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Transforms for the early Kerr metric

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ABSTRACT

The early work of Roy Kerr applied complex variable theory to a system of non-linear partial differential equations. Computer algebra has confirmed that nominal transforms of his original early metric can lead to algebraically exact metrics. The first exact solution is a stationary, extended Schwarzschild formula and the second has an axial singularity that may help to model the jets from black holes.

Key words. Accretion - Gravitation - Relativistic processes - Galaxies: nuclei - Quasars: general - Stars: rotation

The General Theory of Relativity (GTR) was established in 1915 and it was not until 1963 that a model of a rotating astronomical mass was found. Since then researchers have tried to model these masses; pulsars, dwarf stars, neutron stars, stellar black holes and galactic black holes.

The GTR is an implicit type of theory, in that a metric is required before one can study the dynamics of motion or mass distribution. A collection of metrics has become established in the community, mostly derived by mathematical transformation of the Kerr metric. This letter describes two metrics that have not been found in the literature but use some of the easier transforms originating in 1963.

The metric itself may be described in several different ways. All are related to a formula for the spacetime interval between two events or rather the square of that value. The interval may otherwise be known as the line element within that separation formula. The metric tensor of the GTR may be referred to as a matrix rather than as a tensor and the word 'metric' may loosely refer to the separation formula itself.

It was quite appropriate and generous for the man who found the early metric to record his memory of the discovery. It was his 2008 summary of the derivation of the original Kerr metric that has prompted the current work. In particular, it showed the use of complex variable theory to derive exact (algebraic) solutions to a system of non-linear partial differential equations. After this technical part of the derivation, certain easier geometric transformations lead to more familiar coordinate systems. It is also when approximations may have missed an exact solution.

Subsequent to the study of Kerr's 2008 paper, and development of models for several metrics, a search was made for anything similar to the results. The nearest content was from 2015 and did contain a version of one of the relevant transformations. It also had some explanation of the sign changes below.

After the seminal work with complex variable, Kerr obtained his metric in terms of the variable, u. The separation formula was not originally split into the sum of 'base' and 'mass' terms. Sign changes have occured in the splitting of the 1963 formula and the natural transformation, $u = r + t$ has been lost. The line element was later given by the base metric + the mass term in the 2008 summary.

The base metric was given by...

$$\mathrm{d}s_0^2 = (r^2 + a^2 * cos(\theta)^2) * (\mathrm{d}\theta^2 + sin(\theta)^2 * \mathrm{d}\phi^2) - \mathrm{d}s_\mathrm{u}^2 \tag{1}$$

where

$$\mathrm{d}s_\mathrm{u}^2 = (2 * \mathrm{d}r + \mathrm{d}u - a * sin(\theta)^2 * \mathrm{d}\phi) * (\mathrm{d}u + a * sin(\theta)^2 * \mathrm{d}\phi) \tag{2}$$

The mass term was given by...

$$\mathrm{d}s_\mathrm{M}^2 = 2 * m * r * (\mathrm{d}u + a * sin(\theta)^2 * \mathrm{d}\phi)^2 / (r^2 + a^2 * cos(\theta)^2) \tag{3}$$

where the 'line elements' have been named and the letter 'd' notation has been used for the differentials.

The author has created a repository on GitHub for this study and the project uses the type of notebook generated by wxMaxima. Each notebook source file is an active document, in that it has a commentary interleaved with the calculation code and results are embedded next to that code when executed. The active documents for the two algebraic metrics will be uploaded to the open source repository if this letter is published in a journal. They have also contributed to the accuracy of the formulae within this letter.

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The transform strategy can now be described; for two different sequences of transformations, the base term and the mass term defined by Equations (1) to (3) were transformed separately for clarity. For each sequence the two resulting matrices were summed to give the metric for that sequence. The early Kerr metric was transformed to both a spherical system and also to a 'spherical-cylindrical' system of coordinates.

