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Transforms for the early Kerr metric

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Abstract. The early work of Roy Kerr applied complex variable theory to a system of non-linear partial differential equations. Computer algebra has confirmed that nominal transforms of his original early metric lead to an algebraically exact metric with an axial singularity that may allow the jets from black holes to be modeled.

1. Introduction

The General Theory of Relativity (GTR) was established in 1915 and it was not until 1963 that a model of a rotating astronomical mass was found. Since then, with the general expansion of the human race, researchers have tried to model these masses; bright stars, dwarf stars, neutron stars, stellar black holes and galactic black holes. The GTR is an implicit type of theory, in that a metric is required before one can study the dynamics of motion or mass distribution. A collection of metrics has become established in the community, mostly derived by mathematical transformation of the Kerr metric. This letter describes two metrics that have not been found in the literature but use some of the methods originating in 1963.

2. The history

It was quite appropriate and generous for the man who found the early metric to document his memory of the discovery [1]. It was this summary of the derivation of the original Kerr metric that has prompted the curent work. In particular, it shows the use of complex variable theory to derive exact (algebraic) solutions to a system of non-linear partial differential equations. After this technical part of the derivation, certain easier geometric transformations lead to more familiar coordinate systems. It is also when approximations may have missed an exact solution.

3. The separation formula

The metric may be described in several different ways. All are related to a formula for the spacetime interval between two events or rather the square of that value. The interval may otherwise be known as the line element within that separation formula. The metric tensor of the GTR may be referred to as

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a matrix rather than as a tensor and the word 'metric' may loosely refer to the separation formula itself.

4. GTR metrics

Subsequent to the study of Kerr's 2008 paper, and development of models for several metrics, a search was made for anything similar to the results. The nearest content that was found was actually from this century [2]. It is a good summary and it does contain a version of one of the relevant transformations.

5. The early Kerr metric

After the seminal work with complex variable, Kerr obtained his equations 1.25 and 1.26. The separation formula was originally split into 'base' and 'mass' formulae. The line element was given by the base term (Equ.1.26) + the mass term (part of Equn.1.25)

Definition 1. Equn.1.26 (what was called the base metric) is given by...

$$ds_0^2 = (r^2 + a^2 * cos(\theta)^2) * (\delta\theta^2 + sin(\theta)^2 * \delta\phi^2) - ds_{0u}^2 \tag{1}$$

where

$$ds_{0u}^2 = (2 * dr + du - a * sin(\theta)^2 * \delta\phi) * (du + a * sin(\theta)^2 * \delta\phi) \tag{2}$$

Definition 2. Equn.1.25 (part) has been called the mass term and is given by...

$$ds_m^2 = 2 * m * r * (du + a * sin(\theta)^2 * \delta\phi))^2 / (r^2 + a^2 * cos(\theta)^2) \tag{3}$$

where the line elements have been named and the 'delta' notation has been used for the angular differentials, rather than 'd'.

6. The transform strategy

For each of two different sequences of transformations, the two line elements defined above were transformed separately for clarity. For each sequence the two resulting matrices were summed to give the early Kerr metric as transformed to each of two different algebraic metrics.

7. Active documents

The author has created a repository on GitHub for his retirement project and the URL is given at the end of this letter. The project uses the type of notebook generated by wxMaxima. Each notebook source file is an active document, in that it has a commentary interleaved with the calculation code and results are embedded next to that code when executed. The active documents for the two algebraic metrics will be uploaded to the open source repository if this letter is published in a journal.

8. Two exact solutions

At least three transformations were used in Kerr's 2008 paper including: $r * cos(\theta) = z$ and $r + u = -t$; Section 8.1 below uses only the transformation: $r + u = -t$ and Section 8.2 uses both of the above transformations.

8.1. The spherical metric. To begin with, $[du = -dt - dr]$ has been used to substitute for du since u itself does not occur explicitly in Equation (2). The base metric then becomes...

$$ds_0^2 = +\delta\theta^2 * (r^2 + a^2 * cos(\theta)^2) + \delta\phi^2 * (r^2 + a^2) * sin(\theta)^2 + ds_{0t}^2 \tag{4}$$

where

$$ds_{0t}^2 = -2 * a * \delta\phi * dr * sin(\theta)^2 - dt^2 + dr^2 \tag{5}$$

The same substitution was applied to Equation (3) for the mass term. The denominator does not contain u or du and can be isolated for further calculations.

$$den_m = a^2 * cos(\theta)^2 + r^2 \tag{6}$$

For clarity, it is displayed as a multiplier for the transformed mass term.

$$\begin{aligned} ds_m^2 * den_m = +2 * a^2 * \delta\phi^2 * m * r * sin(\theta)^4 \\ -4 * a * \delta\phi * dt * m * r * sin(\theta)^2 - 4 * a * \delta\phi * dr * m * r * sin(\theta)^2 \\ +2 * dt^2 * m * r + 4 * dr * dt * m * r + 2 * dr^2 * m * r \end{aligned} \tag{7}$$

The two formulae were then summed.

$$ds^2 = ds_0^2 + ds_m^2 \tag{8}$$

The usual vector of coordinates for the spacetime spherical system is given by (r, θ, ϕ, t). The extraction of the coefficients into the metric was consistent with a matrix row, $(dr, \delta\theta, \delta\phi, dt)$. The extraction included the off-diagonal coefficients for $\delta\phi * dr$, $\delta\phi * dt$ and $dr * dt$, symmetrically about the diagonal.

All of the non-zero elements of the metric had the same denominator, den_m, except the coefficient of $\delta\theta^2$. The value of that element was den_m itself.

