \frac{4\pi^3 w \bar{\varepsilon}^{2/3} k^{-11/3}}{V} \tag{58}$$

using Eq. (10).

In Fourier space, the turbulence-induced magnetic stress-energy tensor is given by the convolution of the magnetic field [see Eq. (2.9) of Ref. [27]]:

$$T_{ij}^{(B)}(\mathbf{k}, t_*) = \frac{V}{(2\pi)^3}\frac{1}{4\pi}\int d\mathbf{q}\left[B_i(\mathbf{q}, t_*)B_j(\mathbf{k}-\mathbf{q}, t_*) - \frac{1}{2}\delta_{ij}B_l(\mathbf{q}, t_*)B_l(\mathbf{k}-\mathbf{q}, t_*)\right], \tag{59}$$

where the explicit t_* dependence is to remind ourselves of the assumption that the turbulence-induced magnetic fields are generated almost instantaneously during the time of the phase transition. In addition, we have neglected the induced electric field due to the fact that the early Universe is highly conductive. The source for gravitational radiation is given by the transverse-traceless projection of this stress tensor, Eq. (24).

In the absence of any inverse-cascade mechanism, magnetic fields are just frozen into the plasma and evolve by simply redshifting with the Universe's expansion until they are damped away by neutrino viscosity. Therefore, magnetic fields act on a longer time scale than the turbulent fluid velocities. To facilitate the computation, we introduce a comoving quantity $\Pi_{ij}^{(B)}(\tilde{\mathbf{k}})$ corresponding to $\Pi_{ij}^{(B)}(\mathbf{k}, t_*)$ via

$$\Pi_{ij}^{(B)}(\tilde{\mathbf{k}}) \equiv \Pi_{ij}^{(B)}(\mathbf{k}, t_*)a_*^4, \tag{60}$$

where $\tilde{\mathbf{k}}$ is the comoving wave vector corresponding to the physical wave vector $\mathbf{k}$ at the time of the phase transition.

B. Gravitational Radiation Power Spectrum

During the radiation-dominated epoch, $a \propto \eta$ and the homogeneous solutions to Eq. (25) are the zero-order spherical Bessel functions, i.e. $j_0(\tilde{k}\eta)$ and $y_0(\tilde{k}\eta)$. Defining $x \equiv \tilde{k}\eta$ and $x_* \equiv \tilde{k}\eta_*$, where η_* is the conformal time corresponding to the turbulent source generating the magnetic field, the usual Green function technique yields the following inhomogeneous solution for the radiation-dominated epoch:

$$h_{ij}^{(B)}(\tilde{\mathbf{k}}, \eta) = \frac{8\pi G \Pi_{ij}^{(B)}(\tilde{\mathbf{k}})}{\tilde{k}^2}\int_{x_*}^{x} dx'\, \frac{j_0(x')y_0(x) - y_0(x')j_0(x)}{a^2 W(x')}, \tag{61}$$

where W is the Wronskian of the homogeneous solutions

$$W(x) = j_0(x)\frac{d}{dx}y_0(x) - y_0(x)\frac{d}{dx}j_0(x) = \frac{1}{x^2}. \tag{62}$$

Note that in the turbulence case, the time dependence of the turbulent source is known only statistically. The magnetic field, however, is a *coherent* source, and it evolves by frozen flux until being damped away by neutrino viscosity. Therefore in writing down the gravitional wave equation inhomogeneous solution in Eq. (61), the explicit time dependence of the magnetic source is known and we can immediately perform the time integral, unlike the turbulence case. Substituting Eq. (62) into Eq. (61), using the explicit expressions for the zero-order spherical Bessel functions, i.e. $j_0(x) = \sin x/x$ and $y_0 = -\cos x/x$, and the approximation for the scale factor in the radiation-dominated epoch $a(\eta) \simeq H_0\eta\sqrt{\Omega_{\rm rad}}$, we obtain

$$h_{ij}^{(B)}(\tilde{\mathbf{k}}, \eta) \simeq \frac{8\pi G \Pi_{ij}^{(B)}(\tilde{\mathbf{k}})}{\tilde{k}\eta H_0^2 \Omega_{\rm rad}}\xi(\tilde{k}, \eta_*, \eta), \qquad \eta \le \eta_{\tilde{k}}, \tag{63}$$

where $\eta_{\tilde{k}}$ corresponds to the conformal time at which the magnetic perturbation comoving wave number $\tilde{k}$ is damped away by neutrino viscosity. Here we have abbreviated

$$\xi(\tilde{k}, \eta_*, \eta) = \xi(\tilde{k}\eta_*, \tilde{k}\eta) \equiv \int_{\eta_*}^{\eta} d\eta'\, \frac{\sin[\tilde{k}(\eta - \eta')]}{\eta'}. \tag{64}$$

It is simple to see that ξ is an oscillating function with a monotonically decreasing amplitude of oscillation; the amplitude decays more slowly than the η^{-1} dependence of free gravitational waves, since the wave is continually sourced by the magnetic field.

