Desargues_theorem.wxm

An application document for Geometric Algebra using wxMaxima

The example of projective geometry given in the Survey, section 4.1.4, page 25

Initialization

```
(%i1) ext:["wxm"]$
      file_type_maxima:append(ext,file_type_maxima)$
      batchload("initialize_fns")$
```

the pseudoscalar and its inverse
the lowest useable dimension pseudoscalar should be {e1,e2} i.e. Plen = 2
e.g. for four dimensions edit Pseudos:{e1,e2,e3}$ to Pseudos:{e1,e2,e3,e4}$

```
(%i1) Pseudos:{e1,e2,e3,e4}$
      Pvar:listofvars(Pseudos)$
      Plen:length(Pvar)$
      I:Pseudos$
      ni:(Plen-1)*Plen/2$
      Ii:(-1)^ni*I$
      kill(ni)$
      ldisplay(Pvar)$

(%t8) Pvar=[e1,e2,e3,e4]

(%i9) batchload("initialize_lsts")$

(%t9) lstblds=[[{e1},{e2},{e3},{e4}],[{e1,e2},{e1,e3},{e1,e4},{e2,e3},{e2,e4}
,{e3,e4}],[{e1,e2,e3},{e1,e2,e4},{e1,e3,e4},{e2,e3,e4}],[{e1,e2,e3,e4}]]
(%t10) allblds=[{e1},{e2},{e3},{e4},{e1,e2},{e1,e3},{e1,e4},{e2,e3},{e2,e4},{
e3,e4},{e1,e2,e3},{e1,e2,e4},{e1,e3,e4},{e2,e3,e4},{e1,e2,e3,e4}]
(%t11) invblds=[{e1},{e2},{e3},{e4},-{e1,e2},-{e1,e3},-{e1,e4},-{e2,e3},-{e2
,e4},-{e3,e4},-{e1,e2,e3},-{e1,e2,e4},-{e1,e3,e4},-{e2,e3,e4},{e1,e2,e3,e4}]
```

end of Initialization

Projective geometry starts here

use {e1} as the unit vector, {e}, orthogonal to 3D space {e2,e3,e4} and form vectors a,b,c;
also confine the vertices to a particular plane by setting the coefficient of {e3} to zero;
construct both a,b,c and a',b',c' where e.g. ad represents ad(ashed)

```
(%i12) a:{e1}+a2*{e2}+a4*{e4}$
       ad:{e1}+ad2*{e2}+ad4*{e4}$
       b:{e1}+b2*{e2}+b4*{e4}$
       bd:{e1}+bd2*{e2}+bd4*{e4}$
       c:{e1}+c2*{e2}+c4*{e4}$
       cd:{e1}+cd2*{e2}+cd4*{e4}$
       ldisplay(a,b,c)$
       ldisplay(ad,bd,cd)$

(%t18) a=a4*{e4}+a2*{e2}+{e1}
(%t19) b=b4*{e4}+b2*{e2}+{e1}
(%t20) c=c4*{e4}+c2*{e2}+{e1}
(%t21) ad=ad4*{e4}+ad2*{e2}+{e1}
(%t22) bd=bd4*{e4}+bd2*{e2}+{e1}
(%t23) cd=cd4*{e4}+cd2*{e2}+{e1}
```

construct P,Q,R as the lines connecting corresponding vertices of the triangles

```
(%i24) a&^ad$
       P:facsum(%,allblds)$
       b&^bd$
       Q:facsum(%,allblds)$
       c&^cd$
       R:facsum(%,allblds)$
       ldisplay(P,Q,R)$

(%t30) P=(a2*ad4-a4*ad2)*{e2,e4}+(ad4-a4)*{e1,e4}+(ad2-a2)*{e1,e2}
(%t31) Q=(b2*bd4-b4*bd2)*{e2,e4}+(bd4-b4)*{e1,e4}+(bd2-b2)*{e1,e2}
(%t32) R=(c2*cd4-c4*cd2)*{e2,e4}+(cd4-c4)*{e1,e4}+(cd2-c2)*{e1,e2}
```

prescribe the pseudoscalar, K of the (homogeneous model) subspace of the triangles
and find the pseudoscalar inverse, Ki

```
(%i33) K:{e1,e2,e4}$
       ni:(Plen-1-1)*(Plen-1)/2$
       Ki:(-1)^ni*K$
       ldisplay(K,Ki)$

(%t36) K={e1,e2,e4}
(%t37) Ki=-{e1,e2,e4}
```

