FUNCTIONAL CALCULUS FOR DUAL QUATERNIONS

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ABSTRACT. We give a formula for $f(\eta)$, where $f : \mathbb{C} \to \mathbb{C}$ is a continuously differentiable function satisfying $f(\bar{z}) = \overline{f(z)}$, and η is a dual quaternion. Note this formula is straightforward or well known if η is merely a dual number or a quaternion. If one is willing to prove the result only when f is a polynomial, then the methods of this paper are elementary.

A *quaternion* is a quadruple of real numbers, written as $A = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, with the algebraic operations $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$. Its *conjugate* is $\bar{A} = A = w - x\boldsymbol{i} - y\boldsymbol{j} - z\boldsymbol{k}$. Its *norm* is $|A| = (w^2 + x^2 + y^2 + z^2)^{1/2}$. It is called a *pure* quaternion if $w = 0$. We identify 3-vectors with pure quaternions, by identifying $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ with the three standard unit vectors. A *dual number* is a pair of real numbers, written as $\alpha = a + \epsilon b$, with the algebraic operation $\epsilon^2 = 0$. A *dual quaternion* is a pair of quaternions, written as $\eta = A + \epsilon B$, again with $\epsilon^2 = 0$, and ϵ commuting with all other elements. The *conjugate* of this dual quaternion is $\bar{\eta} = \bar{A} + \epsilon \bar{B}$. The notion of dual quaternion goes back to Clifford [3]. For more information, including how they are used by the graphics card industry and in robotics, we refer the reader to [1, 2, 5, 7, 8, 9, 10, 12, 13, 18].

The symbol i denotes one of the square roots of -1, and in this paper won't be identified with $\boldsymbol{i}$.

We say that a function $f : \Omega \to \mathbb{C}$ is *valid* if Ω is a subset of $\mathbb{C}$ such that $z \in \Omega \Leftrightarrow \bar{z} \in \Omega$, and

$$f(\bar{z}) = \overline{f(z)} \quad \text{for } z \in \Omega. \tag{1}$$

Note that if $f(x + iy)$ is a polynomial in x and y, equivalently a polynomial in $z = x + iy$ and $\bar{z}$:

$$f(x + iy) = \sum_{m,n} r_{m,n} x^m (iy)^n = \sum_{m,n} s_{m,n} z^m \bar{z}^n. \qquad (2)$$

then f is valid if and only if the $r_{m,n}$ are real if and only if the $s_{m,n}$ are real.

We show how to extend the definition of f, at least when f is appropriately smooth, to dual quaternions in such a way that it is correct for polynomials. In all of the statements of our results, when we say that a class of functions extends to a subset of dual quaternions, we mean the following standard definition of a functional calculus.

(1) The class of functions contains $f(z) = 1$, $f(z) = z$, and if f and g are in this class, and a is a real number, then $f + g$, fg, $\bar{f}$, and af are also in this class.

(2) Given any dual quaternion η from the prescribed subset, there is a map from the class to dual quaternions, $f \mapsto f(\eta)$, such that $f(z) = 1$ maps to 1, $f(z) = z$ maps to η, $f + g$ maps to $f(\eta) + g(\eta)$, fg maps to $f(\eta)g(\eta)$, $\bar{f}$ maps to $\overline{f(\eta)}$, and af maps to $af(\eta)$.

(3) If any topology is specified on the class of functions, then the map $\eta \mapsto f(\eta)$ is continuous.

All of the results of this paper could be verified for polynomials simply by looking at the individual monomials, and indeed this is how Lemmas 9 and 10 are best proved. Another method is to verify the results if $f(z) = 1$ and $f(z) = z$, and show that the set of functions for which it is true is closed under addition, multiplication, conjugation, and multiplication by real numbers. However, for Lemma 11, these straightforward approaches are nevertheless quite mysterious, and we believe that our approach is more intuitive.

The class of functions, with the exception of Theorem 2 will be a subset of functions that have one continuous derivative on an open neighborhood of a set $\Omega \subset \mathbb{C}$. Using the following version of the Stone-Weierstrass Theorem, which may be found in [11, Section 1.6.2], we see that it is sufficient to prove our results for polynomials.

