# Euclidean Distance Completion

Eucidean Distance Matrices (EDM) are a special class of symmetric matrices that are generated from distance tables. We say $E \in \mathbb{R}^{n \times n}$ is an EDM if there is a latent variable $X \in \mathbb{R}^{n \times d}$ such that $E_{ij} = \|X_i - X_j\|_2^2$ where $X_i$ is the $i^{\text{th}}$ row of the matrix *X*. We will restrict to the type of EDM where *d* is fixed, and in particular, $d = 2$, as these are the EDM that are generated from squared distances in the plane.

## More information

- Liberti, L., Lavor, C., Maculan, N., & Mucherino, A. (2014). Euclidean distance geometry and applications. *Siam Review*, *56*(1), 3-69
- Dokmanic, I., Parhizkar, R., Ranieri, J., & Vetterli, M. (2015). Euclidean distance matrices: essential theory, algorithms, and applications. *IEEE Signal Processing Magazine*, *32*(6), 12-30.
- Boyd, S. and Vandenberghe, L (2006). Convex Optimization. Cambridge University Press (see section 8.3.3, and exercise 2.36)

## Our formulation

Writing $K(Y) = 1\text{diag}(Y)^T + \text{diag}(Y)1^T - 2Y$, we have that *E* is an EDM if $E = K(XX^T)$ for some *X* as above. Suppose we only observe $E_{i,j}$ for some $(i, j) \in \Omega$, and we wish to recover *E* (or more accurately, we wish to recover the underlying *X* -- note that this is only determined up to rotations).

We can solve $\min_{XX} \sum_{(i,j)\in\Omega} (K(XX)_{i,j} - E_{i,j})^2$ subject to $XX \succcurlyeq 0$ and $\text{rank}(XX) = 2$ (for the $d = 2$ case).

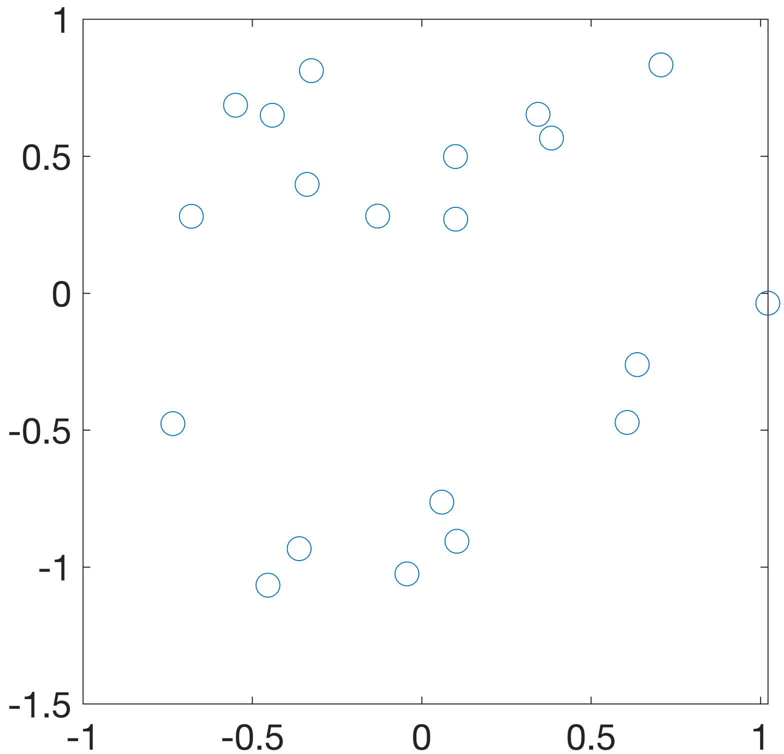
This is intractable, but if we drop the rank constraint, we have a SDP, which we can solve via CVX.

## Setup a problem

```
n   = 20; % number of points
d   = 2;  % dimension
o   = ones(n,1);
K   = @(X) diag(X)*o' + o*diag(X)' - 2*X;

% Make some sample data in [-1,1]x[-1,1]
rng(1);
X0  = 2*(rand(n,d)-.5);
% Put center-of-mass at zero
X0 = X0 - repmat( ones(1,n)/n*X0, n, 1 );

% Plot it:
figure(1); clf;
h=plot( X0(:,1), X0(:,2) , 'o','MarkerSize',12);
% h.MarkerFaceColor = h.Color;
set(gca,'fontsize',18)
hold all
axis square
```