In Figure 10, show that the upper triangle, T, is in the same subspace as that above

```
(%i38) J:a&^b&^c$
       T:facsum(%,allblds)$
       ldisplay(T)$

(%t40) T=(b2*c4-a2*c4-b4*c2+a4*c2+a2*b4-a4*b2)*{e1,e2,e4}
```

construct p,q,r as the intersections of the extensions of the sides of the triangles,
using the dual within the subspace of the triangles (i.e. using Ki)

```
(%i41) ((b&^c)&*Ki)&.(bd&^cd)$
       p:facsum(%,allblds)$
       ldisplay(p)$

(%t43) p=-(bd2*c4*cd4-b2*c4*cd4+b4*c2*cd4-b4*bd2*cd4-bd4*c4*cd2+b4*bd4*cd2
+b2*bd4*c4-b4*bd4*c2)*{e4}-(bd2*c2*cd4-b2*bd2*cd4-b2*c4*cd2-bd4*c2*cd2+b4
*c2*cd2+b2*bd4*cd2+b2*bd2*c4-b4*bd2*c2)*{e2}-
(c2*cd4-b2*cd4-c4*cd2+b4*cd2+bd2*c4-bd4*c2+b2*bd4-b4*bd2)*{e1}

(%i44) ((c&^a)&*Ki)&.(cd&^ad)$
       q:facsum(%,allblds)$
       ldisplay(q)$

(%t46) q=-(ad2*c4*cd4-a2*c4*cd4+a4*c2*cd4-a4*ad2*cd4-ad4*c4*cd2+a4*ad4*cd2
+a2*ad4*c4-a4*ad4*c2)*{e4}-(ad2*c2*cd4-a2*ad2*cd4-a2*c4*cd2-ad4*c2*cd2+a4
*c2*cd2+a2*ad4*cd2+a2*ad2*c4-a4*ad2*c2)*{e2}-
(c2*cd4-a2*cd4-c4*cd2+a4*cd2+ad2*c4-ad4*c2+a2*ad4-a4*ad2)*{e1}

(%i47) ((a&^b)&*Ki)&.(ad&^bd)$
       r:facsum(%,allblds)$
       ldisplay(r)$

(%t49) r=-(ad2*b4*bd4-a2*b4*bd4+a4*b2*bd4-a4*ad2*bd4-ad4*b4*bd2+a4*ad4*bd2
+a2*ad4*b4-a4*ad4*b2)*{e4}-(ad2*b2*bd4-a2*ad2*bd4-a2*b4*bd2-ad4*b2*bd2+a4
*b2*bd2+a2*ad4*bd2+a2*ad2*b4-a4*ad2*b2)*{e2}-
(b2*bd4-a2*bd4-b4*bd2+a4*bd2+ad2*b4-ad4*b2+a2*ad4-a4*ad2)*{e1}
```

again find J and also Jd(ashed) and then display their product

```
(%i50) J:a&^b&^c$
       Jd:ad&^bd&^cd$
       JJd:J&*Jd$
       ldisplay(JJd)$

(%t53)/R/ JJd=(((-bd2+ad2)*cd4+(bd4-ad4)*cd2-ad2*bd4+ad4*bd2)*b2+
((bd2-ad2)*cd4+(-bd4+ad4)*cd2+ad2*bd4-ad4*bd2)*a2)*c4+(((bd2-ad2
((-bd2+ad2)*cd4+(bd4-ad4)*cd2-ad2*bd4+ad4*bd2)*a4)*c2+
((-bd2+ad2)*cd4+(bd4-ad4)*cd2-ad2*bd4+ad4*bd2)*a2*b4+
((bd2-ad2)*cd4+(-bd4+ad4)*cd2+ad2*bd4-ad4*bd2)*a4*b2
```