Theorem 1. *Let $f : \Omega \to \mathbb{R}$ be a k times continuously differentiable function, where Ω is an open subset of $\mathbb{R}^n$. Let K be a compact subset of Ω. Then given $\delta > 0$, there exists a polynomial $p : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\left| \frac{\partial^{\alpha_1 + \cdots + \alpha_n} f}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}(x_1, \ldots, x_n) - \frac{\partial^{\alpha_1 + \cdots + \alpha_n} p}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}(x_1, \ldots, x_n) \right| \leq \delta \quad (3)$$

for all $(x_1, \ldots, x_n) \in \Omega$, *and non-negative integers* $\alpha_1, \ldots, \alpha_n$ *satisfying* $\alpha_1 + \cdots \alpha_n \leq k$.

Note that the restriction that p be valid presents no difficulty, since one can always replace $p(z)$ by $\frac{1}{2}(p(z) + \overline{p(\bar{z})})$, and obtain the same approximation.

We should mention that the formula for dual numbers is well known, and we cite it as Lemma 9.

We also want to mention the remarkable, and rather different, approach taken by Selig to this problem [14].

If f is continuously differentiable, denote

$$f_x(x + iy) = \frac{\partial}{\partial x} f(x + iy), \quad f_{iy}(x + iy) = -i \frac{\partial}{\partial y} f(x + iy). \quad (4)$$

(Thus the Cauchy-Riemann conditions can be stated as f is analytic if and only if $f_x = f_{iy}$.) Define the following real valued, continuous, functions, which are even in y:

$$g(x + iy) = \frac{f(x + iy) + f(x - iy)}{2}, \quad (5)$$

$$h(x + iy) = \begin{cases} \dfrac{f(x + iy) - f(x - iy)}{2iy} & \text{if } y \neq 0 \\ f_{iy}(x) & \text{if } y = 0, \end{cases} \quad (6)$$

so that

$$f(x + iy) = g(x + iy) + iyh(x + iy). \quad (7)$$

First we state how to extend f to the quaternions. The proof is straightforward, because any quaternion that is a unit 3-vector behaves formally exactly like i in $\mathbb{C}$.

Theorem 2. *Suppose that* $f : \Omega \to \mathbb{C}$ *is a valid function. Then* f *extends to quaternions*

$$f(a_0 + \boldsymbol{a}_1) = \begin{cases} g(a_0 + i|\boldsymbol{a}_1|) + h(a_0 + i|\boldsymbol{a}_1|)\boldsymbol{a}_1 & \text{if } \boldsymbol{a}_1 \neq 0 \\ f(a_0) & \text{if } \boldsymbol{a}_1 = 0, \end{cases} \quad (8)$$

where a_0 *is real and* $\boldsymbol{a}_1$ *is a 3-vector, whenever* $a_0 + i|\boldsymbol{a}_1| \in \Omega$. *Furthermore the norm is preserved in that*

$$|f(a_0 + \boldsymbol{a}_1)| = |f(a + i|\boldsymbol{a}_1|)|. \quad (9)$$

If f is a valid polynomial, then a formula that extends f to all dual quaternions follows from Taylor's series. If A and B are quaternions, then

$$f(A + \epsilon B) = f(A) + \epsilon \left. \frac{d}{dr} f(A + rB) \right|_{r=0}. \quad (10)$$

If A and B commute, this implies Lemma 10 below. But if A and B do not commute, it is not immediately apparent how to use this formula. Thus we now state the main result of this paper.