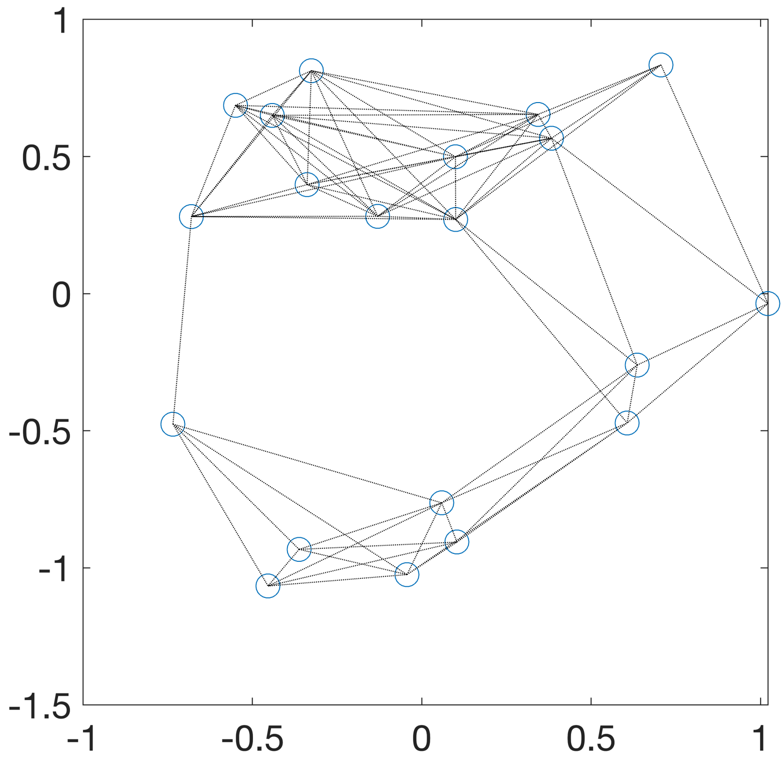



## Observe only some pairwise distances

```
figure(1); clf;
h=plot( X0(:,1), X0(:,2) , 'o','MarkerSize',12);
% h.MarkerFaceColor = h.Color;
set(gca,'fontsize',18)
hold all
axis square

% Only observe some of these
E    = K(X0*X0');

min_dist    = 0.86; % this should give perfect recovery
% min_dist    = 0.85; % this should not
E_thresh    = ( E < min_dist );
E_thresh    = E_thresh - tril(E_thresh);
for i = 1:n
    for j = find( E_thresh(i,:) )
        if ~isempty(j)
            hold all
            plot( [X0(i,1),X0(j,1)], [X0(i,2),X0(j,2) ], 'k:' );
        end
    end
end
```




```
omega   = find( E_thresh(:) );
obs     = @(y) y(omega);
fprintf('Using %d pairwise observations\n',length(omega))
```

```
Using 74 pairwise observations
```

## Solve the SDP using CVX

```
% Solve SDP using CVX
cvx_begin quiet
    variable XX(n,n) symmetric
%    minimize norm( obs(vec(K(XX))) - obs(vec(E)) )
    minimize trace( XX )
    subject to
        XX*o == zeros(n,1)
        XX   == semidefinite(n)
        obs( vec(K(XX))) == obs(vec(E)) % if we know there is no noise
cvx_end

numericalRank   = @(X) length( find(svd(X) > 1e-3) );
fprintf('Approximate rank of solution is %d\n', numericalRank(XX) );
```

```
Approximate rank of solution is 2
```

```
s = svd(XX);
disp(s(1:5))
```

```
    8.6551
    4.6465
    0.0000
    0.0000
    0.0000
```

```
disp(s(5))
```

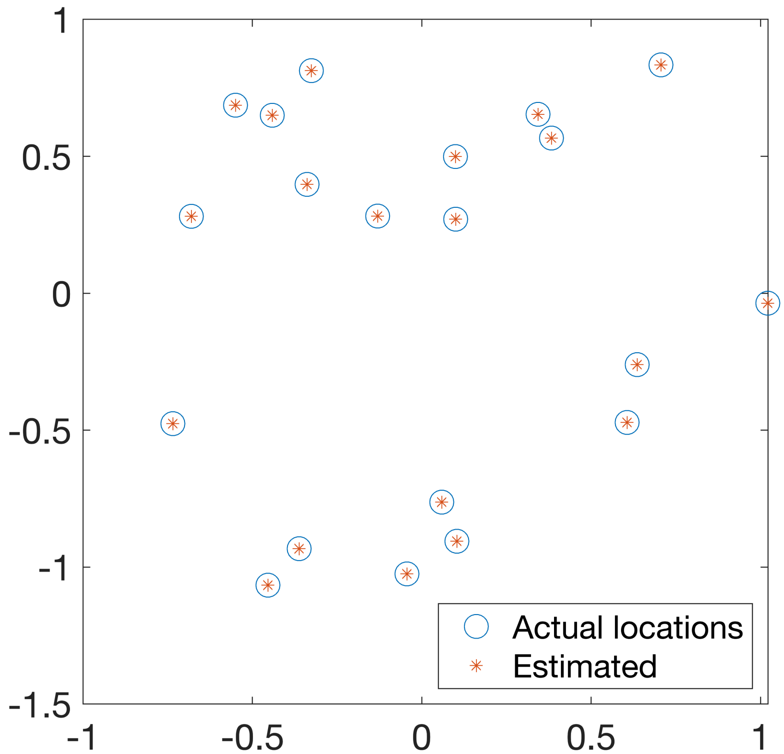
```
   1.3488e-09
```

## Recreate data from XX

```
% We want to find the *right* X such that X*X' = XX
[V,D]    = eig( XX );
[eigv,ind] = sort( diag(D), 'descend' );
X       = V(:,ind(1:2))*diag( sqrt(eigv(1:2)) );

% Solve Procrustes problem to find best rotation
[u,~,v] = svd(X0'*X,'econ');
R       = u*v';
X       = X*R';

figure(1); clf;
h=plot( X0(:,1), X0(:,2) , 'o','MarkerSize',12);
% h.MarkerFaceColor = h.Color;
set(gca,'fontsize',18)
hold all
axis square
hold all
hh  = plot( X(:,1), X(:,2), '*' );
legend([h,hh],'Actual locations','Estimated','location','southeast');
```



1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
Actual locations
Estimated