display all the terms of the product of the bivectors P, Q and R

```
(%i54) P&*Q&*R$
       PQR:facsum(%,allblds)$
       ldisplay(PQR)$

(%t56) PQR=-(a2*ad4*b2*bd4*c2*cd4-a4*ad2*b2*bd4*c2*cd4+ad4*bd4*c2*cd4-a4*
bd4*c2*cd4-a2*ad4*b4*bd2*c2*cd4+a4*ad2*b4*bd2*c2*cd4+ad2*bd2*c2*cd4-a2*bd2
*c2*cd4-ad4*b4*c2*cd4+a4*b4*c2*cd4-ad2*b2*c2*cd4+a2*b2*c2*cd4-ad4*b2*bd4*
cd4+a4*b2*bd4*cd4+a2*ad4*bd4*cd4-a4*ad2*bd4*cd4+ad4*b4*bd2*cd4-a4*b4*bd2*
cd4-a2*ad4*b4*cd4+a4*ad2*b4*cd4-a2*ad4*b2*bd4*c4*cd2+a4*ad2*b2*bd4*c4*cd2-
ad4*bd4*c4*cd2+a4*bd4*c4*cd2+a2*ad4*b4*bd2*c4*cd2-a4*ad2*b4*bd2*c4*cd2-ad2
*bd2*c4*cd2+a2*bd2*c4*cd2+ad4*b4*c4*cd2-a4*b4*c4*cd2+ad2*b2*c4*cd2-a2*b2*
c4*cd2-ad2*b2*bd4*cd2+a2*b2*bd4*cd2+ad2*b4*bd2*cd2-a2*b4*bd2*cd2+a2*ad4*
bd2*cd2-a4*ad2*bd2*cd2-a2*ad4*b2*cd2+a4*ad2*b2*cd2+ad4*b2*bd4*c4-a4*b2*bd4
*c4-a2*ad4*bd4*c4+a4*ad2*bd4*c4-ad4*b4*bd2*c4+a4*b4*bd2*c4+a2*ad4*b4*c4-a4
*ad2*b4*c4+ad2*b2*bd4*c2-a2*b2*bd4*c2-ad2*b4*bd2*c2+a2*b4*bd2*c2-a2*ad4*
bd2*c2+a4*ad2*bd2*c2+a2*ad4*b2*c2-a4*ad2*b2*c2)*{e2,e4}-(ad4*b2*bd4*c2*cd4
-a4*b2*bd4*c2*cd4-a2*ad4*bd4*c2*cd4+a4*ad2*bd4*c2*cd4-ad4*b4*bd2*c2*cd4+a4
*b4*bd2*c2*cd4+a2*ad4*b4*c2*cd4-a4*ad2*b4*c2*cd4+a2*ad4*b2*bd4*cd4-a4*ad2*
b2*bd4*cd4+ad4*bd4*cd4-a4*bd4*cd4-a2*ad4*b4*bd2*cd4+a4*ad2*b4*bd2*cd4+ad2*
bd2*cd4-a2*bd2*cd4-ad4*b4*cd4+a4*b4*cd4-ad2*b2*cd4+a2*b2*cd4-ad4*b2*bd4*c4
*cd2+a4*b2*bd4*c4*cd2+a2*ad4*bd4*c4*cd2-a4*ad2*bd4*c4*cd2+ad4*b4*bd2*c4*
cd2-a4*b4*bd2*c4*cd2-a2*ad4*b4*c4*cd2+a4*ad2*b4*c4*cd2-ad2*bd4*cd2+a2*bd4*
cd2+ad4*bd2*cd2-a4*bd2*cd2+ad2*b4*cd2-a2*b4*cd2-ad4*b2*cd2+a4*b2*cd2-a2*
ad4*b2*bd4*c4+a4*ad2*b2*bd4*c4-ad4*bd4*c4+a4*bd4*c4+a2*ad4*b4*bd2*c4-a4*
ad2*b4*bd2*c4-ad2*bd2*c4+a2*bd2*c4+ad4*b4*c4-a4*b4*c4+ad2*b2*c4-a2*b2*c4+
ad2*bd4*c2-a2*bd4*c2-ad4*bd2*c2+a4*bd2*c2-ad2*b4*c2+a2*b4*c2+ad4*b2*c2-a4*
b2*c2)*{e1,e4}-(ad2*b2*bd4*c2*cd4-a2*b2*bd4*c2*cd4-ad2*b4*bd2*c2*cd4+a2*b4
*bd2*c2*cd4-a2*ad4*bd2*c2*cd4+a4*ad2*bd2*c2*cd4+a2*ad4*b2*c2*cd4-a4*ad2*b2
*c2*cd4+ad2*bd4*cd4-a2*bd4*cd4-ad4*bd2*cd4+a4*bd2*cd4-ad2*b4*cd4+a2*b4*cd4
+ad4*b2*cd4-a4*b2*cd4-ad2*b2*bd4*c4*cd2+a2*b2*bd4*c4*cd2+ad2*b4*bd2*c4*cd2
-a2*b4*bd2*c4*cd2+a2*ad4*bd2*c4*cd2-a4*ad2*bd2*c4*cd2-a2*ad4*b2*c4*cd2+a4*
ad2*b2*c4*cd2+a2*ad4*b2*bd4*cd2-a4*ad2*b2*bd4*cd2+ad4*bd4*cd2-a4*bd4*cd2-
a2*ad4*b4*bd2*cd2+a4*ad2*b4*bd2*cd2+ad2*bd2*cd2-a2*bd2*cd2-ad4*b4*cd2+a4*
b4*cd2-ad2*b2*cd2+a2*b2*cd2-ad2*bd4*c4+a2*bd4*c4+ad4*bd2*c4-a4*bd2*c4+ad2*
b4*c4-a2*b4*c4-ad4*b2*c4+a4*b2*c4-a2*ad4*b2*bd4*c2+a4*ad2*b2*bd4*c2-ad4*
bd4*c2+a4*bd4*c2+a2*ad4*b4*bd2*c2-a4*ad2*b4*bd2*c2-ad2*bd2*c2+a2*bd2*c2+
ad4*b4*c2-a4*b4*c2+ad2*b2*c2-a2*b2*c2)*{e1,e2}+ad2*bd4*c2*cd4-a2*bd4*c2*
cd4-ad4*bd2*c2*cd4+a4*bd2*c2*cd4-ad2*b4*c2*cd4+a2*b4*c2*cd4+ad4*b2*c2*cd4-
a4*b2*c2*cd4-ad2*b2*bd4*cd4+a2*b2*bd4*cd4+ad2*b4*bd2*cd4-a2*b4*bd2*cd4+a2*
ad4*bd2*cd4-a4*ad2*bd2*cd4-a2*ad4*b2*cd4+a4*ad2*b2*cd4-ad2*bd4*c4*cd2+a2*
bd4*c4*cd2+ad4*bd2*c4*cd2-a4*bd2*c4*cd2+ad2*b4*c4*cd2-a2*b4*c4*cd2-ad4*b2*
c4*cd2+a4*b2*c4*cd2+ad4*b2*bd4*cd2-a4*b2*bd4*cd2-a2*ad4*bd4*cd2+a4*ad2*bd4
*cd2-ad4*b4*bd2*cd2+a4*b4*bd2*cd2+a2*ad4*b4*cd2-a4*ad2*b4*cd2+ad2*b2*bd4*
c4-a2*b2*bd4*c4-ad2*b4*bd2*c4+a2*b4*bd2*c4-a2*ad4*bd2*c4+a4*ad2*bd2*c4+a2*
ad4*b2*c4-a4*ad2*b2*c4-ad4*b2*bd4*c2+a4*b2*bd4*c2+a2*ad4*bd4*c2-a4*ad2*bd4
*c2+ad4*b4*bd2*c2-a4*b4*bd2*c2-a2*ad4*b4*c2+a4*ad2*b4*c2
```