Theorem 3. *Let $f : \Omega \to \mathbb{C}$ be a valid continuously differentiable function, where Ω is open in $\mathbb{C}$. Define h by equation (6). Then f can be extended to a continuous function on all dual quaternions as follows. Given quaternions A and B, decompose $A = a_0 + \boldsymbol{a}_1$ and $B = b_0 + \boldsymbol{b}_1 + \boldsymbol{b}_2$, where a_0 and b_0 are real, $\boldsymbol{a}_1$, $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are 3-vectors, $\boldsymbol{b}_1$ is parallel to $\boldsymbol{a}_1$, and $\boldsymbol{b}_2$ is perpendicular to $\boldsymbol{a}_1$. Write $B_1 = b_0 + \boldsymbol{b}_1$. If $a_0 + i|\boldsymbol{a}_1| \in \Omega$, then*

$$f(A + \epsilon B) = g(A) + \epsilon f_x(A)b_0 + h(A)(1 + \epsilon \boldsymbol{b}_2) + \epsilon f_{iy}(A)\boldsymbol{b}_1 \qquad (11)$$

$$= f(A) + \epsilon f_x(A)b_0 + \epsilon f_{iy}(A)\boldsymbol{b}_1 + \epsilon h(A)\boldsymbol{b}_2. \qquad (12)$$

Note that in the statement of the theorem, there is no restriction on the value of B.

Before proving this result, let us provide some examples. First something simple, where we show that the formulas created by the functional calculation agree with the standard definitions given in the literature.

Corollary 4. *With the hypotheses of Theorem 3, the conjugate and norm functions extend to dual quaternions as*

$$\overline{A + \epsilon B} = \overline{A} + \epsilon \overline{B}, \qquad (13)$$

$$|A + \epsilon B| = |A| + \epsilon \frac{\text{Re}(A\overline{B})}{|A|} \quad (A \neq 0). \qquad (14)$$

Next, we compute the exponential function. Formulas for the exponential and logarithm are given in [16], but we believe ours are more explicit. Another formula for the exponential and the logarithm is given in [14], with a correction for the logarithm in [17]. While their formula gives the same result for the exponential, this is not immediately obvious, and we haven't checked for the logarithm. A formula for the exponential and logarithm is also given in [5], but we believe that their formula only works if A and B commute.

Corollary 5. *With the hypotheses of Theorem 3 we have*

$$\exp(A + \epsilon B)$$
$$= e^{a_0} \left(\left(\cos(|\boldsymbol{a}_1|) + \frac{\sin(|\boldsymbol{a}_1|)}{|\boldsymbol{a}_1|} \boldsymbol{a}_1 \right) (1 + \epsilon B_1) + \epsilon \frac{\sin(|\boldsymbol{a}_1|)}{|\boldsymbol{a}_1|} \boldsymbol{b}_2 \right), \quad (15)$$

where if $\boldsymbol{a}_1 = 0$, we set $\sin(|\boldsymbol{a}_1|)/|\boldsymbol{a}_1| = 1$.

We can compute the logarithm in the same way, but then we can only get the principal value. Instead we define a logarithm as a right inverse to the exponential function: $\exp(\log(\eta)) = \eta$.

Corollary 6. *Assume the hypotheses of Theorem 3, with $A \neq 0$. Let t be the angle for a choice of polar coordinates for $(a_0, |\boldsymbol{a}_1|)$. If $\boldsymbol{a}_1 \neq 0$, then a choice of $\log(A + \epsilon B)$ is*

$$\log(|A|) + \frac{t}{|\boldsymbol{a}_1|}(\boldsymbol{a}_1 + \epsilon \boldsymbol{b}_2) + \epsilon \frac{1}{|A|^2} \bar{A}_1 B_1. \tag{16}$$

In the case $\boldsymbol{a}_1 = 0$, and $a_0 \neq 0$, we can assume $t = n\pi$ for some integer n, and $\boldsymbol{b}_2 = 0$. Let $\boldsymbol{p}$ be any 3-vector perpendicular to $\boldsymbol{b}_1$ if $n \neq 0$, and $\boldsymbol{p} = 0$ if $n = 0$. Then a choice of $\log(A + \epsilon B)$ is

$$\log(|a_0|) + n\pi \frac{\boldsymbol{b}_1}{|\boldsymbol{b}_1|} + \epsilon \frac{1}{a_0} B_1 + \epsilon \boldsymbol{p}, \tag{17}$$

where if $\boldsymbol{b}_1 = 0$, we interpret $\boldsymbol{b}_1 / |\boldsymbol{b}_1|$ as any unit 3-vector.