The spherical metric below was derived using only the transformation: $r + u = -t$. The differential, [$\mathrm{d}u = -\mathrm{d}t - \mathrm{d}r$] has been used to substitute for du since u itself does not occur explicitly in Equation (2). The base metric then becomes...

$$\mathrm{d}s_0^2 = +\mathrm{d}\theta^2 * (r^2 + a^2 * cos(\theta)^2) + \mathrm{d}\phi^2 * (r^2 + a^2) * sin(\theta)^2 + \mathrm{d}s_\mathrm{t}^2 \tag{4}$$

where

$$\mathrm{d}s_\mathrm{t}^2 = -2 * a * \mathrm{d}\phi * \mathrm{d}r * sin(\theta)^2 - \mathrm{d}t^2 + \mathrm{d}r^2 \tag{5}$$

The same substitution was applied to Equation (3) for the mass term. The denominator does not contain u or $\mathrm{d}u$ and can be isolated for further calculations.

$$den = a^2 * cos(\theta)^2 + r^2 \tag{6}$$

In order to avoid long quotients, the transformed mass term for the spherical metric has been displayed using the denominator, den as a multiplier.

$$\begin{aligned}\mathrm{d}s_\mathrm{M}^2 * den = {} & +2 * a^2 * \mathrm{d}\phi^2 * m * r * sin(\theta)^4 \\ & -4 * a * \mathrm{d}\phi * \mathrm{d}t * m * r * sin(\theta)^2 \\ & -4 * a * \mathrm{d}\phi * \mathrm{d}r * m * r * sin(\theta)^2 \\ & +2 * \mathrm{d}t^2 * m * r + 4 * \mathrm{d}r * \mathrm{d}t * m * r + 2 * \mathrm{d}r^2 * m * r\end{aligned} \tag{7}$$

The two formulae were then summed.

$$\mathrm{d}s^2 = \mathrm{d}s_0^2 + \mathrm{d}s_\mathrm{M}^2 \tag{8}$$

The usual vector of coordinates for the spacetime spherical system is given by (r, θ, ϕ, t). The extraction of the coefficients into the metric was consistent with a matrix row, $(\mathrm{d}r, \mathrm{d}\theta, \mathrm{d}\phi, \mathrm{d}t)$. The extraction included the off-diagonal coefficients for $\mathrm{d}\phi * \mathrm{d}r$, $\mathrm{d}\phi * \mathrm{d}t$ and $\mathrm{d}r * \mathrm{d}t$, symmetrically about the diagonal.

All of the non-zero elements of the metric had the same denominator, den, except the coefficient of $\mathrm{d}\theta^2$. The value of that element was den itself.

The following is an inspection of the Kerr metric in spherical coordinates. Firstly, only the diagonal terms for time and space, respectively, are examined here for zero rotation ($a = 0$). This was for comparison with the Schwarzschild metric...

$$g_{44} = (-a^2 * cos(\theta)^2 - r^2 + 2 * m * r)/(a^2 * cos(\theta)^2 + r^2) \tag{9}$$

setting $a = 0$,

$$g_{44} = (2 * m)/r - 1 \tag{10}$$

$$g_{11} = (a^2 * cos(\theta)^2 + r^2 + 2 * m * r)/(a^2 * cos(\theta)^2 + r^2) \tag{11}$$

again setting $a = 0$,

$$g_{11} = (2 * m)/r + 1 \tag{12}$$

The $\mathrm{d}r^2$ element, $1 + 2m/r$, is algebraic (no approximation); it only occurs in the Schwarzschild metric after the binomial approximation to the singular quotient, $1/(1 - 2m/r)$.

Now, even for the zero rotation case, there was a non-zero off-diagonal element. This retains a time asymmetry so the metric is stationary but not static...

$$g_{14} = (4 * m)/r \tag{13}$$

The non-rotating spherical Kerr metric is not equivalent to the Schwarzschild metric, it is something else...

$$\mathrm{d}s_\mathrm{A}^2 = (1 + 2 * m/r) * \mathrm{d}r^2 + (8 * m/r) * \mathrm{d}r * \mathrm{d}t - (1 - 2 * m/r) * \mathrm{d}t^2 + \mathrm{d}\Omega^2 \tag{14}$$

where, in a typical notation,

$$d\Omega^2 = r^2 * d\theta^2 + r^2 * sin(\theta)^2 * d\phi^2 \tag{15}$$

For the spherical-cylindrical metric, the same transform strategy was followed and the sequence was similar to the previous metric. The two tranformations were applied in turn to Equation (1) and then to Equation (3). The first transformation was: $r+u = -t$ and the second was $r * cos(\theta) = z$. After the first of these, the resulting equations are Equation (4) for the base metric and Equation (7) for the mass term. Before applying the second transformation, it was required to form the differential of that transformation...

$$cos(\theta) * dr - r * sin(\theta) * d\theta = dz \tag{16}$$

Again, before the substitution of the circular functions themselves, the contribution of functions from the differential were included as...