As in the turbulence case, we are interested in the average power spectrum of the waves, so we consider the quantity

$$\langle h_{ij}^{(B)}(\tilde{\mathbf{k}},\eta) h_{ij}^{(B)*}(\tilde{\mathbf{k}}',\eta)\rangle \simeq \left[\frac{8\pi G}{\tilde{k}\eta H_0^2\Omega_{\rm rad}}\xi(\tilde{k},\eta_*,\eta)\right]^2 \langle \Pi_{ij}^{(B)}(\tilde{\mathbf{k}})\Pi_{ij}^{(B)*}(\tilde{\mathbf{k}}')\rangle. \tag{65}$$

From Eq. (60), we have

$$\langle \Pi_{ij}^{(B)}(\tilde{\mathbf{k}})\Pi_{lm}^{(B)*}(\tilde{\mathbf{k}}')\rangle = a_*^8 \langle \Pi_{ij}^{(B)}(\mathbf{k},t_*)\Pi_{lm}^{(B)*}(\mathbf{k}',t_*)\rangle. \tag{66}$$

As in Eq. (2.20) of Ref. [27], the two-point correlation function $\langle \Pi_{ij}^{(B)}(\mathbf{k},t_*)\Pi_{lm}^{(B)*}(\mathbf{k}',t_*)\rangle$ at the time of the phase transition can be written as:

$$\langle \Pi_{ij}^{(B)}(\mathbf{k},t_*)\Pi_{lm}^{(B)*}(\mathbf{k}',t_*)\rangle \equiv \frac{\mathcal{M}_{ijlm}(\hat{\mathbf{k}})}{V}|\Pi^{(B)}(k,t_*)|^2\delta(\mathbf{k}-\mathbf{k}'), \tag{67}$$

where the tensor structure $\mathcal{M}_{ijlm}$ is [Eq. (2.21) of Ref. [27]]

$$\begin{aligned}\mathcal{M}_{ijlm}(\hat{\mathbf{k}}) &\equiv P_{il}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}}) + P_{im}(\hat{\mathbf{k}})P_{jl}(\hat{\mathbf{k}}) - P_{ij}(\hat{\mathbf{k}})P_{lm}(\hat{\mathbf{k}})\\ &= \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl} - \delta_{ij}\delta_{lm} + \hat{k}_i\hat{k}_j\hat{k}_l\hat{k}_m\\ &\quad + \delta_{ij}\hat{k}_l\hat{k}_m + \delta_{lm}\hat{k}_i\hat{k}_j - \delta_{il}\hat{k}_j\hat{k}_m - \delta_{jm}\hat{k}_i\hat{k}_l - \delta_{im}\hat{k}_j\hat{k}_l - \delta_{jl}\hat{k}_i\hat{k}_m\end{aligned} \tag{68}$$

and satisfies $\mathcal{M}_{ijij} = 4$ and $\mathcal{M}_{iilm} = \mathcal{M}_{ijll} = 0$. Then using Eqs. (59), (24), and (6), a similar calculation as in the previous section (see also the Appendix of Ref. [27]) gives

$$\langle \Pi_{ij}^{(B)}(\mathbf{k},t_*)\Pi_{ij}^{(B)*}(\mathbf{k}',t_*)\rangle = \frac{1}{(4\pi)^2}\delta(\mathbf{k}-\mathbf{k}')\int d\mathbf{q}\, P_B(q)P_B(|\mathbf{k}-\mathbf{q}|)(1+\gamma^2)(1+\beta^2), \tag{69}$$

where as in Eq. (37) we have defined $\gamma = \hat{\mathbf{k}}\cdot\hat{\mathbf{q}}$ and $\beta = \hat{\mathbf{k}}\cdot\widehat{\mathbf{k}-\mathbf{q}}$. In deriving Eq. (69), we have assumed the turbulence-induced magnetic field to be Gaussian, as in the case of the turbulent fluid velocities, and hence we can apply Wick's theorem. Comparing with Eq. (67), replacing γ^2 and β^2 by their average values over the integral of 1/2, and using Eq. (58) gives

$$|\Pi^{(B)}(k,t_*)|^2 \simeq \frac{9V}{16}w^2\int d\mathbf{q}\, P(q)P(|\mathbf{k}-\mathbf{q}|). \tag{70}$$

This integral has already been done in Eq. (42), except that now the lower limit for the physical wave number is k_B instead of k_S; hence

$$|\Pi^{(B)}(k,t_*)|^2 \simeq \frac{27\pi^5}{8V}w^2\bar{\varepsilon}^{4/3}k^{-11/3}k_B^{-2/3}. \tag{71}$$