find the grade zero part of the product of the bivectors P, Q and R

```
(%i57) arr:grader(PQR)$
       den:arr[Plen+1]$
       PQR0:arr[0]/den$
       ldisplay(PQR0)$

(%t60)/R/ PQR0=(((ad2-a2)*bd4+(-ad4+a4)*bd2+(-ad2+a2)*b4+(ad4-a4)*b2)*c2+
(-ad2+a2)*b2*bd4+((ad2-a2)*b4+a2*ad4-a4*ad2)*bd2+(-a2*ad4+a4*ad2)*b2)*cd4+(
((ad4-a4)*b2-a2*ad4+a4*ad2)*bd4+(-ad4+a4)*b4*bd2+(a2*ad4-a4*ad2)*b4)*cd2+
((ad2-a2)*b2*bd4+((-ad2+a2)*b4-a2*ad4+a4*ad2)*bd2+(a2*ad4-a4*ad2)*b2)*c4+
(((-ad4+a4)*b2+a2*ad4-a4*ad2)*bd4+(ad4-a4)*b4*bd2+(-a2*ad4+a4*ad2)*b4)*c2
```

confirm the validity of the last equation (the identity) in section 4.1.4
this may take a while

```
(%i61) lhs:(p&^q&^r)&*Ki$
       rhs:JJd*PQR0$
       is(equal(lhs,rhs))$
       ldisplay(%)$

(%t64) %=true
```

N.B. the dual of p^q^r is found using the pseudoscalar inverse, Ki, for
the (homogeneous model) subspace of the triangles

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