The remainder of this subsection is an inspection of the Kerr metric in spherical coordinates. Firstly, just the two main terms of the metric for a=0 (zero rotation) revealed an algebaic solution.

$$g_{44} = (-a^2 * cos(\theta)^2 - r^2 + 2 * m * r)/(a^2 * cos(\theta)^2 + r^2) \tag{9}$$

setting $a = 0$,

$$g_{44} = (2 * m)/r - 1 \tag{10}$$

The dt^2 element is algebraic; it is identical to that of the Schwarzchild metric;

$$g_{11} = (a^2 * cos(\theta)^2 + r^2 + 2 * m * r)/(a^2 * cos(\theta)^2 + r^2) \tag{11}$$

again setting $a = 0$,

$$g_{11} = (2 * m)/r + 1 \tag{12}$$

The dr^2 element, $1 + 2m/r$, is also algebraic (no approximation); it only occurs in the Schwarzchild metric after the binomial approximation to the singular quotient, $1/(1 - 2m/r)$

8.2. The spherical-cylindrical metric. The transform strategy given in Section 6 was followed and the sequence was similar to that of Section 8.1. The two tranformations were applied in turn to Equation (1) and then to Equation (3). The first transformation was: $r + u = -t$ and the second was $r * cos(\theta) = z$. After the first of these, the resulting equations are found in Section 8.1. They are Equations (4) for the base metric and Equation (7) for the mass term. Firstly, it was required to form the differential of the second transformation...

$$cos(\theta) * dr - r * sin(\theta) * \delta\theta = dz \tag{13}$$

Before the substitution of the circular functions themselves, the contribution of functions from the differential were included as...

$$[\delta\theta = (dr * cos(\theta) - dz)/(r * sin(\theta))] \tag{14}$$

The substitution for the circular functions was...

$$[cos(\theta) = z/r, sin(\theta) = +sqrt(1 - (z/r)^2)] \tag{15}$$

Since the $sin(\theta)$ occured as a square, there was no sign ambiguity due to the +ve sqrt() for the sine. The cosine lies in the interval [+1,-1], and its sign is still present in the matrix elements for $dr * dz$, in Equation (17) below, as +ve and -ve values of z. The angle, θ is measured from the spin axis (z-axis) in the range $[0, \pi]$.

The base metric in Equation (4) was transformed and the denominator was isolated.

$$den_b = r^4 * (z - r) * (z + r) \tag{16}$$

For clarity, the denominator is used as a multiplier for the transformed base metric of the spherical-cylindrical (sc) system .

$$\begin{aligned} ds^2_{0sc} * den_b = &-\delta\phi^2 * r^2 * (r^2 + a^2) * (z - r)^2 * (z + r)^2 \\ &+2 * a * \delta\phi * dr * r^2 * (z - r)^2 * (z + r)^2 \\ &-dr^2 * (a^2 * z^4 + r^6) + 2 * dr * dz * r * z * (a^2 * z^2 + r^4) \\ &-dz^2 * r^2 * (a^2 * z^2 + r^4) - dt^2 * r^4 * (z - r) * (z + r) \end{aligned} \tag{17}$$

The angular coordinate difference, $\delta\theta$ does not occur in Equation (7) and the circular functions within the mass term were transformed directly.

$$den_{msc} = r * (a^2 * z^2 + r^4) \tag{18}$$

Again for clarity, the above denominator is used as a multiplier for the transformed mass term system .

$$\begin{aligned} ds_{msc}^2 * den_{msc} = &+2 * m * a^2 * \delta\phi^2 * (r^2 - z^2)^2 \\ &-4 * m * a * \delta\phi * dt * r^2 * (r^2 - z^2) \\ &-4 * m * a * \delta\phi * dr * r^2 * (r^2 - z^2) \\ &+2 * m * dt^2 * r^4 \\ &+4 * m * dr * dt * r^4 \\ &+2 * m * dr^2 * r^4 \end{aligned} \tag{19}$$

The formulae for the base metric and the mass term were then summed.

$$ds^2 = ds_{0sc}^2 + ds_{msc}^2 \tag{20}$$

The vector of coordinates for a spacetime cylindrical system would be given by (ρ, ϕ, z, t), however, for the spherical-cylindrical system they are (r, ϕ, z, t) with a spherical radius, r.

The extraction of the coefficients into the metric from Equation (20) was consistent with a matrix row, $(dr, \delta\phi, dz, dt)$. This included the off-diagonal coefficients for $\delta\phi * dr$, $\delta\phi * dt$, $dr * dz$ and $dr * dt$ to form a symmetric matrix.

Denominators that occured within the final matrix were either simple multiples of the isolated denominators or r^2.

The remainder of this subsection is an inspection of the Kerr metric in spherical-cylindrical coordinates. Firstly, just the two main terms of the metric for a=0 (zero rotation) again revealed an algebaic solution.

$$g_{44} = (2 * m * r^3)/(a^2 * z^2 + r^4) - 1 \tag{21}$$

setting $a = 0$,

$$g_{44} = (2 * m)/r - 1 \tag{22}$$

The dt^2 element is algebraic; it has the same coordinate zero as the Schwarzchild metric at $r = 2 * m$

$$g_{11} = (2 * m * r^3)/(a^2 * z^2 + r^4) + (-a^2 * z^4 - r^6)/(r^4 * (z - r) * (z + r)) \tag{23}$$

again setting $a = 0$,

$$g_{11} = (2 * m)/r - r^2/(z^2 - r^2) \tag{24}$$

Having applied the zero rotation condition, the Schwarzchild metric value was found only in the plane $z = 0$. There is, in general, no coordinate singularity due to this element at $r = 2 * m$

Remark 1. *However, there is a singularity at $r = +/\text{-}z$, all along the spin axis. It is possible that what we call jets emanating from black holes may be spin axes accretion regions, even, or perhaps mainly, as we approach zero rotation.*

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References

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