Next we give a formula for powers. For $\alpha \in \mathbb{R}$ and $z \in \mathbb{C} \setminus (-\infty, 0]$, let z^α denote the principle value.

Corollary 7. *With the hypotheses of Theorem 3, if $A \notin (-\infty, 0]$, then we have*

$$(A + \epsilon B)^\alpha = A^\alpha + \epsilon \alpha A^{\alpha-1} B_1 + \epsilon \frac{\text{Im}(A^\alpha)}{\text{Im}(A)} \boldsymbol{b}_2, \tag{18}$$

where if $\text{Im}(A) = 0$, we set $\text{Im}(A^\alpha)/\text{Im}(A)$ to $\alpha A^{\alpha-1}$.

Note that for $\alpha = -1$, it can be shown that this is equivalent to the well known formula

$$(A + \epsilon B)^{-1} = A^{-1} - \epsilon A^{-1} B A^{-1} \quad (A \neq 0). \tag{19}$$

Finally, we reproduce Selig's formula for the Cayley Transform [14].

Corollary 8. *With the hypotheses of Theorem 3, if $A \neq 1$, then we have*

$$\begin{aligned}
\frac{1 + A + \epsilon B}{1 - (A + \epsilon B)} &= \frac{1 + A}{1 - A} + \epsilon \frac{2}{(1-A)^2} \boldsymbol{b}_1 + \epsilon \frac{2}{|1-A|^2} \boldsymbol{b}_2 \\
&= \frac{1 + A}{1 - A} + 2\epsilon \frac{1}{(1-A)} B \frac{1}{(1-A)}.
\end{aligned} \tag{20}$$

Now we start the proof of Theorem 3. As stated above, these first two results come straight from equation (10). Lemma 9 is well known, and is a special case of Lemma 10.

Lemma 9. *Let* $f : \Omega \to \mathbb{R}$ *be a continuously differentiable function, where* Ω *is open in* $\mathbb{R}$. *Then* f *extends to dual numbers* $a + \epsilon b$ *with* $a \in \Omega$ *by*

$$f(a + \epsilon b) = f(a) + \epsilon f'(a)b. \tag{21}$$

Lemma 10. *Let* $f : \Omega \to \mathbb{C}$ *be a valid continuously differentiable function, where* Ω *is open in* $\mathbb{C}$. *Then* f *extends to dual quaternions* $A + \epsilon B$ *where* A *and* B *commute with*

$$f(A + \epsilon B) = f(A) + \epsilon f_x(A)b_0 + \epsilon f_{iy}(A)\boldsymbol{b}_1, \tag{22}$$

where $A = a_0 + \boldsymbol{a}_1$, $B = b_0 + \boldsymbol{b}_1$, *with* a_0 *and* b_0 *being real,* $\boldsymbol{a}_1$ *and* $\boldsymbol{b}_1$ *being 3-vectors, and* $a_0 + i|\boldsymbol{a}_1| \in \Omega$.

Proof. Note that A and B commute if and only if $\boldsymbol{a}_1$ and $\boldsymbol{b}_1$ commute. It is sufficient to prove this in the cases that $f(x+iy) = x^m(iy)^n$, where m and n are non-negative integers, when it follows by an application of the binomial theorem. $\qquad\square$

Lemma 11. *Let* $f : \Omega \to \mathbb{C}$ *be a valid continuously differentiable function, where* Ω *is an open subset of* $i\mathbb{R}$. *Define the continuous real valued function*

$$h(iy) = \begin{cases} \dfrac{f(iy) - f(-iy)}{2iy} & \text{if } y \neq 0 \\ f_{iy}(0) & \text{if } y = 0. \end{cases} \tag{23}$$

Then f *extends to dual quaternions of the form* $\eta = \boldsymbol{a} + \epsilon\boldsymbol{b}_1 + \epsilon\boldsymbol{b}_2$, *where* $\boldsymbol{a}$, $\boldsymbol{b}_1$, *and* $\boldsymbol{b}_2$ *are pure,* $\boldsymbol{b}_1$ *and* $\boldsymbol{a}$ *are parallel,* $\boldsymbol{b}_2$ *and* $\boldsymbol{a}$ *are perpendicular, and* $|\boldsymbol{a}| \in \Omega$, *by the formula*

$$f(\eta) = f(\boldsymbol{a} + \epsilon\boldsymbol{b}_1) + \epsilon h(\boldsymbol{a})\boldsymbol{b}_2, \tag{24}$$