Equations (65), (66), (67), and (71) together then give

$$\langle h_{ij}^{(B)}(\tilde{\mathbf{k}},\eta) h_{ij}^{(B)*}(\tilde{\mathbf{k}}',\eta)\rangle \simeq \frac{864\pi^7 G^2}{\tilde{V}^2}\frac{w^2\bar{\varepsilon}^{4/3}\tilde{k}^{-17/3}\tilde{k}_B^{-2/3}a_*^{28/3}}{a^2H_0^2\Omega_{\rm rad}}\xi^2(\tilde{k},\eta_*,\eta)\delta(\tilde{\mathbf{k}}-\tilde{\mathbf{k}}'), \tag{72}$$

where we have approximated $a \simeq H_0\eta\sqrt{\Omega_{\rm rad}}$ while the Universe is radiation dominated, and we have converted to the comoving quantities $\tilde{V} = V/a_*^3$, $\tilde{k} = ka_*$, $\delta(\tilde{\mathbf{k}}-\tilde{\mathbf{k}}') = \delta(\mathbf{k}-\mathbf{k}')/a_*^3$. As in the previous section, we evaluate the real-space correlation function to make contact with measurable quantities:

$$\begin{aligned}\langle h_{ij}^{(B)}(\tilde{\mathbf{x}},\eta) h_{ij}^{(B)}(\tilde{\mathbf{x}},\eta)\rangle &= \frac{\tilde{V}^2}{(2\pi)^6}\int d\tilde{\mathbf{k}}d\tilde{\mathbf{k}}'\, e^{i(\tilde{\mathbf{k}}'-\tilde{\mathbf{k}})\cdot\tilde{\mathbf{x}}}\langle h_{ij}^{(B)}(\tilde{\mathbf{k}},\eta) h_{ij}^{(B)*}(\tilde{\mathbf{k}}',\eta)\rangle\\ &\simeq \frac{54\pi^2G^2w^2\bar{\varepsilon}^{4/3}\tilde{k}_B^{-2/3}a_*^{28/3}}{a^2H_0^2\Omega_{\rm rad}}\int_{\tilde{k}_B}^{\tilde{k}_D} d\tilde{k}\,\tilde{k}^{-11/3}\xi^2(\tilde{k},\eta_*,\eta)\end{aligned} \tag{73}$$

C. Relic Gravitational Radiation

As in Eq. (46), we define the characteristic gravitational wave amplitude $h_c^{(B)}(\tilde{f})$ per unit logarithmic comoving frequency interval via

$$\langle h_{ij}^{(B)}(\tilde{\mathbf{x}},\eta)h_{ij}^{(B)}(\tilde{\mathbf{x}},\eta)\rangle \equiv \frac{1}{2}\int_0^\infty \frac{d\tilde{f}}{\tilde{f}} h_c^{(B)2}(\tilde{f},\eta), \tag{74}$$

and the statistical average on the left side implies that the time dependence on the right side is not the exact time dependence of the gravitational wave but only the time dependence of its amplitude (i.e., the oscillations are averaged over). Since Eq. (63) gives the exact time dependence of the gravitational radiation (as opposed to the turbulence case, when we only know the statistically averaged time dependence), the standard wave dispersion relation holds: $\tilde{k} = 2\pi\tilde{f}$. (In contrast, for the turbulence source, we only know the statistically averaged time dependence, so we use only the approximate dispersion relation $f = \tau_L^{-1}$.) Then comparing with Eq. (73) gives

$$h_c^{(B)}(\tilde{f},\eta_0) \simeq \left(\frac{3^{3/2}}{2^{2/3}\pi^{2/3}}\right)\frac{Gw\bar{\varepsilon}^{2/3}\tilde{f}_B^{-1/3}a_*^{14/3}}{H_0\sqrt{\Omega_{\rm rad}}}\tilde{f}^{-4/3}\bar{\xi}(\tilde{f}\eta_*,\tilde{f}\eta_{\rm end}), \tag{75}$$

which depends on the function $\bar{\xi}$ which we define as the amplitude of the oscillations in $\xi(\eta)$, times the numerical factor $\sqrt{2}/2$ to convert to a root-mean-square value, in accordance with the definition of the characteristic amplitude Eq. (74). In deriving Eq. (75), we have used the fact that after the conformal time $\eta_{\rm end}$ when the magnetic fields are damped away via viscosity and cease to be an efficient source of gravitational radiation, the characteristic amplitude $h_c^{(B)}(\tilde{f},\eta)$ will simply scale inversely with a. Note that $h_c^{(B)}(\tilde{f},\eta_0)$ is only weakly dependent on $\eta_{\rm end}$, which occurs through the upper limit of the integral in $\xi(\tilde{f},\eta_*,\eta_{\rm end})$. The time $\eta_{\rm end}$ technically depends on the scale considered, but for simplicity we simply use the saturation scale L_B at which the gravitational radiation peaks.

The function $\bar{\xi}(\tilde{k}\eta_*,\tilde{k}\eta)$ cannot be expressed in terms of elementary functions, but it is simple to obtain an upper bound. Since the amplitude of the oscillations in ξ is monotonically decreasing, the amplitude at the initial time gives

$$\bar{\xi}(\tilde{f}\eta_*,\tilde{f}\eta) < \frac{\sqrt{2}}{4\pi\tilde{f}\eta_*}, \tag{76}$$

which is useful for constraining the gravitational wave amplitude.