$$[d\theta = (dr * cos(\theta) - dz)/(r * sin(\theta))] \tag{17}$$

The substitution for the circular functions was...

$$[cos(\theta) = z/r, sin(\theta) = +sqrt(1 - (z/r)^2)] \tag{18}$$

Since the $sin(\theta)$ occured as a square, there was no sign ambiguity due to the +ve $sqrt()$ for the sine. The cosine lies in the interval [+1,-1], and its sign is still present in the matrix coefficient for $dr*dz$, in Equation (20) below, as +ve and -ve values of z. The angle, θ is measured from the spin axis (z-axis) in the range $[0, \pi]$.

The base metric in Equation (4) was transformed and the denominator was isolated.

$$den_b = r^4 * (z - r) * (z + r) \tag{19}$$

For clarity, the transformed base metric of the spherical-cylindrical system has been displayed using the denominator den_b as a multiplier.

$$\begin{aligned} ds_b^2 * den_b = &-d\phi^2 * r^2 * (r^2 + a^2) * (z - r)^2 * (z + r)^2 \\ &+2 * a * d\phi * dr * r^2 * (z - r)^2 * (z + r)^2 \\ -dr^2 * (a^2 * z^4 + r^6) &+ 2 * dr * dz * r * z * (a^2 * z^2 + r^4) \\ -dz^2 * r^2 * (a^2 * z^2 + r^4) &- dt^2 * r^4 * (z - r) * (z + r) \end{aligned} \tag{20}$$

The angular coordinate difference, $d\theta$ does not occur in Equation (7) and the circular functions within the mass term were transformed directly.

$$den_m = r * (a^2 * z^2 + r^4) \tag{21}$$

Again for clarity, the transformed mass term for the spherical-cylindrical metric has been displayed using the denominator, den_m as a multiplier.

$$\begin{aligned} ds_m^2 * den_m = &+2 * m * a^2 * d\phi^2 * (r^2 - z^2)^2 \\ &-4 * m * a * d\phi * dt * r^2 * (r^2 - z^2) \\ &-4 * m * a * d\phi * dr * r^2 * (r^2 - z^2) \\ &+2 * m * dt^2 * r^4 \\ &+4 * m * dr * dt * r^4 \\ &+2 * m * dr^2 * r^4 \end{aligned} \tag{22}$$

The formulae for the base metric and the mass term were then summed.

$$ds^2 = ds_b^2 + ds_m^2 \tag{23}$$

The vector of coordinates for a spacetime cylindrical system would be given by (ρ, ϕ, z, t), however, for the spherical-cylindrical system they are (r, ϕ, z, t) with a spherical radius.

The extraction of the coefficients into the metric from Equation (23) was consistent with a matrix row, $(dr, d\phi, dz, dt)$. This included the off-diagonal coefficients for $d\phi * dr$, $d\phi * dt$, $dr * dz$ and $dr * dt$ to form a symmetric matrix. Denominators within the final matrix were formed from Equation (19) and Equation (21).

The remainder of this letter is an inspection of the Kerr metric in spherical-cylindrical coordinates. Just the diagonal terms for time and space, respectively, are examined here for $a = 0$ (zero rotation).

$$g_{44} = (2 * m * r^3)/(a^2 * z^2 + r^4) - 1 \tag{24}$$

setting $a = 0$,

$$g_{44} = (2 * m)/r - 1 \tag{25}$$

The dt^2 element is algebraic; it has the same coordinate zero as the Schwarzschild metric at $r = 2 * m$

$$g_{11} = (2 * m * r^3)/(a^2 * z^2 + r^4) + (-a^2 * z^4 - r^6)/(r^4 * (z - r) * (z + r)) \tag{26}$$

again setting $a = 0$,

$$g_{11} = (2 * m)/r - r^2/(z^2 - r^2) \tag{27}$$

Having applied the zero rotation condition, the Schwarzschild metric value was found only in the plane $z = 0$. There is, in general, no coordinate singularity due to this element at $r = 2 * m$.

However, there is a singularity at r = +/-z, all along the spin axis. The axial singularity is also present for non-zero rotations and for some other elements of the metric matrix. It is possible that active black holes may be modeled with this metric. We do not yet know whether the non-orthogonal system is creating a coordinate singularity or representing an actual singularity. It is tempting to conclude the former since for zero rotation the axis is arbitrary, however, this is a very special limiting condition.