Proof. Without loss of generality, we may assume that f is a polynomial. Define the function

$$g(iy) = \frac{f(iy) + f(-iy)}{2}. \tag{25}$$

Since g and h are even polynomials, we can create polynomials k and l on an open subset of $[-\infty, 0]$ by

$$k(z) = g(\sqrt{z}), \quad l(z) = h(\sqrt{z}). \tag{26}$$

Then we have the identities

$$f(z) = k(z^2) + l(z^2)z, \quad k(z^2) = g(z), \quad l(z^2) = h(z). \tag{27}$$

Let $\alpha = \boldsymbol{a} + \epsilon\boldsymbol{b}_1$, and $\beta = \epsilon\boldsymbol{b}_2$. Use equation (10) to replace z with η in equations (27). Note that $\eta^2 = \alpha^2$. Hence

$$\begin{aligned} f(\eta) &= k(\eta^2) + l(\eta^2)\eta = k(\alpha^2) + l(\alpha^2)\eta \\ &= g(\alpha) + h(\alpha)(\alpha + \beta) = f(\alpha) + h(\alpha)\beta. \end{aligned} \tag{28}$$

Finally, by Lemma 10

$$h(\alpha)\beta = (h(\boldsymbol{a}) + \epsilon h_{iy}(\boldsymbol{a})\boldsymbol{b}_1)\epsilon\boldsymbol{b}_2 = \epsilon h(\boldsymbol{a})\boldsymbol{b}_1, \tag{29}$$

since $\epsilon^2 = 0$. $\qquad\square$

Proof of Theorem 3. Without loss of generality, assume f is a polynomial. Define the functions $u, v : (\Omega - a_0) \cap i\mathbb{R} \to \mathbb{C}$ by

$$u(iy) = f(a_0 + iy), \tag{30}$$
$$v(iy) = f_x(a_0 + iy)b_0, \tag{31}$$

so that by Lemma 9, applied to the real and imaginary parts of $x \mapsto f(x + iy)$, and replacing x by $a_0 + \epsilon b_0$, we have

$$f(a_0 + \epsilon b_0 + iy) = u(iy) + \epsilon v(iy). \tag{32}$$

Applying Lemmas 10 and 11 to u, and setting

$$h_u(iy) = h(a_0 + iy), \tag{33}$$

we obtain

$$\begin{aligned} u(\boldsymbol{a}_1 + \epsilon\boldsymbol{b}_1 + \epsilon\boldsymbol{b}_2) &= u(\boldsymbol{a}_1 + \epsilon\boldsymbol{b}_1) + \epsilon h_u(\boldsymbol{a}_1)\boldsymbol{b}_2 \\ &= u(\boldsymbol{a}_1) + \epsilon u_{iy}(\boldsymbol{a}_1)\boldsymbol{b}_1 + \epsilon h_u(\boldsymbol{a}_0)\boldsymbol{b}_2 \\ &= f(A) + \epsilon f_{iy}(A)\boldsymbol{b}_1 + \epsilon h(A)\boldsymbol{b}_2. \end{aligned} \tag{34}$$

We obtain a simular equation for v, but since $\epsilon^2 = 0$, this simplifies to

$$\epsilon v(\boldsymbol{a}_0 + \epsilon\boldsymbol{b}_0 + \epsilon\boldsymbol{b}_1) = \epsilon v(\boldsymbol{a}_0) = \epsilon f_x(A)b_0. \tag{35}$$

(Here a second derivative of f, which appears when initially applying Lemma 10 to v, disappears in the final result.) The result follows by combining the last two equations. $\qquad\square$