In an analogous calculation to the previous section, writing $\tilde{f}_B = f_B a_*$ and $f_B = L_B^{-1}$ and approximating $H_* \simeq H_0\sqrt{\Omega_{\rm rad}}a_*^{-2}$, we obtain

$$h_c^{(B)}(\tilde{f},\eta_0) \simeq 1.9\times10^{-16}\left(\frac{\kappa\rho_{\rm vac}}{w}\right)^{2/3}\left(\frac{\tau}{H_*^{-1}}\right)^{-2/3}\left(\frac{L_B}{H_*^{-1}}\right)^{5/3}\left(\frac{100\,{\rm GeV}}{T_*}\right)\left(\frac{100}{g_*}\right)^{1/3}\left(\frac{\tilde{f}}{\tilde{f}_B}\right)^{-4/3}\bar{\xi}(\tilde{f},\eta_*,\eta_{\rm end}). \tag{77}$$

This characteristic amplitude is valid for $\tilde{f} > \tilde{f}_B$, where [using Eq. (51)]

$$\tilde{f}_B = 1.65\times10^{-5}\,{\rm Hz}\left(\frac{L_B}{H_*^{-1}}\right)^{-1}\left(\frac{T_*}{100\,{\rm GeV}}\right)\left(\frac{g_*}{100}\right)^{1/6} \tag{78}$$

The corresponding energy density in gravitational waves per logarithmic frequency interval in units of current critical density is

$$\Omega_{\rm GW}^{(B)}(\tilde{f})h^2 = 6.0\times10^{-6}\left(\frac{\kappa\rho_{\rm vac}}{w}\right)^{4/3}\left(\frac{\tau}{H_*^{-1}}\right)^{-4/3}\left(\frac{L_B}{H_*^{-1}}\right)^{4/3}\left(\frac{g_*}{100}\right)^{-1/3}\left(\frac{\tilde{f}}{\tilde{f}_B}\right)^{-2/3}\bar{\xi}^2(\tilde{f},\eta_*,\eta_{\rm end}). \tag{79}$$

V. FIRST-ORDER COSMOLOGICAL PHASE TRANSITIONS

The most likely mechanism for creating turbulence with a large energy density is a first-order phase transition. Such a transition is controlled by an effective potential for some quantity which functions as the order parameter of the phase transition. Initially, the Universe sits in a minimum of the effective potential. As the Universe expands and cools, the effective potential develops a local minimum at a different value of the order parameter; this new local minimum eventually evolves to be the true minimum energy state. Then the order parameter wants to evolve to the new minimum. If a potential energy barrier exists between the old local minimum and the new true minimum, the phase transition must occur via quantum tunnelling through the barrier or thermal fluctuations over the barrier. As a result, bubbles of the low-temperature phase are nucleated at random places in the high-temperature phase. The energy difference between the two phases creates an effective outward force on the bubble, causing it to expand. Once this outward force from the energy difference balances the inward hydrodynamic force from pushing plasma outwards, the bubble reaches an equilibrium and expands at a constant velocity. We will consider only the case of quantum tunnelling, applicable to a strong first-order phase transition with a high barrier between the two phases. In this case the nucleated bubbles are spherical and negligibly small compared to the horizon scale [28]. The more complex case of thermally activated bubbles has been considered in [29].

A. Turbulence

In general, the rate for nucleating a bubble will be the exponential of some tunnelling action, $\Gamma \propto \exp(S(t))$. As a simple model of a phase transition, we expand the action S into a power series in time and keep only the constant and linear terms. This gives a characteristic bubble nucleation rate per unit volume [30]

$$\Gamma = \Gamma_0 e^{\beta t} \tag{80}$$

so the quantity β^{-1} sets the characteristic time scale for the phase transition. Numerical calculations show that the largest bubbles reach a size of order $\beta^{-1} v_b$ by the end of the phase transition [31], where v_b is the bubble expansion velocity, assuming the bubbles remain spherical as they expand. In general, β is expected to be of the order $4\ln(m_{\rm Pl}/T)H \simeq 100H$ for a Hubble rate H [30].

A first-order phase transition is generically described by several parameters: (i) $\alpha \equiv \rho_{\rm vac}/\rho_{\rm thermal} = 4\rho_{\rm vac}/3w$, the ratio of the vacuum energy associated with the phase transition to the thermal density of the Universe at the time (which characterizes the strength of the phase transition); (ii) κ, an efficiency factor which gives the fraction of the available vacuum energy which goes into the kinetic energy of the expanding bubble walls, as opposed to thermal energy; (iii) β, which sets the characteristic time scale for the phase transition; (iv) v_b, the velocity of the expanding bubble walls, which set the characteristic length scale of the phase transition; (v) T_*, the temperature at which the phase transition occurs.

Once the bubbles expand and percolate, much of their kinetic energy will be converted to turbulent bulk motions of the primordial plasma (for an illustration, see the numerical evolution of two scalar field bubbles in Ref. [6]). The energy density contained in a bubble wall of radius r scales with r^3, the bubble volume. As the phase transition ends, far more small bubbles have been nucleated than large ones, but the energy density in the large ones dominates the total energy density [30]. We therefore make the approximation that turbulent energy is injected on a stirring scale $L_S \simeq v_b \beta^{-1}$ corresponding to the size of the largest bubbles. The stirring will last for roughly $\tau_{\rm stir} = \beta^{-1}$, the duration of the phase transition. The duration τ of the turbulence then follows from Eqs. (20) and (18) as

$$\tau = \beta^{-1} \max\left[1, \frac{3\sqrt{2}}{2}\frac{v_b}{(\kappa\alpha)^{1/2}}\right]. \tag{81}$$

The fundamental symmetry breaking mechanism which drives the phase transition determines some effective potential for bubble nucleation. The difference in energy density between the two phases and the bubble nucleation rate are both determined by this mechanism. Thus the parameters T_*, β, and α are all determined directly by the underlying physics, and are precisely calculable to some given order in the various particle interaction strengths. On the other hand, the bubble velocity v_b and the fraction of kinetic energy into the bubbles κ depend on the detailed microphysics involved in the bubble propagation through the relativistic plasma and are not determined from general properties of the effective potential. Generally, the larger the vacuum energy density driving phase transition, the higher bubble wall velocities v_b will be obtained.

The hydrodynamic boundary between a lower-energy phase and a higher-energy one can propagate via two modes, detonation and deflagration. Details of these modes in the case of spherical geometry are known [32]. For a detonation front, the velocity of the phase boundary exceeds the sound speed in the fluid, so that a shock forms at the burning front. In the opposite case, a deflagration propagates slower than the sound speed and piles up an overdensity of fluid in front of it, like a snowplow. The boundary conditions for a detonation are more restrictive, so that once the energy densities and pressures are specified in each phase, the complete solution for the propagating detonation is determined. In this case, we have [32]

$$v_b(\alpha) = \frac{1/\sqrt{3} + (\alpha^2 + 2\alpha/3)^{1/2}}{1+\alpha} \tag{82}$$

and the approximate form [9]

$$\kappa(\alpha) = \frac{1}{1 + A\alpha}\left[A\alpha + \frac{4}{27}\left(\frac{3\alpha}{2}\right)^{1/2}\right] \tag{83}$$

with $A = 0.72$. If the bubbles propagate as a deflagration front, no such general relations apply. However, it has been argued that for relativistic plasmas, instabilities in the bubble shape will accelerate the bubble walls and the hydrodynamic expansion mode is unstable to becoming a detonation. For this reason, in the following analysis, we will assume Eqs. (82) and (83) hold. We also assume that $\alpha \ll 1$ to simplify further Eqs. (82) and (83), which will generally hold for realistic phase transition models; for unusual cases with very strong detonations and $\alpha \gtrsim 1$, the following formulas must be corrected. The duration of the turbulence is then given by the second term in Eq. (81), becoming

$$\tau = \left(\frac{3}{2}\right)^{9/4} \beta^{-1}\alpha^{-3/4}. \tag{84}$$

B. Relic Radiation from the Phase Transition

The characteristic gravitational wave amplitude from turbulence becomes

$$h_c(\tilde{f}) \simeq 3.8 \times 10^{-18}\alpha^{9/8}\left(\frac{H_*}{\beta}\right)^2\left(\frac{100\,\mathrm{GeV}}{T_*}\right)\left(\frac{100}{g_*}\right)^{1/3}\left(\frac{\tilde{f}}{\tilde{f}_S}\right)^{-11/4}, \tag{85}$$

with the characteristic frequency

$$\tilde{f}_S \simeq 5.7 \times 10^{-6}\,\mathrm{Hz}\,\alpha^{3/4}\left(\frac{\beta}{H_*}\right)\left(\frac{T_*}{100\,\mathrm{GeV}}\right)\left(\frac{g_*}{100}\right)^{1/6}. \tag{86}$$

The corresponding energy density per logarithmic frequency interval is

$$\Omega_{\mathrm{GW}}h^2 \simeq 2.7 \times 10^{-10}\alpha^{15/4}\left(\frac{H_*}{\beta}\right)^2\left(\frac{g_*}{100}\right)^{-1/3}\left(\frac{\tilde{f}}{\tilde{f}_S}\right)^{-7/2}. \tag{87}$$

In a first-order phase transition, the expanding, colliding bubbles are themselves a potent source of gravitational radiation [9]. For our idealized model phase transition with spherical expanding bubbles, the ratio of the maximum amplitude of gravitational radiation due to turbulence to the maximum amplitude due to bubble collisions is approximately

$$\frac{h_{\mathrm{turb}}(\tilde{f}_S)}{h_{\mathrm{bub}}(\tilde{f}_{\mathrm{max}})} \simeq 0.18\alpha^{-3/8}, \tag{88}$$

so only for $\alpha < 0.01$ will the amplitude of the turbulent signal be larger (although in this case, turbulent damping due to the expansion of the Universe is significant and our estimate for the turbulence gravitational wave amplitude may be significantly too large). For realistic models with interesting gravitational wave production, the turbulence amplitude will be subdominant to the bubble amplitude, but non-negligible. This is in contrast to the naive dimensional estimate of the turbulence gravitational radiation in Ref. [9] which gave a somewhat larger value. The frequencies at which these maximum amplitudes occur scale differently with the parameters:

$$\frac{\tilde{f}_S}{\tilde{f}_{\mathrm{max}}} = 1.1\alpha^{3/4}. \tag{89}$$

The different scaling arises because the duration of the phase transition τ_{stir} sets the characteristic frequency for the radiation from expanding bubbles, while the circulation time on the stirring scale τ_S sets the characteristic frequency for the radiation from turbulence.