During the writing of this paper, we came across a different approach to proving Theorem 3 motivated by Chasles' Theorem and the approach taken in [16]. It can be shown that any dual quaternion may be factored as

$$a + \boldsymbol{a}_1 + \epsilon(b + \boldsymbol{b}_1) = (1 + \epsilon\boldsymbol{r})(a + \boldsymbol{a}_1 + \epsilon\tilde{B})(1 - \epsilon\boldsymbol{r}), \tag{36}$$

where $\tilde{B}$ commutes with $\boldsymbol{a}_1$, and

$$\boldsymbol{r} = \begin{cases} \frac{\boldsymbol{b}_1\boldsymbol{a}_1 - \boldsymbol{a}_1\boldsymbol{b}_1}{4|\boldsymbol{a}_1|^2} & \text{if } \boldsymbol{a}_1 \neq 0 \\ 0 & \text{if } \boldsymbol{a}_1 = 0 \end{cases} \tag{37}$$

is a 3-vector. (A similar formula is also in [14]). Also, if $f : \Omega \to \mathbb{C}$ is a valid polynomial, then

$$f\big((1 + \epsilon \boldsymbol{r})(A + \epsilon B)(1 - \epsilon \boldsymbol{r})\big) = (1 + \epsilon \boldsymbol{r})f(A + \epsilon B)(1 - \epsilon \boldsymbol{r}). \qquad (38)$$

In this way, the computation is reduced to Lemma 10.

Finally, we note that the methods of the proof of Lemma 11 give a cute, albeit simple, result for the Dunford-Riesz functional calculus of analytic functions on a complex Banach algebra with unity $\mathcal{A}$. Recall from [4], that given $A \in \mathcal{A}$, and Ω an open subset of $\mathbb{C}$ containing the spectrum of A, that the Dunford-Riesz functional calculus maps analytic functions $f : \Omega \to \mathbb{C}$ to $f(A) \in \mathcal{A}$, and satisifes

(1) $f(z) = 1$ maps to the unit of $\mathcal{A}$, $f(z) = z$ maps to A, $f + g$ maps to $f(A) + g(A)$, fg maps to $f(A)g(A)$, and af maps to $af(A)$.

(2) if $f_n \to f$ uniformly on the spectrum of A, then $f_n(A) \to f(A)$ in norm.

Proposition 12. *Suppose* $f : \Omega \to \mathbb{C}$ *is an analytic function on an open subset* Ω *of* $\mathbb{C}$ *satisfying* $z \in \Omega \Leftrightarrow -z \in \Omega$. *Define the analytic functions on* $\{z^2 : z \in \Omega\}$ *by*

$$k(z) = \frac{f(\sqrt{z}) + f(-\sqrt{z})}{2}, \qquad (39)$$

$$l(z) = \begin{cases} \dfrac{f(\sqrt{z}) - f(-\sqrt{z})}{2\sqrt{z}} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0, \end{cases} \qquad (40)$$

$$\qquad (41)$$

If A *and* B *are anti-commuting elements of a complex algebra such that their spectra* $\sigma(A) \cup \sigma(B) \cup \sigma(A + B) \subset \Omega$, *then*

$$f(A + B) = k(A^2 + B^2) + (A + B)l(A^2 + B^2). \qquad (42)$$

Applying this to $(A + B)^n$, where n is a non-negative integer, gives the coefficients of the so called Pauli-Pascal triangle [6, 15], that is, if A and B anti-commute, and n is a non-negative integer, then

$$(A + B)^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} A^{n-k} B^k, \qquad (43)$$

where we define for non-negative integers n and k:

$$\begin{Bmatrix} 2n \\ 2k \end{Bmatrix} = \begin{Bmatrix} 2n + 1 \\ 2k \end{Bmatrix} = \begin{Bmatrix} 2n + 1 \\ 2k + 1 \end{Bmatrix} = \binom{n}{k}, \qquad \begin{Bmatrix} 2n \\ 2k + 1 \end{Bmatrix} = 0. \qquad (44)$$

Statements and Declarations

The author has no relevant financial or non-financial interests to disclose. The author has no competing interests to declare that are relevant to the content of this article. The authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript, except that this is related to consulting work performed by the author for the NASA Johnson Space Center, Houston, Texas, USA. The author has no financial or proprietary interests in any material discussed in this article.

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