Note that the gravitational radiation in the bubble case has a long tail in the amplitude, $h_c(f) \propto f^{-1/3}$, while turbulence driven at a single scale drops off very quickly like $h_c(f) \propto f^{-11/4}$. The tail for bubble collisions arises in the case of bubble collisions because at any given moment, the characteristic frequency of radiation from the collision of two bubbles is v_b/d, where d is the size of the colliding region. Since d ranges from zero to the maximum size of the smaller bubble as the bubbles expand, the gravitational radiation is produced over a wide range of frequencies. This tail of the frequency spectrum is somewhat model-dependent, and will be modified if the bubbles are not spherical. Departures from sphericity could arise from thermal activation over the potential barrier, resulting in non-spherical nucleation, or from shape instabilities as the bubble expands. The results for expanding bubbles also depend on the thin-wall approximation, namely that the width of the bubble wall is small compared to the radius of the bubble. While this approximation will be very good for relativistic detonations, it will not be as good for deflagrations.

The gravitational wave signal from turbulence from a single stirring scale L_S is somewhat more generic, although if the phase transition does not proceed via detonation the specific expressions for κ and v_b in Eqs. (83) and (82) will not hold. However, the single-scale assumption obviously will never be exactly correct; any realistic source like a phase transition will deposit bulk kinetic energy over a range of scales. The energy density in bubble walls of a given size will generically peak at a scale comparable to $v_b\beta^{-1}$ that we have taken for L_S, because the kinetic energy in the wall of a bubble of radius r scales like r^3 so the energy distribution is heavily weighted towards the largest bubbles. Analytic expressions for the size distribution of bubbles, the fraction of space taken up by bubbles, and related quantities are given in Ref. [30]. On the other hand, the stirring scale appropriate to the collision of two bubbles of unequal radius is not entirely clear: some turbulence will clearly be created on the scale of the smaller bubble, but since the larger bubble has greater energy density in the wall, a significant part of the energy will remain in coherent motion determined by the larger bubble.

In realistic cases, the gravitational wave amplitude spectrum in Eq. (85) must be convolved over a range of stirring scales. A specific model of the distribution of stirring scales in a first-order phase transition is beyond the scope of this paper. However, we can make a rough estimate of its effect. Assume that the actual turbulence source stirs the plasma over a range of frequencies Δf_S. The actual bubble size distribution has a significant tail towards larger bubbles [30]. If the same total energy goes into gravitational radiation as in the single stirring scale case, then the characteristic amplitude $h_c(f_S)$ will be reduced by a factor of order $(f_S/\Delta f_S)^{1/2}$. This very crude estimate neglects the strong dependence of the amplitude on the stirring scale and employs only a box-shaped energy density spectrum, but the general scaling is correct. Generically, the distribution of bubble sizes in a model phase transition points to $\Delta f_S/f_S$ on the order of a few (see [30]), but a more precise estimate requires a detailed model of stirring in a phase transition. As a rule of thumb, when estimating the gravitational radiation background from turbulence arising from a phase transition with a single stirring-scale model of the turbulence, the resulting amplitude may be overestimated by a modest factor.

C. Relic Radiation from the Induced Magnetic Fields

For the magnetic fields from the turbulent dynamo mechanism, the characteristic gravitational wave amplitude becomes

$$h_c^{(B)}(\tilde{f},\eta_0) \simeq 1.0\times10^{-17}\alpha^{3/2}\left(\frac{H_*}{\beta}\right)\left(\frac{L_B}{L_S}\right)^{5/3}\left(\frac{100\,\mathrm{GeV}}{T_*}\right)\left(\frac{100}{g_*}\right)^{1/3}\left(\frac{\tilde{f}}{\tilde{f}_B}\right)^{-4/3}\bar{\xi}(\tilde{f}\eta_*,\tilde{f}\eta_{\mathrm{end}}), \tag{90}$$

with the characteristic frequency

$$\tilde{f}_B \simeq 2.9\times10^{-5}\,\mathrm{Hz}\left(\frac{L_S}{L_B}\right)\left(\frac{\beta}{H_*}\right)\left(\frac{T_*}{100\,\mathrm{GeV}}\right)\left(\frac{g_*}{100}\right)^{1/6}. \tag{91}$$

The corresponding energy density per logarithmic frequency interval is

$$\Omega_{\mathrm{GW}}^{(B)}(\tilde{f})h^2 \simeq 4.8\times10^{-8}\alpha^3\left(\frac{L_B}{L_S}\right)^{4/3}\left(\frac{g_*}{100}\right)^{-1/3}\left(\frac{\tilde{f}}{\tilde{f}_B}\right)^{-2/3}\bar{\xi}^2(\tilde{f}\eta_*,\tilde{f}\eta_{\mathrm{end}}). \tag{92}$$

The ratio of the maximum amplitude of gravitational radiation due to turbulence-induced magnetic fields to the maximum amplitude due to the turbulent fluid today is approximately

$$\frac{h_c^{(B)}(\tilde{f}_B)}{h_c^{(\mathrm{turb})}(\tilde{f}_S)} \simeq 2.7\alpha^{3/8}\left(\frac{\beta}{H_*}\right)\left(\frac{L_B}{L_S}\right)^{5/3}\bar{\xi}(\tilde{f}_B\eta_*,\tilde{f}_B\eta_{\mathrm{end}}). \tag{93}$$

The ratio of the frequencies at which these maximum amplitudes occur is

$$\frac{\tilde{f}_B}{\tilde{f}_S} \simeq 5.1\alpha^{-3/4}\left(\frac{L_S}{L_B}\right). \tag{94}$$

The scaling with α arises because the circulation time on the stirring scale τ_S sets the characteristic frequency for the radiation from turbulence, whereas for magnetic fields, $f_B = L_B^{-1}$.

The value of $\eta_{\rm end}$ corresponding to the scale L_B can be determined via consideration of the neutrino viscosity (see [33]) but $\bar{\xi}(\tilde{k}\eta_*, \tilde{k}\eta_{\rm end})$ is only weakly dependent on $\eta_{\rm end}$ so we do not compute it here. Instead, we derive an upper bound on the amplitude. The approximate relation $a_* H_* \simeq 1/\eta_*$, valid during radiation domination, gives $\tilde{f}_B\eta_* \simeq H_*^{-1}/L_B$. Then Eq. (76) combined with the frequency dependence in Eq. (85) gives

$$\frac{h_c^{(B)}(\tilde{f}_B)}{h_c^{(\rm turb)}(\tilde{f}_B)} < 27\alpha^{-27/16} v_b \left(\frac{L_S}{L_B}\right)^{1/12}. \tag{95}$$

As discussed above, $L_S/L_B \simeq 300$ generically, so the peak characteristic amplitude from the magnetic field at frequency $\tilde{f}_B$ will always be negligible compared to the peak characteristic amplitude from the turbulence at frequency $\tilde{f}_S$. The turbulence gravitational waves drop so quickly with frequency, however, that the magnetic field gravitational waves will give a larger characteristic amplitude at $\tilde{f}_B$.

VI. POTENTIAL DETECTABILITY

The detectability of a given stochastic background depends on both its characteristic frequency and its amplitude. The Laser Interferometer Gravitational-wave Observatory (LIGO) [34] is nearing the commencement of scientific observations; it is comprised of two facilities in the United States, each essentially a Michelson interferometer with an arm length of 4 kilometers. LIGO has sensitivity to gravitational radiation in the frequency range from 10 to 1000 Hz. Seismic noise prevents useful gravitational wave detection from the surface of the Earth at frequencies lower than about 10 Hz. Cross-correlation of the two LIGO detectors, along with several smaller laser interferometers and bar detectors at other sites around the world, allow a clean detection of stochastic signals, since widely separated detectors have no correlated sources of noise. Detailed estimates shows that in this frequency range, LIGO will be able to detect stochastic gravitational wave backgrounds with a characteristic amplitude of around $h_c(\tilde{f}) \simeq 3\times10^{-23}$ at $\tilde{f} \simeq 100$ Hz after integrating for four months [35–38]. These levels will hopefully be obtained within three years. Planned technical improvements are projected to reduce this threshold amplitude by another factor of 10 on the time scale of a decade.

The other major gravitational wave observation program, the Laser Interferometer Space Antenna (LISA) [8], is a cornerstone mission of the European Space Agency in partnership with NASA. Current design studies envision three spacecraft arrayed in an equilateral triangle with an arm length of around 5×10^6 kilometers with laser interferometry between each of the three pairs of arms; the spacecraft configuration will trail the Earth's orbit by about 20°. LISA will likely be sensitive to a frequency range from around 0.0001 Hz to 0.1 Hz. The detection of stochastic backgrounds with LISA is more complicated than with LIGO, because any pair of interferometers formed by LISA's arms share one arm in common, so it is not possible to cross-correlate two independent interferometers with uncorrelated noise. It was originally believed that this limited detection of a stochastic background to the level of the instrument noise power because there would be no way to distinguish between instrumental noise and a background signal. This noise level corresponds to a stochastic background amplitude of around $h_c(\tilde{f}) = 10^{-21}$ at 0.01 Hz. It has now been realized that if the complete time series data for positions of 6 independent test masses are recorded, so-called Sagnac observables can be synthesized which are highly insensitive to various kinds of noise in the system [39], including one which is largely independent of low-frequency stochastic gravitational wave backgrounds, allowing a direct measurement of the system noise [40]. This results in a significant improvement in the ability of the system to measure stochastic backgrounds [41]. For one year of observation, this kind of analysis could in principle give sensitivities comparable to two independent Michelson interferometers, reducing the threshold $h_c(\tilde{f})$ by a factor of $(\tilde{f}t)^{1/4}$ for observation over a time t, or $h_c(\tilde{f}) \simeq 4\times10^{-23}$ at 0.01 Hz over one year of observing. Such a sensitivity level depends on a precise understanding of the system noise properties and elimination of other correlated noise sources between the various arms of the detector, which is only partially practicable. Flying and cross-correlating two independent LISA-like detectors [42,43] is still clearly preferable for detecting stochastic backgrounds.

For stochastic background detection at LISA frequencies, raw sensitivity is not the only issue. White dwarf binaries in our galaxy will produce an approximately stochastic gravitational wave background which probably becomes comparable to the LISA sensitivity limits for frequencies below about 10^{-3} Hz [44]. Detection of such a signal will be interesting in its own right, but will effectively provide a lower limit of around 10^{-4} Hz to the stochastic background signals which are detectable, until gravitational wave detectors improve to the point of having enough directional sensitivity to distinguish sources in the galactic plane from sources distributed isotropically.

The characteristic gravitational wave frequency for turbulence from known phase transitions is not promising for detection in the near future. For the electroweak phase transition at $T_* \simeq 100$ GeV, Eq. (86) shows that α, the ratio of the vacuum energy density to the thermal density at the time of the phase transition, must be of order 0.1 for the frequency maximum to be as high as $\tilde{f}_S = 10^{-4}$ Hz, if β/H_* takes its characteristic value of 100. This frequency is the lower limit to what LISA might be able to detect. The amplitude at this frequency for $\alpha = 0.1$ would be $h_c(\tilde{f}_S) \simeq 2.8 \times 10^{-23}$, more than two orders of magnitude smaller than a LISA Sagnac configuration could detect at this frequency. Any push towards higher frequencies via a shorter phase transition further reduces the characteristic amplitude, since $h_c(\tilde{f}) \propto (H_*/\beta)^2$. For an extreme case with $\alpha = 1$ and $\beta = 1000H_*$, the characteristic frequency is near LISA's maximum sensitivity, $\tilde{f}_S \simeq 5.7 \times 10^{-3}$ Hz, with a characteristic amplitude $h_c(\tilde{f}_S) \simeq 3.8 \times 10^{-24}$. This amplitude is an order of magnitude smaller than the LISA Sagnac sensitivity at this frequency. An analysis of the electroweak effective potential in a large class of supersymmetric models [45,46] shows that for models with large values of α, generally $\beta < 100H_*$, and α is never as large as unity [47]. Other well-motivated extensions of the standard model may result in a very strong electroweak phase transition (e.g., [48]).

Satellite missions to probe stochastic backgrounds to lower frequencies have been discussed [49], which would involve multiple spacecraft arrayed at separations on the order of 1 a.u. Such configurations would be a more natural match for the frequency scale of electroweak turbulence, although dealing with the binary foreground signal would still be a major hurdle.

Phase transitions at lower temperatures, like the QCD phase transitions, have larger characteristic length scales and thus even lower frequencies for gravitational radiation. Speculative phase transitions could occur at energy scales higher than the electroweak scale, resulting in higher characteristic frequencies. However, a higher energy scale also translates into a smaller characteristic amplitude, and it is not possible to give a set of parameters with $\alpha \lesssim 1$ for which cosmological turbulence would be detectable in LIGO. LISA could detect the turbulence from a range of imagined phase transitions at energy scales above the weak scale, but at present no compelling theoretical motivation for such phase transitions is at hand.

In contrast to turbulent sources, the expanding bubbles in a first-order phase transition, which drive the turbulence, are themselves a strong source of gravitational radiation [9] and are a much more promising source of detectable signals from the electroweak phase transition. The difference between the detectability of the two sources is essentially the factor of $\alpha^{3/4}$ in Eq. (89), arising from the different time scales of the sources. The characteristic frequency for the expanding bubbles is set by the phase transition time scale β^{-1} because the bubbles expand and percolate in this time. For turbulence, the time scale is instead the circulation time on the stirring scale. The turbulent fluid velocities are significantly smaller than the bubble expansion velocities unless the turbulent flows are near the sound speed, giving lower characteristic frequencies. (Extremely strong turbulence with relativistic fluid velocities would likely produce gravitational radiation in a more detectable range of frequencies and amplitudes, but the amount of energy in turbulent motions is limited by shock formation and heating, and turbulence is not understood in this regime.)

Our results in this work supercede the dimensional estimates in Ref. [9], which predicted that the turbulence signal could be significantly larger than the bubble signal at similar frequencies. Resulting optimistic calculations of turbulent signals from the electroweak phase transition, e.g. Ref. [47], unfortunately do not hold up to more detailed analysis. We emphasize, however, that the results presented here apply in a generic way to any turbulence in the early Universe, and the search for stochastic gravitational radiation backgrounds in the frequency range from 10^{-4} Hz to 1000 Hz is in part a search for unanticipated, dramatic physics at energies above the electroweak scale. Perhaps we will be lucky.

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